ON A THEOREM OF SCHUR AND ON FRACTIONAL INTEGRALS OF PURELY IMAGINARY ORDER

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1. Let \( L^p(a, b) \) be the space of all functions \( f(y) \) whose \( p \)th power is integrable over \((a, b)\) or which are measurable and essentially bounded over \((a, b)\) for \( 1 \leq p < \infty \) or \( p = \infty \) respectively, with the norm

\[
|f|_p = \left\{ \int_a^b |f(y)|^p dy \right\}^{1/p} \quad [1 \leq p < \infty],
\]

\[
|f|_p = \text{ess. u.b. } |f(y)| \quad [p = \infty];
\]

let \( p' = p/(p-1) \) and

\[
L^p = L_p(0, \infty).
\]

The following theorem is in substance due to I. Schur(1):

Let \( K(x, y) \) be homogeneous of degree \(-1\) and \( K(x, y) \geq 0 \) for \( 0 < x < \infty, 0 < y < \infty \), let \( K(x, y)x^{-1/2} \in L_1 \), and let \( f(x) \in L_2 \); then

\[
\left| \int_0^\infty K(x, y)f(x)dx \right|_2 \leq \kappa |f(y)|_2,
\]

where

\[
\kappa = \int_0^\infty K(x, 1)x^{-1/2}dx = \int_0^\infty K(1, y)y^{-1/2}dy.
\]

The constant \( \kappa \) is the best possible.

Of course the inequality is true when \( K(x, y) \) takes negative or even complex values also, if we replace \( \kappa \) by

\[
= \int_0^\infty |K(x, 1)| x^{-1/2}dx = \int_0^\infty |K(1, y)| y^{-1/2}dy.
\]

However \( \tilde{\kappa} \) is not the best possible constant any more. We shall give a better

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(1) Journal für die reine und angewandte Mathematik, vol. 140 (1911), pp. 1–28. The corresponding theorem for \( f \in L_p \), \( 1 < p < \infty \), was proved by G. H. Hardy, J. E. Littlewood, and G. Pólya, vide Inequalities, Cambridge, 1934, Theorem 319.

160
theorem for this case and we shall use it to deal with fractional integrals the order of which is an imaginary number, thus filling a gap in the literature.

Throughout this paper we denote constants depending on the given parameters by the single symbol $C$; $\alpha$ and $\beta$ denote finite numbers such that $\Re(\alpha) > 0$, $\Re(\beta) = 0$.

2. Theorem I. Let (i) $K(x, y)$ be homogeneous of degree $-1$, (ii) $K(x, 1)x^{-1/2} \subseteq L_1$, (iii) $f(x) \subseteq L_2$; then the function

$$Wf = \int_1^\infty K(x, y)f(x)dx$$

exists for almost all values of $y$ in $(0, \infty)$, and

$$|Wf|_2 \leq \kappa_0 |f(\gamma)|_2 = \max_{-\infty < \gamma < \infty} \omega(\tau) \cdot |f(\gamma)|_2,$$

where

$$\omega(\tau) = \int_0^\infty K(x, 1)x^{-1/2 + \tau}dx = \int_0^\infty K(1, y)y^{-1/2 + \tau}dy, \quad \kappa_0 = \max_{-\infty < \gamma < \infty} |\omega(\tau)|.$$

The constant is the best possible.

Obviously $\kappa_0 \leq \bar{\kappa}$; when $K(x, y) \geq 0$ then $\kappa_0 = \kappa$, as we may see taking $\tau = 0$.

Proof. Without loss of generality we may suppose $K(x, y)$ to be no null-function; then $\kappa_0 > 0$. Let $1 < a < \infty, f(x, a) = f(x)$ in $(a^{-1}, a), f(x, a) = 0$ otherwise, and let

$$M\phi = \operatorname{l.i.m. sq.} \int_0^N \phi(x)x^{-1/2 + \tau}dx \quad [\phi \subseteq L_2].$$

In consequence of Schur's theorem $W$ is a bounded linear transformation in $L_2$; the Mellin transform $M$ is a bounded linear transformation from $L_2(0, \infty)$ into $L_2(-\infty, \infty)$. We have

$$\int_0^\infty y^{-1/2 + \tau}W\{f(x, a)\}dy = \int_0^\infty y^{-1/2 + \tau}dy \int_0^\infty K(x, y)f(x)dx$$

$$= \int_0^\infty f(x)x^{-1/2 + \tau}dx \int_0^\infty K(1, y)y^{-1/2 + \tau}dy$$

when we put $y = vx$ and make use of the homogeneousness of $K(x, y)$; the interchanging of the integrations is justified by absolute convergence of the right-hand repeated integral. Since the left-hand integral exists, it must be equal to $MW\{f(x, a)\}$, therefore we have

$$MW\{f(x, a)\} = \omega(\tau)M\{f(x, a)\}.$$
Since \( |f(x, a) - f(x)| \to 0 \; [a \to \infty] \), and \( |\omega(\tau)| \leq \kappa_0 < \infty \), by the continuity of the operations \( M \) and \( W \) we get

\[
(2.2) \quad MWf = \omega(\tau)Mf;
\]

therefore \( |MWf|_2 \leq \kappa_0 |Mf|_2 \) in \( L_2(-\infty, \infty) \). Now the operator \((2\pi)^{-1/2}M\) is isometric, and so we obtain the first assertion of the theorem.

