ON 0-REGULAR SURFACE TRANSFORMATIONS*

BY

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1. The continuous transformation $T(M) = M'$, where $M$ is a compact metric space, is said to be 0-regular† provided that for each sequence of points $\{x'_n\}$ converging to $x'$ in $M'$, the sets $T^{-1}(x'_n)$ converge 0-regularly‡ to $T^{-1}(x')$. This is equivalent to a continuous transformation sending open sets into open sets, while the inverse sets as a collection are uniformly locally connected (that is, for each $\epsilon > 0$ a $\delta > 0$ exists such that every two points $x$ and $y$ of any inverse set $X$ whose distance apart is less than $\delta$ lie in a connected subset of $X$ of diameter less than $\epsilon$). This characterization suggests the projection of a convex euclidean set onto a plane. For example, the orthogonal projection of a solid circle onto a diameter is a 0-regular transformation. It is not 0-regular on the circumference, however, because of the folding about the diameter’s end points. That there exist other types of 0-regular transformations is illustrated by the identification of diametrically opposite points of a 2-sphere to obtain a projective plane. A suggestive property of a 0-regular transformation is that the inverse sets must all contain the same number of components.†

In this paper a study is made of 0-regular transformations defined on 2-dimensional pseudo-manifolds. It is shown that if $M$ is a 2-dimensional pseudo-manifold and $T(M) = M'$ is a monotone 0-regular transformation, then either $T$ is topological, or $M'$ is an arc or a simple closed curve. Moreover, it is shown that $T$ must be topological or $M'$ must be degenerate except in the following cases: (i) The sphere, 2-cell, and circular ring may be mapped onto an arc. (ii) The torus, Klein bottle, circular ring, Möbius band, pinched sphere, and 2-cell with two boundary points identified may be mapped onto a simple closed curve. In each of these cases the possible transformations are characterized. For example, it is shown that the only non-topological monotone 0-regular transformation of a sphere onto a nondegenerate image space is equivalent to an orthogonal projection onto a diameter.

* Presented to the Society, April 8, 1939, under the title The images of 2-dimensional surfaces under 0-regular transformations; received by the editors August 29, 1939.
‡ A convergent sequence of closed sets $\{X_n\}$ is said to converge 0-regularly to $X$ provided that for each $\epsilon > 0$ there exist positive numbers $\delta$ and $N$ such that if $n > N$, any pair of points $x, y$ of $X_n$ with $\rho(x, y) < \delta$ lie together in a continuum in $X_n$ of diameter less than $\epsilon$. See G. T. Whyburn, Fundamenta Mathematicae, vol. 25 (1935), pp. 408–426.
In §6 the above results are stated in terms of possible monotone 0-regular retracting transformations on pseudo-manifolds, while in §7 R. L. Moore's* self-compact equicontinuous collections of curves are used in stating the results. In the concluding section the possible images of pseudo-manifolds under general 0-regular transformations are considered.

2. Throughout this section the following notation will be used: Let $M$ denote a 2-dimensional pseudo-manifold, that is, a 2-dimensional manifold or surface (with or without boundary) among $q$ points of which identifications have been performed so as to produce $r$ local separating points† of $M$. Let $B$ be the boundary (that is, a finite number of simple closed curves) of $M$, and denote the finite set of local separating points of $M$ by $S$. Finally, let $T(M) = M'$ be a monotone 0-regular transformation and assume $M'$ is non-degenerate.

2.1. If $x$ is a point of $S$, then $T^{-1}T(x) = x$.

Proof. Since $M$ is a locally connected continuum and $x$ is a local separating point of $M$, there exists a connected neighborhood $U(x)$ of $x$ such that $U(x) = L_1 + L_2 + \cdots + L_\lambda$ ($\lambda \geq 2$), where the $L_i$ are mutually separated open sets each having $x$ as a limit point. Assume the assertion is false; then there exists an $L_k$ such that $T^{-1}T(x) \cdot L_k$ contains $x$. Let $\{x_i\}$ be a sequence of points in $L_n - T^{-1}T(x)$ ($n$ not $k$) converging to $x$. Now $\{T^{-1}T(x_i)\}$ converges to $T^{-1}T(x)$. Hence for each sufficiently large $i$ there exists a point $y_i$ of $L_k$ such that $T(y_i) = T(x_i)$ and $\{y_i\}$ converges to $x$, since $T^{-1}T(x) \cdot L_k$ contains $x$. But any connected set in $T^{-1}T(x_i)$ containing $x_i$ and $y_i$ must extend outside $U(x)$. Hence $\{T^{-1}T(x_i)\}$ does not converge 0-regularly to $T^{-1}T(x)$ contrary to the hypothesis that $T$ is a 0-regular transformation.

2.11. If $x$ is a point of $S$, then $T(x)$ is a local separating point, but not a cut point of $M'$.

Proof. Since $T^{-1}T(x) = x$, $T^{-1}T(x)$ locally separates $M$. Hence by a known theorem‡ on monotone transformations $T(x)$ locally separates $M'$. However, $T(x)$ cannot be a cut point of $M'$ since $M - x$ is connected.


† The point $x$ of a continuum $M$ is called a local separating point of $M$ provided there exists a neighborhood $U(x)$ of $x$ in $M$ such that $x$ separates $U(x)$ between some pair of points of the component of $(Ux)$ containing $x$. See G. T. Whyburn, Local separating points of continua, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 305–314.

‡ The theorem and proof given here are valid if $M$ is any locally connected continuum and $x$ is a local separating point of $M$.

2.2. If $x'$ is any point of $M'$, then $T^{-1}(x')$ is either an arc or a simple closed curve.

**Proof.** Since $T$ is interior, $T^{-1}(x')$ can contain no open set. Hence either $T^{-1}(x')$ is a single point (that is, a degenerate arc or simple closed curve) or $T^{-1}(x')$ is a 1-dimensional continuum. Moreover, $T^{-1}(x')$ is locally connected.† Therefore, in order to establish the assertion it must be shown that every point $x$ of $X=T^{-1}(x')$ has an order not greater than 2 in $X$. Suppose $x$ has an order greater than 2 in $X$; then there exist nondegenerate arcs $\alpha_1$, $\alpha_2$, $\alpha_3$ in $X$ which are disjoint except for $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = x$, an end point of each $\alpha_i$. Since, after 2.1, $x$ cannot be a point of $S$, there exists a neighborhood $U(x)$ of $x$ in $M$ such that $\overline{U(x)}$ is a 2-cell. It may be assumed each $\alpha_i$ is disjoint‡ with $F(U(x))$ except for the other end point. Now $U(x) - \sum \alpha_i$ must contain at least two components $L_1$, $L_2$ such that $L_1$ contains $x$. (Observe that one cannot say three components here because $x$ may be a point of $B$.) Moreover, it may be assumed $\alpha_1 \cdot L_1 = x$, since $\overline{U(x)}$ is a 2-cell. There exists a sequence of points $\{y_i\}$ of $L_1 \cdot (M - X)$ converging to $x$, since $X$ contains no open subset of $M$. Now if $\{Y_i\} = \{T^{-1}T(y_i)\}$ converges to $x$, there must exist for all sufficiently large $i$ points $z_i$ of $Y_i$ not contained in $L_1$, since $\alpha_1 \cdot L_1 = x$. It may be assumed $\{z_i\}$ converges to $x$. Therefore $\{Y_i\}$ does not converge 0-regularly to $X$, since any connected subset of $Y_i$ containing $y_i$ and $z_i$ must extend outside $\overline{U(x)}$. Thus the assumption that the order of $x$ is greater than 2 has led to the contradiction that $T$ is not 0-regular.

