HARMONIC SURFACES IN RIEMANN METRIC*

BY

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If x^i $(i=1, \cdots, n)$ is a vector representing a map from a domain Q in euclidean (u, v)-space into the euclidean x^i-space, the Euler equations of the problem

$$\iint (x_u^i x_u^i + x_v^i x_v^i) du dv = \min, \qquad \text{summation over } i, \tag{1}$$

have the simple form

$$\Delta x^i \equiv x_{uu}^i + x_{vv}^i = 0. \tag{2}$$

The boundary value problem reduces trivially to the classical Dirichlet problem in its simplest form. If the x^i-space is a Riemannian space S with a positive definite metric, the generalization of (1) is

$$\iint g_{pq}(x)(x_u^p x_u^q + x_v^p x_v^q) du dv = \min,$$

and the Euler equations are

$$\frac{\partial g_{pq}}{\partial x^i}(x_u^p x_u^q + x_v^p x_v^q) - 2\frac{\partial}{\partial u}(g_{iq} x_u^q) - 2\frac{\partial}{\partial v}(g_{iq} x_v^q) = 0.$$

Putting

$$\frac{\partial g_{pq}}{\partial x^i} = g_{p\alpha}\Gamma_{iq}^{\alpha} + g_{\alpha q}\Gamma_{pi}^{\alpha}, \tag{3}$$

we obtain the equivalent equations

$$\Delta x^i + \Gamma_{pq}^i (x_u^p x_u^q + x_v^p x_v^q) = 0. \tag{4}$$

Any solution of (4) will be called a *harmonic surface*.

Equations (4) are nonlinear in the first derivatives of x^i and the boundary value problem is now decidedly more difficult than in the classical case (2). There is no direct generalization of Poisson's integral and the results obtained directly by Picard's approximation method are geometrically very restricted.†

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† G. Giraud, *Sur le problème de Dirichlet généralisé*, II, Annales de l'École Normale Supérieure, vol. 46 (1929), pp. 131–245.

However S. Bernstein's elaborate method of majorization* can be shown to be effective in our case, and in the present note we shall offer some first results in this direction. In §1 we shall state some elementary properties of harmonic surfaces and in §2 a uniqueness theorem relative to the boundary value problem; and in §3 we shall prove a theorem on compactness of families of harmonic surfaces.

1. **Subharmonic character of harmonic surfaces.** Equations (4) are an extension, from one to two parameters, of the equations for geodesics:

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds} = 0. \tag{5}$$

As in the latter case, equations (4) can be easily seen to be formally invariant under transformations of coordinates in the x^i-space, the expressions on the left side of (4) being the components of a vector relative to such transformations. Other quantities which are formally invariant are the coefficients

$$E = g_{pq}\overset{p}{x}_u\overset{q}{x}_u, \qquad F = g_{pq}\overset{p}{x}_u\overset{q}{x}_v, \qquad G = g_{pq}\overset{p}{x}_v\overset{q}{x}_v \tag{6}$$

of the fundamental form $Edu^2+2Fdudv+Gdv^2$. These coefficients are scalars.

As in the euclidean case, the expression

$$E - G - 2F(-1)^{1/2} = g_{pq}(x)(\overset{p}{x}_u - (-1)^{1/2}\overset{p}{x}_v)(\overset{q}{x}_u - (-1)^{1/2}\overset{q}{x}_v)$$

is an analytic function of the complex variable $u+iv$. In other words we have the Cauchy-Riemann equations

$$E_u - G_u = -2F_v, \qquad E_v - G_v = 2F_u, \tag{7}$$

$$\Delta E = \Delta G, \qquad \Delta F = 0. \tag{8}$$

Relations (7) are formal identities. They hold for expressions (6), subject to relations (3) and (4). The differentiations involved can be reduced to a minimum by the use of geodesic coordinates. The first derivatives of E, F, G with respect to u and v involve the coefficients g_{pq}, Γ^i_{pq} but no derivatives of Γ^i_{pq}. In order to prove (7) for a fixed value (u_0, v_0) we may select geodesic coordinates originating at the point $x^i(u_0, v_0)$, for which, at the origin,

$$\Gamma^i_{pq} = 0, \qquad g_{pq} = \delta_{pq}. \tag{9}$$

Thus the proof of (7) consists of proving the same trivial identities as in the euclidean case.

