MEAN MOTIONS AND ALMOST PERIODIC FUNCTIONS*

BY

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Introduction. A continuous function $F(t)=U(t)+iV(t)$, $(-\infty<t<+\infty)$, is said to possess a mean motion μ if it has a representation of the form

$$F(t) = r(t) \exp 2\pi i\phi(t), \qquad -\infty < t < +\infty, \tag{1}$$

such that $r(t)$, $\phi(t)$ are real-valued continuous functions and

$$\phi(t)/t \to \mu(\phi(t) = \mu t + o(t)) \qquad t \to \infty. \tag{2}$$

The problem of the existence and determination of this constant μ for functions $F(t)$ of the type

$$F(t) = \sum_{k=1}^{n} a_k \exp 2\pi i(\Lambda_k t + \alpha_k), \tag{3}$$

where Λ_k, α_k are real and $a_k>0$, goes back to Lagrange's approximative treatment of the secular perturbations of the major planets. The earliest result in this direction is that if the amplitudes a_k satisfy Lagrange's relation, that is, if for some j,

$$a_j > a_1 + \cdots + a_{j-1} + a_{j+1} + \cdots + a_n, \tag{4}$$

so that

$$|F(t)| > c, \qquad -\infty < t < +\infty, \tag{5}$$

for a constant $c>0$, then the mean motion μ exists and

$$\mu = \Lambda_j. \tag{6}$$

Bohl [1] has proved the existence of μ if $n=3$; Weyl [12] has treated the case $n=4$ when the frequencies $\Lambda_1, \cdots, \Lambda_n$ are linearly independent. In the case of a general n, it has been shown (Hartman, van Kampen, and Wintner [5]) that if the numbers $a_1, \cdots, a_n$ do not satisfy a relation of the type

$$\sum_{k=1}^{n} e_k a_k = 0, \qquad e_k = \pm 1,$$

and if the frequencies $\Lambda_1, \cdots, \Lambda_n$ and the amplitudes $a_1, \cdots, a_n$ are fixed,

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then the mean motion μ exists whenever the phases $\alpha_1, \cdots, \alpha_n$ do not belong to a certain zero set (which may be empty) in the $(\alpha_1, \cdots, \alpha_n)$-space. Actually, this was stated explicitly only in the case that $\Lambda_1, \cdots, \Lambda_n$ are linearly independent, but it is clear from the proof that this restriction is unnecessary. It was also shown that if $(\alpha_1, \cdots, \alpha_n)$ does not belong to the exceptional set and if the frequencies are linearly independent, then the mean motion μ possesses an explicit integral representation. More recently, Weyl [13] has shown that if the frequencies are linearly independent, then the exceptional zero set is empty.

It is known* that if $F(t)$ is an arbitrary almost periodic function† satisfying the condition (5), then $\phi(t)=\mu t+\omega(t)$, where $\omega(t)$ is almost periodic. Also (Hartman and Wintner [7]), in this case the mean motion possesses an explicit integral representation.

Let

$$(7) \qquad f(s) = f(\sigma + it) = u(\sigma, t) + iv(\sigma, t), \qquad \alpha < \sigma < \beta; -\infty < t < +\infty,$$

be an analytic almost periodic function in the strip $\alpha<\sigma<\beta$. In this paper the mean motions of the functions

$$(8) \qquad F_\sigma(t) = f(\sigma + it)$$

will be investigated. The method will be that of considering σ as a varying parameter, so that a given $F(t)$ is thought of as embedded into a sheaf of functions (8) depending on σ.

According to Jessen [9], there is associated with every function (7) a Jensen function $\psi(\sigma)$, $(\alpha<\sigma<\beta)$, such that $\psi(\sigma)$ is convex and if $\psi(\sigma)$ is differentiable at $\sigma=\alpha'$, $\sigma=\beta'$ for $\alpha<\alpha'<\beta'<\beta$, then the frequency $H(\alpha', \beta')$ of the zeros of $f(s)$ in the strip $\alpha'<\sigma<\beta'$ exists and $2\pi H(\alpha', \beta')=\psi'(\beta')-\psi'(\alpha')$, where $\psi'=d\psi/d\sigma$. In §1, it will be proved that the mean motion $\mu(\sigma)$ exists for every σ, $(\alpha<\sigma<\beta)$, at which $\psi'(\sigma)$ exists, and $2\pi\mu(\sigma)=\psi'(\sigma)$. Since $\psi(\sigma)$ is convex, it has a derivative at every point σ with the possible exception of a denumerable set. The connection between the mean motion and the derivative of the Jensen function is established by an adaptation of the methods used by Jessen [9] to prove the existence and the properties of the Jensen function. This connection, when combined with simple examples of limit periodic functions mentioned by Jessen [9], show that on the one hand $\mu(\sigma)$ may exist even though $\psi'(\sigma)$ does not, while on the other hand $\mu(\sigma)$ need not exist for all σ.

A criterion is obtained in §2 for the existence of $\mu(\sigma)$ for all σ in the inter-

* This result was conjectured by Wintner and proved by Bohr [3].

† Throughout this paper, almost periodicity will be meant in the sense of Bohr.

val $\alpha<\sigma<\beta$. The criterion, in the case of an arbitrary function (7), is obtained by a generalization of the methods used by Jessen [10] in the study of zeros of those functions (7) having linearly independent Fourier exponents. However, this general criterion takes a simpler form for a large class of analytic almost periodic functions. An application of this simplified criterion shows that if (7) is a trigonometric polynomial

$$f(s) = \sum_{k=1}^{n} a_k \exp 2\pi(\Lambda_k s + i\alpha_k), \tag{9}$$

then all of the corresponding functions (8), with the possible exception of a finite set of σ, possess mean motions $\mu(\sigma)$ (§3). The question whether the finite set of exceptional σ is necessarily empty will remain open. It will be shown, however, that if the polynomial (9) has a decomposition

$$f(s) = f_1(s) + f_2(s) \tag{10}$$

into the sum of two polynomials which are not both periodic and whose frequencies are contained in linearly independent moduli, then $\mu(\sigma)$ exists for all σ.