2.3. If $x'$ is a point of $M'$ such that $T^{-1}(x')$ is nondegenerate and not contained in $B$, then $x'$ is a local separating point of $M'$.

**Proof.** Since $X = T^{-1}(x')$ is nondegenerate and not contained in $B$, it follows from 2.2 and 2.1 that there exists a subarc $\alpha$ of $X$ which is disjoint with $B+S$. Let $x$ be an interior point of $\alpha$ and let $\alpha_1$, $\alpha_2$ be arcs such that $\alpha = \alpha_1 + \alpha_2$ while $\alpha_1 \cdot \alpha_2 = x$. There exists a neighborhood $U(x)$ of $x$ which is an open 2-cell and does not contain either of the $\alpha_i$. Let $x_1x_2x_3 = \beta$ be the subarc of $\alpha$ containing $x$ such that $\beta \cdot F(U(x)) = x_1 + x_2$; then $U(x) - \beta = L_1 + L_2$ is a separation such that $\overline{L_1 \cdot L_2} = \beta$. Now let $\{U_i(x)\}$ be a sequence of connected neighborhoods contained in $U(x)$ and closing down on $x$. Then for each $i$, $U_i(x) - X = L_{ii} + L_{ia}$, where $L_{ii} = L_{i} \cdot (U_i(x) - X)$, is a separation, since $X$ contains no open subset of $M$. Suppose $x'$ is not a local separating point of $M'$;
then for each $i$, $T(U_i(x)-X) = T(U_i(x)) - x' = 0'$ must be a connected set.* Hence there must exist a point $y'_i$ of $0'$ for each $i$ such that $T^{-1}(y'_i) \cdot L_{ij}$ contains a point $y_{ij}$ for $j=1, 2$. Now $\{y'_i\}$ converges to $x'$ and $\rho(y_{il}, y_{ir})$ converges to zero, since the $U_i(x)$ close down on $x$. But from the definition of the $L_{ij}$ it follows that any connected set in $T^{-1}(y'_i)$ containing $y_{il}$ and $y_{ir}$ must go outside $U(x)$. Therefore $\{T^{-1}(y'_i)\}$ does not converge 0-regularly to $X$, contrary to the hypothesis that $T$ is a 0-regular transformation.

2.31. Under the conditions of 2.3, $x'$ locally separates $M'$ into exactly two components.

**Proof.** Suppose $x'$ locally separates $M'$ into more than two components; then there exists a connected neighborhood $V(x')$ such that $V(x') - x' = L'_i + L'_j + L'_k + \cdots$, where the $L'_i$ are mutually separated connected open sets with $x'$ a point of $F(L'_j)$ for each $j$. For each $j$, $T^{-1}(L'_j)$ is a connected set, since connectedness is invariant under the inverse of a monotone transformation. Thus, since $T$ is interior, $T^{-1}(x')$ is on the boundary of at least three mutually separated connected open sets in $M$. This is impossible since $T^{-1}(x')$ is a nondegenerate arc or simple closed curve.

2.32. If $x'$ is an end point of $T^{-1}(x')$ for some point $x'$ of $M'$, then $x$ belongs to $B$.

2.33. If $x'$ is not a local separating point of $M'$, then $T^{-1}(x')$ is a single point or is contained in $B$.

2.4. If $x'$ is a point of $M'$ such that $B \cdot T^{-1}(x')$ contains a nondegenerate continuum $K$, then $T^{-1}(x')$ is contained in $B$ and $x'$ is an end point of $M'$.

**Proof.** If $T^{-1}(x')$ is not contained in $B$, it follows from 2.31 that $x'$ locally separates $M'$ into exactly two components. Thus there exists a connected neighborhood $V(x')$ of $x'$ such that $V(x') - x' = L'_i + L'_j$, where $L'_i$ and $L'_j$ are mutually separated connected open sets with $x'$ a point of $L'_i \cdot L'_j$. Hence $T^{-1}(L'_i)$ and $T^{-1}(L'_j)$ are disjoint connected open sets in $M$ whose boundaries have $T^{-1}(x')$ in common, consequently have $K$ in common. This is impossible since $T^{-1}(x')$ is locally connected and $K$ is contained in $B$.

Let $x'$ be a point of $M'$ such that $T^{-1}(x')$ is nondegenerate and contained in $B$. Now $T^{-1}(x')$ can contain no point of $S$. Hence for any point $x$ of $T^{-1}(x')$ which is not an end point there exists a neighborhood $U(x)$ such that $\overline{U(x)}$ is a 2-cell. Moreover, $U(x)$ may be taken so small that $x$ is an interior point of an arc lying in $T^{-1}(x')$ with its end points not in $U(x)$.

Thus there exists a neighborhood $V(x')$ such that for each point $y'_i$ of $V(x') - x'$ the set $T^{-1}(y'_i)$ contains an arc $Y_i$, with its endpoints only in $F(U(x))$, which separates $U(x)$ into exactly two components, since $T$ is monotone 0-regular. In case $\{y'_i\}$ is any sequence of points converging to $x'$, the corresponding $Y_i$ may be so selected that if $L_i$ denotes the component of $U(x) - Y_i$ containing $x$, then $x$ is a point of $L = \bigcup L_i$ which is contained in $T^{-1}(x')$. Finally, every sufficiently small neighborhood $W(x)$ must have the property that $T^{-1}(x') \cdot W(x)$ is contained in $L$ and for every point $y'_i$ of $T(W(x)) - x'$ the product $L_i \cdot W(x) \cdot T^{-1}(y'_i)$ is empty. Suppose $x'$ is not an end point of $M'$. Then it is an interior point of an arc $z'_1 x'_2 z'_1$. Now $T^{-1}(z'_1 x'_2)$ ($i = 1, 2$) is locally connected.* Hence there exist arcs $y'_1 x_i$ in $W(x)$ such that $T(y'_1 x_i) = y'_1 x'_i$ is contained in $z'_i x'_i$, where it is assumed $y_i$ is the last point of $y'_1 x_i$ in $T^{-1}(y'_i)$ and $x_i$ is the first point in $T^{-1}(x')$. From the choice of $W(x)$ it follows that $y_i$ is contained in $Y_i$ and $x_i$ in $L$. Since $\overline{U(x)}$ is a 2-cell it may be assumed $Y_1$ separates $Y_1$ and $L$ in $U(x)$ and consequently in $W(x)$. Thus $y'_1 x_i$ contains a point of $Y_2$ which contradicts the fact that $T(y'_1 x_i)$ is contained in $z'_1 x'_1$.

2.5. If for a point $x'$ of $M'$ the set $T^{-1}(x')$ is not contained in $B$, then $B \cdot T^{-1}(x')$ can contain only end points of $T^{-1}(x')$.