The function \( \omega(\tau) \) is continuous in consequence of (ii) and attains its maximum value at a finite point \( \tau \), since, by the Riemann-Lebesgue theorem, \( \omega(\tau) \to 0 \; [\tau \to \pm \infty] \). Now let \( \lambda \) be any positive number smaller than \( \kappa_0 \). Then we can easily show the existence of functions \( f(x) \subseteq L_2 \) such that \( |Wf|_2 > \lambda |f|_2 \).

Let \( E \) be a set of measure \( m(E) > 0 \) such that \( |\omega(\tau)| > \lambda \) in \( E \) and \( E \) is included in some finite interval. Take \( \phi(\tau) = 1 \) in \( E \) and \( \phi(\tau) = 0 \) otherwise, and let \( f = M^{-1}\phi \). Then from (2.2) we have

\[
\int_{-\infty}^{\infty} |MWf|^2 d\tau = \int_{E} |\omega(\tau)|^2 d\tau > \lambda^2 m(E)
\]

\[
= \lambda^2 \int_{-\infty}^{\infty} |\phi(\tau)|^2 d\tau = 2\pi \lambda^2 \int_{0}^{\infty} |f(x)|^2 dx,
\]

\[
\int_{0}^{\infty} |Wf|^2 dy > \lambda^2 \int_{0}^{\infty} |f(x)|^2 dx.
\]

Hence the theorem is proved.

3. We could give an alternative proof by the theory of “general transforms,” without making use of Schur’s theorem. Let \( V \) be a transformation of the form

\[
Vf = \int_{0}^{\infty} L(x, y) f(x) dx,
\]

the infinite integral being defined in some sense. Then it turns out that, roughly speaking, the class of all transformations which are representable in the form \( V_1 V_2 \) is identical with the class of the transformations

\[
Wf = \int_{0}^{\infty} K(x, y) f(x) dx,
\]

where \( K(x, y) \) is homogeneous of degree \(-1\). We leave that proof of I to the reader.\(^{(*)}\)

\(^{(*)}\) \( W \) belongs to the so-called “product-class.” We need the lemmas:

A. Let \( \gamma^+ \chi(y) \subseteq L_x \), let \( \omega(\tau) = (\frac{1}{2} - i\tau) M \{ \gamma^+ \chi(y) \} \) be essentially bounded in \((-\infty, \infty)\), and let \( \chi(y) \) have the form \( \chi(y) = \int_{\gamma} H(\xi) d\xi + c \), where \( c = \chi(1) \) is an arbitrary constant. Then, for any \( f \subseteq L_x \), the function

\[
g(y) = Wf = \lim_{N \to \infty} \int_{1/N}^{N} H(\frac{y}{x}) f(x) \frac{dx}{x}
\]
4. We replace the customary operators

\[ f_\alpha(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y - x)^{\alpha-1} f(x) dx, \]

\[ \tilde{f}_\alpha(y) = \frac{1}{\Gamma(\alpha)} \int_y^\infty (x - y)^{\alpha-1} f(x) dx \]

by the more general ones

\[ f_{\eta,\alpha}(y) = \frac{y^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^y (y - x)^{\alpha-1} x^{\eta} f(x) dx, \]

\[ \tilde{f}_{\eta,\alpha}(y) = \frac{y^{\eta}}{\Gamma(\alpha)} \int_y^\infty (x - y)^{\alpha-1} x^{-\eta} f(x) dx, \]

where \( \eta \) is a given parameter. Obviously

\[ f_{\alpha}(y) = y^{\eta} f_{\eta,\alpha}(y), \quad \tilde{f}_{\alpha}(y) = y^{-\eta} \tilde{f}_{\eta,\alpha}(y). \]

In another paper we have proved that \( I_{\eta,\alpha}^+ f \) and \( J_{\eta,\alpha}^- f \) are bounded linear transformations in \( L_p \) for \( 1 \leq p \leq \infty \) when \( \Re(\alpha) > 0 \) and when \( \Re(\eta) > -1/p \) or \( \Re(\eta) > -1/\beta \) respectively. Obviously the definitions above have no meaning at all when we replace \( \alpha \) by an imaginary number \( \beta \), but we shall show that the operators \( I_{\eta,\alpha}^+ f \) and \( J_{\eta,\alpha}^- f \) exist in some sense for any \( f \in L_\infty \). Those definitions are of importance in the theory of Hankel transforms, as will be shown in a joint paper of A. Erdélyi and myself.

exists, and \( Mg = \omega(\tau) Mf \).


B. Let \( \psi(x) \in L_1 \) and \( \psi(y) = y^{1/2} \psi(x)x^{1/2} dx \); then \( y^{1/2} \psi(y) \to 0 \) for \( y \to 0 \) and \( y \to \infty \), and

\[ |\psi|_{1/2} \leq |\phi|. \]

C. Let \( y^{-1} \chi(y) \in L_1 \), let \( y^{-1} \chi(y) \to 0 \) for \( y \to 0 \) and \( y \to \infty \), and let \( \chi(y) \) have the form as in Lemma A; then

\[ (1 - i\tau)M |y^{-1} \chi(y)| = \lim_{N \to \infty} \int_{1/N}^N H(y)y^{1/2+\tau} dy, \]

if the right-hand limit exists.

Cf. H. Kober, loc. cit., Theorem 3(i).