In §4 the smoothness of the function μ of σ is discussed. It is shown, in particular, that in the case of a trigonometric polynomial (9) with linearly independent frequencies, $\mu(\sigma)$ is a regular analytic function at every point σ for which there is no relation of the type

$$\sum_{k=1}^{n} e_k a_k \cdot \exp 2\pi\Lambda_k\sigma = 0, \qquad e_k = \pm 1,$$

while $\mu(\sigma)$ possesses p continuous derivatives for $-\infty<\sigma<+\infty$, if $n\geqq 3+2p$.

In §5 the methods are extended so as to apply to the Riemann ζ-function for $1/2<\sigma\leqq 1$. As is to be expected, $\mu(\sigma)$ exists for all $\sigma>1/2$ and $\mu(\sigma)\equiv 0$.

1. **Mean motions and the Jensen function.** In the sequel, it will be supposed that $f(s)\equiv f(\sigma+it)\not\equiv 0$ is a regular almost periodic function in the strip $\alpha<\sigma<\beta$, where $-\infty\leqq\alpha<\beta\leqq+\infty$. It will first be shown that every function (8) can be represented in the form (1), where the corresponding function $\phi(t)$ is an analytic function of the real variable t.

LEMMA 1. *For every σ in $\alpha<\sigma<\beta$ there exists a unique function $\phi_\sigma(t)$ satisfying the conditions*:

(i) *$\phi_\sigma(t)$ is a regular analytic function of the real variable t for $-\infty<t<+\infty$.*

(ii) $2\pi\phi_\sigma(t)\equiv\arg F_\sigma(t) \pmod{\pi}$ *for* $-\infty<t<+\infty$.

(iii) $0\leqq\phi_\sigma(0)<1/2$.

This lemma has been proved by Bohl [1] for the case of polynomials (9) when the word "continuous" replaces "regular analytic" in (i). This proof, however, is valid for the case of an arbitrary regular function (7). In order to prove (i) itself, consider (where R is the real part)

$$d\phi_\sigma(t)/dt = (1/2\pi)R\{d \log f(\sigma + it)/ds\}, \tag{11}$$

if $F_\sigma(t) = f(\sigma+it) \neq 0$, where $\log f(s)$ is any branch of the logarithm of $f(s)$. Elementary considerations show that the function on the right-hand side of (11) is a continuous, even a regular analytic function for all t, including those t for which $F_\sigma(t) = 0$. From this the analyticity of $d\phi_\sigma(t)/dt$ and, consequently, the analyticity of $\phi_\sigma(t)$ are easily deduced.

Now, according to Jessen [9], the function

$$\psi(\sigma; T) = T^{-1}\int_0^T \log |F_\sigma(t)|\, dt = T^{-1}\int_0^T \log |f(\sigma + it)|\, dt, \tag{12}$$

for $\alpha < \sigma < \beta$, $0 < T < \infty$, exists and is a continuous function of σ. The functions (12) tend uniformly to a limit function $\psi(\sigma)$ in every closed subinterval of $\alpha < \sigma < \beta$ as $T \to \infty$,

$$\psi(\sigma) = \lim_{T\to\infty} \psi(\sigma; T). \tag{13}$$

The limit function $\psi(\sigma)$, which is called the Jensen function associated with $f(s)$, is convex and has the following property: If $N(\alpha', \beta'; T)$, where $\alpha < \alpha' < \beta' < \beta$, denotes the number of zeros of $f(s)$ in the rectangle $\alpha' < \sigma < \beta'$, $0 < t < T$, and if $\psi(\sigma)$ is differentiable at $\sigma = \alpha'$ and $\sigma = \beta'$, then the limit

$$\lim_{T\to\infty} N(\alpha', \beta'; T)/T = H(\alpha', \beta') \tag{14}$$

exists and

$$2\pi H(\alpha', \beta') = \psi'(\beta') - \psi'(\alpha'), \tag{15}$$

where $\psi' = d\psi/d\sigma$. The number $H(\alpha', \beta')$ is called the frequency of the zeros of $f(s)$ in the strip $\alpha' < \sigma < \beta'$. Since a convex function is not differentiable at most on an enumerable set of points, the relation (15) holds with the possible exception of a countable set of α', β' in the interval $\alpha < \sigma < \beta$.

It will be shown that these facts concerning Jensen's function can be transformed into corresponding facts concerning mean motions as follows:

THEOREM I. *If $f(\sigma+it)$ is a regular almost periodic function in the strip $\alpha < \sigma < \beta$, where $-\infty \leqq \alpha < \beta \leqq +\infty$, then for every σ at which $\psi(\sigma)$ has a derivative, the function*

$$F_\sigma(t) = f(\sigma + it)$$

possesses a mean motion $\mu(\sigma)$ *and*

$$\mu(\sigma) = (1/2\pi)\psi'(\sigma). \tag{16}$$

Proof.† Let $\alpha<\alpha_1<\alpha'<\beta'<\beta_1<\beta$, and $F_{\alpha'}(t)\neq 0$, $F_{\beta'}(t)\neq 0$ for $-\infty<t<+\infty$. Since $f(s)$ has only a countable set of zeros in the strip $\alpha<\sigma<\beta$, it is clear that $F_\sigma(t)=f(\sigma+it)\neq 0$, $(-\infty<t<+\infty)$, with the possible exception of an enumerable set of σ. Let $t=0$ and $t=T$ be such that

$$|f(\sigma)|>c>0, \qquad |f(\sigma+iT)|>c>0, \qquad \alpha'\leqq\sigma\leqq\beta'. \tag{17}$$