Proof. Suppose the assertion is not true. Then there exists a point $x$ of $B \cdot T^{-1}(x')$ which is not an end point of $T^{-1}(x')$. Now $x$ is not a point of $S$, since $T^{-1}(x')$ must be nondegenerate. Thus there exists a neighborhood $U(x)$ such that $\overline{U(x)}$ is a 2-cell. It may be assumed there exist points of $T^{-1}(x')$ which are not in $\overline{U(x)}$. Let $x_1, x_2$ be points of $T^{-1}(x')$ different from $x$ such that the arc $x_1 x_2 x_1$ is contained in $U(x) \cdot T^{-1}(x')$, and furthermore, let $y, z$ be points of $B$ different from $x$ such that the arc $y z z$ is contained in $B \cdot U(x)$. Then $(x_1 x_2 y) \cdot (y z z) = x$; for suppose the product set contained another point $x_3$. Then $x_1 x_2 + y z z$ contains a simple closed curve $J$, since no subcontinuum of $T^{-1}(x')$ can be contained in $B$ because of 2.4. Thus $J$ separates $\overline{U(x)}$ into exactly two components and there is one component $C$ such that $C \cdot [M - U(x)] = 0$, since $J$ is contained in $U(x)$. Let $\{w_i\}$ be a sequence of points in $C$ converging to a point $x_0$ of $J \cdot T^{-1}(x')$; then $T^{-1}T(w_i)$ is contained in $C = C + J$ for each $i$. Hence $\{T^{-1}T(w_i)\}$ cannot converge to $T^{-1}T(x_0) = T^{-1}(x')$, since the latter set contains points outside $\overline{U(x)}$. Thus the four arcs $x x, x x x, y x, z x$ are contained in the 2-cell $\overline{U(x)}$ and have by pairs only $x$ in common. Hence there exists a neighborhood $W(x)$ in $U(x)$ which is separated into three components by $x x$, such that $x$ is on the boundary of

each, and only one can have both $x_1x$ and $x_2x$ on its boundary. Let $L_i$ be the component such that $L_i$ contains $x_1x$ but not $x_2x$, and let $\{w_n\}$ be a sequence of points in $L_i$ converging to $x$. Then $\{T^{-1}(w_n)\}$ cannot converge 0-regularly to $T^{-1}(x) = T^{-1}(x')$, since they must go outside $W(x)$ to converge to the arc $xx$.

2.6. If the sequence $\{x'_n\}$ of local separating points of $M'$ converges to a point $x'$ of $M' - T(S)$ and $T^{-1}(x')$ is degenerate, then $x'$ is an end point of $M'$.

**Proof.** There exists a neighborhood $U(x)$ of $x = T^{-1}(x')$ such that $U(x)$ is a 2-cell, since $x$ is not a point of $S$. Moreover, it may be assumed that each $T^{-1}(x'_n)$ is contained in $U(x)$, since $\{T^{-1}(x'_n)\}$ converges to $x$. Since $x'_n$ is a local separating point of $M'$, $T^{-1}(x'_n)$ locally separates $M$ and, consequently, separates $U(x)$ because it is a 2-cell. But since $T^{-1}(x'_n)$ is contained in $U(x)$ it follows that $T^{-1}(x'_n)$ separates $M$; that is, $M - T^{-1}(x'_n) = L_n + N_n$ is a separation and it is assumed $x$ is a point of $L_n$. Now $F(L_n)$ is contained in $T^{-1}(x'_n)$, whence $F[T(L_n)]$ contains at most the single point $x'_n$. Thus $T(L_n)$ is an open set containing $x'$ whose boundary consists of at most a single point. Thus in order to complete the proof it remains to be shown that $T(L_n)$ is a sequence of sets closing down on $x'$. It may be assumed that for each $n > k$, $T^{-1}(x'_n)$ is contained in $L_k$. Moreover, because the transformation is interior, $F(L_n) = F(N_n) = T^{-1}(x'_n)$. Hence $x$ is a point of $L_n + 1$ which is contained in $L_n$, and consequently $x$ is a point of $L = \bigcap L_n$. But $N = \sum N_n$ is open and $F(N) = x$, since for $n < k$, $N_n$ is contained in $N_k$, $F(N_n) = T^{-1}(x'_n)$, and $\{T^{-1}(x'_n)\}$ converges to $x$. Thus if $L_n$ does not close down on $x$, that is, $x$ is not $L$, then $M - x = (L - x) + N$ is a separation. But this is impossible, since $M$ is a 2-dimensional pseudo-manifold. Therefore $x = L$, and the proof is complete.

3. We next prove the following theorem.

**Theorem.** If $M$ is a 2-dimensional pseudo-manifold and $T(M) = M'$ is a monotone 0-regular transformation, then $T$ is topological, or $M'$ is either an arc or a simple closed curve.

**Proof.** In case $M'$ is degenerate there is nothing to prove. Thus assume $M'$ is nondegenerate and let $K$ be the set of all points of $M$ on which $T$ is one-to-one. Then $K$ is closed, since $T$ is interior. Let $G = M - K$; then by 2.3 and 2.4, $T(G)$ consists of local separating points and end points of $M'$. Suppose $G$ is not empty, and let $x$ be a point of $F(G)$. If $x$ is not a point of $S$, then it follows from 2.6 that $T(x)$ is an end point of $M'$. If $x$ is a point of $S$, then it follows from 2.11 that $T(x)$ is a local separating point of $M'$. Hence $T(\overline{G})$ consists of local separating points and end points of $M'$. Suppose $M - \overline{G}$ is not

empty; then there exist points \( y \) of \( (M-S)
abla (M-G) \) and \( z \) of \( (M-S)
abla G \), since \( M-G \) and \( G \) are both open and \( S \) is finite. Now \( M-S \) is a region in the locally connected continuum \( M \). Hence there exists an arc \( yz \) in \( M-S \) which must intersect \( F(G) \). Let \( x \) be the first such point from \( y \) to \( z \). Now the arc \( yx \) is contained in \( K \), and consequently \( T(yx) \) is topological. Therefore no point \( x' \) of \( T(yx) \) can be a local separating point of \( M' \), since no point of \( M-S \) locally separates \( M \). But \( T(x) \) is an end point of \( M' \), which is a contradiction. Thus either \( G \) is empty or \( M-G \) is empty. In the first case \( T \) is topological, while in the second \( M' \) consists of end points and local separating points and consequently is a 1-dimensional continuum. But \( T \) is interior and for each point \( x' \) of \( M' \), \( T^{-1}(x') \) is locally connected. Hence it follows from a known theorem\(^*\) that \( M' \) is either an arc or a simple closed curve.

4. It is proposed in this section to show that a monotone 0-regular transformation on a 2-dimensional pseudo-manifold must be topological except in a few specific cases. The notation is that used in §2.

4.1. In order that \( T \) be topological it is sufficient that either (a) \( S \) contains a point which locally separates \( M \) into at least three components, or (b) \( S \) contains more than one point.

**Proof.** Suppose \( T \) is not topological; then \( M' \) is either an arc or a simple closed curve. Now after 2.11 the image of a point \( x \) of \( S \) cannot be an end point of \( M' \). Hence \( x'=T(x) \) must locally separate \( M' \) into exactly two components; that is, there exists a connected neighborhood \( V(x') \) in \( M' \) such that \( V(x')-x'=L'+N' \), where \( L' \), \( N' \) are open arcs with \( x'=L'
abla N' \).

Suppose \( x \) locally separates \( M \) into more than two components; then there exists a connected neighborhood \( U(x) \) such that \( T(U(x)) \) is contained in \( V(x') \) and \( U(x)-x=M_1+M_2+\ldots+M_k (k\geq3) \), where, for each integer \( i<k \), \( M_i \) is a component and \( x \) is a point of \( M_i \). Thus, since \( T^{-1}(x')=x \), it may be assumed \( L'.T(M_1).T(M_2) \) contains a sequence of points \( \{x_n'\} \) converging to \( x' \). But \( \{T^{-1}(x_n')\} \) does not converge 0-regularly to \( x \), since for each \( n \), \( T^{-1}(x_n') \cdot M_1 \) is not empty and \( T^{-1}(x_n') \cdot M_2 \) is not empty. Thus for (a) \( T \) must be topological. Now under the assumption that \( T \) is not topological it follows that if \( y \), \( z \) are points of \( S \), then \( T(y+z) \) separates \( M' \). Hence \( y+z \) separates \( M', \) which is impossible. Thus for (b), \( T \) is also topological.

