

INTEGRATION IN A CONVEX LINEAR TOPOLOGICAL SPACE*

BY

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This paper is concerned with a theory of integration for functions with values in a convex linear topological space. We consider an integral which is essentially an extension to this general space of the integral studied by Garrett Birkhoff [1] in a Banach space. By imposing different convex neighborhood topologies on a Banach space, we obtain as instances of our integral those defined by Birkhoff [1], Dunford [2], Gelfand [3], and Pettis [4].

Let $f(s)$ be a function on an abstract set S to the real numbers, and let $\alpha(\sigma)$ be a nonnegative measure function on an additive family of subsets \mathcal{S} of S . A necessary and sufficient condition for the Lebesgue integral to exist is that for each $\epsilon > 0$ there exist a partition Δ_ϵ of S into a denumerable set of sets (σ_i) such that for any two orderings of these sets, (σ_i^1) and (σ_i^2) ,

$$\left| \limsup_n \sum \alpha(\sigma_i^1) \cdot \sup_{s_i \in \sigma_i^1} f(s_i) - \liminf_n \sum \alpha(\sigma_i^2) \cdot \inf_{s_i \in \sigma_i^2} f(s_i) \right| < \epsilon.$$

We have introduced upper and lower limits in order to stress the fact that unconditional convergence is an unnecessary concept in the definition of the Lebesgue integral. In fact by avoiding unconditional convergence we have been able to extend this integral to a linear convex topological space. This further permits a simplification in the concepts involved.

In the first section we consider a few relations between linear continuous operations and convex neighborhoods which have been found useful in the study of the integral. In §2 the integral is defined and some essential properties of unconditional summability are established. This is followed in §3 by a discussion of the basic properties of the integral. The integral of an integrable function x is an absolutely continuous and completely additive set function depending linearly on x . It is found in §4 that the integrability conditions can be relaxed when the space satisfies a certain type of completeness condition. A demonstration that the linear continuous transform of an integrable function is itself integrable occupies §5. Section 6 treats convergent sequences of integrable functions, while §7 deals with relations to other integrals. A theorem on differentiation is proved in §8. In §9 an application is made of this theory to an existence theorem in differential equations. Finally in §10

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some examples are given to demonstrate properties of the integral. Example 10.2 shows that the Birkhoff and Pettis integrals are different. As a consequence our development leads to the definition of a class of integrals for the theory of integration on a Banach space.

1. **On convex linear topological spaces.** We will restrict ourselves in this paper to the type of convex linear topological space X defined by Kolmogoroff [5, p. 29] to satisfy the following axioms:

1. X is a linear space relative to real numbers [6, p. 26].
2. There exists an operation of closure, \bar{B} , for any subset B of X with the following properties:
 - 2a. If B consists of a single element of X , then $\bar{B} = B$.
 - 2b. $\overline{\bar{B}} = \bar{B}$.
 - 2c. $\overline{B+C} = \bar{B} + \bar{C}$ (+ stands for set addition).
3. Addition of elements and multiplication by numbers are continuous.
4. There exists an equivalent neighborhood system of convex* neighborhoods \mathcal{U} . Kolmogoroff has shown that Axioms 1, 2, 3 are sufficient to make X a regular Hausdorff space with a uniform topology.† We will therefore use only neighborhoods of the null vector.

In a convex space *pseudo-norms* have been defined by von Neumann [7, pp. 18–19] as follows: Let $\|x\|_U^+$ equal the greatest lower bound of $a > 0$ such that $x \in aU$, and let $\|x\|_U = \max [\|x\|_U^+, \|-x\|_U^+]$. He has established the following properties by arguments that are applicable to our space:

$$\|x + y\|_U \leq \|x\|_U + \|y\|_U, \quad \|ax\|_U = |a| \cdot \|x\|_U, \quad \|\theta\|_U = 0,$$

$\|x\|_U$ is a continuous function of x , and the sets $\|x\|_U < \delta$, $U \in \mathcal{U}$, $\delta > 0$ form a complete system of neighborhoods for θ . The norm has the additional property: $\|x\| = 0$ if and only if $x = \theta$. This is not a property of the pseudo-norm. In the remainder of this paper we will use the system of neighborhoods defined by the pseudo-norms and designate by U the set of all x such that $\|x\|_U < 1$. Hence $U_1 \supset U_2$ is equivalent to the statement that $\|x\|_{U_2} < 1$ implies $\|x\|_{U_1} < 1$. Since the pseudo-norm is homogeneous and continuous, this is in turn equivalent to $\|x\|_{U_1} \leq \|x\|_{U_2}$.

Let us now set up a one-to-one correspondence between the set of all linear continuous operations‡ on X and an abstract class T of the same cardinal power; say, \bar{x}_i corresponds to $t \in T$. We proceed to prove that the neighborhoods of X can be defined by linear continuous operations on X .

* A set B is convex if, whenever $x, y \in B$, $a, b \geq 0$, and $a+b=1$, then $ax+by \in B$.

† X is said to have a *uniform topology* if, when $\mathcal{U}(x)$ and $\mathcal{U}(\theta)$ are the neighborhood systems for x and θ respectively, $x + \mathcal{U}(\theta)$ is equivalent to $\mathcal{U}(x)$.

‡ The term "operation" is used for a numerical function.

THEOREM 1.1. *For each $U \in \mathcal{U}$ there exists a subset T_U of T such that $[x | x \in X, |\bar{x}_t(x)| < 1 \text{ for every } t \in T_U] \equiv [x | x \in X, \|x\|_U < 1]$.**

Wehausen [8, p. 162, Theorem 8] has demonstrated that given any $x_0 \in X$ and $U \in \mathcal{U}$, there exists a linear continuous operation \bar{x}_0 on X with the property that $|\bar{x}_0(x)| \leq \|x\|_U$ and $\bar{x}_0(x_0) = \|x_0\|_U$. To each $x \in X$ corresponds in this fashion \bar{x}_t . To the set of $x \in X$ for which $\|x\|_U = 1$ will then correspond a subset $T_U \equiv [t_x]$ of T . For $t \in T_U$, $|\bar{x}_t(x)| \leq \|x\|_U$, and hence $[x | x \in X, |\bar{x}_t(x)| < 1 \text{ for every } t \in T_U] \supset [x | x \in X, \|x\|_U < 1]$. Further if $|\bar{x}_t(x)| < 1$ for every $t \in T_U$, then either $\|x\|_U = 0$, or $\|x\|_U \neq 0$, in which case there exist $t_{ax} \in T_U$ and $a > 1$ such that $|\bar{x}_{t_{ax}}(ax)| = 1 = \|ax\|_U$ and $\|x\|_U = 1/|a| < 1$. The inclusion therefore holds both ways.

We will use interchangeably the three symbols U , $[x | x \in X, \|x\|_U < 1]$, and $[x | x \in X, |\bar{x}_t(x)| < 1 \text{ for every } t \in T_U]$ since they represent the same subset of X . The last expression can be considered as an analysis of the neighborhood U just as the neighborhoods U furnish a decomposition of the neighborhood system \mathcal{U} .

It is instructive to investigate the relation between the set of all linear continuous operations on X and that subset used to define the neighborhood system. This can be conveniently described by means of the algebraic notion of a factor group. Essentially the factor group permits us to study a linear convex topological space by means of a set of linear normed vector spaces.

If Θ is a closed linear manifold in X , then, being a subgroup of X , Θ defines a factor group Y whose elements y correspond biuniquely to the cosets. Y is again a linear space. Now for a given $U \in \mathcal{U}$ we will define $\Theta_U \equiv [x | x \in X, \|x\|_U = 0]$ or what is equivalent $\Theta_U \equiv [x | x \in X, \bar{x}_t(x) = 0 \text{ for every } t \in T_U]$. Then as Θ_U is linear and closed, it defines the factor group Y_U . We define a norm on Y_U to be $\|y\| = \|x\|_U$ where x is any element of the coset y . This is clearly independent of the particular $x \in$ coset y used and satisfies all of the properties of a norm. The transformation $y = T(x)$, where $x \in$ coset y , is a linear continuous transformation on X to Y_U with its norm topology. A linear continuous operation $\bar{y} \in \bar{Y}_U$, the space adjoint to Y_U , defines a linear continuous operation $\bar{x} = \bar{T}(\bar{y})$, namely $\bar{x}(x) = \bar{y}(y)$ where $x \in$ coset y . We will use this correspondence in the statement of the following theorem:

THEOREM 1.2. *The set of all linear continuous operations on X is the linear extension of the set of operations which correspond to the adjoint spaces \bar{Y}_U for all $U \in \mathcal{U}$.*

If $\bar{y} \in \bar{Y}_U$, then $|\bar{x}(x)| = |\bar{y}(y)| \leq \|\bar{y}\| \cdot \|y\| = \|\bar{y}\| \cdot \|x\|_U$. In other words

* We will designate by $[y|P]$ the class of elements y with the properties P .

$|\bar{x}(x)| < \epsilon$ if $x \in U/\|\bar{y}\|$. On the other hand if \bar{x} is continuous on X , then by a theorem due to Wehausen [8, pp. 162–163, Theorem 9], there exists a $U \in \mathcal{U}$ and a positive constant M for which $|\bar{x}(x)| \leq M \cdot \|x\|_U$. The correspondence $\bar{y}(y) = \bar{x}(x)$ where $x \in$ coset y makes \bar{x} correspond to an element of \bar{Y}_U .

THEOREM 1.3. *If the set $[\bar{x}_t | t \in T_U]$ has n and only n linearly independent elements, then Y_U has n and only n linearly independent elements.*

Suppose first that there exist $n+1$ linearly independent vectors y_1, \dots, y_{n+1} in Y_U , and that $\bar{x}_1, \dots, \bar{x}_n \in [\bar{x}_t | t \in T_U]$ are linearly independent. If \bar{y}_t corresponds to \bar{x}_t , then the system of n homogeneous linear equations $\sum_{i=1}^{n+1} a_i \bar{y}_t(y_i) = 0$ ($t=1, \dots, n$) has a non-trivial solution $\{a_i\}$. Let x_i be an element of coset y_i . Since $\sum_{i=1}^{n+1} a_i \bar{x}_t(x_i) = \sum_{i=1}^{n+1} a_i \bar{y}_t(y_i) = 0$ ($t=1, \dots, n$) and since any other \bar{x}_t for $t \in T_U$ is a linear combination of $\bar{x}_1, \dots, \bar{x}_n$, it follows that $\bar{x}_t(\sum_{i=1}^{n+1} a_i x_i) = 0$ for all $t \in T_U$. Hence $\sum_{i=1}^{n+1} a_i x_i \in \Theta_U$ which is contrary to our supposition. On the other hand, if there do not exist n linearly independent elements in Y , then some linear combination of every set of n elements of X belongs to Θ_U . Hence for every set x_1, \dots, x_n , the n rowed determinant $|\bar{x}_t(x_j)| = 0$ ($t, j=1, \dots, n$). Let r be the largest rank attained by $|\bar{x}_t(x_j)|$ for any choice of $\{x_j\}$ —say $|\bar{x}_t(x_j)| \neq 0$ ($t, j=1, \dots, r$). Then $|\bar{x}_t(x_j)| = 0$ ($t, j=1, \dots, r+1$) for any $x_{r+1} \in X$ and hence there exist $\{c_t\}$ such that $\sum_{t=1}^{r+1} c_t \bar{x}_t(x) = 0$ for all $x \in X$. But this means that $\bar{x}_1, \dots, \bar{x}_{r+1}$ are linearly dependent.

COROLLARY 1.4. *If each of the sets $[\bar{x}_t | t \in T_U]$ has only a finite number of linearly independent elements, then the set of continuous linear operations on X is just the linear extension of $[\bar{x}_t | t \in T_U, U \in \mathcal{U}]$.*

Since an n -dimensional normed vector space has an n -dimensional adjoint space, this is an immediate consequence of Theorems 1.2 and 1.3.

If a weak topology is defined on a linear vector space by means of all finite subsets of a certain class X^* of linear operations on the space, then, by the corollary, the set of all linear continuous operations on this space is exactly the linear extension of X^* .

2. The integral and unconditional summability. Let S be an abstract set possessing a sigma-field \mathfrak{S} of “measurable” subsets σ . We will study the integration of multiple-valued set functions $x(\sigma)$ having the property that the set $x(\sigma_1)$ contains the set $x(\sigma_2)$ if $\sigma_1 \supset \sigma_2$. A point function $x(s)$ on S to X defines a set function of this type $x(\sigma)$ where $x(\sigma)$ denotes the set of all elements $x(s)$ for $s \in \sigma$. The function $\alpha(\sigma)$ will be a single-valued, nonnegative, bounded, completely additive measure function on \mathfrak{S} . We will designate by Δ a subdivision of S into a finite or denumerable set of sets (σ_i) such that

$\sigma_i \in S, \sigma_i \cdot \sigma_j = 0$ if $i \neq j$, and $\sum \sigma_i = S$. The expression $\Delta_1 \geq \Delta_2$ will mean that every set σ^1 of Δ_1 is a subset of some σ^2 of Δ_2 , and $\Delta_1 \cdot \Delta_2$ will be the subdivision of sets $(\sigma_i^1 \cdot \sigma_j^2), \sigma^i$ of Δ_i . The convex extension of the subset B of X which is the set of all $\sum_1^n a_i x_i$ where $a_i \geq 0, \sum_1^n a_i = 1, x_i \in B$, will be written as $\text{cvx}(B)$. π will be a finite set of positive integers. $\pi_1 \geq \pi_2, \pi_1 + \pi_2, \pi_1 \cdot \pi_2$, and $\pi_1 - \pi_2$ will denote respectively inclusion, the integers in π_1 or π_2 , the integers in π_1 and π_2 , the integers in π_1 not in π_2 . $\sum_\pi B_i$ will be the set of all $\sum_\pi x_i$ summed over π where x_i is an element of the subset B_i of X . Let $\{B_i\}$ be a finite or denumerable sequence of subsets of X . We will designate by $[\sum_\pi B_i]$ the set of all sums $\sum_\pi B_i$. In this notation $[\sum_\pi B_i]$ will be called unconditionally summable with respect to U if there exists π_U such that if $\pi_i \geq \pi_U (i=1, 2)$, then $\|\sum_{\pi_1} B_i - \sum_{\pi_2} B_i\|_U < 1$; † and unconditionally summable to the value $x \in X$ with respect to U if there exists π_U such that if $\pi \geq \pi_U$, then $\|\sum_\pi B_i - x\|_U < 1$. If the B_i are real numbers, then unconditional summability of $[\sum_\pi B_i]$ is equivalent to the absolute convergence of $\sum_{i=1}^\infty B_i$ [9, p. 63].