The almost periodicity of $f(s)$ implies that there exists a number $\tau>0$ such that in every t-interval $[t^*, t^*+\tau]$, $(-\infty<t^*<+\infty)$, of length τ, there exist values of $t=T$ satisfying (17). Since $f(s)\neq 0$ on the boundary of the rectangle $\alpha'\leqq\sigma\leqq\beta'$, $0\leqq t\leqq T$, one has

$$2\pi N(\alpha', \beta'; T) = \int_0^T \frac{d\log f(\beta'+it)}{ds}\,dt - \int_0^T \frac{d\log f(\alpha'+it)}{ds}\,dt$$
$$+ i\int_{\alpha'}^{\beta'} \frac{d\log f(\sigma+iT)}{ds}\,d\sigma - i\int_{\alpha'}^{\beta'} \frac{d\log f(\sigma)}{ds}\,d\sigma.$$

It follows that if $\log f(s)$ denotes any fixed branch of the logarithm of $f(s)$ in the neighborhood of the line segments $\sigma=\alpha'$, $(0\leqq t\leqq T)$, and $\sigma=\beta'$, $(0\leqq t\leqq T)$, then

$$2\pi N(\alpha', \beta'; T)/T = d\Phi(\beta'; T)/d\sigma - d\Phi(\alpha'; T)/d\sigma + \mathrm{P}(\alpha', \beta'; T), \tag{18}$$

where

$$\Phi(\sigma; T) = T^{-1}\int_0^T \log f(\sigma+it)dt, \tag{19}$$

and P denotes a remainder term such that

$$|\mathrm{P}(\alpha', \beta'; T)| \leqq \frac{2C}{Tc}(\beta'-\alpha'), \tag{20}$$

if C denotes the upper bound of $|f'(s)|$ in $\alpha'\leqq\sigma\leqq\beta'$.

By the Cauchy-Riemann differential equations and (11), one has in the neighborhood of the lines $\sigma=\alpha'$, $(0\leqq t\leqq T)$, and $\sigma=\beta'$, $(0\leqq t\leqq T)$,

$$\frac{1}{2\pi}\,d\log|f(\sigma+it)|/d\sigma = (1/2\pi)R\{d\log f(\sigma+it)/ds\} = d\phi_\sigma(t)/dt. \tag{21}$$

Thus, by (12) and (21)

† The first part of this proof is a modification of Jessen's proof of (13), (14), and (15); Jessen [9].

$$\frac{1}{2\pi} T d\psi(\alpha'; T)/d\sigma = \phi_{\alpha'}(T) - \phi_{\alpha'}(0). \tag{22}$$

A similar relation holds if α' is replaced by β'.

The function $\Psi(\sigma; T)$ defined by

$$\Psi(\sigma; T) = \psi(\sigma; T) + \frac{C}{Tc}\sigma^2, \qquad \alpha_1 < \sigma < \beta_1, \tag{23}$$

possesses a continuous derivative with respect to σ in a neighborhood of $\sigma = \alpha'$ and $\sigma = \beta'$. Now the real part of the function (19) is $\psi(\sigma; T)$, so that by (18), (20), and (23),

$$2\pi N(\alpha', \beta'; T)/T = d\Psi(\beta'; T)/d\sigma - d\Psi(\alpha'; T)/d\sigma + \rho(\alpha', \beta'; T), \tag{24}$$

where

$$0 \geqq \rho(\alpha', \beta'; T) \geqq -\frac{4C}{Tc}(\beta' - \alpha'). \tag{25}$$

It follows from (24) and (25) that

$$d\Psi(\beta'; T)/d\sigma \geqq d\Psi(\alpha'; T)/d\sigma.$$

This inequality is clearly valid for all points α', β' in the neighborhood of which $\Psi(\sigma; T)$ has a continuous derivative. Thus, for a fixed T satisfying (17), $\Psi(\sigma; T)$ is convex for $\alpha_1 < \sigma < \beta_1$. In virtue of (13) and (23), one has, uniformly in any closed subinterval of $\alpha_1 < \sigma < \beta_1$,

$$\Psi(\sigma; T) \to \psi(\sigma), \qquad T \to \infty.$$

Since $\psi(\sigma)$ is convex,

$$\lim_{T\to\infty} d\Psi(\alpha'; T)/d\sigma = \psi'(\alpha'), \qquad \lim_{T\to\infty} d\Psi(\beta'; T)/d\sigma = \psi'(\beta') \tag{26}$$

if $\psi'(\alpha')$ and $\psi'(\beta')$ exist and if T satisfies the condition (17).

On the other hand (22), (23), (26) imply that if T satisfies (17), then

$$\lim_{T\to\infty} \phi_{\alpha'}(T)/T = (1/2\pi)\psi'(\alpha'), \qquad \lim_{T\to\infty} \phi_{\beta'}(T)/T = (1/2\pi)\psi'(\beta'). \tag{27}$$

In virtue of the remark following (17), to show that (16) holds for $\sigma = \alpha'$ and $\sigma = \beta'$, it is sufficient to prove that there exists a constant M such that

$$|\phi_\sigma(t) - \phi_\sigma(t')| < M, \qquad |t - t'| \leqq \tau. \tag{28}$$

Let $F_\sigma(t) = U_\sigma(t) + iV_\sigma(t)$. It is seen from the geometrical relation of $\phi_\sigma(t)$ to the curve $x = U_\sigma(t)$, $y = V_\sigma(t)$ that if t', t'' are any two points in an interval in which $F_\sigma(t) \neq 0$, then a necessary condition for

$$|\phi_\sigma(t') - \phi_\sigma(t'')| \geqq 1/2$$

is that both $U_\sigma(t)$ and $V_\sigma(t)$ vanish in $t' \leqq t \leqq t''$. Since there exists† an integer N such that the number of zeros of $f(s)$ in any rectangle $\alpha' \leqq \sigma \leqq \beta'$, $t^* \leqq t \leqq t^* + \tau$, for $-\infty < t^* < +\infty$, does not exceed N, the statement (28) follows by an application of the following lemma to the set of functions $z(s) = f(s+it^*)$, $(-\infty < t^* < +\infty)$:

LEMMA 2. *Let Q be an open set containing the closed rectangle S: $\alpha_1 \leqq \sigma \leqq \beta_1$, $0 \leqq t \leqq \tau$, and let $\alpha_1 \leqq \sigma_0 \leqq \beta_1$. For every set Σ of functions $z(s) = x(\sigma, t) + iy(\sigma, t)$ which are regular and uniformly bounded in any closed subset of Q and which do not possess the function $z(s) \equiv 0$ as a limit function, there exists an integer K such that the number of zeros of either $x(\sigma_0, t)$ or $y(\sigma_0, t)$ on the interval $0 \leqq t \leqq \tau$ does not exceed K.*

This lemma is an immediate consequence of well known properties of normal families.

This completes the proof that if $\psi'(\alpha')$ exists and that if $F_{\alpha'}(t) \neq 0$ for all t, then $\mu(\alpha')$ exists and $2\pi\mu(\alpha') = \psi'(\alpha')$. Actually, the condition $F_{\alpha'}(t) \neq 0$ is not needed. The existence of $\psi'(\alpha')$ implies that $n(\alpha'; T)/T \to 0$, $T \to \infty$, where $n(\alpha'; T)$ is the number of zeros of $F_{\alpha'}(t)$ in the interval $0 < t < T$. Suppose that $F_{\alpha'}(t)$ has a zero of order k at $t = t_0$ and that $\eta > 0$ is such that $f(s)$ has no other zeros in $|s - (\alpha' + it_0)| \leqq \eta$. Then

$$d \log f(s)/ds = k/[s - (\alpha' + it_0)] + g(s),$$

where $g(s)$ is regular in $|s - (\alpha' + it_0)| \leqq \eta$. Thus,‡

$$R\{d \log f(\alpha' + it)/ds\} = R\{g(\alpha' + it)\},$$

so that integration from $\alpha' + i(t_0 - \eta)$ to $\alpha' + i(t_0 + \eta)$ along a semicircle $(\sigma \geqq \alpha')$ gives

$$R\left\{-i \int [d \log f(s)/ds] ds\right\} = \phi_{\alpha'}(t_0 + \eta) - \phi_{\alpha'}(t_0 - \eta) + k\pi. \tag{29}$$

Denoting by $L(\alpha'; T)$ the real part of $\{i \int [d \log f(s)/ds] ds\}$ where the integral extends from $(\alpha' + i0)$ to $(\alpha' + iT)$ along a path consisting of segments of the line $\sigma = \alpha'$ and semicircles $(\sigma \geqq \alpha')$ in which there are no zeros other than those on the line $\sigma = \alpha'$, it follows from (15) and the differentiability of $\psi(\sigma)$ at $\sigma = \alpha'$ that

$$L(\alpha'; T)/T \to \psi'(\alpha'), \qquad T \to \infty.$$

† Cf. Bohr and Jessen [4]; or Jessen [9, Lemma 1].

‡ Cf. Lemma 1.

In virtue of (29), (11), and the fact that $n(\alpha'; T)/T \to 0$, $T \to \infty$, the mean motion $\mu(\alpha')$ exists and is equal to $\psi'(\alpha')/2\pi$.

2. **A criterion for the existence of $\mu(\sigma)$ for every σ.** It is clear from the proof of Theorem I that if $N(\alpha', \beta'; T)/T$, $n(\alpha'; T)/T$, $n(\beta'; T)/T$ each has a limit as $T \to \infty$ and if $\mu(\beta')$ exists, then $\mu(\alpha')$ exists. If, in particular, $N(\alpha', \beta'; T)$ has a limit for every α', β', $\alpha < \alpha' < \beta' < \beta$, and if $n(\sigma; T)/T \to 0$, $T \to \infty$, for every σ, then $\mu(\sigma)$ exists for all σ, $(\alpha < \sigma < \beta)$, and the frequency (14) satisfies

$$H(\alpha', \beta') = \mu(\beta') - \mu(\alpha') \tag{30}$$

for every α', β'. In order to investigate under what conditions there are no exceptional α', β', it is convenient to consider the function $Z(\theta_1, \theta_2, \cdots; \sigma)$ defined for every σ, $(\alpha < \sigma < \beta)$, on a finite or infinite dimensional θ-torus Θ and associated with $F_\sigma(t)$ in the usual manner (Bohr [2]). Consider those functions $f(s)$ for which there exists a finite or infinite sequence of real, linearly independent numbers $\lambda_1, \lambda_2, \cdots$ and, correspondingly, Θ represents a finite or infinite dimensional torus† on which the continuous function $Z(\theta_1, \theta_2, \cdots; \sigma)$ is defined for each $F_\sigma(t)$ such that

$$F_\sigma(t) = Z(\lambda_1 t, \lambda_2 t, \cdots; \sigma), \qquad -\infty < t < +\infty, \tag{31}$$

where the numbers $\lambda_i t$ in (31) are reduced modulo 1. This restriction on $f(s)$ excludes, for example, limit periodic functions for which $\mu(\sigma)$ need not exist‡ for all σ.

Suppose that, for every point $(\theta_1, \theta_2, \cdots)$ of the torus Θ, the continuous function $Z(\theta_1, \theta_2, \cdots; \sigma)$ is a regular analytic function of the real variable σ. Thus, the torus function Z is still defined if σ is replaced by the complex variable $s = \sigma + it$. Suppose further that the relation

$$\begin{aligned} Z(\theta_1, \theta_2, \cdots; \sigma + it) &= Z(\theta_1 + \lambda_1 t, \theta_2 + \lambda_2 t, \cdots; \sigma), \\ &\qquad \alpha < \sigma < \beta; -\infty < t < +\infty, \end{aligned} \tag{32}$$

is satisfied.