The following statement follows immediately from 2.11:

4.11. In order that \( M' \) be an arc it is necessary that \( S \) be empty.

4.2. If \( T \) is not topological, then either \( T^{-1}(x') \) is an arc for every point \( x' \)

of $M'$ or every $T^{-1}(x')$ is a simple closed curve. In neither case can more than two $T^{-1}(x')$ be degenerate.

**Proof.** Let $L'$ be the set of all points $x'$ of $M'$ such that $T^{-1}(x')$ is degenerate. Since $T$ is not topological either $x'$ must be an end point of $M'$ or $x = T^{-1}(x')$ must be a point of $S$. After 4.11 it follows that $S$ is empty when $M'$ has end points. Thus, since $M'$ is either an arc or simple closed curve and consequently $S$ consists of not more than one point, $L'$ can contain at most two points and $M' - L'$ is connected. Let $N'_1$ be the set of all points $x'$ of $M'$ such that $T^{-1}(x')$ is an arc, and $N'_2$ the set of all points such that $T^{-1}(x')$ is a simple closed curve. From 2.2 it follows that $M' - L' = N'_1 + N'_2$. But $N'_1 \cdot N'_2 = N'_1 \cdot N'_2 = 0$ since the 0-regular limit of a sequence of arcs (simple closed curves) is an arc (simple closed curve).* Hence either $N'_1 = 0$ or $N'_2 = 0$, since $M' - L'$ is connected.

4.21. For each point $x'$ of $M'$ let $T^{-1}(x')$ be a simple closed curve, at least one of which is nondegenerate. If $J$ is any simple closed curve of $B$, then some $T^{-1}(x') = J$.

**Proof.** From 2.5 it follows that if $B \cdot T^{-1}(x')$ is not empty then $T^{-1}(x')$ is contained in $B$. Moreover, it follows from 4.2 that every $T^{-1}(x')$, except possibly two, is a nondegenerate simple closed curve. Thus there exists a nondegenerate simple closed curve $T^{-1}(x')$ contained in $B$ such that $J \cdot T^{-1}(x')$ is not empty. But $J \cdot (B - J)$ is contained in $S$ (of course, may be empty), while $S \cdot T^{-1}(x') = 0$ by 2.1. It follows that $T^{-1}(x')$ is contained in $J$ and is therefore $J$.

The following assertion also comes out of the above proof:

4.22. Under the hypotheses of 4.21, $S \cdot B = 0$.

Let $p_n^m(M)$ denote the first Betti number (mod $m$) of $M$, for $m \geq 0$. Also, if $N$ is a closed subset of $M$, let $p_n^m(N, M)$ denote the first Betti number (mod $m$) of $N$ relative to $M$, that is, the number of independent cycles in $N$ relative to homologies (mod $m$) in $M$. In another paper† the writer has shown that

(i) If $x', y'$ are any two points of $M'$, then

$$p_n^m \left[ T^{-1}(x'), M \right] = p_n^m \left[ T^{-1}(y'), M \right].$$

(ii) If $x'$ is any point of $M'$, then

$$p_n^1(M) = p_n^1(M') + p_n^1 \left[ T^{-1}(x'), M \right].$$

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† *On regular transformations* (offered for publication to Duke Mathematical Journal).
which in the case considered here may be written

\[(2')\quad 0 \leq p_n^1(M) - p_n^1(M') = p_n^1[T^{-1}(x'), M],\]

since all the numbers involved are finite.

The two relations above, along with 2.2, give the following assertions:

4.31. In any case \(0 \leq p_n^1(M) - p_n^1(M') \leq 1\) (\(m \geq 0\)).

4.32. If, for any \(m \geq 0\), \(p_n^1(M) > 2\), then \(T\) is topological.

4.33. If, for some point \(x'\) of \(M', T^{-1}(x')\) is degenerate or an arc, then
\(p_n^1(M) = p_n^1(M') (m \geq 0)\).

4.34. In order that \(M'\) be an arc it is necessary that \(p_n^1(M) \leq 1\) (\(m \geq 0\)).

4.35. In order that \(M'\) be a simple closed curve it is necessary that
\(1 \leq p_n^1(M) \leq 2\) (\(m \geq 0\)).

The assertions 4.1 and 4.11 may be obtained from 4.33, 2.1, and 2.2 as follows: Let \(M\) be a pseudo-manifold with \(S = y_1 + y_2 + \cdots + y_\lambda\); then it may be assumed \(M\) was obtained from a manifold \(L\), which contains no identifications, by identifying \(\mu_i\) points to obtain \(y_i\). Thus it follows from the Euler-Poincaré formula that

\[(3)\quad p_n^1(M) = p_n^1(L) + \sum_{i=1}^{\lambda} (\mu_i - 1) (m \text{ prime}).\]

Therefore, if \(\lambda > 1\) or some \(\mu_i > 2\) it follows that \(p_n^1(M) \geq 2\). Thus it follows from 2.1 and 4.33 that \(p_n^1(M) = p_n^1(M') \geq 2\), since \(S\) is not empty. Hence \(T\) must be topological, since \(M'\) cannot be an arc of a simple closed curve.

By the same reasoning it follows that if \(S\) contains a single point then
\(p_n^1(M') = p_n^1(M) \geq 1\), and therefore \(M'\) cannot be an arc.

A 2-dimensional closed surface \(M\) (that is, \(S = 0, B = 0\)) can possess no 0- or 2-dimensional torsion,* and if \(M\) is orientable (that is, a sphere, or torus, and so on) it can possess no 1-dimensional torsion. However, if \(M\) is not orientable (that is, projective plane, Klein bottle, and so on) its 1-dimensional torsion group is cyclic of order 2.† Hence it follows from a known theorem‡ that if \(M\) is a 2-dimensional closed surface, then

\[(4a)\quad p_1^1(M) = p_1^1(M)\text{ when }M\text{ is orientable, and}
\[(4b)\quad p_1^2(M) = p_1^1(M) + 1\text{ when }M\text{ is non-orientable.}

Therefore it follows† that

\[(5a)\quad p_1^1(M) = p_1^1(M) = 0,\text{ if }M\text{ is a sphere;}\]

† See Topologie I, paragraph 10, pp. 266–269.
(5b) $p_1(M) = 0$, $p_1(M) = 1$, if $M$ is a projective plane;
(5c) $p_1(M) = 1$, $p_1(M) = 2$, if $M$ is a Klein bottle;
(5d) $p_1(M) = p_1(M) = 2$, if $M$ is a torus; and
(5e) $p_1(M) > 2$, if $M$ is any other 2-dimensional closed surface.

Now let $M$ be a 2-dimensional surface with boundary (that is, let $S = 0$, $B = J_1 + J_2 + \cdots + J_\beta$, where the $J_i$ are disjoint simple closed curves); then $M$ may be thought of as a closed 2-dimensional surface $L$ with $\beta$ open 2-cells cut out. Thus if $\beta$ is not 0, then*

(6a) $M$ possesses no torsion;
(6b) $p_1(M) = p_1(L) + \beta - 1$ when $L$ is orientable; and
(6c) $p_1(M) = p_1(L) + \beta$ when $L$ is non-orientable.

The relations (3), (5), and (6) are enough to determine the first Betti number (mod 0 or 2) of any 2-dimensional pseudo-manifold with which this paper hereafter is concerned.