We are now in a position to define the \mathcal{U} -integral. Let $J_\Delta(x, \sigma) = [\sum_\pi x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i)]$ where $\Delta = (\sigma_i)$.

DEFINITION 2.1. $x(\sigma)$ will be said to be \mathcal{U} -integrable if for every $\sigma \in S$ there exists a $J(x, \sigma) \in X$ such that given $U \in \mathcal{U}$ there exists $\Delta_{U\sigma}$ for which $J_{\Delta_{U\sigma}}(x, \sigma)$ is unconditionally summable to the value $J(x, \sigma)$ with respect to U . We define $J(x, \sigma)$ to be the value of the integral on the set σ ; that is, $J(x, \sigma) = \int_\sigma x d\alpha$.

The uniqueness of $J(x, \sigma)$ will be a consequence of Theorem 2.3 to follow.

The essential difference exhibited by this definition is that it does not require the sums $\sum_1^\infty x_i \alpha(\sigma_i)$ where $x_i \in x(\sigma_i)$ to converge. Instead it requires the sums of a subdivision corresponding to a given neighborhood U to be contained in $J(x, \sigma) + U$ whenever the set of integers over which the sum is taken includes a certain finite set. We are thus freed from convergence problems. On the other hand to have demanded the unconditional convergence of the sums $\sum x(\sigma_i) \alpha(\sigma_i)$ would have placed unnecessary restrictions either on $x(\sigma)$ or on X . Finally it has been necessary to postulate the existence of $J(x, \sigma)$ because a linear topological space need not be complete.

THEOREM 2.2. If, for a given $\Delta, J_\Delta(x, S)$ is unconditionally summable with respect to U , then $J_\Delta(x, \sigma)$ is unconditionally summable with respect to $3U$ uniformly in σ (that is, π_{3U} is the same for all σ).

By hypothesis there exists π_U such that if $\pi_i \geq \pi_U (i=1, 2)$, then $[\sum_{\pi_1} x(\sigma_i) \alpha(\sigma_i) - \sum_{\pi_2} x(\sigma_i) \alpha(\sigma_i)] \subset U$. Since U is convex, we may write the relationship

† If B is a subset of X , then $\|B\|_U < 1$ means that $B \subset U$.

$$U \supset \frac{\alpha(\sigma \cdot \sigma_{i_1})}{\alpha(\sigma_{i_1})} \left[\sum_{\pi_U} [x(\sigma_i)\alpha(\sigma_i) - x(\sigma_i)\alpha(\sigma_i)] \right] \\ + \left[1 - \frac{\alpha(\sigma \cdot \sigma_{i_1})}{\alpha(\sigma_{i_1})} \right] \left[\sum_{\pi_U} [x(\sigma_i)\alpha(\sigma_i) - x(\sigma_i)\alpha(\sigma_i)] \right]$$

where $i_1 \in \pi_U$. Let us take any element of $[x(\sigma_i)\alpha(\sigma_i) - x(\sigma_i)\alpha(\sigma_i)]$ in the first term on the right and take the same element of $[x(\sigma_i)\alpha(\sigma_i) - x(\sigma_i)\alpha(\sigma_i)]$ in the second except in the case of i_1 where we always take $\theta \in [x(\sigma_{i_1})\alpha(\sigma_{i_1}) - x(\sigma_{i_1})\alpha(\sigma_{i_1})]$ in the second term. This will give

$$U \supset \sum_{\pi_U - i_1} [x(\sigma_i)\alpha(\sigma_i) - x(\sigma_i)\alpha(\sigma_i)] + [x(\sigma_{i_1})\alpha(\sigma \cdot \sigma_{i_1}) - x(\sigma_{i_1})\alpha(\sigma \cdot \sigma_{i_1})].$$

In a like manner we obtain

$$U \supset \sum_{\pi_U - i_1 - i_2} [x(\sigma_i)\alpha(\sigma_i) - x(\sigma_i)\alpha(\sigma_i)] + \sum_{i_1 + i_2} [x(\sigma_i)\alpha(\sigma \cdot \sigma_i) - x(\sigma_i)\alpha(\sigma \cdot \sigma_i)]$$

where $i_2 \in \pi_U$. Treating successively the integers of π_U , we will finally arrive at

$$(1) \quad U \supset \sum_{\pi_U} [x(\sigma_i)\alpha(\sigma \cdot \sigma_i) - x(\sigma_i)\alpha(\sigma \cdot \sigma_i)] \supset \sum_{\pi_U} [x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i) - x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i)].$$

If $\pi_0 \cdot \pi_U = 0$, then

$$(2) \quad \sum_{\pi_0} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i) \subset \sum_{\pi_0} x(\sigma_i)\alpha(\sigma \cdot \sigma_i) \subset \sum_{\pi_0} \text{cvx} [x(\sigma_i)\alpha(\sigma_i) + \theta] \\ \subset \text{cvx} \left[\sum_{\pi} x(\sigma_i)\alpha(\sigma_i) \mid \text{finite set of } \pi\text{'s where } \pi \cdot \pi_U = 0 \right] \subset U.$$

(1) and (2) together give $(\sum_{\pi_1} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i) - \sum_{\pi_2} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i)) \subset 3U$ for all $\sigma \in S$ where $\pi_i \geq \pi_U$ ($i = 1, 2$).

THEOREM 2.3. *If $J_{\Delta_0}(x, \sigma)$ is unconditionally summable to $J(x, \sigma)$ with respect to U , then for any $\Delta \geq \Delta_0$, $J_{\Delta}(x, \sigma)$ is unconditionally summable to $J(x, \sigma)$ with respect to $4U$.*

Without loss of generality we may consider only the case $\sigma = S$. By hypothesis there exists π_0 such that if $\pi \geq \pi_0$, then $(\sum_{\pi} x(\sigma_i)\alpha(\sigma_i) - J(x, S)) \subset U$ where $\Delta_0 = (\sigma_i)$. Let (σ_j^i) be a subdivision of σ_i . Then $\Delta = (\sigma_j^i)$ ($i, j = 1, 2, \dots$) is a subdivision of S . We first propose to prove a lemma.

LEMMA. *Let $x(\sigma_i)\alpha(\sigma_i) + Y_i \subset b_i U$ where Y_i is a subset of X and $b_i > 0$. Then there exists an n_i such that if $n \geq n_i$ then*

$$\left\| \left[\sum_{j=n+1}^{\infty} \alpha(\sigma_j^i) \right] x(\sigma_i) \right\|_U < \frac{1}{2^i}$$

and

$$\sum_{j=1}^n x(\sigma_j^i)\alpha(\sigma_j^i) + Y_i \subset b_i U + \frac{1}{2^i} U.$$

Since U is convex,

$$b_i U \supset \text{cvx} [x(\sigma_i)\alpha(\sigma_i) + Y_i] \supset \frac{1}{c_n^i} \sum_{j=1}^n x(\sigma_j^i)\alpha(\sigma_j^i) + Y_i$$

where $c_n^i = \sum_{j=1}^n \alpha(\sigma_j^i)/\alpha(\sigma_i) \leq 1$. Hence

$$\sum_{j=1}^n x(\sigma_j^i)\alpha(\sigma_j^i) + c_n^i Y_i \subset c_n^i b_i U_i \subset b_i U.$$

As $\|x(\sigma_i)\alpha(\sigma_i) + Y_i\|_U < b_i$, there exists an M_i for which $\|x(\sigma_i)\alpha(\sigma_i)\|_U + \|Y_i\|_U < M_i$. Now $c_n^i \rightarrow 1$ as $n \rightarrow \infty$ so that there will be an n_i such that if $n \geq n_i$ then $\|(1 - c_n^i)Y_i\|_U < 1/2^i$ and $\|(1 - c_n^i)x(\sigma_i)\alpha(\sigma_i)\|_U < 1/2^i$. Therefore $\sum_{j=1}^n x(\sigma_j^i)\alpha(\sigma_j^i) + Y_i < b_i U + U/2^i$ for $n \geq n_i$.

We will now apply this lemma successively to the elements of π_0 . If i_n is the n th integer of π_0 so chosen, then

$$Y_{i_n} = \sum_{\pi_0 - \pi_n - i_n} x(\sigma_i)\alpha(\sigma_i) + \sum_{\pi_n} \left[\sum_{j=1}^{n_i} x(\sigma_j^i)\alpha(\sigma_j^i) \right] - J(x, S),$$

and $b_{i_n} = \sum_{0+\pi_n} 1/2^i$ where $\pi_n = (i_1, i_2, \dots, i_{n-1})$. We finally obtain

$$\left\{ \sum_{\pi_0} \left[\sum_{j=1}^{n_i} x(\sigma_j^i)\alpha(\sigma_j^i) \right] - J(x, S) \right\} \subset \left(\sum_{0+\pi_0} \frac{1}{2^i} \right) U \subset 2U.$$

Let ν be a finite set of number pairs (i, j) which designate the superscript and subscript respectively of σ_j^i . Define $\nu' = [(i, j) | i \in \pi_0, j = 1, \dots, n_i]$. If $\nu \geq \nu'$, then the terms of $\sum_{\nu} x(\sigma_j^i)\alpha(\sigma_j^i)$ not included in the above sum may be split into two partial sums Σ' and Σ'' where Σ' contains only terms from $x(\sigma_i)$, $i \notin \pi_0$, and Σ'' contains only terms from $x(\sigma_i)$, $i \in \pi_0$. There exists π_1 ($\pi_1 \cdot \pi_0 = 0$) such that

$$\begin{aligned} \Sigma' &\subset \sum_{i \notin \pi_0, i \in \pi_1} \text{cvx} [x(\sigma_i)\alpha(\sigma_i) + \theta] \\ &\subset \text{cvx} \left[\sum_{\pi} x(\sigma_i)\alpha(\sigma_i) \mid \text{finite set of } \pi\text{'s where } \pi \cdot \pi_0 = 0 \right] \subset U. \end{aligned}$$

Also

$$\Sigma'' \subset \sum_{i \in \pi_0, (i, j) \text{ of } \Sigma''} x(\sigma_{a_j^i})\alpha(\sigma_{a_j^i}) \subset \sum_{\pi_0} \left[\sum_{(i, j) \text{ of } \Sigma''} x(\sigma_{a_j^i})\alpha(\sigma_{a_j^i}) \right].$$

Now in the (i, j) of Σ'' we have $j > n_i$ so that $[\sum_{(i, j) \text{ of } \Sigma''} \alpha(\sigma_{a_j^i})]x(\sigma_i) \subset U/2^i$.

By convexity, $\sum_{(i, j) \text{ of } \Sigma''} x(\sigma_i)\alpha(\sigma_{aj}) \subset U/2^i$, and hence $\Sigma'' \subset \sum_{\pi_0} U/2^i \subset U$. Finally if $\nu \geq \nu'$,

$$\sum_{\nu} x(\sigma_j^i)\alpha(\sigma_j^i) - J(x, S) \subset 2U + U + U \subset 4U.$$

COROLLARY 2.4. *If $J_{\Delta_1}(x, \sigma)$ and $J_{\Delta_2}(x, \sigma)$ are unconditionally summable to $J(x, \sigma)$ with respect to U_1 and U_2 respectively, then $J_{\Delta_1 \cdot \Delta_2}(x, \sigma)$ is unconditionally summable to $J(x, \sigma)$ with respect to both $4U_1$ and $4U_2$.*

3. Basic properties of the integral. A set function $y(\sigma)$ on \mathfrak{S} to X will be said to be *absolutely continuous* if for every $U \in \mathcal{U}$ there exists a $\delta_U > 0$ such that if $\alpha(\sigma) < \delta_U$ then $y(\sigma) \subset U$. Absolute continuity is clearly relative to $\alpha(\sigma)$.

THEOREM 3.1. *If $x(\sigma)$ is \mathcal{U} -integrable, then $\int x d\alpha$ is absolutely continuous.*

Since $x(\sigma)$ is \mathcal{U} -integrable, it follows from Theorem 2.2 that to each $U \in \mathcal{U}$ there corresponds a Δ_U and π_U such that $\|\sum_{\pi_U} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i) - J(x, \sigma)\|_U < 1/2$ for all $\sigma \in \mathfrak{S}$. As $\|x(\sigma_i)\|_U < \infty$ there exists a $\delta_U > 0$ for which if $\alpha(\sigma) < \delta_U$ then $\|x(\sigma_i)\alpha(\sigma)\|_U < 1/2^{i+1}$ ($i \in \pi_U$). But then $\|\sum_{\pi_U} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i)\|_U < 1/2$ and hence if only $\alpha(\sigma) < \delta_U$,

$$\|J(x, \sigma)\|_U \leq \left\| \sum_{\pi_U} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i) - J(x, \sigma) \right\|_U + \left\| \sum_{\pi_U} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i) \right\|_U < 1.$$

THEOREM 3.2. *If $x(\sigma)$ is \mathcal{U} -integrable, then $\int x d\alpha$ is additive.*

It follows from Theorem 2.2 that to each $U \in \mathcal{U}$ there corresponds a $\Delta_U = (\sigma_i)$ and π_U such that if $\pi \geq \pi_U$, then $\|\sum_{\pi} x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i) - J(x, \sigma)\|_U < 1/10$ for all $\sigma \in \mathfrak{S}$. Suppose $\sigma^0 = \sigma^1 + \sigma^2$ and $\sigma^1 \cdot \sigma^2 = 0$. The subdivision $(\sigma_i^0) = \{(\sigma^1 \cdot \sigma_i), (\sigma^2 \cdot \sigma_i)\}$ of σ^0 is greater than (\geq) the subdivision $(\sigma^0 \cdot \sigma_i)$ of σ^0 . Hence by Theorem 2.3 there exists π_0 such that if $\pi \geq \pi_0$ then

$$\left\| \sum_{\pi} x(\sigma_i^0)\alpha(\sigma_i^0) - J(x, \sigma^0) \right\|_U < 2/5.$$

When we combine these statements, taking $\pi \geq \pi_0$ sufficiently large so that among the sets σ_i^0 ($i \in \pi$) are the sets $(\sigma^1 \cdot \sigma_i)$ and $(\sigma^2 \cdot \sigma_i)$ for $i \in \pi_U$, we obtain

$$\begin{aligned} & \|J(x, \sigma^1) + J(x, \sigma^2) - J(x, \sigma^0)\|_U \\ & \leq \sum_{k=0}^2 \left\| J(x, \sigma^k) - \sum_{\pi} x(\sigma^k \cdot \sigma_i^0)\alpha(\sigma^k \cdot \sigma_i^0) \right\|_U \\ & \quad + \sum_{k=1}^2 \left\| \sum_{\pi} [x(\sigma^k \cdot \sigma_i^0)\alpha(\sigma^k \cdot \sigma_i^0) - x(\sigma^k \cdot \sigma_i^0)\alpha(\sigma^k \cdot \sigma_i^0)] \right\|_U < 1. \end{aligned}$$

Therefore $J(x, \sigma^0) = J(x, \sigma^1) + J(x, \sigma^2)$.