Let \( \Re(\eta) > -1/2 \) and (4)
\[
K(x, y) = \begin{cases} 
\{\Gamma(\alpha)\}^{-1}(y-x)^{\alpha-1}y^{-\alpha} & \{0 < x < y\}, \\
0 & \{x > y\};
\end{cases}
\]
then \( K(x, y) \) satisfies the hypotheses of Theorem I, and we have
\[
\omega(\tau) = \int_0^\infty K(x, 1)x^{-1/2-ir}dx = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1}x^{1/2-ir}dx
\]
\[
= \frac{\Gamma(\eta + \frac{1}{2} - ir)}{\Gamma(\eta + \alpha + \frac{1}{2} - ir)},
\]
\[
I_{\eta, \alpha}^+ = \int_0^\infty K(x, y)f(x)dx [f \subset L_2],
\]
therefore by the theorem
\[
| I_{\eta, \alpha}^+ f |_2 \leq \max_{-\infty < r < \infty} \left| \frac{\Gamma(\eta + \frac{1}{2} - ir)}{\Gamma(\eta + \alpha + \frac{1}{2} - ir)} \right| | f |_2 = \kappa_0 | f |_2.
\]
When we take \( |\alpha| < C \), then, in consequence of a well known property of the gamma function, \( \kappa_0 \) is uniformly bounded for \( \Re(\alpha) > 0 \); therefore
\[
| I_{\eta, \alpha}^+ f |_2 \leq C | f |_2,
\]
where \( C \) depends on \( \eta \) only. Let \( \beta \) be any fixed imaginary number or zero; then, by a well known theorem on weak convergence, a sequence \( \alpha_1, \alpha_2, \alpha_3, \ldots \) and a function \( \phi(y) \subset L_2 \) exist such that \( I_{\eta, \alpha_n}^+ f \) converges weakly to \( \phi(y) \) when \( \alpha_n \) tends to \( \beta \) [\( n \to \infty \)].

A similar argument applies to \( J_{\eta, \alpha}^- f \), and we now define
\[
I_{\eta, \beta}^+ f = \text{weak limit } I_{\eta, \alpha}^+ f \quad \text{[}\Re(\eta) > -\frac{1}{2}, f \subset L_2]\]
\[
J_{\eta, \beta}^- f = \text{weak limit } J_{\eta, \alpha}^- f
\]
for some sequences \( \{\alpha_n\}, \{\alpha_m\} \).

5. Strong convergence. Starting from \( I_{\eta, \beta}^+ f \) and \( J_{\eta, \beta}^- f \) for step-functions \( \psi \) we can show that \( I_{\eta, \alpha}^+ f \) or \( J_{\eta, \alpha}^- f \) converges to \( I_{\eta, \beta}^+ f \) or \( J_{\eta, \beta}^- f \) in the strong sense also for any \( f \subset L_2 \) when \( \alpha \) tends to \( \beta \). We can also proceed in a shorter way. By (2.2) we have
\[
MI_{\eta, \alpha}^+ f = \Gamma(\eta + \frac{1}{2} - ir)\Gamma(\eta + \alpha + \frac{1}{2} - ir)^{-1}Mf
\]
and, taking...
\[ K(x, y) = \begin{cases} 0 & [0 < x < y] \\ \{\Gamma(\alpha)\}^{-1}(x - y)^{\alpha - 1}x^{-\alpha}y^\alpha & [x > y], \end{cases} \]
we get
\[(5.2) \quad MF_{\eta, \alpha}f = \Gamma(\eta + \frac{1}{2} + ir)\{\Gamma(\eta + \alpha + \frac{1}{2} + ir)\}^{-1}Mf.\]

Let
\[Mf = g(\tau); \quad \omega(\tau; \alpha) = \frac{\Gamma(\eta + \frac{1}{2} - ir)}{\Gamma(\eta + \alpha + \frac{1}{2} - ir)}.\]

Then
\[
\int_{-\infty}^{\infty} \left| MI_{\eta, \alpha}f - \omega(\tau; \beta)g(\tau) \right|^2 d\tau = \int_{-\infty}^{\infty} \left| g(\tau) \right|^2 \omega(\tau; \alpha) - \omega(\tau; \beta) \left| d\tau \right|^2 \\
= \int_{-\infty}^{-N} + \int_{N}^{\infty} + \int_{-N}^{N} = Z_1 + Z_2 + Z_3.
\]

Since \(\omega(\tau; \alpha)\) is bounded in \((-\infty, \infty)\) uniformly when \(\Re(\alpha) > 0\) and \(|\alpha| < C\), and since \(g(\tau) \in L_2(-\infty, \infty)\), we can fix \(N\) sufficiently large such that \(Z_1 < \epsilon/3\), \(Z_2 < \epsilon/3\) uniformly in \(\alpha\) for any given \(\epsilon > 0\). Now it is easy to show that \(Z_3 < \epsilon/3\) when \(|\alpha - \beta|\) is sufficiently small. Hence \(MI_{\eta, \alpha}f\) converges strongly to the function \(\omega(\tau; \beta)g(\tau)\) and, by the property of the Mellin transformation mentioned above, \(I_{\eta, \alpha}^+f\) to \(M^{-1}\{\omega(\tau; \beta)g(\tau)\}\). By the same argument we get the corresponding result for \(J_{\eta, \alpha}^-f\), and so we have

**Theorem II.** Let \(\Re(\eta) > -1/2\), let \(\Re(\alpha) > 0\) and \(\Re(\beta) = 0\) and \(\alpha \rightarrow \beta\), and let \(f \in L_2\); then the functions \(I_{\eta, \alpha}^+f\) and \(J_{\eta, \alpha}^-f\) converge strongly to \(I_{\eta, \beta}^+f\) and \(J_{\eta, \beta}^-f\) respectively, where

\[(5.3) \quad I_{\eta, \alpha}^+f = M^{-1}\left\{\frac{\Gamma(\eta + \frac{1}{2} - ir)}{\Gamma(\eta + \beta + \frac{1}{2} - ir)}Mf\right\},\]
\[(5.4) \quad J_{\eta, \alpha}^-f = M^{-1}\left\{\frac{\Gamma(\eta + \frac{1}{2} + ir)}{\Gamma(\eta + \beta + \frac{1}{2} + ir)}Mf\right\}.
\]

Evidently \(I_{\eta, \alpha}^+f = f_{\eta, \alpha}^+f\), \(J_{\eta, \alpha}^-f = f_{\eta, \alpha}^-f\).