4.4. If $M'$ is a nondegenerate arc, then $M$ must be either a sphere, a 2-cell, or a circular ring.

**Proof.** Since $M'$ is an arc, it follows from 4.11 that $S = 0$, and from 4.34 that $p_1(M) \leq 1$. Hence, besides those surfaces given in the theorem, $M$ may be either a projective plane or a Möbius band, both of which have $p_1(M) = 1$. Suppose $M$ to be one of these and $M'$ to be an arc; then it follows from 4.33 that $T^{-1}(x')$ is a nondegenerate simple closed curve for each point $x'$ of $M'$. But let $y'$, $z'$ be the end points of $M'$; then it follows from 2.33 that $T^{-1}(y') + T^{-1}(z')$ is contained in $B$. This is impossible, since in the first case $B = 0$ while in the second $B$ consists of a single simple closed curve.

4.5. If $M'$ is a nondegenerate simple closed curve, then $M$ is either a torus, a Klein bottle, a circular ring, a Möbius band, a pinched sphere (that is, a sphere with two points identified), or a 2-cell with two boundary points identified.

**Proof.** Since $M'$ is a simple closed curve it follows from 4.35 that $1 \leq p_1(M) \leq 2$ for $m = 0$ or 2. Thus the torus and Klein bottle are the only closed surfaces which can possibly transform into a simple closed curve. However, if $S = 0$ and $M$ has boundary, it is possible, besides the circular ring and Möbius band, that $M$ may be either a 2-cell with two holes, a Möbius band with a hole, a torus with a hole, or a Klein bottle with a hole. In each of these cases $p_1(M) = 2$ and $\beta \geq 1$. Suppose $M$ is one of these; then from 4.33 it follows that $T^{-1}(x')$ is a nondegenerate simple closed curve for each point $x'$ of $M'$. Hence after 4.21 it follows that there exists a $T^{-1}(x')$ contained in $B$. Thus this $x'$ is an end point of $M'$ because of 2.4. This is impossible under

* See Topologie I, paragraph 11, pp. 269–270.
the assumption that $M'$ is a simple closed curve. If $S$ is not empty, then it follows from 2.1 and 4.33 that $\rho_{m}(M) = \rho_{m}(M') = 1$, for $m = 0$ or 2. Hence the only possibilities here are the pinched sphere and 2-cell with two points identified. It remains to show that in the case of the 2-cell the two points which are identified must be on the boundary. Let $x_1$, $x_2$ be the two points of a 2-cell, at least one of which is not on the boundary, which are identified to give the point $x = S$. Then there exists a neighborhood $U(x)$ such that $U(x) - x$ has two components, one of which, say $C$, is such that $\overline{C}$ is a 2-cell. Let $\{y_i\}$, contained in $C$, converge to $x$. Then for sufficiently large $i$, $T^{-1}(y_i)$ is contained in $C$ and is disjoint with $F(C)$. Now each of these $T^{-1}(y_i)$ must locally separate $M$ and consequently locally separate $\overline{C}$. Thus each must be a nondegenerate simple closed curve. Therefore, $T^{-1}(x')$ is a simple closed curve for each $x'$ of $M'$ after 4.2, and consequently some $T^{-1}(x')$ which is nondegenerate is contained in $B$. But this $x'$ would be an end point of $M'$ by 2.4, which is impossible since by assumption $M'$ is a simple closed curve.

5. In 4.4 and 4.5 it is shown that only a few of the 2-dimensional pseudo-manifolds can possibly be transformed into a nondegenerate arc or simple closed curve by a monotone 0-regular transformation. In this section it is shown that each of these transformations is possible. Moreover, the transformations are completely characterized.

A transformation $T(M) = M'$ is said to be topologically equivalent* or simply equivalent to a transformation $W(N) = N'$ provided one can write $T(M) = h WH(M) = M'$, where $H(M) = N$, $h(N') = M'$ are homeomorphisms.

5.1. If $M$ is a 2-cell and $M'$ is a nondegenerate arc, then $T(M) = M'$ is equivalent to one of the transformations $W(N) = N'$, where $N'$ is the interval $0 \leq \xi' \leq 1$ and either (a) $N$ is the square

$$0 \leq \xi \leq 1, \quad 0 \leq \eta \leq 1$$

with $\xi' = \xi$, (b) $N$ is the triangle

$$0 \leq \xi \leq 1, \quad 0 \leq \eta \leq \xi$$

with $\xi' = \xi$, (c) $N$ is the triangle

$$0 \leq \xi \leq 1/2, \quad 0 \leq \eta \leq \xi; \quad 1/2 \leq \xi \leq 1, \quad 0 \leq \eta \leq 1 - \xi$$

with $\xi' = \xi$, or (d) $N$ is the solid circle

$$0 \leq \xi^2 + \eta^2 \leq 1$$

with $\xi' = (\xi^2 + \eta^2)^{1/2}$.

Proof. Suppose, after 4.2, that the inverse of every point of \( M' = x', x' \) is an arc. Then there are three possibilities: (i) neither of the continua \( X_i = T^{-1}(x_i) \) \((i = 0 \text{ or } 1)\) is degenerate, (ii) one, say \( X_0 = x_0 \), is degenerate, and (iii) both \( X_0 = x_0 \) and \( X_1 = x_1 \) are degenerate. It will be shown that the transformations arising from these possibilities are equivalent to (a), (b), and (c) respectively.

Since in (i) \( X_0 \) and \( X_1 \) are nondegenerate arcs, it follows from 2.33 that they lie in \( B \). Hence \( B = \alpha_0 + X_0 + \alpha_1 + X_1 \), where each \( \alpha_i \) is an arc disjoint with \( X_0 \) and \( X_1 \) except for one end point in each. Now if \( x' \) is an interior point of \( M' \), the arc \( X = T^{-1}(x') \) must separate \( X_0 \) and \( X_1 \) in \( M \). Moreover, after 2.5, \( B \) can contain only end points of \( X \). Hence \( \phi = [X] \), where \( X = T^{-1}(x') \) for some \( x' \) of \( M' \), is an equicontinuous (since it is \( 0 \)-regular) collection\(^*\) of arcs satisfying a theorem of R. L. Moore.\(^\dagger\) Thus there exists a self-compact\(^*\) collection \( G = [g] \) of mutually disjoint arcs such that \( \sum g = M \) and for each \( X \) of \( \phi \) and \( g \) of \( G \) the product \( X \cdot g \) is a single point. Let \( h(N') = M' \) be a topological transformation, and for \( N \) of (a) in the theorem let \( H_0(M) = N \) be a topological transformation such that \( H_0(X_i) = W^{-1}h^{-1}(x_i) \) \((i = 0, \text{ or } 1)\). Now \( \phi_0 = [H_0(X)] \) and \( G_0 = [H_0(g)] \) are self-compact collections of arcs filling up \( N \). Let \( \phi_t = [X_t] \) and \( G_t = [g_t] \) be countable subcollections of \( \phi_0 \) and \( G_0 \) respectively such that

\[
\sum_{t=1}^{\infty} X_t = N = \sum_{t=1}^{\infty} g_t.
\]

These subcollections may be used in order to set up a sequence \( H_s(N) = N \) of topological transformations, each of which is the identity on \( \xi = 0, 0 \leq n \leq 1 \), whose limit is the homeomorphism \( H_\infty(N) = N \) with the property \( H_\infty H_0(X) = W^{-1}h^{-1}(x') \) for every point \( x' \) of \( M' \). Now define \( H = H_\infty H_0; \) then \( T(M) = H(W(M)) \).