Let $\nu(\theta_1, \theta_2, \cdots; \alpha', \beta')$ denote the number of zeros of $Z(\theta_1, \theta_2, \cdots; s)$ in the rectangle

$$S: \quad \alpha' < \sigma < \beta', \qquad 0 < t < 1. \tag{33}$$

Then $\nu(\theta_1, \theta_2, \cdots; \alpha', \beta')$ is a bounded function on Θ. For otherwise there

† For a theory of limits, measure, and integration on the infinite dimensional torus, see Jessen [10].

‡ Cf. Jessen [9, example 1]. In view of the connection between mean motions and the distribution of zeros, it is easily seen that this function is such that $\mu(0)$ does not exist.

would exist a sequence of points $\{(\theta_1^n, \theta_2^n, \cdots)\}$ such that

$$(\theta_1^n, \theta_2^n, \cdots) \to (\theta_1^*, \theta_2^*, \cdots), \qquad \nu(\theta_1^n, \theta_2^n, \cdots; \alpha', \beta') \to \infty, \qquad n \to \infty.$$

This would imply that $Z(\theta_1^*, \theta_2^*, \cdots; s) \equiv 0$, since for reasons of continuity $Z(\theta_1^n, \theta_2^n, \cdots; s) \to Z(\theta_1^*, \theta_2^*, \cdots; s)$ holds as $n \to \infty$ uniformly in S. This is impossible unless $f(s) \equiv 0$. Also, from the continuity of the function Z, it follows by Rouché's theorem that if $Z(\theta_1^0, \theta_2^0, \cdots; s)$ has no zeros on the boundary of (33), then $\nu(\theta_1, \theta_2, \cdots; \alpha', \beta') = \nu(\theta_1^0, \theta_2^0, \cdots; \alpha', \beta')$ for all points in a sufficiently small vicinity of $(\theta_1^0, \theta_2^0, \cdots)$ on the torus Θ. Thus, the discontinuity points of ν are among those points $(\theta_1, \theta_2, \cdots)$ for which $Z(\theta_1, \theta_2, \cdots; s)$ vanishes on the boundary of S.† To insure that the set of discontinuity points of $\nu(\theta_1, \theta_2, \cdots; \alpha', \beta')$ is a zero set on Θ assume the following:

(A) *The set of all points* $(\theta_1, \theta_2, \cdots)$ *of* Θ *satisfying either*

$$\text{(A, i)} \qquad Z(\theta_1, \theta_2, \cdots; \sigma) = 0$$

for some σ, $\alpha' < \sigma < \beta'$, *or at least one of the two relations*

$$\text{(A, ii)} \qquad Z(\theta_1, \theta_2, \cdots; \alpha' + it) = 0, \qquad Z(\theta_1, \theta_2, \cdots; \beta' + it) = 0,$$

for some t, $0 \leqq t \leqq 1$, *is a zero set.* (*A condition on* $Z(\theta_1, \theta_2, \cdots; \sigma + 1i)$ *similar to* (A, i) *is unnecessary in virtue of* (32).)

Thus, under the condition (A), $\nu(\theta_1, \theta_2, \cdots; \alpha', \beta')$ is Riemann integrable over Θ, so that, by the Kronecker-Weyl approximation theorem,

$$\text{(34)} \qquad \lim_{T\to\infty} T^{-1} \int_0^T \nu(\lambda_1 t, \lambda_2 t, \cdots; \alpha', \beta') dt = \int_\Theta \nu(\theta_1, \theta_2, \cdots; \alpha', \beta') d\Theta.$$

Since $\nu(\lambda_1 t^*, \lambda_2 t^*, \cdots; \alpha', \beta')$ is, by (31), (32), and the definition of ν, the number of zeros of $f(s)$ in the rectangle $\alpha' < \sigma < \beta'$, $t^* < t < t^* + 1$, it is clear from (34) that

$$\text{(35)} \qquad \lim_{T\to\infty} N(\alpha', \beta'; T)/T = \int_\Theta \nu(\theta_1, \theta_2, \cdots; \alpha', \beta') d\Theta.$$

By a slight modification of this argument, it follows from the condition (A, ii) that

$$n(\alpha'; T)/T \to 0, \qquad n(\beta'; T)/T \to 0 \quad \text{as} \quad T \to \infty.$$

Thus, if condition (A) is satisfied for all α', β', $\alpha < \alpha' < \beta' < \beta$, the mean motion $\mu(\sigma)$ exists for all σ and satisfies (30).

† The above arguments are used by Jessen [10, §27], in discussing the zeros of functions $f(s)$ with linearly independent Fourier exponents.

This criterion for the existence of $\mu(\sigma)$ can be transformed into a slightly different form if Θ is a finite dimensional torus. Let Z be a function on a finite dimensional torus Θ, say of dimension $m>1$. Suppose further that $Z(\theta_1, \theta_2, \cdots, \theta_m; \sigma)$ is a regular analytic function of its $m+1$ arguments, in addition to satisfying (32). A necessary and sufficient condition for the conditions (A) to be satisfied for all α', β' is the following:

(B) *There does not exist an* $(m-1)$*-dimensional manifold on* Θ *on which*

$$X(\theta_1, \cdots, \theta_m; \sigma) + iY(\theta_1, \cdots, \theta_m; \sigma) \equiv Z(\theta_1, \cdots, \theta_m; \sigma) = 0 \tag{36}$$

for some σ, $(\alpha<\sigma<\beta)$.