**6. The inversions of the operators** \(I_{\eta, \alpha}^+f\), \(J_{\eta, \alpha}^-f\): The operators \((I_{\eta, \beta}^+)^{-1}\) and \((J_{\eta, \beta}^-)^{-1}\) are also bounded linear transformations in \(L_2\). We have

**Theorem III.** Let \(\Re(\eta) > -1/2\), \(\Re(\beta) = 0\), let \(f(x) \subset L_2\), and let

\[(6.1) \quad I_{\eta, \alpha}^+f = g(y); \quad J_{\eta, \alpha}^-f = h(y).\]

Then
\[(6.2) \quad f = I_{\eta, \alpha}^+g; \quad f = J_{\eta, \alpha}^-h.\]
The proof follows from (5.3) and (5.4), immediately; for instance,

\[ M I_{\eta+\beta}^{+} g = \frac{\Gamma(\eta + \beta + \frac{1}{2} - it)}{\Gamma(\eta + \beta - \beta + \frac{1}{2} - it)} M g = M f. \]

From (5.3) and (5.4) we may also see that both the domain and the range of \((I_{\eta}^{\beta})^{-1}\) and \((I_{\eta}^{\beta})^{-1}\) are \(L_2\), since \(\{\omega(\tau; \beta)\}^{-1}\) is bounded in \((-\infty, \infty)\).

Of course the operators \(I_{\eta}^{\alpha}\) and \(J_{\eta}^{\alpha}\) do not possess this simply property.

7. Application to the customary fractional integrals, to that of Riemann-Liouville and to that of Weyl. Let the operators \(X_{af}=f_{a}^{+}(y)\) and \(Y_{af}=f_{a}^{-}(y)\) be defined by

\begin{align}
X_{af} &= y^{\beta} I_{0,\beta}^{+} f, & Y_{af} &= y^{\beta} J_{-\beta,0}^{-} f & [f \in L_2] 
\end{align}

when they are of imaginary order, in accordance with (4.5). Since \(|y^\beta|=1\), \(X_{\beta}\) and \(Y_{\beta}\) are bounded linear transformations in \(L_2\) with domain \(L_2\), and it is not difficult to show that, for \(a-\infty\),

\[ |y^{-a} \tilde{f}_{a} - X_{\beta}|_2 \to 0, \quad |y^{-a} \tilde{f}_{a} - Y_{\beta}|_2 \to 0. \]

The semi-group property of \(f_{a}^{+}\) in \(L_\phi(0, a)\) for \(1 \leq \phi \leq \infty, 0 < a < \infty\) is well known(6). Here we shall prove

**Theorem IV.** The transformations \(X_{\beta}\) or \(Y_{\beta}\) form a group in \(L_2\).

Since \(X_{af}=I_{a0}^{0} f=f\) and \(Y_{af}=J_{a0}^{0} f=f\), we have only to prove that, for any imaginary numbers \(\beta, \gamma\)

\begin{align}
X_{\beta} X_{\gamma} &= X_{\beta+\gamma}, & Y_{\beta} Y_{\gamma} &= Y_{\beta+\gamma}.
\end{align}

We need the following lemmas:

**Lemma 1.** When \(f \in L_2, \Re(\eta) > -1/2, \Re(\lambda) \geq 0, \Re(\mu) \geq 0,\)

\begin{align}
I_{\eta,\lambda,0}^{+} I_{\eta,\mu}^{+} &= I_{\eta,\lambda+\mu}^{+}, & J_{\eta,\lambda}^{-} J_{\eta,\mu}^{-} &= J_{\eta,\lambda+\mu}^{-}, & J_{\eta,\lambda}^{-} I_{\eta,\mu}^{+} &= J_{\eta,\lambda}^{-} I_{\eta,\mu}^{+}.
\end{align}

**Lemma 2.** When \(f \in L_2, \Re(\lambda) \geq 0, \Re(\nu) = 0,\)

\begin{align}
I_{\eta,\lambda}^{+} f &= y^{y^\lambda} I_{\eta,\nu}^{+} \{x^{-\nu} f(x)\}, & J_{\eta,\lambda}^{-} f &= y^{-\nu} J_{\eta,\nu}^{-} \{x^{-\nu} f(x)\}.
\end{align}

We can easily prove Lemma 1 by taking the Mellin transforms of both sides and employing (5.1)-(5.4).

The proof of Lemma 2 follows from the definitions (4.3) and (4.4) immediately when \(\Re(\lambda) > 0\), since \(x^{-\nu} f(x) \subset L_2, x^{\nu} f(x) \subset L_2\). Taking \(\Re(\lambda) > 0, \nu \to \beta\), we have

\[ |I_0^+ - I_1^+|_2 \to 0; \]
\[ |y^\gamma I_{+\gamma}^+ \{x^{-\gamma}f(x)\} |_2 = |I_{+\gamma}^+ \{x^{-\gamma}f(x)\} - I_{+\gamma}^+ \{x^{-\gamma}f(x)\}|_2 \to 0, \]
and so (7.3) is true for \( \lambda = \beta \) also.