* S. Bernstein, *Sur la généralisation du problème de Dirichlet*, Mathematische Annalen, vol. 69 (1910), pp. 82–136, §1.

If S is a space whose Riemannian curvature is nonpositive, then relations (8) can be supplemented by the further relation

$$\Delta E \geqq 0, \qquad \Delta G \geqq 0. \tag{10}$$

We shall derive a more comprehensive result relating to harmonic "surfaces" in several dimensions. If we replace the two parameters u, v by parameters u^α, $\alpha=1, \cdots, m$, and put

$$\Delta x = \frac{\partial^2 x}{\partial u^\alpha \partial u^\alpha} \equiv \sum_{\alpha=1}^{m} \frac{\partial^2 x}{\partial u^\alpha \partial u^\alpha}, \qquad E_{\beta\gamma} = g_{pq}\frac{\partial x^p}{\partial u^\beta}\frac{\partial x^q}{\partial u^\gamma},$$

then the equations

$\ast$
$$\Delta x^i + \Gamma^i_{pq}\frac{\partial x^p}{\partial u^\alpha}\frac{\partial x^q}{\partial u^\alpha} = 0$$

are the Euler equations of the problem

$$\int\int E_{\alpha\alpha}\, du_1 du_2 \cdots du_m \equiv \min.$$

If for given values u^α, we select a coordinate system which is geodesic at the point $x^i(u^\alpha)$, then we obtain

$$\frac{\partial E_{\beta\gamma}}{\partial u^\alpha} = g_{pq}\left(\frac{\partial^2 x^p}{\partial u^\beta \partial u^\alpha}\frac{\partial x^q}{\partial u^\gamma} + \frac{\partial x^p}{\partial u^\beta}\frac{\partial^2 x^q}{\partial u^\gamma \partial u^\alpha}\right) + g_{p\alpha}\Gamma^\alpha_{qr}\left(\frac{\partial x^p}{\partial u^\beta}\frac{\partial x^q}{\partial u^\gamma} + \frac{\partial x^p}{\partial u^\gamma}\frac{\partial x^q}{\partial u^\beta}\right)\frac{\partial x^r}{\partial u^\alpha}$$

and hence, putting $\Gamma^i_{pq}=0$,

$$\Delta E_{\beta\gamma} = g_{pq}\left(\Delta\frac{\partial x^p}{\partial u^\beta}\cdot\frac{\partial x^q}{\partial u^\gamma} + \frac{\partial x^p}{\partial u^\beta}\cdot\Delta\frac{\partial x^q}{\partial u^\gamma}\right) + 2g_{pq}\frac{\partial^2 x^p}{\partial u^\beta \partial u^\alpha}\frac{\partial^2 x^q}{\partial u^\gamma \partial u^\alpha}$$
$$+ g_{p\alpha}\frac{\partial \Gamma^\alpha_{qr}}{\partial x^s}\frac{\partial x^s}{\partial u^\alpha}\left(\frac{\partial x^p}{\partial u^\beta}\frac{\partial x^q}{\partial u^\gamma} + \frac{\partial x^p}{\partial u^\gamma}\frac{\partial x^q}{\partial u^\beta}\right)\frac{\partial x^r}{\partial u^\alpha}.$$

Now, using the formula from $\ast$ and $\Gamma^p_{ab}=0$

$$\frac{\partial}{\partial u^\beta}\Delta x^p = -\frac{\partial \Gamma^p_{ab}}{\partial x^q}\frac{\partial x^q}{\partial u^\beta}\frac{\partial x^a}{\partial u^\alpha}\frac{\partial x^b}{\partial u^\alpha}$$

and a similar formula for $\partial\Delta x^q/\partial u^\gamma$ and putting $R_{pqrs} = g_{p\alpha}(\partial\Gamma^\alpha_{rs}/\partial x^q - \partial\Gamma^\alpha_{rq}/\partial x^s)$, we finally obtain

$$\Delta E_{\beta\gamma} = 2g_{pq}\frac{\partial^2 x^p}{\partial u^\beta \partial u^\alpha}\frac{\partial^2 x^q}{\partial u^\gamma \partial u^\alpha} - R_{pqrs}\left(\frac{\partial x^p}{\partial u^\beta}\frac{\partial x^q}{\partial u^\gamma} + \frac{\partial x^q}{\partial u^\beta}\frac{\partial x^p}{\partial u^\gamma}\right)\frac{\partial x^r}{\partial u^\alpha}\frac{\partial x^s}{\partial u^\alpha}.$$