In possibility (ii) \( X_0 = x_0 \) is a single point and \( X_1 \) is a nondegenerate arc. Just as above \( X_1 \) must be contained in \( B \). Moreover, \( x_0 \) is contained in \( B \), for suppose it were not. Then there exists a neighborhood \( U(x_0) \) disjoint with \( B \) and a point \( x' \) interior to \( x'_0 \) such that \( T^{-1}(x') \) is contained in \( U(x_0) \) and separates \( X_1 \) and \( x_0 \) in \( M \). This is impossible, since in the case considered \( T^{-1}(x') \) is an arc. Now let \( \{y_i\} \) be a sequence of points converging to \( x'_0 \). Then on \( M_i = T^{-1}(x'_i y_i) \), and generally on \( M_i = T^{-1}(y_{i-1} y_i) \) the transformation behaves as in (1). Hence just as for (1) there exists for each \( i \) a self-compact collection \( G^i = [g^i] \) of mutually disjoint arcs such that \( \sum g^i = M_i \) and for each \( X = T^{-1}(x') \) of \( M_i, g^i \cdot X \) is a single point. Now it may be assumed that

\* Foundations of Point Set Theory, pp. 396–397.
\dagger Foundations of Point Set Theory, Theorem 1, p. 397.
for each $i$, $y_i'_{i+1}$ precedes $y_i'$ in $x_0 x_1'$. For any point $y_1$ of $T^{-1}(y'_1)$ let $g_1$ be the arc of $G_1$ which has $y_1$ for an end point, and let $g_2$ be the arc of $G_2$ which has $y_1$ for an end point. Then the other end point of $g_2$ is a point of $T^{-1}(y'_2)$. Step by step for each $i$ let $g_i$ be the arc of $G_i$ having $y_{i-1}$ of $T^{-1}(y'_i)$ for one end point and denote the other, which must be in $T^{-1}(y'_i)$, by $y_i$. Define

$$g = \sum_{i=1}^{\infty} g_i;$$

then $g$ is an arc. Thus as $y_1$ ranges over $T^{-1}(y'_1)$ it generates a self-compact collection $G = [g]$ of arcs such that $\sum g = M$ and each $g$ intersects any $X = T^{-1}(x')$ of $M$ in a single point. As for the previous case, this collection along with $\phi = [X]$ may be used in connection with an arbitrary homeomorphism $h(N') = M'$, where $N$ is given by (b), to obtain a topological transformation $T(M) = hWH(M)$.

By the argument used above it follows that in possibility (iii) both the points $x_0 = X_0$ and $x_1 = X_1$ lie in $B$. Moreover, if $x'$ is an interior point of $M' = x_0 x_1'$, then $T$ behaves on $M_1 = T^{-1}(x_0 x_1')$ as in (ii). Hence if $h(N') = M'$ is an arbitrary homeomorphism, where $N$ and $W$ are given by (c), it follows from (b) that there exist homeomorphisms $H_i(M_i) = W^{-1} h_1^{-1}(x'_i x')$ such that $T(M_i) = hWH_i(M_i)$. Moreover, the $H_i(M_i)$ may be so defined that $H_0 T^{-1}(x') = H_1 T^{-1}(x')$. Define $H(x) = H_i(x)$ for $x$ a point of $M_i$; then $T(M) = hWH(M)$. Thus all three possibilities for $T^{-1}(x')$ an arc are characterized.

If for each $x'$ of $M'$, $T^{-1}(x')$ is a simple closed curve, then it follows from 4.21 and 2.33 that the inverse of one end point of $M' = x_0 x_1'$, say $T^{-1}(x_1')$, is the whole of $B$ while the inverse of the other end point $T^{-1}(x_0') = x_0$ is a single point in the interior of $M$. Since no point separates $M$, it therefore follows that the collection $\phi = [T^{-1}(x')]$ for all points $x'$ of $M'$ satisfies a theorem of KerékJártó.* Hence the collection $\phi$ is homeomorphic with a collection of concentric circles filling a circle. Thus for $N$ and $W$ of (d) there exists a topological transformation $H(M) = N$ such that $WHT^{-1}(x')$ is a point of $N'$ for every point $x'$ of $M'$ and conversely. Thus for each point $a'$ of $N'$ define $h(a') = x'$, where $WHT^{-1}(x') = a'$. Then $h(N') = M'$ is topological and such that $T(M) = hWH(M)$.

5.2. If $M$ is a sphere and $M'$ is a nondegenerate arc, then $T(M) = M'$ is equivalent to $W(N) = N'$, where $N$ is the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$, $N$ is the interval $-1 \leq \xi' \leq 1$, and $W$ is the transformation $\xi' = \xi$.

**Proof.** Since no arc separates the sphere, it follows from 4.2 and 2.33

---

that the inverse of every point \( x' \) of \( M' = x'_0 x'_1 \) is a nondegenerate simple closed curve except for the end points \( x'_0, x'_1 \), which must have degenerate inverses. Let \( y' \) be any interior point of \( M' \), \( M'_i = x'_i y'_i, M_i = T^{-1}(M_i' \setminus N'_i) \) the interval \([0, (-1)^i]\), and \( N_i = W^{-1}(N'_i) \) for \( i = 0 \) and \( 1 \). Then it follows from 5.1 (d) that there exist homeomorphisms \( h_i(N'_i) = M'_i \) and \( H_i(M_i) = N_i \) such that \( T(M_i) = h_i W H_i(M_i) \). It may be assumed \( h_0(0) = h_i(0) \) and \( H_i T^{-1}(y') = H_i T^{-1}(y') \). Define \( H(x) = H_i(x) \) for \( x \) a point of \( M_i \) and \( h(N') = M' \) accordingly. Then \( T(M) \equiv h W H(M) \).

5.3. If \( M \) is a circular ring and \( M' \) is a nondegenerate arc, then \( T(M) = M' \) is equivalent to \( W(N) = N' \) where \( N \) is the ring \( 1 \leq r^2 + \eta^2 \leq 2, N' \) is the interval \( 0 \leq \xi' \leq 1, \) and \( W \) is the transformation \( \xi' = (\xi^2 + \eta^2)^{1/2} - 1 \).

**Proof.** Since \( \rho^i(M) \) and \( \rho^j(M') \) are not equal, it follows from 4.33 that the inverse of every point \( x' \) of \( M' \) is a nondegenerate simple closed curve. Let \( B = J_0 + J_1 \), where \( J_i \) is a simple closed curve. Then it follows from 4.21 and 2.33 that \( J_i = T^{-1}(x'_i) \), where \( M' = x'_0 x'_1 \). Let \( H_0(M) = N \) be topological and suppose \( N^* \) to be the solid circle \( 0 \leq \xi^2 + \eta^2 \leq 1 \). Let this be filled with the family of concentric circles \( G = [g] \). Then the families \( G \) and \( \phi = [H_i T^{-1}(x')] \) for all \( x' \) of \( M' \) satisfy KerékJártói's condition \( \dagger \) that there exists a homeomorphism \( H_1(N + N^*) = N + N^* \) such that \( [H_1(g)] \) and \( [H_i H_0 T^{-1}(x')] \) together form a family of concentric circles filling \( N + N^* \). Moreover, it may be assumed that for the circle \( X_0 \) (that is, for \( \xi^2 + \eta^2 = 1 \)) \( H_1(X_0) = X_0 \). Define \( H = H_1 H_0 \); then for each \( x' \) of \( M' \), \( W H T^{-1}(x') = a' \), a point of \( N' \), and conversely. Now for each point \( a' \) of \( N' \) define \( h(a') = x' \), where \( W H T^{-1}(x') = a' \); then \( h(N') = M' \) is topological and \( T(M) = h W H(M) \).