In order to see this, note that, in virtue of the analyticity of Z in all variables together, the sets involved in (A) are a finite set of manifolds (with possible singularities); so that a necessary and sufficient condition for them to be zero sets is that their dimension numbers be less than m. Now, under the condition (B), the set of points $(\theta_1, \cdots, \theta_m; \sigma)$ satisfying (A, i) are manifolds in the $(\theta_1, \cdots, \theta_m; \sigma)$-space with dimension numbers not exceeding $(m-1)$. It follows that the projection of this set on the $(\theta_1, \cdots, \theta_m)$-space Θ is a set of manifolds with dimension numbers not exceeding $(m-1)$, so that it is certainly a zero set. Similar arguments, using (32), show that (B) is necessary as well as sufficient for the condition (A, ii) to be satisfied for all α', β'.

3. **Trigonometric polynomials.** As an application of the above criterion for the existence of $\mu(\sigma)$ for all σ in an interval, consider a general trigonometric polynomial

$$f(s) = \sum_{k=1}^{n} a_k \exp 2\pi(\Lambda_k s + i\alpha_k), \tag{37}$$

where Λ_k, α_k are real and $a_k>0$. It may be supposed that $f(\sigma+it)$ is not a periodic function of t, for this case is trivial. Thus there exist m (greater than 1) linearly independent numbers $\lambda_1, \cdots, \lambda_m$ such that

$$\Lambda_k = \sum_{j=1}^{m} n_{kj}\lambda_j, \qquad k = 1, \cdots, n, \tag{38}$$

where the n_{kj} are, for $k=1, \cdots, n$ and $j=1, \cdots, m$, integers and the matrix (n_{kj}) is of rank m (less than or equal to n). Thus

$$\begin{aligned} X + iY \equiv Z &= Z(\theta_1, \cdots, \theta_m) \\ &= \sum_{k=1}^{n} a_k \exp 2\pi\left[\Lambda_k\sigma + i\left(\sum_{j=1}^{m} n_{kj}\theta_j + \alpha_k\right)\right]. \end{aligned} \tag{39}$$

Since $\nu(\theta_1, \theta_2, \cdots; \alpha, \beta)$ is uniformly bounded on Θ for any α, β,

there exists an integer N such that the $N+1$ functions $Z(\theta_1, \cdots, \theta_m; \sigma)$, $\partial Z/\partial\sigma, \cdots, \partial^N Z/\partial\sigma$ do not vanish simultaneously for $0 \leqq \theta_j < 1$, $\alpha \leqq \sigma \leqq \beta$. If one introduces the jacobian

$$\frac{\partial(X, Y)}{\partial(\sigma, \theta_l)} = 4\pi^2 \sum_{k=1}^{n} \sum_{p=1}^{n} a_k a_p \Lambda_k n_{pl} \cdot \exp 2\pi\sigma(\Lambda_k + \Lambda_p) \cdot \cos\left[\sum_{j=1}^{m} (n_{kj} - n_{pj})\theta_j + (\alpha_k - \alpha_p)\right], \tag{40}$$

(38), (39), and (40) show that

$$J_0 \equiv \sum_{l=1}^{m} \lambda_l \frac{\partial(X, Y)}{\partial(\sigma, \theta_l)} = \left|\frac{\partial Z}{\partial \sigma}\right|^2 = \left(\frac{\partial X}{\partial \sigma}\right)^2 + \left(\frac{\partial Y}{\partial \sigma}\right)^2. \tag{41_0}$$

Similarly, if one places $X_p = \partial^p X/\partial\sigma^p$, $Y_p = \partial^p Y/\partial\sigma^p$, then

$$J_p \equiv \sum_{l=1}^{m} \lambda_l \frac{\partial(X_p, Y_p)}{\partial(\sigma, \theta_l)} = \left|\frac{\partial^{p+1} Z}{\partial \sigma^{p+1}}\right|^2 = X_{p+1}^2 + Y_{p+1}^2, \qquad p = 0, 1, \cdots. \tag{41_p}$$

If $\alpha \leqq \sigma \leqq \beta$, the functions (39), $(41_0), \cdots, (41_{N-1})$ do not vanish simultaneously, so that the set of points $(\theta_1, \cdots, \theta_m; \sigma)$, $\alpha \leqq \sigma \leqq \beta$, at which (39) vanishes is a finite set of disjoint, connected, analytic manifolds, whose dimension numbers do not exceed $(m-1)$. It follows that there are in the interval $\alpha \leqq \sigma \leqq \beta$ at most a finite number of values σ_0 such that the intersection of these manifolds and the hyperplane $\sigma = \sigma_0$ contains a manifold with a dimension number greater than $(m-2)$. Hence, there is at most a finite number of such exceptional hyperplanes, $-\infty < \sigma < +\infty$, since α, β are arbitrary and the function (39) does not vanish if $|\sigma|$ is sufficiently large; for if the number Λ is chosen so that some of the numbers $\Lambda + \Lambda_1, \cdots, \Lambda + \Lambda_n$ are positive and some negative, then $|\exp 2\pi\Lambda\sigma \cdot Z(\theta_1, \cdots, \theta_m; \sigma)| \to \infty$ as $|\sigma| \to \infty$ uniformly in $(\theta_1, \cdots, \theta_m)$.

For arbitrary functions Z of the type (39), this statement is the most general; for example, the hyperplane $\sigma = 0$ is exceptional for $Z(\theta_1, \theta_2; \sigma) = \exp 2\pi(\lambda_1\sigma + i\theta_1) + \exp 2\pi(\lambda_2\sigma + i\theta_2)$; also, trigonometric polynomials of the type

$$f(s) = \prod_{j=1}^{k} (a_{1j} \exp 2\pi\Lambda_{1j}s + a_{2j} \exp 2\pi\Lambda_{2j}s)$$

for properly chosen $a_{1j}, a_{2j}, \Lambda_{1j}, \Lambda_{2j}$ lead to torus functions Z having k exceptional values of σ associated with them.