Now by (7.1)
\[ X_\beta X_\gamma f = y^\beta I_0^+ \{x^\gamma I_0^+ f\}, \]
and \( I_{0,\beta}^+ \{x^\gamma \phi\} = y^\gamma I_{0,\beta}^+ \phi \) by Lemma 2. Hence, by Lemma 1,
\[ X_\beta X_\gamma f = y^\beta y_\gamma I_\gamma I_\beta f = y^\beta + y_\gamma f = X_\beta + X_\gamma f. \]

Similarly we have
\[ Y_\beta Y_\gamma f = y^\beta Y_\gamma \{x^\gamma Y_\gamma f\} = y^\beta + Y_\gamma f = Y_\beta + Y_\gamma f. \]

**Corollary.** The transformations \((X_\beta)^{-1}\) and \((Y_\beta)^{-1}\) are linear and bounded in \( L_2 \) with domain \( L_2 \), and \((X_\beta)^{-1} = X_{-\beta} \), \((Y_\beta)^{-1} = Y_{-\beta} \).

8. We shall now deal with the corresponding problems in \( L_p \) for \( p \geq 1 \). We do not know if Theorem I can be generalized in some way for \( p \neq 2 \). Therefore we cannot extend the results of §§4–7 to the general case \( f \subset L_p \) \([1 \leq p \leq \infty]\). We have to restrict ourselves to certain subspaces of \( L_p \) or, as in Theorem VI, to the case when \( \alpha \) tends to zero under certain conditions. Moreover we shall discuss the characteristic values (§10).

Let \( 0 < a < \infty \) and let the step-function \( \phi_a(x) \) be defined by \( \phi_a(x) = 1 \) for \( 0 \leq x \leq a \), \( \phi_a(x) = 0 \) otherwise. When \( \Re(\xi) > -1 \), we easily find
\[
\begin{align*}
\mathcal{I}_{+,\alpha}^+ := & \frac{\Gamma(\xi + 1)}{\Gamma(\xi + \alpha + 1)} \left\{ \mathcal{I}_{\alpha} \int_0^{a/\gamma} (1 - \xi)^{\alpha - 1} d\xi \right\} \quad \text{or} \quad \frac{1}{\Gamma(\xi)} \int_0^{a/\gamma} (1 - \xi)^{\alpha - 1} d\xi, \\
\mathcal{J}_{\eta,\alpha}^+ := & \frac{1}{\Gamma(\alpha + 1)} \left\{ \left( 1 - \frac{y}{a} \right)^{\alpha} \left( \frac{y}{a} \right)^{-1} \right\} \\
& + \left\{ \eta - 1 \right\} \int_{y/a}^1 (1 - \xi)^{\alpha - 2} d\xi. 
\end{align*}
\]

(8.11)

for \( 0 < y < a \) or \( a < y < \infty \) respectively; therefore, in these open intervals, \( \mathcal{I}_{\alpha}^+ \phi_a \) and \( \mathcal{J}_{\eta,\alpha}^+ \phi_a \) certainly exist and are continuous, also when we replace \( \alpha \) by a purely imaginary number \( \beta \), and \( \mathcal{I}_{\alpha}^+ \phi_a \to \mathcal{I}_{\beta}^+ \phi_a \), \( \mathcal{J}_{\eta,\alpha}^+ \phi_a \to \mathcal{J}_{\eta,\beta}^+ \phi_a \) as \( \alpha \to -\beta \). Hence, for any step-function \( f \), \( \mathcal{I}_{\alpha}^+ f \) and \( \mathcal{J}_{\eta,\alpha}^+ f \) exist and are continuous almost everywhere in \((0, \infty)\), and \( \mathcal{I}_{\alpha}^+ f \equiv \mathcal{J}_{\eta,\alpha}^+ f \equiv f \), as we can easily deduce from (8.11) and (8.12). The following theorem holds:

**Theorem V.** Let \( 1 \leq p < \infty \) and \( \Re(\xi) > -1/p' = 1 - 1/p \), \( \Re(\eta) > -1/p \), let
$\mathcal{R}(\lambda) \geq 0$, and let $f(x)$ be any step-function. Then, as $\alpha \to \lambda$,

\begin{align}
(8.21) \quad & |I_{t, \alpha}^+ f - I_{t, \lambda}^+ f|_p \to 0, \\
(8.22) \quad & |J_{t, \alpha}^- f - J_{t, \lambda}^- f|_p \to 0.
\end{align}

To prove (8.21), we simply take $f = \phi_0$ and $|\alpha - \lambda| < 1$. Then

$$
\int_0^\infty |I_{t, \alpha}^+ \phi_0 - I_{t, \lambda}^+ \phi_0|^p \, dy \leq \int_0^\alpha \left| \frac{\Gamma(\xi + 1)}{\Gamma(\xi + \alpha + 1)} - \frac{\Gamma(\xi + 1)}{\Gamma(\xi + \lambda + 1)} \right|^p \, dy
$$

say, where

$$
(8.3) \quad \psi(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^{\alpha/y} (1 - t)^{\alpha-1} \, dt.
$$

Obviously $V_1 \to 0$ as $\alpha \to \lambda$. When $a < y < \infty$ and $y$ is fixed, then $\psi(\alpha, y) \to \psi(\lambda, y)$ by the Lebesgue convergence theorem, since $|(1-t)^{\alpha-1}t| \leq (1-a/y)^{-r} \mathcal{R}(t)$ and $\mathcal{R}(t) \subset L_1(0, 1)$. To prove $V_2 \to 0$ we need only show that $|\psi(\alpha, y)| \leq U(y)$, where $U(y)$ does not depend on $\alpha$ and belongs to $L^p(\alpha, \infty)$.