5.4. If \( M \) is a circular ring and \( M' \) is a nondegenerate simple closed curve, then \( T(M) = M' \) is equivalent to the transformation \( \xi' = \cos \theta, \eta = \sin \theta \) on the circular ring \( N' \):

\[
\xi = r \cos \theta, \quad \eta = r \sin \theta \quad (0 \leq \theta \leq 2\pi, 1 \leq r \leq 2).
\]

**Proof.** Since \( S = 0 \) and \( M' \) has no end points, \( T^{-1}(x') \) is nondegenerate for each \( x' \) of \( M' \) and is not contained in \( B = J_0 + J_1 \). Thus, after 4.21, each \( T^{-1}(x') \) is a nondegenerate arc, and \( B \cdot T^{-1}(x') \) consists of the end points of \( T^{-1}(x') \) because of 2.32 and 2.5. Moreover, the end points of \( T^{-1}(x') \) must lie one in \( J_0 \) and one in \( J_1 \), for if both were contained in \( J_i \), \( T^{-1}(x') \) would separate \( M \) and consequently \( x' \) would separate \( M' \). Let \( y'_0 \) and \( y'_1 \) be any two points of \( M' \); then \( M' = a_0' + a_1' \), where \( a_0' \) is an arc and \( a_0' \cdot a_1' = y'_0 + y'_1 \). Define \( M_i = T^{-1}(a_i') \). Let \( W(N) = N' \) designate the transformation of the theorem and let \( a_i', a_1' \) be any two points of \( N' \). Express \( N' \) as the sum of

\(\dagger\) Topologie I, p. 246.
two arcs, \( N' = \beta_1' + \beta_2' \) where \( \beta_1' \cdot \beta_2' = a_1' + a_2' \). Define \( N'_i = W^{-1}(\beta'_i) \); then after 5.1 (a) there exist homeomorphisms \( h_i(\beta'_i) = \alpha'_i \) and \( H_i(M_i') = N_i \) such that \( T(M_i) = h_i W H_i(M_i) \). Moreover, the homeomorphisms may be so defined that \( h_1(a'_1) = h_2(a'_1) \) and \( H_i T^{-1}(y'_i) = H_i T^{-1}(y'_i) \). Let \( H(x) = H_i(x) \) on \( M_i \), and \( h(a') = h_i(a') \) on \( \beta_i \); then \( T(M) = h W H(M) \).

5.5. If \( M \) is a Möbius band and \( M' \) is a nondegenerate simple closed curve, then \( T(M) = M' \) is equivalent to the transformation \( W: \xi' = \cos \theta, \eta' = \sin \theta, \zeta' = 0 \), on the Möbius band \( N' \):

\[
\begin{align*}
\xi &= (2 + r \cos \theta/2) \cos \theta, & \eta &= (2 + r \cos \theta/2) \sin \theta, & \zeta &= r \sin \theta/2 \\
& (0 \leq \theta < 2\pi, -1 \leq r \leq 1).
\end{align*}
\]

**Proof.** Just as in 5.4 it follows that the inverse of each point \( x' \) of \( M' \) must be a nondegenerate arc with its end points only in \( B \). Again just as in 5.4, 5.1 (a) may be used to define the homeomorphisms \( h \) and \( H \) such that \( T(M) = h W H(M) \), where \( W(N) = N' \) is the analytical transformation of the theorem.

5.6. If \( M \) is a 2-cell with two boundary points identified and \( M' \) is a nondegenerate simple closed curve, then \( T(M) = M' \) is equivalent to the transformation \( W(N) = N' \): \( \xi' = \cos \theta, \eta' = \sin \theta \), where \( N \) is defined by

\[
\begin{align*}
\xi &= r \cos \theta, & \eta &= r \sin \theta, & 0 \leq \theta \leq 2\pi, 1 \leq r \leq (1/2)(3 - \cos \theta).
\end{align*}
\]

**Proof.** Here \( S = y_1 \) is a single point. Thus it follows from an argument similar to that used in 5.4 that except for \( T^{-1}(y_1) \) the inverse of every point \( x' \) of \( M' \) is a nondegenerate arc with its end points in \( B \) and separated in \( B \) by \( y_1 \). Express \( M' \) as the sum of two nondegenerate arcs, that is, as \( M' = \alpha_1' + \alpha_2' \), where \( \alpha_1' \cdot \alpha_2' = y_1' + y_2' \). Then 5.1 (b) may be applied here as 5.1 (a) was in 5.4 to give homeomorphisms such that \( T(M) = h W H(M) \).

5.7. If \( M \) is a pinched sphere and \( M' \) is a nondegenerate simple closed curve, then \( T(M) = M' \) is equivalent to the transformation \( W(N) = N' \): \( \xi' = \cos \theta, \eta' = \sin \theta \), \( \zeta' = 0 \), where \( N \) is defined by

\[
\begin{align*}
\xi &= (2 + \sin^2 (\theta/2) \cos \phi) \cos \theta, & \eta &= (2 + \sin^2 (\theta/2) \cos \phi) \sin \theta, \\
\zeta &= \sin^2 (\theta/2) \sin \phi & (0 \leq \theta, \phi \leq 2\pi).
\end{align*}
\]

**Proof.** Here again \( S = y_1 \) is a single point, Just as in 5.2 it follows that the inverse of every point of \( M' \), except \( T(y_1) \), is a nondegenerate simple closed curve. Express \( M' \) as the sum of two nondegenerate arcs one common end point of which is \( T(y_1) \). Then 5.1 (d) may be used to define homeomorphisms such that \( T(M) = h W H(M) \).
5.8. If \( M \) is a torus and \( M' \) is a nondegenerate simple closed curve, then \( T(M) = M' \) is equivalent to the transformation \( W(N) = N' \): \( \xi' = \cos \theta, \eta' = \sin \theta, \zeta' = 0 \), where \( N \) is defined by

\[
\xi = (2 + \cos \phi) \cos \theta, \quad \eta = (2 + \cos \phi) \sin \theta, \quad \zeta = \sin \phi \quad (0 \leq \theta, \phi \leq 2\pi).
\]

**Proof.** Since \( p_2^k(M) > p_2^k(M') \) it follows from 4.32 and 4.2 that the inverse of every point of \( M' \) is a nondegenerate simple closed curve. Let \( M' = \alpha_1' + \alpha_2' \), where \( \alpha_1' \) and \( \alpha_2' \) are nondegenerate arcs with common end points (that is, \( \alpha_1' \cdot \alpha_2' = y_1' + y_2' \)). Now \( M_i = T^{-1}(\alpha_i') \) is a 2-dimensional manifold with \( B = T^{-1}(y_1') + T^{-1}(y_2') \). Moreover, \( T(M_i) = \alpha_i' \), an arc. Thus \( M_i \) must be a circular ring, since this is the only 2-dimensional surface with \( B \) consisting of two simple closed curves, which maps into an arc by a monotone 0-regular transformation. Express \( N' = \beta_1' + \beta_2' \), where \( \beta_1' \) and \( \beta_2' \) are nondegenerate arcs with end points only in common. Define \( N_i = W^{-1}(\beta_i') \). Then 5.3 gives homeomorphisms \( h_i(\beta_i') = \alpha_i', \ H_i(M_i) = N_i \) such that \( T(M_i) = h_iWH_i(M_i) \). Moreover, the \( h_i \) may be so chosen that \( h_i(\beta_i' \cdot \beta_2') = h_2(\beta_1' \cdot \beta_2') \) and the \( H_i \) so chosen that \( H_i(M_1 \cdot M_2) = H_2(M_1 \cdot M_2) \). As several times before define \( H(x) = H_i(x) \) for \( x \) a point of \( M_i \) and \( h(a') = h_i(a') \) for \( a' \) a point of \( \beta_i' \); then \( T(M) = hWH(M) \).