THEOREM 3.3. *If $x(\sigma)$ is \mathcal{U} -integrable, then $\int_{\sigma} x d\alpha$ is completely additive and $\sum_i \int_{\sigma_i} x d\alpha$ is unconditionally convergent.*

Let $\{\sigma_i\}$ be a sequence of disjoint sets of \mathcal{S} . Since $\sum \alpha(\sigma_i) < \infty$, Theorems 3.1 and 3.2 imply $\lim_n \|\int_{\sum \sigma_i} x d\alpha - \sum_{i=1}^n \int_{\sigma_i} x d\alpha\|_U = 0$. As the same considerations hold for any subsequence of $\{\sigma_i\}$, the sum is by definition unconditionally convergent.

THEOREM 3.4. *If $x(\sigma)$ and $y(\sigma)$ are \mathcal{U} -integrable and m is a real number, then $m \cdot x(\sigma)$ and $x(\sigma) + y(\sigma)$ are \mathcal{U} -integrable, $mJ(x, \sigma) = J(mx, \sigma)$, and $J(x + y, \sigma) = J(x, \sigma) + J(y, \sigma)$.*

The conclusion about $m \cdot x(\sigma)$ follows from the fact that when $J_{\Delta}(x, \sigma)$ is unconditionally summable with respect to U to the value $J(x, \sigma)$, then $J_{\Delta}(mx, \sigma)$ is unconditionally summable to $mJ(x, \sigma)$ with respect to mU . Finally if $J_{\Delta_1}(x, \sigma)$ and $J_{\Delta_2}(x, \sigma)$ are unconditionally summable with respect to U to the values $J(x, \sigma)$ and $J(y, \sigma)$ respectively, then $J_{\Delta}(x + y, \sigma)$ where $\Delta \supseteq \Delta_1 \cdot \Delta_2$ is unconditionally summable to $J(x, \sigma) + J(y, \sigma)$ with respect to $8U$ by Theorem 2.3.

4. The integral in a complete space. The conditions for integrability can be relaxed when X satisfies a certain type of completeness property. It is convenient to define this property by means of the general limit notion of E. H. Moore and H. L. Smith [10, p. 103] on a class D with a relation R on DD which is transitive and compositive. Each element of D will represent a unique subdivision Δ of S and neighborhood $U \in \mathcal{U}$, and will be designated by ΔU . We define $\Delta_1 U_1 R \Delta_2 U_2$ to mean $\Delta_1 \supseteq \Delta_2$ and $U_1 \subset U_2$. A sequence $x_n \in X$ will be called a *fundamental sequence* if for each $U \in \mathcal{U}$ there is an N_U such that if $m, n \geq N_U$ then $(x_n - x_m) \in U$. X will be said to be *sequentially complete* if every fundamental sequence is convergent. X will be said to be *complete with respect to D* if every "fundamental D -sequence," $x(\Delta U) \subset X$ (that is, every sequence such that for each $U \in \mathcal{U}$ there is a ΔU such that $\Delta_i U_i R \Delta U$ ($i = 1, 2$) implies $[x(\Delta_1 U_1) - x(\Delta_2 U_2)] \in U$) is "convergent" (that is, an $x \in X$ exists such that for each $U \in \mathcal{U}$ there is a ΔU such that $\Delta_1 U_1 R \Delta U$ implies $[x(\Delta_1 U_1) - x] \in U$).

THEOREM 4.1. *If X is complete with respect to D and $x(\sigma)$ has the property that for every $U \in \mathcal{U}$ there exists a Δ_U such that $J_{\Delta_U}(x, S)$ is unconditionally summable with respect to U , then $x(\sigma)$ is \mathcal{U} -integrable.*

By hypothesis there exists Δ_U and π_U such that if $\pi_i \supseteq \pi_U$ then

$$\left\| \sum_{\pi_1} x(\sigma_i) \alpha(\sigma_i) - \sum_{\pi_2} x(\sigma_i) \alpha(\sigma_i) \right\|_U < 1.$$

Let us designate by $\mathcal{F}_{\Delta U}(x, \sigma)$ some one element of the set $\sum_{\pi_U} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i)$. Theorem 2.2 implies that $\|\sum_{\pi} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i) - \mathcal{F}_{\Delta U}(x, \sigma)\|_U < 3$ for all $\sigma \in \mathcal{S}$ if $\pi \geq \pi_U$. By Theorem 2.3, if $\Delta_j = (\sigma_i^j) \geq \Delta_U$, then there exists π_j such that for $\pi \geq \pi_j$, $\|\sum_{\pi} x(\sigma \cdot \sigma_i^j) \alpha(\sigma \cdot \sigma_i^j) - \mathcal{F}_{\Delta U}(x, \sigma)\|_U < 12$. Finally when $\Delta_j U_j \mathcal{R} \Delta U$, we have

$$\begin{aligned} \left\| \sum_{\pi} x(\sigma \cdot \sigma_i^j) \alpha(\sigma \cdot \sigma_i^j) - \mathcal{F}_{\Delta_j U_j}(x, \sigma) \right\|_U & \\ \leq \left\| \sum_{\pi} x(\sigma \cdot \sigma_i^j) \alpha(\sigma \cdot \sigma_i^j) - \mathcal{F}_{\Delta_j U_j}(x, \sigma) \right\|_{U_j} & < 1 \end{aligned}$$

if only $\pi \geq \pi_{U_j}$. Combining these statements gives the result that for $\Delta_j U_j \mathcal{R} \Delta U$ ($j=1, 2$),

$$\|\mathcal{F}_{\Delta_1 U_1}(x, \sigma) - \mathcal{F}_{\Delta_2 U_2}(x, \sigma)\|_U \leq \sum_{j=1}^2 \|\mathcal{F}_{\Delta_j U_j}(x, \sigma) + \mathcal{F}_{\Delta U}(x, \sigma)\|_U < 2 \cdot 13 = 26.$$

Completeness with respect to D then implies the existence of $J(x, \sigma)$ for which if $U \in \mathcal{U}$ there exists a Δ_U and π_U such that if $\pi \geq \pi_U$, then

$$\left\| J(x, \sigma) - \sum_{\pi} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i) \right\|_U < 1 + \|\mathcal{F}_{\Delta U}(x, \sigma) - J(x, \sigma)\|_U < 1 + 26 = 27.$$

LEMMA 4.2. *If X is sequentially complete and satisfies the first countability axiom, then X is complete with respect to D .*

Let $\{U_n\}$ where $U_n \supset U_{n+1}$ be a complete neighborhood system of θ . If $x(\Delta U)$ is a fundamental D -sequence, then for each $U \in \mathcal{U}$ there exists a $\Delta_U U \in D$ such that $\Delta_i U_i \mathcal{R} \Delta_U U$ ($i=1, 2$) implies $\|x(\Delta_1 U_1) - x(\Delta_2 U_2)\|_U < 1$. If we define $\Delta_n = \prod_{i=1}^n \Delta_{U_i}$, then $k_i \geq n$ implies $\Delta_{k_i} U_{k_i} \mathcal{R} \Delta_{U_{k_i}} U_{k_i} \mathcal{R} \Delta_{U_n} U_n$ and $\|x(\Delta_{k_1} U_{k_1}) - x(\Delta_{k_2} U_{k_2})\|_{U_n} < 1$. As X is sequentially complete, there exists $x_0 = \lim_n x(\Delta_n U_n)$. Now given $U \in \mathcal{U}$, there exists $U_{n_0} \subset U$. Let $\Delta' = \Delta_U \cdot \Delta_{n_0}$. Then

$$\begin{aligned} \|x_0 - x(\Delta_U U)\|_U & \leq \|x_0 - x(\Delta_{n_0} U_{n_0})\|_U + \|x(\Delta_{n_0} U_{n_0}) - x(\Delta' U_{n_0})\|_{U_n} \\ & + \|x(\Delta' U_{n_0}) - x(\Delta_U U)\|_U \leq 3. \end{aligned}$$

COROLLARY 4.3. *If X is sequentially complete and satisfies the first countability axiom, and if $x(\sigma)$ has the property that for every $U \in \mathcal{U}$ there exists a Δ_U such that $J_{\Delta_U}(x, S)$ is unconditionally summable with respect to U , then $x(\sigma)$ is \mathcal{U} -integrable.*

This is an immediate consequence of Theorem 4.1 and Lemma 4.2.

5. The continuous transform of an integrable function. Let X and Y be linear topological spaces with the convex neighborhood systems \mathcal{U} and U of their respective null elements. The following theorem can then be proved:

THEOREM 5.1. *If T is a linear continuous transformation on X to Y and if $x(\sigma)$ is \mathcal{U} -integrable, then $T[x(\sigma)]$ is U -integrable and $J(T[x], \sigma) = T[J(x, \sigma)]$.*

Since T is continuous, for every $V \in U$ there exists a $U \in \mathcal{U}$ such that $T(U) \subset V$. The \mathcal{U} -integrability of $x(\sigma)$ implies the existence of $J(x, \sigma)$, Δ_U, π_U such that if $\pi \supseteq \pi_U$ then $\sum_{\pi} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i) - J(x, \sigma) \subset U$. Hence for $\pi \supseteq \pi_U$ we have $\sum_{\pi} T[x(\sigma \cdot \sigma_i)] \alpha(\sigma \cdot \sigma_i) - T[J(x, \sigma)] \subset V$. It follows that $T[x(\sigma)]$ is U -integrable and that $J(T[x], \sigma) = T[J(x, \sigma)]$.

COROLLARY 5.2. *If \bar{x} is a linear continuous operation on X and if $x(\sigma)$ is \mathcal{U} -integrable, then $\bar{x}[x(\sigma)]$ is Lebesgue integrable and $\bar{x}[J(x, \sigma)] = \int_{\sigma} \bar{x}[x] d\alpha$.*

In this case Y is the space of real numbers. For real-valued functions our integral reduces to Fréchet's interpretation of the Lebesgue integral (11). The corollary then follows from Theorem 5.1.

If $x(s)$ is a multiple-valued point function on S to X and if the set function $x(\sigma) \equiv [x(s) | s \in \sigma]$ is \mathcal{U} -integrable to the value $J(x, \sigma)$, then we will say that $x(s)$ is \mathcal{U} -integrable to the value $J(x, \sigma)$. Further a set of linear continuous operations Γ on X will be said to be *total* if when $\bar{x}(x) = 0$ for all $\bar{x} \in \Gamma$, then $x = \theta$ [6, p. 42].

THEOREM 5.3. *If the set of linear continuous operations on X contains a denumerable subset which is total and if $x(s)$ and $y(s)$ are integrable to the same values on \mathcal{S} , then $x(s) = y(s)$ a.e.*

Let $\{\bar{x}_n\}$ be a total denumerable subset of linear continuous operations on X . Then for all $\sigma \in \mathcal{S}$, $\int_{\sigma} \bar{x}_n [x(s) - y(s)] d\alpha = 0$ by Corollary 5.2. Hence $\bar{x}_n [x(s) - y(s)] = 0$ on $S - \sigma_n$ where $\alpha(\sigma_n) = 0$. Therefore $\bar{x}_n [x(s) - y(s)] = 0$ on $S - \sigma_0 = S - \sum \sigma_i$ for all n , and $\alpha(\sigma_0) = 0$. As $\{\bar{x}_n\}$ is total, $x(s) = y(s)$ on $S - \sigma_0$, which proves the theorem.

The conjugate space \bar{X} to a Banach space X has been defined by Dunford to be a fundamentally separable space with a determining manifold Γ in case Γ is a separable linear manifold in \bar{X} such that for every $x \in X$, we have l.u.b. $[|\bar{x}(x)| | \bar{x} \in \Gamma, \|\bar{x}\| = 1] = \|x\|$ [2, p. 310]. It is clear that if X is a Banach space with its weak neighborhood topology or even its norm topology (see §1) and \bar{X} is a fundamentally separable space, then the set of linear continuous operations on X contains a denumerable subset which is total. Theorem 5.3 is therefore applicable to such spaces.

6. Convergence of integrable functions. We prove first the following theorem.

THEOREM 6.1. *If $x_n(\sigma)$ are \mathcal{U} -integrable and if, for all $\sigma \in \mathcal{S}$, $\{\int_{\sigma} x_n d\alpha\}$ is a fundamental sequence, then the integrals are equi-absolutely continuous.*

The argument of Saks (12) is applicable here. With the distance function $(\sigma_1, \sigma_2) = \alpha(\sigma_1 + \sigma_2 - \sigma_1 \cdot \sigma_2)$ the family of sets \mathcal{S} becomes a complete metric space. Let $F_n(\sigma) = \int_{\sigma} x_n d\alpha$ and $\mathcal{S}_q = [\sigma \in \mathcal{S} \text{ such that } \|F_n(\sigma) - F_m(\sigma)\|_U \leq e/3, m, n \geq q]$. Since $F_n(\sigma)$ is absolutely continuous by Theorem 3.1, the sets \mathcal{S}_q are closed. Also $\mathcal{S} = \sum_1^{\infty} \mathcal{S}_q$. Hence by the Baire category theorem there exists an integer q_0 and a sphere $K(\sigma_0, r) \subset \mathcal{S}_{q_0}$. Let $\delta < r$ be such that $\|F_{q_0}(\sigma)\|_U \leq e/3$ if $\alpha(\sigma) < \delta$. Then for all σ such that $\alpha(\sigma) < \delta < r$, $\sigma_1 = \sigma + (\sigma_0 - \sigma)$ and $\sigma_2 = \sigma_0 - \sigma$ are in $K(\sigma_0, r)$ so that for $m, n \geq q_0$,

$$\|F_n(\sigma) - F_m(\sigma)\|_U \leq \|F_n(\sigma_1) - F_m(\sigma_1)\|_U + \|F_n(\sigma_2) - F_m(\sigma_2)\|_U \leq 2e/3.$$

Therefore $\|F_n(\sigma)\|_U \leq e$ if $\alpha(\sigma) < \delta$ and $n \geq q_0$.