For any quantities λ^α, putting $\mu_\alpha{}^p = \lambda^\beta(\partial^2 x^p/\partial u^\beta \partial u^\alpha)$, $\nu^p = \lambda^\beta(\partial x^p/\partial u^\beta)$, we hence obtain

$$\Delta E_{\beta\gamma}\lambda^\beta\lambda^\gamma = 2g_{pq}\mu_\alpha^p\mu_\alpha^q - 2R_{pqrs}\nu^p\nu^q \frac{\partial x^r}{\partial u^\alpha}\frac{\partial x^s}{\partial u^\alpha}.$$

Thus, for a space S of nonpositive curvature, $\Delta E_{\beta\gamma}\lambda^\beta\lambda^\gamma \geqq 0$, and this implies relations (10).

Finally we shall indicate a proof for the assertion that in a space S of nonpositive curvature, the curvature $K(u, v)$ of any harmonic surface is also nonpositive. By a general theorem of Ricci,* K is the sum of two quantities K_1 and K_2; K_1 is the so-called relative curvature of the surface and K_2 is the curvature of S at the point $x^i(u, v)$. Since $K_2 \leqq 0$ by assumption, we have only to show that $K_1 \leqq 0$. Now, if (9) holds, the quantity K_1 can be computed by a formula which is formally the same as in the euclidean case, namely

$$(EG - F^2)K_1 = \left| x_{uu}^i, x_u^i, x_v^i \right| \left| x_{vv}^i, x_u^i, x_v^i \right| - \left| x_{uv}^i, x_u^i, x_v^i \right| \left| x_{uv}^i, x_u^i, x_v^i \right|.$$

Each term on the right side is the (numerical) product of two matrices of 3 rows and n columns each. Now, (9) implies $x_{vv}^i = -x_{uu}^i$ and therefore $K_1 \leqq 0$.

2. **The distance function.** If two solutions of (2) over the same region Q have the same limit values on the boundary B of Q, then they are identical throughout Q. We must not expect a similar theorem to hold without restriction for solutions of (4) since it does not even hold for solutions of the simpler equations (5). However, the restrictions required for equations (4) are exactly the same as those needed for equations (5). It is sufficient to assume that the solutions lie in a region U of the space S in which there are no conjugate points. More precisely we assume that any two points x^i and y^i of U can be joined in S by a geodesic arc whose closure contains no conjugate points and whose geodesic length is the shortest geodesic distance between the points x^i and y^i. Denoting the latter distance by $H(x^i, y^i)$, if the space S belongs to class C_4, then $H(x^i, y^i)$ belongs to class C_2 on the domain $U \times U$ minus the diagonal manifold $x^i = y^i$.

For an arbitrary function $W = W(x^i, y^i)$ and arbitrary numbers ξ^i, η^i we shall consider the expression

$$(W_{x^p x^q} - \Gamma_{pq}^i(x)W_{x^i})\xi^p\xi^q + 2W_{x^p y^q}\xi^p\eta^q + (W_{y^p y^q} - \Gamma_{pq}^i(y)W_{y^i})\eta^p\eta^q$$

(the subscripts of W indicating partial derivations) and we shall denote this expression by

$$P(W) \equiv P(W; \xi, \eta).$$

* L. P. Eisenhart, *Riemannian Geometry*, 1926, p. 174.

We shall say that *a function* $W(x^i, y^j)$ *has positive character, if* $P(W; \xi, \eta) \geqq 0$ *identically in* x^i, y^j; ξ^i, η^j (*assuming* $x^i \neq y^i$, *if necessary*).