It follows from the previous section that the mean motion $\mu(\sigma_0)$ exists whenever $\sigma = \sigma_0$ is not an exceptional hyperplane. Thus, the following theorem has been proved:

THEOREM II. *The function*

$$F_\sigma(t) = \sum_{k=1}^{n} a_k \cdot \exp 2\pi\Lambda_k\sigma \cdot \exp 2\pi i(\Lambda_k t + \alpha_k)$$

possesses a mean motion if σ, $(-\infty < \sigma < +\infty)$, *does not belong to a certain (possibly empty) finite set.*

In some cases, it is certain that the function (37) gives rise to a function Z for which there are no exceptional values of σ. For example, let

$$f(s) = \sum_{k=1}^{m} a_k \exp 2\pi(\lambda_k s + i\alpha_k), \qquad 2 < m < \infty, \tag{42}$$

where again α_k are real, $a_k > 0$, and $\lambda_1, \cdots, \lambda_m$ are real linearly independent* numbers. The corresponding torus function is

$$X + iY \equiv Z = Z(\theta_1, \cdots, \theta_m; \sigma) = \sum_{k=1}^{m} a_k \exp 2\pi[\lambda_k\sigma + i(\theta_k + \alpha_k)]. \tag{43}$$

It is known† that for any fixed σ, the set of points $(\theta_1, \cdots, \theta_m)$ on Θ at which (43) vanishes is either empty or is a finite set of analytic curves if $m=3$; in the case that $m>3$, this set of points, if it is not empty, is an analytic $(m-2)$-dimensional manifold without singularities or with a finite number of singular curves according as at least one relation of the type

$$\sum_{k=1}^{m} e_k a_k \exp 2\pi\lambda_k\sigma = 0, \qquad e_k = \pm 1, \tag{44}$$

does not or does exist. Thus, by the preceding section, $\mu(\sigma)$ exists for every σ.

It is proved similarly that if the trigonometric polynomial $f(s)$ has the form

$$f(s) = f_1(s) + f_2(s), \tag{45}$$

where $f_1(s), f_2(s)$ are each functions of the type (37), such that not both $f_1(s)$, $f_2(s)$ are periodic and such that the moduli determined by their frequencies are linearly independent, then the corresponding torus function (39) satisfies condition (B) for arbitrary α, β. Thus, one has the following theorem:

THEOREM III. *If the trigonometric polynomial*

$$F(t) = \sum_{k=1}^{n} a_k \exp 2\pi i(\Lambda_k t + \alpha_k)$$

* For a different proof that the function (42) gives rise to a torus function Z satisfying (A) in the case for $5 \leqq m \leqq \infty$, cf. Jessen [10, pp. 316–317].

† Hartman, van Kampen, and Wintner [5, pp. 265–266].

can be decomposed into the sum of two trigonometric polynomials F_1, F_2 such that not both F_1, F_2 are periodic and such that the moduli determined by their frequencies are linearly independent, then $F(t)$ possesses a mean motion.

4. **Smoothness of** $\mu(\sigma)$. It is clear from Theorem I that* for any regular analytic almost periodic function $f(s)$, the mean motion $\mu(\sigma)$ is a non-decreasing function on the set on which it is defined. In many cases, certain smoothness properties (for example, differentiability of a given order or analyticity) can be discussed.

Consider first the case (42), where it is known that $\mu(\sigma)$ is defined for all σ. If $5 \leqq m \leqq \infty$, some of these properties can be deduced from the formula (Jessen [10])

$$H(\alpha, \beta) = \int_\alpha^\beta G(\sigma; 0, 0)d\sigma, \tag{46}$$

where $G(\sigma; x, y)$ is the density of the asymptotic distribution function of $F_\sigma(t)$ with respect to the weight function $|dF_\sigma(t)/dt|^2$. By the previous section, (30) holds for all α', β', so that by (46)

$$\mu(\beta) - \mu(\alpha) = \int_\alpha^\beta G(\sigma; 0, 0)d\sigma. \tag{47}$$

Now, the function $G(\sigma; x, y)$ is given by the formula†

$$G(\sigma; x, y) = \frac{1}{4\pi^2}\iint \exp i(xu + vy)\cdot \mathrm{X}(\sigma; \xi)du\, dv, \tag{48}$$

where the integral extends over the entire (u, v)-plane, $\xi = u + iv$, and

$$\begin{aligned} \mathrm{X}(\sigma; \xi) = &\sum_{k=1}^m |\lambda_k b_k|^2 \prod_{j=1}^m J_0(|b_j\xi|) \\ &+ \sum_{\substack{k,l=1\\ k\neq l}}^m \lambda_k\lambda_l b_k b_l \cdot \prod_{\substack{j=1\\ j\neq k,l}}^m J_0(|b_j\xi|)\cdot J_1(|b_k\xi|)J_1(|b_l\xi|), \end{aligned} \tag{49}$$

where $b_j = b_j(\sigma) = a_j \exp 2\pi\lambda_j\sigma$, the functions J_0, J_1 being the Bessel functions. Using the well known properties of the Bessel functions

$$2dJ_n(w)/dw = J_{n-1}(w) - J_{n+1}(w);\ |J_n(w)| \leqq 1; J_n(w) = O(|w|^{-1/2}),\ |w| \to \infty,$$

for $n = 0, \pm 1, \cdots$, we see that if $m \geqq 5 + 2p$, the function $d^k\mathrm{X}/d\sigma^k$ is $O(|\xi|^{-m/2+k})$, so that if $k = 1, \cdots, p$, then $d^k\mathrm{X}/d\sigma^k$ is absolutely integrable

* Compare Jessen [11].