The integrable functions dealt with in the remainder of the paper will be single-valued point functions on S to X . As previously stated the integrability of $x(s)$ depends directly upon the integrability of the set function $x(\sigma) = [x(s) | s \in \sigma]$, and the integral of $x(s)$ is defined to be that of $x(\sigma)$.

We would like to extend to X the notion of *approximate convergence* introduced by Dunford [2, p. 343]. Let $\sigma(n, e, U) = [s | \|x_n(s) - x(s)\|_U \geq e]$. The sequence $x_n(s)$ will be said to approach $x(s)$ approximately on S if for every n, U , and $e > 0$ there is a measurable set $\sigma'(n, e, U) \supset \sigma(n, e, U)$ such that $\lim_n \alpha[\sigma'(n, e, U)] = 0$.

We are now in a position to prove the following result:

THEOREM 6.2. *Let X be sequentially complete, let $x_n(s)$ be \mathcal{U} -integrable, and let $x_n(s) \rightarrow x(s)$ approximately. Then the following statements are equivalent:*

- (1) $x(s)$ is \mathcal{U} -integrable and

$$\lim_n \int_{\sigma} x_n d\alpha = \int_{\sigma} x d\alpha$$

uniformly in σ .

- (2) $\lim_n \int_{\sigma} x_n d\alpha$ exists for every $\sigma \in \mathcal{S}$.
- (3) $\int_{\sigma} x_n d\alpha$ are equi-absolutely continuous.

That (1) implies (2) is obvious, while Theorem 6.1 proves that (2) implies (3). There remains only the proof that (3) implies (1). For this purpose we demonstrate

LEMMA 6.3. *If $x(s)$ is \mathcal{U} -integrable and $\|x(s)\|_U \leq e, a.e.,$ then $\|\int_{\sigma} x d\alpha\|_U \leq e\alpha(\sigma)$.*

For given η and U there exist $\Delta_{\eta U}$ and $\pi_{\eta U}$ such that $\|\sum_{\pi_{\eta U}} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i) - J(x, \sigma)\|_U < \eta$ where $\Delta_{\eta U} = (\sigma_i)$ and $\sigma_1 \equiv [s | \|x(s)\|_U > e]$. But

$$\left\| \sum_{\pi_{\eta U}} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i) \right\|_U \leq \sum_{\pi_{\eta U}} \|x(\sigma \cdot \sigma_i)\|_U \alpha(\sigma \cdot \sigma_i) \leq e\alpha(\sigma).$$

Hence $\|J(x, \sigma)\|_U \leq e\alpha(\sigma) + \eta$ where η is arbitrarily small.

To continue the proof of the theorem, we will first show that $\lim_n \int_{\sigma} x_n d\alpha$ exists uniformly in σ . By hypothesis, given U and $e > 0$ there exists δ_{Ue} such that if $\alpha(\sigma) < \delta_{Ue}$, then $\|\int_{\sigma} x_n d\alpha\|_U < e$, and further, given U and $\delta > 0$ there exists $N_{U\delta}$ such that if $n \geq N_{U\delta}$ then $\alpha[\sigma'(n, \delta, U)] < \delta$. Let δ_0 be the smaller of e and δ_{Ue} . Then for $m, n \geq N_{U\delta_0}$ we have $\|x_n(s) - x_m(s)\|_U < 2\delta_0$ on $S - \sigma'(n, \delta_0, U) - \sigma'(m, \delta_0, U)$. If $\sigma_{nm} = \sigma'(n, \delta_0, U) + \sigma'(m, \delta_0, U)$, then using Lemma 6.3,

$$\begin{aligned} \left\| \int_{\sigma} [x_n(s) - x_m(s)] d\alpha \right\|_U &\leq \left\| \int_{\sigma - \sigma_{nm}} [x_n(s) - x_m(s)] d\alpha \right\|_U \\ &+ \left\| \int_{\sigma_{nm}} [x_n(s) - x_m(s)] d\alpha \right\|_U \leq 2\delta_0\alpha(S) + 2e \leq (2\alpha(S) + 2)e. \end{aligned}$$

This together with the fact that X is sequentially complete implies that $\lim_n \int_{\sigma} x_n d\alpha$ exists uniformly in σ .

Given $U \in \mathcal{U}$, let us select a subsequence, which for notational convenience we will designate as $\{x_n(s)\}$, having the properties: (1) $\|\int_{\sigma} x_n d\alpha - \lim_n \int_{\sigma} x_n d\alpha\|_U < 1/2^n$ for all $\sigma \in \mathcal{S}$, and (2) $\|x_n(s) - x(s)\|_U < 1/(2^n\alpha(S))$ on $S - \tau_n$, where $\tau_n \in \mathcal{S}$, $\tau_n \supset \tau_{n+1}$, and $\alpha(\tau_n) < 1/2^n$. Relation (2) is a consequence of approximate convergence. We will now show that $x(s)$ is \mathcal{U} -integrable to the value $J(\sigma) = \lim_n \int_{\sigma} x_n d\alpha$. As $x_n(s)$ is integrable, there exists a Δ_n and a π_n such that if $\pi \geq \pi_n$ then

$$\left\| \sum_{\pi} x_n(\sigma \cdot \sigma_i^n) \alpha(\sigma \cdot \sigma_i^n) - J(x_n, \sigma) \right\|_U < \frac{1}{2^n}$$

where $\Delta_n = (\sigma_i^n)$. Hence for $\sigma \in S - \tau_n$ and $\pi \geq \pi_n$,

$$\begin{aligned} \left\| \sum_{\pi} x(\sigma \cdot \sigma_i^n) \alpha(\sigma \cdot \sigma_i^n) - J(\sigma) \right\|_U &\leq \text{l.u.b.}_{s_i \in \sigma \cdot \sigma_i^n} \left[\left\| \sum_{\pi} [x(s_i) - x_n(s_i)] \alpha(\sigma \cdot \sigma_i^n) \right\|_U \right. \\ &+ \left. \left\| \sum_{\pi} x_n(s_i) \alpha(\sigma \cdot \sigma_i^n) - J(x_n, \sigma) \right\|_U + \|J(x_n, \sigma) - J(\sigma)\|_U \right] \\ (1) \quad &< \text{l.u.b.}_{s_i \in \sigma \cdot \sigma_i^n} \left[\sum_{\pi} \|x(s_i) - x_n(s_i)\|_U \alpha(\sigma \cdot \sigma_i^n) + 1/2^n + 1/2^n \right] \\ &\leq \frac{\alpha(\sigma)}{2^n\alpha(S)} + \frac{1}{2^{n-1}} \leq \frac{1}{2^{n-2}}. \end{aligned}$$

Statement (3) postulates for a given U , a $\delta_U > 0$ such that if $\alpha(\sigma) < \delta_U$ then $\|\int_{\sigma} x_n d\alpha\|_U < 1/2^3$ for all n . We now choose $n_0 > 5$ such that $\alpha(\tau_{n_0}) < \delta_U$. Let $\bar{\sigma}_k^n = \sigma_k^n \cdot (\tau_{n-1} - \tau_n)$ for $n > n_0$ and let $\bar{\Delta}$ be the subdivision $(\bar{\sigma}_k^n)$ of τ_{n_0} . Then for any finite set $n_i, k_i, \|\sum_{i=1}^p x(\bar{\sigma}_{k_i}^{n_i})\alpha(\bar{\sigma}_{k_i}^{n_i})\|_U < 1/2$. This can be demonstrated as follows: By the inequality (1), if $\pi \geq \pi_n$, then

$$\left\| \sum_{\tau} [x(\bar{\sigma}_k^n)\alpha(\bar{\sigma}_k^n) - x(\bar{\sigma}_k^n)\alpha(\bar{\sigma}_k^n)] \right\|_U < \frac{1}{2^{n-3}}$$

and a fortiori

$$\left\| \sum_{n_i=n} [x(\bar{\sigma}_{k_i}^{n_i})\alpha(\bar{\sigma}_{k_i}^{n_i}) - x(\bar{\sigma}_{k_i}^{n_i})\alpha(\bar{\sigma}_{k_i}^{n_i})] \right\|_U < \frac{1}{2^{n-3}}.$$

Therefore

$$(2) \quad \left\| \sum_{i=1}^p [x(\bar{\sigma}_{k_i}^{n_i})\alpha(\bar{\sigma}_{k_i}^{n_i}) - x(\bar{\sigma}_{k_i}^{n_i})\alpha(\bar{\sigma}_{k_i}^{n_i})] \right\|_U < \frac{1}{2^{n_0-3}} \leq \frac{1}{2^3}.$$

Let l be the greater of $n_i (i=1, 2, \dots, p)$. Clearly there exists $\pi' \geq \pi_l$ for which

$$\left\| \left[\sum_{t \in \pi} \alpha(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) - \alpha(\bar{\sigma}_{k_i}^{n_i}) \right] x(\bar{\sigma}_{k_i}^{n_i}) \right\|_U < \frac{1}{2^3 p}$$

whenever $\pi \geq \pi'$. Hence if $\pi \geq \pi'$,

$$\begin{aligned} & \sum_{i=1}^p \left\{ \left[\sum_{t \in \pi} x(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i})\alpha(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) \right] - x(\bar{\sigma}_{k_i}^{n_i})\alpha(\bar{\sigma}_{k_i}^{n_i}) \right\} \\ & \subset \sum_{i=1}^p \left[\sum_{t \in \pi} x(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i})\alpha(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) \right] - \sum_{i=1}^p \left[\sum_{t \in \pi} x(\bar{\sigma}_{k_i}^{n_i})\alpha(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) \right] \\ & \quad + \sum_{i=1}^p x(\bar{\sigma}_{k_i}^{n_i}) \left[\sum_{t \in \pi} \alpha(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) - \alpha(\bar{\sigma}_{k_i}^{n_i}) \right] \\ (3) \quad & = \sum_{i=1}^p \left\{ \sum_{t \in \pi} [x(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) - x(\bar{\sigma}_{k_i}^{n_i})]\alpha(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) \right\} \\ & \quad + \sum_{i=1}^p x(\bar{\sigma}_{k_i}^{n_i}) \left[\sum_{t \in \pi} \alpha(\sigma_t^l \cdot \bar{\sigma}_{k_i}^{n_i}) - \alpha(\bar{\sigma}_{k_i}^{n_i}) \right] \\ & \subset \sum_{i=1}^p \text{cvx} [\{x(\bar{\sigma}_{k_i}^{n_i}) - x(\bar{\sigma}_{k_i}^{n_i})\}\alpha(\bar{\sigma}_{k_i}^{n_i})] + \frac{1}{2^3} U \\ & \subset \frac{1}{2^3} U + \frac{1}{2^3} U \subset \frac{1}{2^2} U. \end{aligned}$$

The next to last inclusion follows because the inequality (2) holds for all subsets of $(1, 2, \dots, p)$. Finally since the subdivision $\{(\sigma_i^l \cdot \bar{\sigma}_{k_i}^{n_i}), i=1, \dots, p\}$ of $\sigma_0 = \sum_{i=1}^p \bar{\sigma}_{k_i}^{n_i}$ is a repartition of the subdivision $(\sigma_0 \cdot \sigma_i^l)$, the inequality (1) and Theorem 2.3 imply the existence of $\pi'' \geq \pi'$ such that

$$(4) \quad \left\| J(\sigma_0) - \sum_{i=1}^p \left\{ \sum_{t \in \pi''} x(\sigma_i^l \cdot \bar{\sigma}_{k_i}^{n_i}) \alpha(\sigma_i^l \cdot \bar{\sigma}_{k_i}^{n_i}) \right\} \right\|_U < \frac{4}{2^{l-2}}.$$

As $\|J(\sigma_0)\|_U < 1/2^3$, we have, after combining (3) and (4),

$$\left\| \sum_{i=1}^p x(\bar{\sigma}_{k_i}^{n_i}) \alpha(\bar{\sigma}_{k_i}^{n_i}) \right\|_U < \frac{1}{2^2} + \frac{1}{2^{l-4}} + \frac{1}{2^3} \leq \frac{1}{2}.$$

We define a subdivision $\Delta_0 = (\sigma_i)$ of S to be the sets of $\{\sigma_i^{n_0}(S - \tau_{n_0})\}$ and of $\bar{\Delta}$. Let π_0 be the π_{n_0} terms of $\{\sigma_i^{n_0} \cdot (S - \tau_{n_0})\}$. Then if $\pi \geq \pi_0$, we have

$$\begin{aligned} \left\| \sum_{\pi} x(\sigma_i) \alpha(\sigma_i) - J(S) \right\|_U &\leq \left\| \sum_{\text{sets of } \Delta_n} x(\sigma_i) \alpha(\sigma_i) - J(S - \tau_{n_0}) \right\|_U + \|J(\tau_{n_0})\|_U \\ &\quad + \left\| \sum_{\text{sets of } \bar{\Delta}} x(\sigma_i) \alpha(\sigma_i) \right\|_U \\ &\leq \frac{1}{2^{n_0}} + \frac{1}{2^3} + \frac{1}{2} < 1. \end{aligned}$$

$x(s)$ is therefore integrable to $J(S)$. Clearly what has been shown for S could similarly be shown for any $\sigma \in S$.

We state the following theorem without proof:

THEOREM 6.4. *If $x_n(s)$ are \mathcal{U} -integrable ($n=0, 1, \dots$) and if $x_n(s) \rightarrow x_0(s)$ approximately, then the following are equivalent:*

- (1) $\lim_n \int_{\sigma} x_n d\alpha = \int_{\sigma} x_0 d\alpha$ exists uniformly in σ .
- (2) $\int_{\sigma} x_n d\alpha$ are uniformly absolutely continuous.

In the next theorem it is convenient to consider a pseudo-norm for the \mathcal{U} -integrable functions defined as $\|J(x)\|_U = \text{l.u.b.}_{\sigma \in S} \|J(x, \sigma)\|_U$.