We shall now prove

LEMMA 1. *The distance function* $H(x^i, y^j)$ *has positive character.*

We take a geodesic arc issuing from x^i, whose direction cosines at the point of issuance have the given values ξ^i, and on it a point of geodesic distance ϵ. Denoting the coordinates of the latter point by $x^i + \xi^i(\epsilon)$ we will have by familiar formulas from the theory of normal coordinates,*

$$\xi^i(\epsilon) = \epsilon\xi^i - \frac{\epsilon^2}{2!}\Gamma^i_{pq}(x)\xi^p\xi^q + o(\epsilon^2) \tag{11}$$

and

$$H(x^i, x^j + \xi^j(\epsilon)) = \epsilon\{g_{pq}(x)\xi^p\xi^q\}^{1/2}.$$

In particular

$$\frac{d}{d\epsilon}H(x^i, x^j + \xi^j(\epsilon)) = 0. \tag{12}$$

Similarly we put

$$\eta^i(\epsilon) = \epsilon\eta^i - \frac{\epsilon^2}{2!}\Gamma^i_{pq}(y)\eta^p\eta^q + o(\epsilon^2) \tag{13}$$

and obtain

$$\frac{d}{d\epsilon}H(y^i, y^j + \eta^j(\epsilon)) = 0. \tag{14}$$

Since there are no conjugate points on the geodesic arc joining x^i and y^j, the second derivative with respect to ϵ of

$$H(x^i, x^j + \xi^j(\epsilon)) + H(x^i + \xi^i(\epsilon), y^j + \eta^j(\epsilon)) + H(y^i, y^j + \eta^j(\epsilon))$$

is greater than or equal to zero for $\epsilon = 0$. On account of (12) and (14), we therefore have

$$\lim_{\epsilon \to 0} \frac{d^2}{d\epsilon^2} H(x^i + \xi^i(\epsilon), y^j + \eta^j(\epsilon)) \geqq 0.$$

Substituting (11) and (13) we hence obtain $P(H; \xi, \eta) \geqq 0$, which is the assertion of our lemma.

LEMMA 2. *If* $\chi(t)$ *is differentiable twice and* $\chi'(t) \geqq 0$ *and* $\chi''(t) \geqq 0$, *and if* $W(x^i, y^j)$ *has positive character, then* $\chi(W(x^i, y^j))$ *has positive character too.*

* T. Y. Thomas, *Differential Invariants of Generalized Spaces*, 1934, p. 7.

In fact,

$$P(\chi(W);\xi,\eta) = \chi'(W)\cdot P(\chi(W);\xi,\eta) + \chi''(W)\cdot(W_{x^p}\xi^p + W_{y^p}\eta^p)^2.$$

In particular, we see that the function $(H(x^i, y^i))^2$ has positive character, and this function has the advantage of belonging to class C_2 throughout $U\times U$, the manifold $x^i=y^i$ not being excepted.

LEMMA 3. *If $W(x^i, y^i)$ has positive character and $x^i(u, v)$, $y^i(u, v)$ are two harmonic surfaces, then the function $f(u, v)=W(x^i(u, v), y^i(u, v))$ is subharmonic, that is $\Delta f\geqq 0$.*

In fact, carrying out the differentiations and replacing Δx^i and Δy^i by $-\Gamma^i_{pq}(x)(x_u{}^p x_u{}^q+x_v{}^p x_v{}^q)$ and $-\Gamma^i_{pq}(y)(y_u{}^p y_u{}^q+y_v{}^p y_v{}^q)$ respectively, we obtain

$$\Delta f = P(W; x_u, y_u) + P(W; x_v, y_v),$$

and therefore $\Delta f\geqq 0$.

Since a subharmonic function attains its maximum on the boundary and since the distance function $H(x^i, y^i)$ vanishes only for $x^i=y^i$, we may conclude the following theorem.