† The formula is obtained by Jessen [10] by methods adapted from Wintner [14].

over the entire $\xi = u + iv$ plane. It follows from (48) that $G(\sigma; x, y)$ has p continuous partial derivatives with respect to σ, so that, by (47), $\mu(\sigma)$ has $p+1$ continuous derivatives in this case. It is clear that if $m = \infty$, then $\mu(\sigma)$ has continuous derivatives of arbitrarily high order.

The formulas (48), (49) were obtained by the use of Fourier transforms, so that it is only possible to decide from them that $G(\sigma; x, y)$ has certain smoothness properties either for all (x, y) or for no (x, y). On the other hand, it is possible to discuss the existence and smoothness of $G(\sigma; x, y)$ by using methods* recently applied to the density $\delta(\sigma; x, y)$ of the ordinary asymptotic distribution function of $F_\sigma(t)$. These methods apply not only to the case (42) but also to the case of a general trigonometric polynomial (37). It is easily seen that $G(\sigma; x, y)$ can be defined for every point for which $\delta(\sigma; x, y)$ is defined. Also, if for a point $(\sigma; x, y)$ the functions $Z(\theta_1, \cdots, \theta_m; \sigma) - (x+iy)$, $\partial(X, Y)/\partial(\theta_k, \theta_j)$, $(k, j = 1, \cdots, m)$, do not vanish simultaneously at any point $(\theta_1, \cdots, \theta_m)$ of the torus Θ, then $G(\sigma; x, y)$ is defined and is a regular analytic function of its real arguments in a neighborhood of this point. It follows from §2 that if $(\sigma_0; 0, 0)$ is such a point, then $\mu(\sigma)$ exists for all σ sufficiently near to σ_0, since condition (B) is satisfied if one places $\alpha = \sigma_0 - \epsilon$, $\beta = \sigma_0 + \epsilon$ for a sufficiently small ϵ. On the other hand, it is clear that the considerations of Jessen [10] may be modified to show that (47) holds for $\sigma_0 - \epsilon < \alpha < \beta < \sigma_0 + \epsilon$ (even though the frequencies are not linearly independent). This proves the following:

THEOREM IV. *Let*

$$f(s) = \sum_{k=1}^{n} a_k \exp 2\pi(\Lambda_k s + i\alpha_k)$$

and let $X + iY \equiv Z = Z(\theta_1, \cdots, \theta_m; \sigma)$ *be the corresponding function* (39). *If, for* $\sigma = \sigma_0$, *the function* $Z(\theta_1, \cdots, \theta_m; \sigma)$ *and the jacobians* $\partial(X, Y)/\partial(\theta_j, \theta_k)$, $(j, k = 1, \cdots, m)$, *do not vanish simultaneously at any point* $(\theta_1, \cdots, \theta_m)$ *of the torus* Θ, *then* $\mu(\sigma)$ *exists and is a regular analytic function for all* σ *sufficiently near to* σ_0.

In the particular case (42), the conditions of Theorem IV are satisfied for all σ for which there is no relation of the type (44). If $m = \infty$ the same is true if (44) is replaced by

$$\sum_{k=1}^{n} e_k a_k \exp 2\pi\lambda_k\sigma - \sum_{k=n+1}^{\infty} a_k \exp 2\pi\lambda_k\sigma = 0, \qquad e_k = \pm 1,$$

* van Kampen and Wintner [8]; Hartman, van Kampen, and Wintner [6]. The formulation of the results in the latter paper can be extended at once to the density $G(\sigma; x, y)$ of the weighted distribution function.

for some n. Summarizing the results for the functions (42), one has the following:

THEOREM V. *If the numbers* $\lambda_1, \lambda_2, \cdots$ *are linearly independent, the function*

$$F_\sigma(t) = \sum_{k=1}^{m} a_k \exp 2\pi(\lambda_k s + i\alpha_k), \qquad 1 \leqq m \leqq \infty,$$

possesses a mean motion $\mu(\sigma)$. *If* $3+2p \leqq m \leqq \infty$, $\mu(\sigma)$ *possesses* p *continuous partial derivatives. If* $m < \infty$, $\mu(\sigma)$ *is a regular analytic function at every point, with the possible exception of those* σ *for which there is a relation of the type*

$$\sum_{k=1}^{m} e_k a_k \exp 2\pi\lambda_k\sigma = 0, \qquad e_k = \pm 1.$$

Finally, if $m = \infty$, $\mu(\sigma)$ *is regular analytic at every point, with the possible exception of those* σ *for which there is a relation of the type*

$$\sum_{k=1}^{n} e_k a_k \exp 2\pi\lambda_k\sigma - \sum_{k=n+1}^{\infty} a_k \exp 2\pi\lambda_k\sigma = 0, \qquad e_k = \pm 1,$$

for some n.

5. **Mean motions of the Riemann ζ-function.** It is clear from Theorem I, and the fact that $\zeta(s)$ is almost periodic (in the sense of Bohr) and does not vanish for $\sigma > 1$, that the mean motion $\mu(\sigma)$ exists for every $\sigma > 1$. Furthermore, $\mu(\sigma)$ is independent of σ; since the real part of $\zeta(s)$ does not vanish when σ is sufficiently large, $\mu(\sigma) = 0$ for all $\sigma > 1$.

Although $\zeta(s)$ is not almost periodic in the sense of Bohr for $1/2 < \sigma \leqq 1$, the methods developed in §1 can be adapted for this case by using the well known fact that $N(\alpha, \beta; T)/T \to 0$ as $T \to \infty$, where $1/2 < \alpha < \beta \leqq \infty$ and $N(\alpha, \beta; T)$ denotes the number of zeros of $\zeta(s)$ in the rectangle $\alpha < \sigma < \beta$, $1 < t < T$. Thus, it may be concluded that $\mu(\sigma) \equiv 0$ for $\sigma > 1/2$.

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