THEOREM 6.5. *If X is sequentially complete, $x(s)$ \mathcal{U} -integrable, and $\phi(s)$ real-valued, measurable, and bounded by M , then $y(s) = \phi(s)x(s)$ is \mathcal{U} -integrable and $\|J(y)\|_U \leq 2M\|J(x)\|_U$.*

Let $\phi_n(s)$ be a step function with the property that $|\phi_n(s) - \phi(s)| < 1/n$. As $x(s)$ is \mathcal{U} -integrable, there exists for a given $U \in \mathcal{U}$ a $\Delta_U = (\sigma_i)$ such that $J_{\Delta_U}(x, S)$ is unconditionally summable with respect to U . Hence for every k , $\|x(s)\|_U$ will be bounded on $\tau_k = \sum_{i=1}^k \sigma_i$ so that $\|(\phi_n(s) - \phi(s))x(s)\|_U < e$ for sufficiently large n on τ_k . Therefore $\phi_n(s)x(s) \rightarrow y(s)$ approximately. Suppose

$\phi_n(s) = c_i^n$ on σ_i^n . Then by a lemma due to Dunford [2, p. 313]

$$\begin{aligned} \left\| \int_{\sigma} \phi_n x_n d\alpha \right\|_U &= \left\| \sum_1^{N_n} c_i^n \int_{\sigma \cdot \sigma_i^n} x d\alpha \right\|_U \\ (1) \quad &\leq 2M \cdot \text{l.u.b.}_{\pi} \left[\left\| \sum_{\pi} \int_{\sigma \cdot \sigma_i^n} x d\alpha \right\|_U \mid \pi \leq (1, 2, \dots, N_n) \right] \\ &\leq 2M \cdot \text{l.u.b.}_{\sigma^*} \left[\left\| \int_{\sigma \cdot \sigma^*} x d\alpha \right\|_U \mid \sigma^* \in \mathfrak{S} \right]. \end{aligned}$$

The absolute continuity of $\int_{\sigma} x d\alpha$ (Theorem 3.1) thus implies the equi-absolute continuity of the integrals $\int_{\sigma} \phi_n x d\alpha$. By Theorem 6.2, $y(s)$ is \mathcal{U} -integrable and $\int_{\sigma} y d\alpha = \lim_n \int_{\sigma} \phi_n x d\alpha$. The remaining part of the theorem results immediately from (1).

7. Instances of the \mathcal{U} -integral. A further insight into the nature of the \mathcal{U} -integral can be gained by displaying some familiar integrals as representations of the \mathcal{U} -integral. We will first consider a very general theorem of this nature. To this end it will be recalled that in Theorem 1.1 we have shown that to every $U \in \mathcal{U}$ there corresponds a set of linear continuous operations and a range T_U such that $U \equiv [x \mid x \in X, |\bar{x}_t(x)| < 1 \text{ for every } t \in T_U]$. Further, since the relation \geq on the set of subdivisions Δ is transitive and compositive, the general limit of E. H. Moore and H. L. Smith [10, p. 103] is applicable to this range.

THEOREM 7.1. *$x(s)$ is \mathcal{U} -integrable if and only if for every $U \in \mathcal{U}$, $\lim_{\Delta} \sum_{i=1}^{\infty} \bar{x}_t [x(\sigma \cdot \sigma_i)] \alpha(\sigma \cdot \sigma_i) = \bar{x}_t [J(x, \sigma)]$ uniformly in $t \in T_U$ and the $\sum_{i=1}^{\infty} \bar{x}_t [x(\sigma \cdot \sigma_i)] \alpha(\sigma \cdot \sigma_i)$ are absolutely convergent uniformly in $t \in T_U$.*

The statement that for every U and σ there exist $\Delta_{U\sigma}$ and $\pi_{U\sigma}$ such that if $\pi \geq \pi_{U\sigma}$ then $\left\| \sum_{\pi} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i) - J(x, \sigma) \right\|_U < 1$ is equivalent by Theorems 1.1 and 2.3 to the statement that for every U and σ there exists $\Delta_{U\sigma}$ such that if $\Delta \geq \Delta_{U\sigma}$ there is a π_{Δ} such that if $\pi \geq \pi_{\Delta}$ then

$$\left| \sum_{\pi} \bar{x}_t [x(\sigma \cdot \sigma_i)] \alpha(\sigma \cdot \sigma_i) - \bar{x}_t [J(x, \sigma)] \right| < 1$$

for all $t \in T_U$.

COROLLARY 7.2. *If X is a Banach space with the weak neighborhood topology defined by all finite sets of elements in the adjoint space (that is, if $U \equiv [x \mid x \in X, |\bar{x}_t(x)| < 1, t = 1, \dots, n]$), then the \mathcal{U} -integral is the integral defined by Pettis [4, p. 280].*

Each \bar{x}_t , element of the adjoint space, will clearly be continuous in the

weak topology and hence by Corollary 5.2 $\bar{x}[J(x, \sigma)] = \int_{\sigma} \bar{x}[x(s)] d\alpha$ for every $\sigma \in \mathcal{S}$. Conversely, if the latter is true, then by Theorem 7.1, the \mathcal{U} -integral exists and is equal to the Pettis integral.

Dunford [2, p. 338] has considered a more general type of integral defined as follows: Let X be a closed linear manifold contained in \bar{Y} , the adjoint space to a Banach space Y . Then a function $y(s)$ on S to Y which has the property that $x[y(s)]$ is summable for every $x \in X$ defines uniquely an element in \bar{X} according to the equation

$$\bar{x}_{\sigma}[x] = \int_{\sigma} x[y(s)] d\alpha.$$

\bar{x}_{σ} is the Dunford integral of $y(s)$ over σ . However each $y \in Y$ corresponds to a unique element $\bar{x} \in \bar{X}$ through the correspondence $\bar{x}(x) = x(y)$ for all $x \in X$. The Dunford integral is unchanged when we replace the integrand $y(s)$ on S to Y by its correspondent $\bar{x}(s)$ on S to \bar{X} (that is, $\int_{\sigma} x[y(s)] d\alpha = \int_{\sigma} \bar{x}(s)[x] d\alpha$ for all $x \in X$). If we make this replacement, we can prove by an argument similar to that used in Corollary 7.2 the following:

COROLLARY 7.3. *If \bar{X} is adjoint to a Banach space X with the weak topology on \bar{X} defined by all finite sets of elements of X , then the \mathcal{U} -integral of a function to \bar{X} is the integral defined by Dunford.*

THEOREM 7.4. *If X is a Banach space and \mathcal{U} its norm topology, then the \mathcal{U} -integral is the integral defined by Birkhoff.*

If $x(s)$ is Birkhoff integrable, then given $\epsilon > 0$ and $\sigma \in \mathcal{S}$ there exists Δ_{ϵ} and $J(x, \sigma)$ such that $J_{\Delta_{\epsilon}}(x, \sigma)$ is unconditionally convergent* to elements in an ϵ -sphere about $J(x, \sigma)$. It is a fortiori unconditionally summable to $J(x, \sigma)$ with respect to an ϵ -sphere and hence \mathcal{U} -integrable to $J(x, \sigma)$. Conversely, suppose $x(s)$ is \mathcal{U} -integrable. Then given $\epsilon/2^n$, there exists $\Delta_n = (\sigma_i^n)$, where $\Delta_{n+1} \supseteq \Delta_n$, and π_n such that if $\pi \supseteq \pi_n$ then $\|\sum_{\pi} x(\sigma_i^n) \alpha(\sigma_i^n) - J(x, S)\| < \epsilon/2^n$ ($n=0, 1, \dots$). If N_n is the greater of the integers in π_n , then $m, i_k \geq N_n$ implies that $\|\sum_1^m x(\sigma_i^n) \alpha(\sigma_i^n) - J(x, S)\| < \epsilon/2^n$ and $\|\sum_{k=1}^p x(\sigma_{i_k}^n) \alpha(\sigma_{i_k}^n)\| < 2\epsilon/2^n$ ($n=0, 1, \dots$). Let P_n be the greater of N_n and M_n where $\sum_{i=M_n}^{\infty} \alpha(\sigma_i^n) < 1/2^n$. We define a sequence of sets $\sigma_1, \sigma_2, \dots, \sigma_n, \dots$ as follows: $\sigma_1 = \sigma_1^0$. Suppose σ_k is of Δ_n . Then σ_{k+1} is the set of lowest subscript of Δ_n which is disjoint from any of the previously chosen sets unless all such sets have subscripts greater than P_n . In the latter case, $R_n = k$ and σ_{k+1} is the set of lowest subscript of Δ_{n+1} which is disjoint from any of the previously chosen sets. $\sum_{i=1}^{R_n} \alpha(\sigma_i) \geq \alpha(S) - 1/2^n$.

* If X is sequentially complete, then $\sum B_i$ where $B_i \subset X$ is said to be unconditionally convergent if given $U \in \mathcal{U}$ there exists N_U such that if $i_k \geq N_U$ then $\|\sum_{k=1}^p B_{i_k}\|_U < 1$.

Hence $\Delta = (\sigma_i)$ is a subdivision of S . Now since $\Delta \geq \Delta_0$, there exists by Theorem 2.3 an N_Δ such that $\|\sum_1^n x(\sigma_i)\alpha(\sigma_i) - J(x, S)\| < 4e$ if only $n \geq N_\Delta$. Finally by our choice of $\{\Delta_n\}$, if $i_k > R_n$ then σ_{i_k} is a subset of some σ_{i^n} ($i \geq N_n$). Hence there exists π such that $\pi \cdot (1, \dots, N_n) = 0$ and

$$\begin{aligned} \left\| \sum_{k=1}^p x(\sigma_{i_k})\alpha(\sigma_{i_k}) \right\| &\leq \left\| \sum_{\tau} \text{cvx} [x(\sigma_{i^n})\alpha(\sigma_{i^n}) + \theta] \right\| \\ &\leq \text{l.u.b.} \left[\left\| \sum_{\tau'} x(\sigma_{i^n})\alpha(\sigma_{i^n}) \right\| \mid \pi' \leq \pi \right] \leq \frac{e}{2^{n-1}}. \end{aligned}$$

$J_\Delta(x, S)$ is therefore unconditionally convergent to elements in an e -sphere about $J(x, S)$ and consequently $x(s)$ is Birkhoff integrable.

In §10 we demonstrate that the Birkhoff integral is distinct from the Pettis integral which in turn is distinct from the Dunford integral. Hence if we limit ourselves to Banach spaces on which we impose all possible convex neighborhood topologies which lie, say, in strength* between the norm topology and the usual weak topology, the corresponding ranges of integrable functions lie between the Birkhoff range and the Pettis range.

8. Differentiation. In the study of differentiation we wish to limit our attention to the subclass of \mathcal{U} -integrable functions on $S = (0, 1)$ to X defined as follows: $x(s)$ will be said to be \mathcal{U}' -integrable if

- (1) $x(s)$ is \mathcal{U} -integrable.
- (2) There exists a sequence of step functions† $\{x_n(s)\}$ such that $\|x_n(s) - x(s)\|_{\mathcal{U}} \rightarrow 0$ a.e. for each $U \in \mathcal{U}$.
- (3) $\|x(s)\|_{\mathcal{U}}$ is summable for each $U \in \mathcal{U}$.

If X is a Banach space, then the \mathcal{U}' -integral is the Bochner integral [13]. On the other hand, if S is a Banach space with the usual weak neighborhood topology, then the \mathcal{U}' -integral is the Pettis integral [4]. This is easily seen from Corollary 7.2 and by writing $x_n(s) = \int_{I_i^n} x(s) ds / |I_i^n|$ if $s \in I_i^n$ where $I_i^n = ((i-1)/n, i/n)$. Thus a peculiarity of this weak neighborhood topology is that the classes of \mathcal{U} - and \mathcal{U}' -integrable functions on $(0, 1)$ to X are the same (see Corollary 7.2).

We define a function $x(I)$ on intervals of $(0, 1)$ to X to be of *bounded variation*‡ if for every $U \in \mathcal{U}$ there exists an M_U such that for every finite set of disjoint intervals I_1, \dots, I_n , $\sum_{i=1}^n \|x(I_i)\|_{\mathcal{U}} \leq M_U$. The function $x(I)$ will

* \mathcal{U} is said to be weaker than U if whenever $x_n \rightarrow x$ in U then $x_n \rightarrow x$ in \mathcal{U} .

† A function on $(0, 1)$ to X will be called a *step function* if and only if it is single-valued on each of a finite number of disjoint intervals whose sum is $(0, 1)$.

‡ This is a generalization of the two notions of strong and weak bounded variation which have previously been introduced (see [3]).

be said to be *pseudo-differentiable** if there exists a $y(s)$ on $(0, 1)$ to X such that for every $U \in \mathcal{U}$, a.e.

$$\lim_{|I| \rightarrow 0} \left\| \frac{x(I)}{|I|} - y(s) \right\|_U = 0$$

where I is an arbitrary interval containing s . The function $y(s)$ is the *pseudo-derivative* of $x(I)$.

THEOREM 8.1. *A necessary and sufficient condition that a single-valued, additive, absolutely continuous function $x(\sigma)$ on measurable sets in $(0, 1)$ to X be pseudo-differentiable and of bounded variation is that the function $x(\sigma)$ be a \mathcal{U} -integral. $x(\sigma)$ is then the \mathcal{U} -integral of its pseudo-derivative.*

Proof of the necessity. Let $x_n(s) = x(I_i^n)/|I_i^n|$ on $I_i^n = (i/2^n, (i+1)/2^n)$ ($i=0, 1, \dots, 2^n-1$). Then by the definition of the pseudo-derivative there exists a $y(s)$ for which $\|x_n(s) - y(s)\|_U \rightarrow 0$ a.e. Since $\|x_n(s)\|_U$ is measurable and $\|x_n(s)\|_U \rightarrow \|y(s)\|_U$ a.e., it follows that $\|y(s)\|_U$ is measurable. As $x(\sigma)$ is of bounded variation, $\int_\sigma \|x_n(s)\|_U ds \leq M_U$. Therefore by Fatou's lemma, $\|y(s)\|_U$ will be summable. It remains only to show that $y(s)$ is \mathcal{U} -integrable to the value $x(\sigma)$. Now given $\epsilon > 0$ and $U \in \mathcal{U}$, there exists $\delta_{\epsilon U} > 0$ such that if $|\sigma| < \delta_{\epsilon U}$ then $\int_\sigma \|y(s)\|_U ds \leq \epsilon/8$. Further, since $\|x_n(s) - y(s)\|_U = \lim_{m \rightarrow \infty} \|x_n(s) - x_m(s)\|_U$ a.e. and is thus measurable, there exists $\sigma_{\epsilon U}$ such that $\|x_n(s) - y(s)\|_U \rightarrow 0$ uniformly on $(0, 1) - \sigma_{\epsilon U} = \tau_{\epsilon U}$ and $|\sigma_{\epsilon U}| < \delta_{\epsilon U}$. Let n be chosen so that $\|x_n(s) - y(s)\|_U < \epsilon/4$ for $s \in \tau_{\epsilon U}$. If σ_i ($i=1, \dots, N$) are the sets on which $x_n(s)$ is constant in $\tau_{\epsilon U}$, then

$$\left\| \sum_{i=1}^N y(\sigma_i) \cdot |\sigma_i| - \sum_{i=1}^N x_n(\sigma_i) \cdot |\sigma_i| \right\|_U < \epsilon/4.$$

Finally let σ_i ($i=N+1, \dots$) be a subdivision of $\sigma_{\epsilon U}$ for which

$$\left\| \sum_{N+1}^{\infty} \|y(\sigma_i)\|_U |\sigma_i| - \int_{\sigma_{\epsilon U}} \|y(s)\|_U ds \right\| \leq \epsilon/8.$$

Then for $i_k > N$,

$$\begin{aligned} \left\| \sum_{k=1}^p y(\sigma_{i_k}) \cdot |\sigma_{i_k}| \right\|_U &\leq \sum_{k=1}^p \|y(\sigma_{i_k})\|_U \cdot |\sigma_{i_k}| \leq \sum_{N+1}^{\infty} \|y(\sigma_i)\|_U |\sigma_i| \\ &\leq \epsilon/8 + \int_{\sigma_{\epsilon U}} \|y(s)\|_U ds \leq \epsilon/4. \end{aligned}$$

* Pettis has used this definition of pseudo-differentiability for a weak neighborhood topology on a Banach space.