THEOREM 1. *If U is a domain of S on which the distance function $H(x^i, y^i)$ has positive character, then for any two harmonic surfaces $x^i(u, v)$, $y^i(u, v)$, the function $H(x^i(u, v), y^i(u, v))$ attains its maximum on the boundary B of Q. If $x^i(u, v)=y^i(u, v)$ on B, then this equality holds throughout $Q+B$.*

If $K(a^i, \sigma)$ is a geodesic sphere $H(a^i, x^i)<\sigma$ in U, then any harmonic surface in U whose boundary lies in $K(a^i, \sigma)$ lies there in its entirety.

The last part of the theorem is a consequence of the first if we put $y^i(u, v)\equiv a^i$.

3. **The Dirichlet integrand on the boundary.** Using polar coordinates $u=\rho\cos\theta$, $v=\rho\sin\theta$, we assume now that Q is the unit circle $\rho<1$, $0\leqq\theta<2\pi$, and we shall denote the same function in Q by $x(u, v)$ and $x(\rho, \theta)$ interchangeably. If a periodic function $\phi(\theta)$ is continuous, and has m continuous derivatives, $m=0, 1, \cdots$, we shall write

$$\|\phi(\theta)\|_m = \max_{0\leqq\theta<2\pi} \left| d^m\phi(\theta)/d\theta^m\right|.$$

In case of a vector $\phi^i(\theta)$ we shall write

$$\|\phi(\theta)\|_m = \sum_{i=1}^{n}\|\phi^i(\theta)\|_m.$$

This symbolism will refer exclusively to the variable denoted by θ.

The theorem to follow will apply to a sufficiently small neighborhood U

of a fixed point on S in which the distance function has positive character. Making U sufficiently small and choosing the coordinate system appropriately, we may first assume that U is the entire cell $-\infty < x^i < \infty$, $i=1, \cdots, n$, the fixed point of U being the origin $x^i=0$. U contains some geodesic sphere $K(0, \sigma')$ whose radius will be also denoted by $3\sigma^0$. Obviously, for $0<\sigma<3\sigma^0$ there exist two functions $r'(\sigma)$ and $r''(\sigma)$, $0<r'<r''$, such that the geodesic sphere $K(0, \sigma)$ contains the euclidean sphere $x^ix^i \leqq r'^2$ and is in its turn contained in the euclidean sphere $x^ix^i \leqq r''^2$. We further observe that for $x^i(u, v)$ in $K(0, 2\sigma^0)$, the quantity

$$E + G = g_{pq}(x)(\overset{p}{x}_u\overset{q}{x}_u + \overset{p}{x}_v\overset{q}{x}_v) \tag{15}$$

is majorized above and below by

$$\overset{i}{x}_u\overset{i}{x}_u + \overset{i}{x}_v\overset{i}{x}_v. \tag{16}$$

Now, for $a^i \subset K(0, 2\sigma^0)$ and $x^i \subset K(0, \sigma^0)$ we put $\xi^i = x^i - a^i$, and we consider the function $W(a^i, \xi^i) = (H(a^i, x^i))^2/2$, and the quantities

$$\eta^i = \eta^i(a^p, \xi^q) = W_{\xi^i}. \tag{17}$$

Since, for $\xi^i=0$, $\partial\eta^p/d\xi^q = g_{pq}(a)$, we may consider the inverse functions

$$\xi^i = \xi^i(a^p, \eta^q). \tag{18}$$

They are defined, and they are inverse to the functions (17), for

$$a^i \subset K(0, 2\sigma^0), \qquad \eta^i\eta^i \leqq r_0^2 \qquad (> 0), \tag{19}$$

the number r_0 being independent of a^i.

We now consider a solution $x^i(u, v)$ of (4) in Q which lies in $K(0, \sigma^0)$ and whose partial derivatives are uniformly continuous in Q. The functions $x^i(u, v)$ and their partial derivatives of first and second order have continuous boundary values on B. The function $f(u, v) = f(\rho, \theta) = W(a^i, x^i(u, v) - a^i)$ has partial derivatives of second order which are uniformly continuous in Q, and satisfies

$$\|f(1, \theta)\|_m \leqq M\|x(1, \theta)\|_m, \qquad m = 0, 1, 2,$$

the constant M being independent of $x^i(u, v)$ but not necessarily the same in all relations in which it will occur. The important property of $f(u, v)$ is that it is subharmonic, $\Delta f \geqq 0$ (see Theorem 1). We now put

$$f(\rho, \theta) = g(\rho, \theta) + h(\rho, \theta)$$

where $g(\rho, \theta)$ is the harmonic function with the boundary values $f(1, \theta)$ and