For $2a < y < \infty$, we have $1 - a/y > 1/2,$

$$
\left| \psi(\alpha, y) \right| \leq \frac{2}{|\Gamma(\alpha)|} \int_0^{\alpha/y} \mathcal{R}(t) \, dt \leq K y^{-\mathcal{R}(t)^{-1}} \subset U_0(y) \subset L_p(2a, \infty).
$$

For $a < y < 2a$, we have

$$
\psi(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^{1/2} \frac{1}{\Gamma(\alpha)} \int_{1/2}^{\alpha/y} \psi_1(\alpha, y) + \psi_2(\alpha, y),
$$

$$
|\psi_1(\alpha, y)| \leq \frac{2}{|\Gamma(\alpha)|} \int_0^{1/2} \mathcal{R}(t) \, dt + C_1 = U_1(y) \subset L_p(a, 2a).
$$

$$
|\psi_2(\alpha, y)| = \frac{1}{|\Gamma(\alpha + 1)|} \left| 2^{-\alpha} - \left(1 - \frac{a}{y}\right)^\alpha \left(\frac{a}{y}\right)^\alpha \right|
$$

$$
+ t \int_0^{\alpha/y} (1 - t)^{\alpha-1} \, dt < C_2 = U_2(y) \subset L_p(a, 2a).
$$

Applying Lebesgue’s theorem again, we have $V_2 \to 0$, which completes the proof.

The proof of (8.22) is similar. Let $|\alpha - \lambda| < 1$ again. We have to take into consideration that $J_{t, \alpha}^- \phi_0$ is bounded uniformly in $\alpha$ and $y$ for $a/2 < y < a$. Furthermore, for $0 < y < a/2$ we have to replace (8.12) by
where $|W_1| \leq K$, $|W_2| \leq K$, $|W_3| \leq K$, $y \in L_\alpha(0, a/2)$, when $-1/p < \eta_1 < \min (0, \Re(\eta))$. For $2 \leq p < \infty$, the theorem also follows from Theorem III (§9).

Theorem V holds also when we suppose that $\xi$ and $\eta$ are not fixed, but that $\xi \rightarrow \xi_0$ and $\eta \rightarrow \eta_0$ as $\alpha \rightarrow \lambda$, where $\Re(\xi_0) > -1/p'$ and $\Re(\eta_0) > -1/p$.

Now for $1 \leq p \leq \infty$ and $\Re(\xi) > -1/p'$, $\Re(\eta) > -1/p$, we have (9)

$$
|I_{\xi, \alpha}^+ g|_p \leq \frac{\Gamma(\Re(\alpha)) \Gamma(\Re(\zeta) + 1/p')}{\Gamma(\alpha) \cdot \Gamma(\Re(\zeta + \alpha) + 1/p')} |g|_p;
$$

(8.4)

$$
|J_{\eta, \alpha}^- g|_p \leq \frac{\Gamma(\Re(\alpha)) \Gamma(\Re(\eta) + 1/p)}{|\Gamma(\alpha) \cdot \Gamma(\Re(\eta + \alpha) + 1/p)|} |g|_p.
$$

When in (8.21) and (8.22) $\Re(\lambda)$ is greater than zero, then $|I_{\xi, \alpha}^+ g|_p$ and $|J_{\eta, \alpha}^- g|_p$ are bounded uniformly in $\alpha$ for $|\alpha - \lambda| < \frac{1}{2} \Re(\lambda)$; by approximating to $g(x)$ by a sequence of step-functions we see that, for $\Re(\lambda) > 0$ and $1 \leq p < \infty$, Theorem V is valid for any $g \in L_\alpha$. It is an open question whether or not it holds for $\Re(\lambda) = 0$ also, when $1 \leq p < 2$ or $2 < p < \infty$ (cf. Theorem II), but it is certainly true in the following sense for $\lambda = 0$:

**Theorem VI.** Let $1 \leq p < \infty$ and $f(x) \in L_\alpha$, let $\Re(\xi) > -1/p'$ and $\Re(\eta) > -1/p'$; let $\Theta$ be any positive number smaller than $\pi/2$, and let $\alpha \rightarrow 0$ with the restriction $|\arg \alpha| \leq \Theta$. Then

$$
|I_{\xi, \alpha}^+ f - f(y)|_p \rightarrow 0; \quad |J_{\eta, \alpha}^- f - f(y)|_p \rightarrow 0.
$$

The proof is an immediate consequence of Theorem V, since in $L_\alpha$ the operations $I_{\xi, \alpha}^+$ and $J_{\eta, \alpha}^-$ are uniformly bounded for $|\arg \alpha| \leq \Theta$, when $|\alpha| < \Theta$.

Let $\alpha = \alpha_1 + i \alpha_2$; then $\alpha_1 > 0$, $|\alpha_2| < \alpha_1 \tan \Theta$, and

$$
\frac{\Gamma(\Re(\alpha))}{|\Gamma(\alpha)|} = \frac{\Gamma(\alpha_1 + 1)}{|\Gamma(\alpha + 1)|} \left| 1 + i \frac{\alpha_2}{|\alpha_1|} \right| \leq \frac{\Gamma(\alpha_1 + 1)}{|\Gamma(\alpha + 1)|} (1 + \tan \Theta) < K.
$$

Therefore, (8.4), the operations have the desired property.