For $\Delta = (\sigma_i)$ and $\pi_i \geq (1, 2, \dots, N)$ we have

$$(1) \quad \left\| \sum_{\pi_1} y(\sigma \cdot \sigma_i) \cdot |\sigma \cdot \sigma_i| - \sum_{\pi_2} y(\sigma \cdot \sigma_i) \cdot |\sigma \cdot \sigma_i| \right\|_U < e.$$

Now by Theorem 1.1 there exist linear continuous operations \bar{x}_t on X such that $U \equiv [x | x \in X, |\bar{x}_t(x)| < 1 \text{ for every } t \in T_U]$. Our hypothesis clearly implies $\bar{x}_t[x(\sigma)] = \int_{\sigma} \bar{x}_t[y(s)] ds$.^{*} There will therefore exist a subdivision $\Delta_t = (\sigma_i^t) \geq \Delta$ for which

$$\left| \sum_i \bar{x}_t[y(\sigma \cdot \sigma_i^t)] \cdot |\sigma \cdot \sigma_i^t| - \bar{x}_t[x(\sigma)] \right| < e.$$

By (1) if $J_t(\sigma)$ is some element of $\sum_{\pi_0} \bar{x}_t[y(\sigma \cdot \sigma_i)] \cdot |\sigma \cdot \sigma_i|$ where $\pi_0 \equiv (1, 2, \dots, N)$, then $|\bar{x}_t[\sum_{\pi} y(\sigma \cdot \sigma_i) \cdot |\sigma \cdot \sigma_i|] - J_t(\sigma)| < e$ for $\pi \geq \pi_0$. Theorem 2.3 then implies

$$\left| \sum_i \bar{x}_t[y(\sigma \cdot \sigma_i^t)] \cdot |\sigma \cdot \sigma_i^t| - J_t(\sigma) \right| < 4e$$

and hence $|J_t(\sigma) - \bar{x}_t[x(\sigma)]| < 5e$. Therefore for $\pi \geq \pi_0$,

$$\left| \bar{x}_t \left[\sum_{\pi} y(\sigma \cdot \sigma_i) \cdot |\sigma \cdot \sigma_i| - x(\sigma) \right] \right| < 6e$$

for every $t \in T_U$ or, what amounts to the same thing,

$$\left\| \sum_{\pi} y(\sigma \cdot \sigma_i) \cdot |\sigma \cdot \sigma_i| - x(\sigma) \right\|_U < 6e.$$

$y(s)$ is thus integrable to the value $x(\sigma)$.

Proof of the sufficiency. We will first prove a lemma.

LEMMA 8.2. *If $x(s)$ is \mathcal{U} -integrable and $\|x(s)\|_U$ is integrable for a given $U \in \mathcal{U}$, then $\|\int_{\sigma} x d\alpha\|_U \leq \int_{\sigma} \|x(s)\|_U d\alpha$.*

Given $e > 0$, there will be a Δ_e and π_e such that

$$\left\| \sum_{\pi_e} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i) - J(x, \sigma) \right\|_U < e/2,$$

$$\left| \sum_{\pi_e} \|x(\sigma \cdot \sigma_i)\| \alpha(\sigma \cdot \sigma_i) - \int_{\sigma} \|x(s)\|_U d\alpha \right| < e/2.$$

* If $x(\sigma)$ defined only on intervals is additive, absolutely continuous, of bounded variation, and has a pseudo-derivative, then this statement and the rest of the necessity remains valid for integrals on intervals. If X is in addition complete with respect to D , the integral can be extended on all measurable sets.

Since we have $\|\sum_{\sigma_i} x(\sigma \cdot \sigma_i) \alpha(\sigma \cdot \sigma_i)\|_U \leq \sum_{\sigma_i} \|x(\sigma \cdot \sigma_i)\|_U \alpha(\sigma \cdot \sigma_i)$, it follows that $\|J(x, \sigma)\| \leq \int_{\sigma} \|x(s)\|_U d\alpha + e$ where the e is arbitrary.

The proof of the sufficiency is now essentially an argument used by Bochner [13, pp. 269–270]. As $y(s)$ is U -integrable, there will exist a sequence $x_n^*(s)$ of step functions such that $\|x_n^*(s) - y(s)\|_U \rightarrow 0$ a.e. Define $x_n(s) = x_n^*(s)$ if $\|x_n^*(s)\|_U \leq \|y(s)\|_U + 1$, $x_n(s) = 0$ otherwise. Then $\|x_n(s) - y(s)\|_U \rightarrow 0$ a.e.; $\|x_n(s) - y(s)\|_U$ is bounded by the summable function $(2\|y(s)\|_U + 1)$; and $\|x_n(s) - y(s)\|_U$ is measurable since $\|x_n(s) - y(s)\|_U = \lim_m \|x_n(s) - x_m(s)\|_U$. Hence making use of the lemma,

$$\left\| \int_{\sigma} [x_n(s) - y(s)] ds \right\|_U \leq \int_{\sigma} \|x_n(s) - y(s)\|_U ds.$$

Finally $\int_{\sigma} x_n(s) ds$ has a pseudo-derivative because $x_n(s)$ assumes only a finite set of values. Therefore

$$\begin{aligned} \lim_{|I| \rightarrow 0} \left\| \frac{\int_I y(s) ds}{|I|} - y(s) \right\|_U &\leq \lim_{|I| \rightarrow 0} \frac{\int_I \|y(s) - x_n(s)\|_U ds}{|I|} \\ &+ \lim_{|I| \rightarrow 0} \left\| \frac{\int_I x_n(s) ds}{|I|} - x_n(s) \right\|_U + \|x_n(s) - y(s)\|_U \\ &\leq 2\|x_n(s) - y(s)\|_U \text{ a.e.} \end{aligned}$$

As $\lim_n \|x_n(s) - y(s)\|_U = 0$ a.e., $y(s)$ is the pseudo-derivative of $x(\sigma)$. Bounded variation follows from the lemma.

This theorem has previously been proved for the special cases where X is a Banach space [14, p. 410, footnote] and where X is a Banach space with its weak neighborhood topology [4, p. 300].

We may generalize the notion of derivative in another way: namely, a function $x(I)$ on intervals of $(0, 1)$ to X is said to be *differentiable* at a point s if $y(s) \in X$ exists such that for every U

$$\lim_{|I| \rightarrow 0} \left\| \frac{x(I)}{|I|} - y(s) \right\|_U = 0$$

where I is an arbitrary interval containing s . $y(s)$ is called the *derivative* of $x(I)$ at s . The vanishing of the limit is here independent of the choice of U . This kind of derivative does not seem to lend itself to a theorem analogous to Theorem 8.1. One should expect the integral of a well behaved integrand to have a derivative a.e. However in Example 10.9 we exhibit an additive, absolutely continuous function on measurable sets of $(0, 1)$ to a Hilbert space with its weak neighborhood topology which is pseudo-differentiable and of

bounded variation but not differentiable a.e. On the other hand its pseudo-derivative is both \mathcal{U}' -integrable and the limit in the norm sense of a sequence of step functions.

Pettis [23] has treated in detail the case $x(I)$ of bounded variation in the norm topology on a Banach space. In some special cases we have been able to demonstrate a few propositions about the derivative in a weak neighborhood topology. Theorem 5.1 of Pettis' paper is similar to Theorem 8.4 below.

In the remainder of this section X_1 will designate a Banach space and X the space X_1 with its weak neighborhood topology defined by all finite sets of linear limited operations on X_1 .

$x(I)$ will be said to satisfy the *weak Lipschitz condition* (w.L.) if a.e. on $(0, 1)$ the norm of $x(I)/|I|$ is not infinite (that is, $\|x(I)/|I|\| < \infty$) for all intervals containing $s \in (0, 1)$.

THEOREM 8.3. *If $x(I)$ on intervals to X is differentiable a.e., then it satisfies w.L.*

As the neighborhoods of X are defined by all finite sets of elements of \bar{X}_1 , adjoint to X_1 , we have a.e. $\lim_{|I| \rightarrow 0} \bar{x}[x(I)/|I|]$ exists and a fortiori $|\bar{x}[x(I)/|I|]| < \infty$ for arbitrary $I \ni s$ and all $\bar{x} \in \bar{X}_1$. The theorem is then an immediate consequence of a theorem due to Dunford on uniform boundedness [2, p. 308, Theorem 2]. In this case the abstract set is the set of all intervals containing a point s at which the derivative exists.

THEOREM 8.4. *If X is sequentially complete with \bar{X} separable or if X_1 is regular and if $x(I)$ on intervals to X is of bounded variation, then a necessary and sufficient condition that $x(I)$ be differentiable a.e. is that $x(I)$ satisfy w.L.*

The necessity has been proved in Theorem 8.3. For the sufficiency argument we first suppose \bar{X}_1 to be separable and obtain a sequence $\{\bar{x}_p\}$ dense in \bar{X}_1 . Since $\bar{x}_p[x(I)]$ is of bounded variation, $\lim_{|I| \rightarrow 0} \bar{x}_p[x(I)]/|I|$ exists on a set σ_p of measure one. W.L. implies the existence of a function $L(s)$ such that $|\bar{x}_p[x(I)]/|I| - L(s)| < \epsilon$ on a set σ'_p of measure one. Therefore $\lim_{|I| \rightarrow 0} \bar{x}_p[x(I)]/|I|$ exists uniformly in p on a set $\sigma_0 = \bigcap \sigma'_p$ of measure one. It follows that $\lim_{|I| \rightarrow 0} \bar{x}[x(I)]/|I|$ exists on σ_0 for all $\bar{x} \in \bar{X}_1$. As X is sequentially complete, the derivative likewise exists on σ_0 . If X_1 is regular and separable, then \bar{X}_1 is likewise separable. There remains only the case for which X_1 is a nonseparable regular Banach space. But here the above argument suffices because (1) $x(I)$ is contained in a separable closed linear subspace of X_1 and (2) every closed linear subspace of X_1 is regular [15, p. 423, Theorem 3]. (1) is a consequence of the following lemma:

LEMMA 8.5. *If $x(s)$ on $(0, 1)$ to a sequentially complete space X is of bounded variation, then $x(s)$ can have at most a denumerable number of points of discontinuity relative to the norm.*

Suppose the contrary to be true. Then there exists a positive number ϵ and a denumerable set of points at which the norm variation of $x(s)$ is greater than ϵ . We can then find a denumerable sequence of pairs of points with the property that $\|x(s_n^1) - x(s_n^2)\| > \epsilon$ and $s_n^1 < s_n^2 < s_{n+1}^1$ (or $s_n^1 > s_n^2 > s_{n+1}^1$). Let $y_n = x(s_n^1) - x(s_n^2)$. Since $x(s)$ is of bounded variation, $\sum_{n=1}^{\infty} |\bar{x}[y_n]| < \infty$ for every $\bar{x} \in \bar{X}_1$. The sequential completeness of X implies that every subseries of $\sum_n y_n$ is convergent in X . By a theorem due to Orlicz and Banach [4, pp. 281–282, Theorem 2.32], $\sum y_n$ is unconditionally convergent in X_1 and hence $\|y_n\| \rightarrow 0$, which is contrary to our choice of y_n 's.

Returning to the theorem, it is clear from the lemma that $x(s) = x(I_0^s)$ has only a denumerable set of points of discontinuity relative to the norm and hence all of its values can be found as limits of a denumerable set of values (again in the sense of the norm). $x(I)$ is therefore contained in the separable linear closed subspace of X_1 determined by this denumerable set of values of $x(s)$. This completes the proof of Theorem 8.4.

COROLLARY 8.6. *If X is sequentially complete with \bar{X}_1 separable or if X_1 is regular, and if $x(I)$ is additive, absolutely continuous, of bounded variation, and of w.L., then $x(I)$ is an integral.*

This follows from Theorem 8.4 and a theorem due to Pettis [4, pp. 300–301].

In the final theorem of this section we suppose X_1 to be a Banach space with a base $\{x_i\}$ [6, p. 110] which satisfies the following postulate:

(M) If a_1, a_2, \dots is any sequence of real numbers such that l.u.b. _{n} $\|\sum_1^n a_i x_i\| < \infty$, then $\sum_1^\infty a_i x_i$ converges.

By redefining the norm we can consider instead of X_1 a space isomorphic [6, p. 180] to X_1 satisfying in addition to (M) the property

$$(N) \quad \left\| \sum_1^n a_i x_i \right\| \leq \left\| \sum_1^{n+1} a_i x_i \right\|$$

for any constants a_1, a_2, \dots [16, pp. 415–416]. The spaces l_p ($p \geq 1$) and L_p ($p > 1$) satisfy (M) [16, p. 419].