$h(\rho, \theta)$ is a subharmonic function with boundary values 0. For $g(\rho, \theta)$ we have*

$$\left|\frac{\partial g(\rho, \theta)}{\partial \rho}\right| + \left|\frac{\partial g(\rho, \theta)}{\partial \theta}\right| \leqq M\|x(1, \theta)\|_2.$$

For the function $h(\rho, \theta)$ we immediately obtain the relation

$$\lim_{\rho\to 1}\left|\frac{\partial h(\rho, \theta)}{\partial \theta}\right| \equiv \left|\frac{\partial h(1, \theta)}{\partial \theta}\right| = 0,$$

which is an immediate consequence of the assumption that $h(\rho, \theta)$ shall vanish on the boundary. For the radial derivative we obtain the one-sided relation

$$\lim_{\rho\to 1}\frac{\partial h(\rho, \theta)}{\partial \rho} \equiv \frac{\partial h(1, \theta)}{\partial \rho} \geqq 0$$

with no absolute values to start with. It follows from the fact that $h(\rho, \theta)$ is zero on the boundary and, being subharmonic, is nonpositive in Q. Altogether we have the estimates

$$\left|\frac{\partial f(1, \theta)}{\partial \theta}\right| \leqq M\|x(1, \theta)\|_2, \qquad -\frac{\partial f(1, \theta)}{\partial \rho} \leqq M\|x(1, \theta)\|_2,$$

and this can also be written in the form

$$\max\left|\eta^i(1, \theta)\,\frac{\partial x^i(1, \theta)}{\partial \theta}\right| - \eta^i(1, \theta)\,\frac{\partial x^i(1, \theta)}{\partial \rho} \leqq M\|x(1, \theta)\|_2.$$

The constant M is independent of θ and a^i, and all combinations (19) are admissible. Hence we obtain the estimate

$$\left|\frac{\partial x^i(1, \theta)}{\partial \theta}\right| + \left|\frac{\partial x^i(1, \theta)}{\partial \rho}\right| \leqq M\|x(1, \theta)\|_2,$$

the second quantity on the left side being also an absolute value. Since

$$\left(\frac{\partial x^i}{\partial u}\right)^2 + \left(\frac{\partial x^i}{\partial v}\right)^2 = \left(\frac{\partial x^i}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial x^i}{\partial \theta}\right)^2,$$

we finally obtain the following theorem.

THEOREM 2. *If S belongs to class C_4, if U is a sufficiently small coordinate neighborhood of the origin, and if $x^i(u, v) = x^i(\rho, \theta)$ is a harmonic surface in U*

* H. Whitney, *Analytic coordinate systems in a manifold*, Annals of Mathematics, (2), vol. 38 (1937), p. 809, Lemma 4.

whose partial derivatives of the second order are uniformly continuous in the unit circle Q, then the inequality

$$\{E(1, \theta) + G(1, \theta)\}^{1/2} \leqq M \sum_{i=1}^{n} \max \left| \frac{\partial^2 x^i(1, \theta)}{\partial \theta^2} \right|$$

holds for a constant M which is independent of the given surface.

If S has nonpositive curvature, then the sharper inequality

$$\{E(u, v) + G(u, v)\}^{1/2} \leqq M \sum_{i=1}^{n} \max \left| \frac{\partial^2 x^i(1, \theta)}{\partial \theta^2} \right| \tag{20}$$

holds for the same constant M.

The second half of the theorem follows from the first since for spaces of nonpositive curvature, $E+G$ attains its maximum on the boundary of Q. But the inequality (20) probably holds for spaces S of any curvature.

From Theorem 2 we could easily deduce the existence theorem* that any vector $\phi^i(\theta)$ in U which has two continuous derivatives is the boundary of a harmonic surface in U.

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* J. Léray et J. Schauder, *Topologie et équations fonctionelles*, Annales de l'École Normale Supérieure, vol. 51 (1934), pp. 45–78.

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