**Observation.** Let \( Z_1 \) and \( Z_2 \) be simple closed curves on \( M \) which, when oriented, may be considered as generators of the Betti group \( B_2(M) \). For each pair of positive integers \( k_1, k_2 \) there exists a monotone 0-regular transformation of \( M \) into a nondegenerate simple closed curve such that for each \( x' \) of \( M' \) the simple closed curve \( T^{-1}(x') \) can be so oriented as to carry a cycle which is homologous to \( k_1Z_1 + k_2Z_2 \). In case \( k_i = 0 \), however, one must choose \( k_i = 1 \).

5.9. While the Klein bottle can be mapped onto a nondegenerate simple closed curve by a monotone 0-regular transformation, \( T(M) = M' \), there is not a convenient analytical description as in the previous cases. However, the possible transformations can be characterized after a fashion. In the first place \( T^{-1}(x') \), for every point \( x' \) of \( M' \), must be a nondegenerate simple closed curve, since \( p_2^k(M) > p_2^k(M') \). Let \( Z_1, Z_2 \) be simple closed curves on \( M \) which, when oriented, may be considered as generators of the Betti group \( B_2(M) \). Since \( M \) is a Klein bottle, it may be assumed that \( 2kZ_2 \sim 0 \) for all integers \( k \). There exist integers \( k_1 \) and \( k_2 \) such that

\[
T^{-1}(x') \sim k_1Z_1 + k_2Z_2,
\]

after \( T^{-1}(x') \) is oriented. However \( k_1 \) must be zero, for if it were not, then \( p_2^k[T^{-1}(x'), M] = 1 \). Consequently
\[ p_1 (M') = p_1 (M) - p_1 \left[ T^{-1}(x'), M \right] = 0, \]

contrary to the fact that \( M' \) is a nondegenerate simple closed curve. Thus for every \( x' \) of \( M' \), \( T^{-1}(x') \sim k_2 Z_2 \), when oriented. Moreover, \( k_2 \) must be odd, since \( p_1 \left[ T^{-1}(x'), M \right] \) cannot be zero.

In order to demonstrate such a mapping suppose \( M \) to arise from the oriented square \( ABCD \) by identifying the oriented sides, \( AB \) with \( DC \) and \( BC \) with \( DA \).\(^*\) Let \( T(M) = M' \) be such that the collection \( [T^{-1}(x')] \) in \( ABCD \) is a collection of straight lines parallel to \( BC \).

6. The continuous transformation \( T(M) = N \), a subset of \( M \), is said to be retracting\(^†\) provided that for each point \( x \) of \( N \), \( T(x) = x \). The following statements are immediate consequences of the results in the preceding sections:

6.1. In order that there exist a monotone 0-regular retracting transformation of the 2-dimensional pseudo-manifold \( M \) onto a nondegenerate arc, it is necessary and sufficient that \( M \) be a 2-cell, a circular ring, or a sphere.

6.2. In order that there exist a monotone 0-regular retracting transformation of the 2-dimensional pseudo-manifold \( M \) onto a nondegenerate simple closed curve, it is necessary and sufficient that \( M \) be a circular ring, a Möbius band, a torus, a Klein bottle, a 2-cell with two boundary points identified, or a pinched sphere.

6.3. There exist no monotone 0-regular retracting transformations of 2-dimensional pseudo-manifolds onto nondegenerate sets except those given by 6.1 and 6.2.

7. A collection \( G \) of continua is said to be equicontinuous\(^‡\) with respect to a given set \( M \) if for every collection \( H \) of open sets covering \( M \) there exists a finite collection \( H' \) of open sets covering \( M \) such that if \( x_1 \) and \( x_2 \) are two points of \( M \) lying in some one set of \( H' \) and belonging to a continuum \( X \) of \( G \), then there exists an arc \( x_1 x_2 \) lying both in \( X \) and in some set of the collection \( H \). The collection \( G \) is said to be self-compact\(^‡\) if every infinite sequence of continua of the collection \( G \) contains an infinite subsequence which converges to some set of the collection \( G \).

Let \( M \) be any compact space and \( T(M) = M' \) be a monotone 0-regular transformation. Then, obviously, the collection \( G = [T^{-1}(x')] \) for all points \( x' \) of \( M' \) is an equicontinuous self-compact collection of mutually disjoint continua filling \( M \). Moreover, it is easily seen that an equicontinuous self-compact collection \( G = [X] \) of mutually disjoint continua filling \( M \) gives rise

\(^*\) See Alexandroff-Hopf, p. 207.
\(^†\) See Borsuk, Fundamenta Mathematicae, vol. 18 (1932), p. 204.
\(^‡\) Foundations of Point Set Theory, pp. 396–397.
to a monotone 0-regular transformation. Thus the following assertions are immediate consequences of the results of §§4 and 5:

7.1. Let $M$ be a 2-dimensional pseudo-manifold and $G = [X]$ be an equicontinuous self-compact collection of mutually disjoint continua filling $M$. If $G$ contains more than one element, then each $X$ is an arc or each $X$ is a simple closed curve.

7.2. In order that a 2-dimensional pseudo-manifold $M$ may be decomposed into an equicontinuous self-compact collection of mutually disjoint arcs, at least one of which is nondegenerate, it is necessary and sufficient that $M$ be a 2-cell, a 2-cell with two boundary points identified, a circular ring, or a Möbius band.

7.3. In order that a 2-dimensional pseudo-manifold $M$ may be decomposed into an equicontinuous self-compact collection of mutually disjoint simple closed curves, at least one of which is nondegenerate, it is necessary and sufficient that $M$ be a 2-cell, a circular ring, a sphere, a pinched sphere, a torus, or a Klein bottle.

8. A. D. Wallace* has shown that if $T(M) = T_2T_1(M)$ is the usual monotone-light factoring of any 0-regular transformation, then $T_1$ is a monotone 0-regular transformation and $T_2$ is a local homeomorphism.† Moreover, he has shown that when the image is nondegenerate any 0-regular transformation on an arc or on a simple closed curve is a homeomorphism or a local homeomorphism respectively. Thus the following assertions are consequences of the results of §§3, 4, and 5:

8.1. If $M$ is a 2-dimensional pseudo-manifold and $T(M) = M'$ is a 0-regular transformation, then $M'$ is a 2-dimensional pseudo-manifold, an arc, or a simple closed curve.

8.2. Let $M$ be a 2-dimensional pseudo-manifold and $T(M) = M'$ a 0-regular transformation. If $M'$ is a nondegenerate arc, then $T$ is monotone and, consequently, is equivalent to one of the transformations in 5.1, 5.2, or 5.3.

Since a local homeomorphism on a simple closed curve is equivalent to the transformation $W': \xi' = \cos k\theta, \eta' = \sin k\theta (k \text{ an integer})$ on the circle $\xi = \cos \theta, \eta = \sin \theta (0 \leq \theta \leq 2\pi)$,‡ the following assertion is immediate:

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* On 0-regular transformations, loc. cit.
† The transformation $T(M) = M'$ is said to be a local homeomorphism if for each point $x$ of $M$ there exists a neighborhood $U(x)$ on which $T$ is topological. See S. Eilenberg, Fundamenta Mathematicae, vol. 24 (1935), p. 35.
8.3. Let $M$ be a 2-dimensional pseudo-manifold and $T(M) = M'$ a 0-regular transformation. If $M'$ is a nondegenerate simple closed curve, then $T$ is equivalent to the transformation $WW'$, where $W'$ is given above and $W$ is one of the transformations in 5.4, 5.5, 5.6, 5.7, 5.8, or 5.9.

The following assertion results from 6.3, 8.2, and 8.3:

8.4. If $M$ is a 2-dimensional pseudo-manifold and $T(M) = N$ is a 0-regular retracting transformation, then $T$ must be monotone.