THEOREM 8.7. *If $x(I)$ is an additive, absolutely continuous function satisfying w.L. on intervals of $(0, 1)$ to X where X_1 satisfies (M) and (N), then $x(I)$ is pseudo-differentiable and is the integral of its pseudo-derivative on intervals.*

Now $x(I) = \sum_1^\infty a_i(I) x_i$ where $a_i(I) = \bar{x}_i[x(I)]$ and hence are additive abso-

lutely continuous functions of intervals with summable derivatives $a_i'(s)$ a.e. As $x(I)$ satisfies w.L.,

$$\left\| \sum_{i=1}^n \frac{a_i(I)}{|I|} x_i \right\| \leq \left\| \frac{x(I)}{|I|} \right\| \leq L_s \text{ a.e.}$$

so that $\|\sum_1^\infty a_i'(s)x_i\| \leq L_s$ a.e. By property (M) $y(s) = \sum_1^\infty a_i'(s)x_i$ exists a.e. Let $x_n(I) = \sum_1^n [\int_I a_i'(s)ds]x_i$. Then $\|x(I)\| \geq \|x_n(I)\|$ implies that the $x_n(I)$ are uniformly absolutely continuous. Hence

$$\begin{aligned} \bar{x}[x(I)] &= \lim_n \bar{x}[x_n(I)] = \lim_n \int_I \left[\sum_1^n a_i'(s)\bar{x}(x_i) \right] ds \\ &= \int_I \left[\sum_1^\infty a_i'(s)\bar{x}(x_i) \right] ds = \int_I \bar{x}[y(s)] ds. \end{aligned}$$

This is equivalent to the desired conclusions.

9. An application to differential equations. We consider in this section an existence theorem for ordinary differential equations in X similar to a real variable theorem due to Carathéodory [17, p. 672].* We limit ourselves to linear convex topological spaces which are sequentially complete and satisfy the first countability axiom. Garrett Birkhoff [17] has shown that such a space has a group metric. X is therefore an F -space [6, p. 34] with a convex neighborhood topology. We will describe the topology by both a metric and the convex neighborhoods as suits our convenience.

$x(s)$ and $\phi(x, s)$ will be functions on $(0, 1)$ and the product space $X \times (0, 1)$ respectively to X . $Dx(s)$ will designate the pseudo-derivative of $x(s)$. Since X satisfies the first countability axiom, the existence of the pseudo-derivative will imply the existence of the derivative a.e. (see §8). $\int \phi(x(s), s) ds$ will be the \mathcal{U}' -integral. The system

$$(1) \quad Dx = \phi(x, s) \quad x(0) = x_0$$

will be said to have a *solution* $x(s)$ if and only if

$$x(s) = x_0 + \int_0^s \phi[x(t), t] dt.$$

$x(s)$ will be called *measurable* if there exists a sequence of step functions $\{x_n(s)\}$ such that $|x_n(s) - x(s)| \rightarrow 0$ a.e. where $|x - y|$ is the distance between x and y . We will say that the sequence of functions $\{x_n(s)\}$ are *equally-continuous* if for every $U \in \mathcal{U}$ there exists $\delta_U > 0$ such that $|s_1 - s_2| < \delta_U$ implies

* The basic ideas of this theorem were also in part suggested by T. H. Hildebrandt.

$\|x(s_1) - x(s_2)\|_U < 1$. Finally $\phi(x, s)$ will be said to satisfy *condition (A)* if it is continuous in x for fixed s and measurable in s for fixed x .

THEOREM 9.1. *A necessary and sufficient condition that $x(s)$ be a solution of (1) is that (a) $x(0) = x_0$, (b) $x(I_{s_1}^{s_2}) = x(s_2) - x(s_1)$ be an additive, absolutely continuous function of bounded variation on intervals of $(0, 1)$ to X , and (c) $Dx(s)$ exist and be equal to $\phi[x(s), s]$.*

This is an immediate consequence of Theorem 8.1 and the footnote on page 133.

LEMMA 9.2. *If $\phi(x, s)$ satisfies (A) and $x(s)$ is measurable on $(0, 1)$ to X , then $\phi[x(s), s]$ is measurable.*

Since $x(s)$ is measurable, there exists a sequence of step functions $\{x_n(s)\}$ such that $|x_n(s) - x(s)| \rightarrow 0$ a.e. and by the continuity of $\phi(x, s)$ in x , $|\phi[x_n(s), s] - \phi[x(s), s]| \rightarrow 0$ a.e. Now $\phi[x_n(s), s]$ is clearly measurable. There will therefore exist a double sequence of step functions $\{y_n^p(s)\}$ such that for every n , $\lim_p |y_n^p(s) - \phi[x_n(s), s]| = 0$ a.e. As the real-valued functions $|y_n^p(s) - \phi[x_n(s), s]|$ are measurable, there will exist the set $y_n^{p_n}$ for which $|y_n^{p_n} - \phi[x_n(s), s]| < 1/2^n$ except on a set of measure less than $1/2^n$. Therefore $|y_n^{p_n} - \phi[x(s), s]| \rightarrow 0$, a.e.

LEMMA 9.3. *If $\{x_n(s)\}$ on $(0, 1)$ to X are equally continuous and compact-valued for each $s \in (0, 1)$, then there exists a subsequence $\{x_{n_i}(s)\}$ which converges uniformly to a continuous function on $(0, 1)$ to X .*

The proof is similar to that of the familiar Arzelas theorem and will be left to the reader.

LEMMA 9.4. *If Γ is a compact subset of X and $M > 0$, then the set of all elements of the type $\sum_1^n a_i x_i$ where $x_i \in \Gamma$ and $\sum_1^n |a_i| \leq M$ form a compact set.*

Let $\{U_n\}$ where $U_n \supset U_{n+1}$ be a complete neighborhood system for θ and let $e = 1/2(M+1)$. There will exist a sequence $\{\eta_i\} \subset \Gamma$ and integers N_n such that for $x \in \Gamma$ there is an η_i ($i \leq N_n$) such that $\|x - \eta_i\|_{U_n} < e$. In the n th step we let correspond to every $x \in \Gamma$ an η_i ($i \leq N_n$ and $\|x - \eta_i\|_{U_n} < e$) and then to every $\sum_1^p a_j x_j$ the element $\sum_1^{N_n} A_{ni} \eta_i$ where $A_i = \sum a_j$ summed over all j such that x_j corresponds to η_i . Let $\{\sum_1^{p_r} a_j^r x_j^r\}$ be a sequence of such elements and let $\{\sum_1^{N_n} A_{ni}^r \eta_i\}$ be their correspondents. Then there exists a subsequence $\{r_k\}$ for which $A_{ni}^r \rightarrow \bar{A}_i^{r_n}$ ($i = 1, \dots, N_n$). Hence there will be an N'_n such that $k \geq N'_n$ implies

$$\left\| \sum_1^{N_n} A_{ni}^{r_k} \eta_i - \sum_1^{N_n} \bar{A}_i^{r_n} \eta_i \right\|_{U_n} \leq \sum_1^{N_n} |A_{ni}^{r_k} - \bar{A}_i^{r_n}| \cdot \|\eta_i\|_{U_n} < e.$$

Clearly we can apply the diagonal process and obtain a subsequence $\{s_k\}$ for which $\|\sum_1^{N_n} A_{n_i}^{s_k} \eta_i - \sum_1^{N_n} A_{n_i} \eta_i\|_{U_n} < e$ if $k \geq N_n''$. Now

$$\left\| \sum_1^{p_{s_k}} a_i^{s_k} x_i^{s_k} - \sum_1^{N_n} A_{n_i}^{s_k} \eta_i \right\|_{U_n} < Me.$$

Therefore

$$\left\| \sum_1^{p_{s_k}} a_i^{s_k} x_i^{s_k} - \sum_1^{p_{s_l}} a_i^{s_l} x_i^{s_l} \right\|_{U_n} < 2(M + 1)e$$

if only $k, l \geq N_n''$. As X is sequentially complete, the sequence $\{\sum_1^{p_{s_k}} a_i^{s_k} x_i^{s_k}\}$ converges to an element of X .

We are now in a position to establish the existence theorem.

THEOREM 9.5. *If $\phi(x, s)$ satisfies (A) and takes on only a compact set of values for $s \in (0, 1)$ and $x \in X$, then the system (1) has a solution.*

We first show that for any measurable function $x(s)$, $\phi[x(s), s]$ can be uniformly approximated by simple functions* assuming values in $\Gamma \equiv [\phi(x, s) | s \in (0, 1), x \in X]$. Let $\{U_n\}$ where $U_n \supset U_{n+1}$ be a complete neighborhood system for θ . There will exist a sequence $\{\eta_i\} \subset \Gamma$ and integers N_n such that for $x \in \Gamma$ there is an η_i ($i \leq N_n$) for which $\|x - \eta_i\|_{U_k} \leq 1$ ($k \leq n$). We now define $S_n^i \equiv [s | s \in (0, 1), \|\eta_i - \phi[x(s), s]\|_{U_n} \leq 1]$, $\sigma_n^i = S_n^i - \sum_{j=1}^{i-1} S_n^j$ ($i = 1, \dots, N_n$), and $y_n(s) = \eta_i$ on σ_n^i . To show that $\{y_n(s)\}$ is the desired sequence we need only to demonstrate that S_n^i is measurable. But this follows from the fact that $\phi[x(s), s]$ is approximable by step functions.

By hypothesis, Γ is compact. Therefore given $U \in \mathcal{U}$, there will exist $M(U)$ such that $\|\Gamma\|_U \leq M(U)$. Further if $x(s)$ is measurable, then $\phi[x(s), s]$ is measurable, $\|\phi[x(s), s]\|_U$ is summable, and hence the \mathcal{U} -integrability of $\phi[x(s), s]$ will imply its \mathcal{U}' -integrability.

Let $x_n(s) = x_0 + \int_0^s \phi[x_n'(s), s] ds$ where $x_n'(s) = x(p/n)$ for $p/n < s \leq (p+1)/n$ ($p = 0, 1, \dots, n-1$). The integral of $\phi[x_n'(s), s]$ exists by Theorem 6.2, since $\phi[x_n'(s), s]$ is uniformly approximable by simple functions whose pseudo-norms with respect to U are bounded by $M(U)$. Now by Lemma 8.2

$$\begin{aligned} (2) \quad \|x_n(s_1) - x_n(s_2)\|_U &\leq \left\| \int_{s_1}^{s_2} \phi[x_n'(s), s] ds \right\|_U \leq \int_{s_1}^{s_2} \|\phi[x_n'(s), s]\|_U ds \\ &\leq M(U) \cdot |s_2 - s_1|. \end{aligned}$$

This clearly implies

$$(3) \quad \|x_n(s) - x_n'(s)\|_U \leq M(U)/n.$$

* A simple function on $(0, 1)$ to X is constant on each of a finite number of disjoint measurable sets whose sum is $(0, 1)$.

As above there exists a double sequence of simple functions $y_p^n(s)$ such that $y_p^n(s) = \eta_i$ on $\sigma_{pi}^n \in \mathcal{S}$ ($\eta_i \in \Gamma$ and $\sum_{i=1}^{N_p} \sigma_{pi}^n = S$) and $\|y_p^n(s) - \phi[x_n'(s), s]\|_{U_p} \leq 1$. It is clear that $\|y_p^n(s) - \phi[x_n'(s), s]\|_{U_k}$ is measurable and converges to zero as $p \rightarrow \infty$. Thus $\|y_p^n(s)\|_U \leq M(U)$ and $y_p^n(s) \rightarrow \phi[x_n'(s), s]$ approximately. By Theorem 6.2 $\lim_p \int_{\sigma} y_p^n(s) ds = \int_{\sigma} \phi[x_n'(s), s] ds$. But $\int_{\sigma} y_p^n(s) ds = \sum_{i=1}^{N_p} \eta_i |\sigma \cdot \sigma_{pi}^n|$ where $\sum_{i=1}^{N_p} |\sigma \cdot \sigma_{pi}^n| \leq 1$. By Lemma 9.4 the set $[\int_{\sigma} y_p^n(s) ds | \sigma \in (0, 1); n; p=1, 2, \dots]$ is compact and therefore $[\int_0^s \phi[x_n'(s), s] ds | s \in (0, 1); n=1, 2, \dots]$ is compact. By (2) and Lemma 9.3 there exists a subsequence $\{x_{n_k}(s)\}$ which approaches the continuous function $x(s)$ uniformly on $(0, 1)$. Then by (3) $x_{n_k}'(s) \rightarrow x(s)$ uniformly on $(0, 1)$. Clearly $\|\phi[x_{n_k}'(s), s] - \phi[x(s), s]\|_U$ is measurable. Therefore $\phi[x_{n_k}'(s), s] \rightarrow \phi[x(s), s]$ approximately as $k \rightarrow \infty$. Also $\|\phi[x_{n_k}'(s), s]\|_U \leq M(U)$. Again by Theorem 6.2

$$x(s) - x_0 = \lim_k \int_0^s \phi[x_{n_k}'(s), s] ds = \int_0^s \phi[x(s), s] ds.$$

10. Examples. The following examples serve to illustrate some of the properties of the U -integral as well as the importance of various hypotheses. Example 10.2 demonstrates that the Birkhoff and Pettis integrals are different. It is therefore possible to define a class of integrals for a Banach space. Example 10.8 shows that the class of Pettis integrable functions does not include a very well behaved set of functions. Example 10.9 proves that the integral of a strongly measurable function may not be weakly differentiable. These three examples answer questions raised by Pettis [4, p. 303].

B will be the space of real bounded functions x_i on $T = (0, 1)$ with norm $\|x_i\| = \text{l.u.b.}_{t \in T} |x_i|$. We will consider functions $x(s)$ on $S = (0, 1)$ to X . Integration on $S = (0, 1)$ is taken with respect to the Lebesgue measure function.

It will be necessary for us to refer to the following theorem whose proof can be found in another paper* by the author:

THEOREM 10.1. *For every bounded additive set function $\beta(\tau)$ on all subsets of $(0, 1)$ there exists a denumerable set $R(\beta)$ and a unique decomposition $\beta(\tau) = \beta_1(\tau) + \beta_2(\tau)$ such that $\beta_i(\tau)$ ($i=1, 2$) are bounded additive set functions, $\beta_1(\tau) = \beta(R \cdot \tau)$, and $\beta_2(\tau) = 0$ if τ is a denumerable subset of $(0, 1)$.*

10.2. Example of a function which is Pettis integrable but not Birkhoff integrable (§7). Let $x(s)$ on $S = (0, 1)$ to B be defined as $x(s) \equiv x_i(s) = 1$ if $t - s = p/2^n$ for $\pm p, n$ positive integers and $x_i(s) = 0$ elsewhere. Thus on the unit square with coordinates s and t , $x_i(s) = 1$ only on a denumerable set of points on any line parallel to either coordinate axis.

* This paper will appear in the Bulletin of the American Mathematical Society.