**Corollary.** Let $\alpha$ be restricted as in Theorem VI and let $f \in L_\alpha$; then

$$
|y^{-\alpha} f_{\alpha}(y) - f(y)|_p \rightarrow 0; \quad |y^{-\alpha} f_{\alpha}(y) - f(y)|_p \rightarrow 0
$$

for $1 < p < \infty$ and $1 \leq p < \infty$ respectively, as $\alpha$ tends to zero.

(9) Cf. our paper cited in Footnote 4.
The case $p = 1$ is not included for $f_+(y)$ by this theorem; some results for $f_+(y)$ $[\alpha \to 0]$ under the hypothesis $f \in L_1(0, A)$ were given by Hardy-Littlewood (loc. cit.) and by J. D. Tamarkin (*)

9. We can state some better theorems for $p > 2$. Let $2 < p \leq \infty$ and let $\mathcal{M}_p$ be the set of all functions $f \in L_p$ which possess a Mellin transform $F(\tau) = Mf$ in the well defined sense that $f(x)$ is representable in the form

$$f(x) = \lim_{\alpha \to 0} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(\tau) x^{-1/p - i\tau} d\tau = M^{-1} F,$$

where $F(\tau) \in L_{p'}(-\infty, \infty)$. In consequence of the well known theory of Fourier transforms, $M^{-1} F$ is a bounded linear transformation from $L_{p'}(-\infty, \infty)$ into $L_p(0, \infty)$ and $\mathcal{M}_p$ a subspace of $L_p$ and smaller than $L_p$. When $p = 2$ obviously $\mathcal{M}_p = L_2(\cdot)$.

**Lemma 3.** Let $2 < p \leq \infty$ and $\Re(\xi) > -1/p'$ and $\Re(\eta) > -1/p$, and let $f \in \mathcal{M}_p$. Then $I_{\tau, \alpha} f$ and $J_{\eta, \alpha} f$ belong to $\mathcal{M}_p$ also, and

$$M I_{\tau, \alpha} f = \omega(\tau, \alpha) Mf;$$

$$M J_{\eta, \alpha} f = \chi(\tau, \alpha) Mf,$$

where

$$\omega(\tau, \alpha) = \frac{\Gamma(\tau + 1/p' - i\eta)}{\Gamma(\tau + \alpha + 1/p' - i\eta)}; \quad \chi(\tau, \alpha) = \frac{\Gamma(\eta + 1/p + i\tau)}{\Gamma(\eta + \alpha + 1/p + i\eta)}.$$

We shall only outline the proof. Let $F_n(\tau) = F(\tau)$ in $(-n, n)$, $F_n(\tau) = 0$ for $|\tau| > n$, and let $f(x, n) = M^{-1} F_n$. Then, for $0 < y < \infty$,

$$I_{\tau, \alpha} \{f(x, n)\} = \frac{\Gamma(\tau - \alpha - 1/p' - i\eta)}{2\pi \Gamma(\tau)} \int_{-n}^{n} (y - x)^{\alpha - 1/p' - i\eta} x^{\tau - 1/p - i\tau} d\tau$$

$$= \frac{1}{2\pi} \int_{-n}^{n} F(\tau) y^{-1/p - i\tau} d\tau \frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1 - \xi)^{\alpha - 1/p' - i\eta} d\xi$$

$$= M^{-1} \{F_n(\tau) \omega(\tau, \alpha)\},$$

and so (9.21) follows by $\int |F - F_n| \tau^{p'} d\tau \to 0$, $|f(x) - f(x, n)|_{p} \to 0 \quad [n \to \infty]$, and by (8.4).


(8) When $2 < p < \infty$, then (9.1) implies

$$F(\tau) = \lim_{\alpha \to \infty} \frac{\Gamma(\alpha)}{2\pi \Gamma(\alpha)} \int_{1/\alpha}^{\infty} f(x) x^{-1/p' - i\tau} dx$$


For Lemma 3 see also our paper cited above.
By (9.21) and (9.22) \( I^+_{t, \alpha} f \) and \( J^-_{\alpha} f \) are defined also when we replace \( \alpha \) by \( \beta \); for \( \omega(\tau, \beta) \) and \( \chi(\tau, \beta) \) are bounded for \(- \infty < \tau < \infty\), therefore \( \omega(\tau, \beta)Mf = \omega(\tau, \beta)F(\tau) \) and \( \chi(\tau, \beta)Mf = \chi(\tau, \beta)F(\tau) \) belong to \( L_{\rho'}(- \infty, \infty) \). Also to every \( g \in \mathbb{M}_p \) corresponds a uniquely determined function \( f \in \mathbb{M}_p \) such that \( I^+_{t, \alpha} f = g \) or \( J^-_{\alpha} f = g \) respectively (cf. (6.1) and (7.2)). By the same reasoning as in §5 and §7, we have the theorems:

**Theorem IIb.** Let \( 2 \leq p \leq \infty \), let \( \Re(\zeta) > -1/p' \), \( \Re(\eta) > -1/p \), and let \( f \in \mathbb{M}_p \). Then

\[
| I^+_{t, \alpha} f - I^+_{t, \beta} f |_p \to 0; \quad | J^-_{\alpha} f - J^-_{\alpha} f |_p \to 0
\]

as \( \alpha \) tends to \( \lambda \), where \( \Re(\lambda) \geq 0 \).