Hildebrandt [19] has shown that for every \bar{x} of the space adjoint to B there exists an additive bounded set function $\beta(\tau)$ on all subsets of $T = (0, 1)$ such that for all $x \in B$, $\bar{x}(x) = \int_T x_t d\beta$. By Theorem 10.1, $\beta(\tau) = \beta_1(\tau) + \beta_2(\tau)$, where $\beta_1(\tau) = \beta(R \cdot \tau)$, R is a denumerable subset of T , and $\beta_2(\tau) = 0$ for all denumerable subsets of T . Hence for each s

$$\bar{x}[x(s)] = \int_T x_t(s) d\beta = \int_T x_t(s) d\beta_1 + \int_T x_t(s) d\beta_2 = \int_T x_t(s) d\beta_1.$$

It follows that $\bar{x}[x(s)] \neq 0$ only if $x_t(s) \neq 0$ for $t \in R$. Let $\delta_t \equiv [s | s \in (0, 1), x_t(s) \neq 0, \text{ fixed } t]$ and $\delta_0 = \sum_{t \in R} \delta_t$. Then δ_0 is the subset of S on which $x_t(s) \neq 0$ for some $t \in R$. Since both δ_t and R are denumerable sets, δ_0 is a denumerable subset of S . Therefore $\bar{x}[x(s)] = 0$ a.e. and $x(s)$ is Pettis integrable to the value $x_\sigma = \theta \in B$ for all measurable sets σ .

On the other hand $x(s)$ is not Birkhoff integrable. Suppose the contrary were true. Then [1, Theorem 13], given $\epsilon > 0$, there exist disjoint sets σ_i such that $\sum \sigma_i = S$, the series $\sum x(\sigma_i) \alpha(\sigma_i)$ is unconditionally convergent, and $s_i, s'_i \in \sigma_i$ implies $|\sum_{i=1}^{\infty} x(s_i) | \sigma_i | - \sum_{i=1}^{\infty} x(s'_i) | \sigma_i || < \epsilon$. We first suppose that for each t there exists a σ_i such that for all $s \in \sigma_i, x_t(s) = 0$. We can then divide T into sets θ_i where $T = \sum \theta_i$, and for all $t \in \theta_i$ and $s \in \sigma_i, x_t(s) = 0$. Since $x_t(s) = 1$ on a diagonal, $S - \sigma_i$ contains the set θ'_i whose coordinate values are those of θ_i . In addition $S - \sigma_i$ contains θ''_i consisting of all points of the type $s \pm p/2^n$ and $s/2^n$ where $s \in \theta'_i$ for all integral p and n . Now θ''_i is a homogeneous point set in the sense of Knopp [20, pp. 411-414] so that its upper measure $|\overline{\theta''_i}| = 1$ or 0 . As there exists an i_0 such that $|\overline{\theta''_{i_0}}| > 0$, it follows that $|S - \sigma_{i_0}| \geq |\overline{\theta''_{i_0}}| = 1$ which is contrary to $|\sigma_{i_0}| > 0$. Our supposition being false, we can select a t_0 with the property that for every i, σ_i contains an s_i such that $x_{t_0}(s_i) = 1$. As $|\sigma_i| > 0$, there is also an $s'_i \in \sigma_i$ such that $x_{t_0}(s'_i) = 0$. But this implies that $|\sum_i x(s_i) | \sigma_i | - \sum_i x(s'_i) | \sigma_i || \geq 1$. The assumption that $x(s)$ is Birkhoff integrable has thus led to a contradiction.

10.3. *Example of a \mathcal{U} -integrable function for which there exists Δ_0 such that for all $\Delta \geq \Delta_0, J_\Delta(x, S)$ is not unconditionally convergent (§2).* One would expect a definition of unconditional convergence for $J_\Delta(x, S)$ at least to demand the convergence of every sum $\sum_1^\infty x(s_i) | \sigma_i |$ ($s_i \in \sigma_i$). Even such a definition would eliminate the function of the present example from the class of integrable functions if (1) we replace unconditional summability in Definition 2.1 by unconditional convergence, and (2) we insist that $J_\Delta(x, S)$ is unconditionally convergent if this is true for $J_{\Delta_0}(x, S)$ where $\Delta \geq \Delta_0$.

Let X be the space B with a weak neighborhood topology defined by all finite sets of linear limited operations on B . Let $x(s) \equiv x_t(s) = 1/s$ for $s - t = p/2^n$ where $\pm p, n$ are positive integers and $x_t(s) = 0$ elsewhere. $x(s)$ can be shown

to be \mathcal{U} -integrable (that is, Pettis integrable) in a manner identical with that used in Example 10.2. Suppose $\Delta_0 = \{\sigma_i^0 = (1/2^i, 1/2^{i-1})\}$ and $\Delta = (\sigma_i) \geq \Delta_0$. As in Example 10.2 there exists t_0 and $s_i \in \sigma_i$ such that $x_{t_0}(s_i) = 1/s_i$. We will now show that some sum of $J_\Delta(x, S)$ does not converge with respect to the neighborhoods $U \equiv [x | x \in X, |\bar{x}_{t_0}(x)| < \delta]$ where $\bar{x}_{t_0}(x) = x_{t_0}$. This is an immediate consequence of

$$\sum_i x_{t_0}(s_i) | \sigma_i | \geq \sum_i \frac{1}{1/2^{i-1}} | \sigma_i^0 | = \sum_i \frac{2^{i-1}}{2^i} = \infty .$$

10.4. *Example of a function $x(s)$ on $S = (0, 1)$ to B which is \mathcal{U} -integrable having the properties (1), (2), (3) given below.*

(1) $\|x(s)\|$ is not measurable. Convergence a.e. will therefore not imply uniform convergence on the complement of sets of arbitrary small measure (approximate convergence in Theorem 6.2).

(2) $\int_\sigma x(s) ds = \theta$ for all σ yet $x(s) \neq 0$ a.e. \bar{B} does not contain a denumerable subset which is total (Theorem 5.2).

(3) $x(s)$ is not \mathcal{U} -integrable. The two integrals are therefore distinct in general (§8).

Let σ_0 be a non-measurable set in $S = (0, 1)$ and define $x(s) = \theta$ for $s \in S - \sigma_0$ and $x(s)$ equal to the characteristic function of the point s for $s \in \sigma_0$ [4, p. 301, Example 9.1].

10.5. *Example illustrating the need for sequential completeness in Corollary 4.3 and Theorem 6.2.* In the Banach space of bounded sequences let m_0 be the linear extension of the set of unit vectors ϕ_n . The space m_0 is not sequentially complete. Let $x^k(s) \equiv x_i^k(s) = \delta_{in} \phi_n$ for $s \in (1/2^n, 1/2^{n-1})$ and $n \leq k$ (δ_{in} is the Kronecker symbol), and vanish elsewhere ($k = 1, 2, \dots, \omega$). Despite the fact that for every $\epsilon > 0$ there exists a $J_\Delta(x^\omega, S)$ which is unconditionally summable with respect to an ϵ -sphere, $x^\omega(s)$ is not \mathcal{U} -integrable in m_0 (Corollary 4.3). On the other hand $x^k(s) \rightarrow x^\omega(s)$ approximately, and the $\int_\sigma x^k(s) ds$ exist and are uniformly absolutely continuous (Theorem 6.2).

10.6. *Example showing the need for sequential completeness in Theorem 6.5.* Let B_0 be the linear subset of B consisting of all functions on $T = (0, 1)$ whose derivatives exist a.e. and take on only a finite set of values. B_0 is not sequentially complete. We define $x(s) \equiv x_t(s) = 1$ for $t \geq s$, and to vanish elsewhere. $x(s)$ is integrable on σ to $(|\sigma \cdot I_0^t|)$. However if $\phi(s) = s$, then $\int_0^1 \phi(s) x(s) ds = (t^2/2)$ which is not in B_0 .

10.7. *Example showing the need of completeness with respect to D in a space which is sequentially complete (Theorem 4.1).* Let l_0 be the linear space of functions x_t on $T = (0, 1)$ such that $\sum_t |x_t| < \infty$ and let \bar{l}_0 be the set of all

bounded measurable functions \bar{x}_t on $T = (0, 1)$. Then $\bar{x}(x) \equiv \sum \bar{x}_t \cdot x_t$ is a linear continuous operation on l_0 . We will define the topology on l_0 by all finite sets of elements from \bar{l}_0 . Any denumerable sequence of elements of l_0 differs on T at only a denumerable set of points. l_0 is sequentially complete since this is equivalent to the weak completeness of l_1 [6, p. 143]. Finally set $x_t(s) = \delta_{st}$ on $S = (0, 1)$ to l_0 . As $\int \sigma \bar{x}(x_t(s)) ds = \int \sigma \bar{x}_t dt$ exists, it follows that for every $U \in \mathcal{U}$ (finite set of \bar{x} 's) there will be a $J_\Delta(x, S)$ which is unconditionally summable with respect to U . On the other hand there exists no $x^0 \in l_0$ such that $\sum_t \bar{x}_t \cdot \bar{x}_t^0 = \int_0^1 \bar{x}_t dt$ for all $\bar{x} \in \bar{l}_0$. $x(s)$ is therefore not \mathcal{U} -integrable.

For convenience let M be the class of functions $x(s)$ on $S = (0, 1)$ to B such that (1) $\bar{x}[x(s)]$ as measurable for all $\bar{x} \in \bar{B}$, and (2) $\int \sigma \bar{x}[x(s)] ds$ are uniformly absolutely continuous with respect to elements of \bar{B} . If $x(s)$ is contained in a separable subspace of B , then it is Pettis integrable [4, p. 293, Theorem 5.3]. It is Dunford integrable in any case.

10.8. *Example of a member of M which is not Pettis integrable.* Sierpiński [21, pp. 9–10] has shown that equivalent to the hypothesis of the continuum is the existence of a set θ in the unit square I such that on every parallel to the s axis θ contains at most a denumerable set of points in I while on every parallel to the t axis $I - \theta$ contains at most a denumerable set of points in I . We define $x(s)$ on $S = (0, 1)$ to B so that $x(s) \equiv x_t(s)$ is the characteristic function of θ . In the notation of Example 10.2,

$$\bar{x}[x(s)] = \int_T x_t(s) d\beta = \int_T x_t(s) d\beta_1 + \int_T x_t(s) d\beta_2.$$

Since for any $t \in (0, 1)$, $x_t(s) = 1$ for only a denumerable set δ_t in S , it follows that $\int_T x_t(s) d\beta_1 \neq 0$ on at most the set $\delta_0 = \sum_{t \in R} \delta_t$ which is denumerable. Let 1_t be the element of B which has the value one for all $t \in T$. As $(1_t - x_t(s))$ has only a denumerable set of nonzero values for each $s \in S$, we have $\int_T (1_t - x_t(s)) d\beta_2 = 0$. Hence $\bar{x}[x(s)] = \int_T 1_t d\beta_2$ for all $s \in (S - \delta_0)$ so that $\bar{x}[x(s)]$ is measurable. The integrals $\int \sigma \bar{x}[x(s)] ds$ are obviously uniformly absolutely continuous with respect to elements of \bar{B} . Thus $x(s) \in M$.

Finally, let us suppose that there exists a $y \in B$ such that $\bar{x}(y) = \int \sigma \bar{x}[x(s)] ds$ for every $\bar{x} \in \bar{B}$. For every $t_0 \in T$ there exists an $\bar{x}_{t_0} \in \bar{B}$ defined by $\bar{x}_{t_0}(x_t) = x_{t_0}$. This set is total on B .

$$y_{t_0} = \bar{x}_{t_0}(y) = \int_S \bar{x}_{t_0}[x_t(s)] ds = \int_S x_{t_0}(s) ds = 0.$$

Now Banach [6, p. 30] has demonstrated the existence of $\bar{x}_0 \in \bar{B}$ for which $\bar{x}_0(x) = L \int \sigma x_t dt$ whenever x_t is Lebesgue integrable. Our supposition then leads to

$$0 = \bar{x}_0(y) = \int_s \bar{x}_0[x_t(s)]ds = \int_s 1ds = 1.$$

Therefore $x(s)$ is not Pettis integrable.

It is to be noticed that $F(\bar{x}) = \int_s \bar{x}[x(s)]ds$ is weakly continuous on \bar{B} . Since there exists no $y \in B$ such that $F(\bar{x}) = \bar{x}(y)$ for all $\bar{x} \in \bar{B}$, the Banach theorem [6, p. 131, Theorem 8] to this effect for separable Banach spaces is not in general true for nonseparable Banach spaces.

10.9. *Example of an additive, absolutely continuous function $x(\sigma)$ on measurable sets of $(0, 1)$ to Hilbert space H which in the weak neighborhood topology is pseudo-differentiable but a.e. not differentiable (§8).* Let $x_{i,j}$ be a doubly infinite set of orthonormal vectors in H . We define $y_i(s) = 2^i x_{i,j}$ on $(j/2^i, 2^{-2i} + j/2^i)$ $j=0, 1, \dots, (2^i - 1)$, and elsewhere set $y_i(s) = 0$. As Birkhoff [1, p. 375, Ex. ample 2] has observed, $y(s) = \sum_{n=1}^\infty y_n(s)$ exists a.e. and is Birkhoff integrable, Pettis has shown [4] that $x(\sigma) = \int_\sigma y(s)ds$ is additive, absolutely continuous and in the weak neighborhood topology pseudo-differentiable. We will now demonstrate that $\|x(I)/|I|\|$ is unbounded a.e. As in Theorem 8.3, this will imply that a.e. the weak derivative does not exist. Let $s = {}_0a_1a_2 \dots a_n \dots$ be the dyadic representation of $s \in (0, 1)$. Hardy and Littlewood [22, p. 189] have proved that any finite sequence of digits will recur an infinite number of times on a set of measure one. In particular, n consecutive zeros will occur infinitely often on σ_n of unit measure. Then $\sigma_0 = \prod \sigma_n$ is of measure one. For each point of σ_0 all numbers of consecutive zeros occur infinitely often. Suppose p is the number of consecutive zeros not exceeding $n/2$ which immediately proceed the $(n+1)$ st digit of ${}_0a_1a_2 \dots a_n \dots = s$. If I_n is the interval ${}_0a_1a_2 \dots a_{n-p} \leq s \leq {}_0a_1a_2 \dots a_{n-p} + 1/2^n$, then

$$\begin{aligned} x(I_n) &= \int_{I_n} \left[\sum_{n-p}^n y_i(s) \right] ds + \int_{I_n} \left[y(s) - \sum_{n-p}^n y_i(s) \right] ds \\ &= \sum_{n-p}^n x_{i,j_i}/2^i + x'(I_n) \end{aligned}$$

where $x'(I_n)$ is orthogonal to (x_{i,j_i}) ($i=n-p, \dots, n$). In this case $\|x(I_n)/|I_n|\| \geq 2^p$. Since for each point of σ_0 , $\lim_n p_n = \infty$, our assertion is proved.

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