**Lemma 4.** The operator \( X_{\alpha} f \), defined by (7.1), exists in \( L_{\rho}(0, \infty) \) with both domain and range \( \mathbb{M}_p \) when \( 2 \leq p \leq \infty \). So does \( Y_{\alpha} f \) when \( 2 \leq p < \infty \).

**Theorem IVb.** Under the restrictions of Lemma 4, the transformations \( X_\beta \) or \( Y_\beta \) form a group in \( L_{\rho} \), and \( | X_\beta - X_\beta \delta |_p \to 0 \), \( | Y_\beta - Y_\beta \delta |_p \to 0 \) as \( \beta \to \beta_0 \).

10. **Characteristic values.** From (7.1) and (9.2) we easily have

\[
MX_{\alpha} f = \frac{\Gamma(1/p' - \beta - i\tau)}{\Gamma(1/p' - i\tau)} F(\tau - i\beta) \quad [2 \leq p \leq \infty],
\]

\[
MY_{\alpha} f = \frac{\Gamma(1/p + \beta + i\tau)}{\Gamma(1/p + i\tau)} F(\tau - i\beta) \quad [2 \leq p < \infty],
\]

where \( f = M^{-1}F \in \mathbb{M}_p \). We shall now deal with the equations

(10.21) \hspace{1cm} X_{\alpha} f = kf,

(10.22) \hspace{1cm} Y_{\alpha} f = kf,

(10.23) \hspace{1cm} X_{\alpha} f = k\bar{f}.

Let \( \beta = ip \), where \( p \) is real. Obviously (10.21) is equivalent to

(10.31) \hspace{1cm} h(\tau + p) = h(\tau); \quad F(\tau) = h(\tau)/\Gamma(1/p' - i\tau) \in L_{\rho'}(- \infty, \infty),

and (10.22) or (10.23) to

(10.32) \hspace{1cm} k(\tau + p) = k(\tau); \quad F(\tau) = k(\tau)/\Gamma(1/p + i\tau) \in L_{\rho'}(- \infty, \infty)

or

(10.33) \hspace{1cm} h(\tau + p) = \bar{h}(- \tau); \quad F(\tau) = h(\tau)/\Gamma(1/p' - i\tau) \in L_{\rho'}(- \infty, \infty).

By means of the well known asymptotic expansion of \( |\Gamma(\sigma + it)| \) for \( t \to \pm \infty \) we arrive at the following results:

The transformation \( X_{\alpha} f \) \( \beta \neq 0 \) with domain \( \mathbb{M}_p \) \( [2 \leq p \leq \infty] \) has no characteristic values at all.
The characteristic values of \( Y_{\beta f} \) [\( \beta \neq 0 \), domain \( \mathbb{M}_p \), \( 2 \leq p < \infty \)] are the set of points \( l \) for which \( \exp \left( -\frac{1}{2} \pi |\beta| \right) < |l| < \exp \left( \frac{1}{2} \pi |\beta| \right) \). To every characteristic value \( l \) corresponds an infinity of characteristic functions.

We construct all these functions by (10.32), taking \( k(\tau) \) in \((0, |\rho|)\) as an arbitrary function belonging to \( L_{\rho'}(0, |\rho|) \).

Also, from the group property of \( Y_\beta \), we can deduce the result:

Let \( \mathcal{R}(\kappa) = \mathcal{R}(\lambda) = 0 \), let \( \kappa/\lambda \) be no rational number, and let \( Y_\sigma f = f \) and \( Y_\lambda f = f \) and \( f \subset \mathbb{M}_p \). Then \( f(x) = ce^{-x} \), and \( Y_{\gamma f} = f \) for any \( \gamma \) such that \( \mathcal{R}(\gamma) \geq 0 \).

Furthermore, by (10.33), we can prove the result:

The equation (10.23) has a solution \( f \subset \mathbb{M}_p \) if, and only if, \( |l| = 1 \). To every number \( l \) of this kind corresponds an infinity of solutions \( f \subset \mathbb{M}_p \).

For instance, \( f(x) = e^{-1/2}x^{\beta-1} \) is a solution of (10.23) for \( l = 1 \).

11. It is not difficult to show that \( I_\sigma f \) and \( J_{\sigma f} \) certainly exist and are integrable for any \( f \subset L_p \) when \( \mathcal{R}(\xi) > -1/2 \) and \( \mathcal{R}(\eta) > -1/p \) and \( 2 < p < \infty \).

We have

**Theorem VII.** Let \( 2 < p < \infty \) and \( f(x) \subset L_p \); let \( \mathcal{R}(\xi) > -1/2, \mathcal{R}(\eta) > -1/p \). Then there exist functions \( I_\sigma f \) and \( J_{\sigma f} \), defined in \((0, \infty)\) and such that, for any positive finite number \( A \), \( I_\sigma f \) and \( J_{\sigma f} \) belong to \( L_2(0, A) \) and that

\[
\int_0^A |I_{\sigma f} - J_{\sigma f}|^2 dy \rightarrow 0; \quad \int_0^A |J_{\sigma f} - J_{\sigma f}|^2 dy \rightarrow 0 \quad [\alpha \rightarrow \beta].
\]

**Proof.** Let \( f_n(x) = f(x) \) or \( f_n(x) = 0 \) for \( 0 \leq x \leq n \) or \( x > n \) respectively \([n = 1, 2, \cdots]\). Then \( f_n(x) \subset L_2(0, \infty) \), and by Theorem II, the function \( g_{\alpha}(y, n) = g_{\alpha f_n} \) belongs to \( L_2(0, \infty) \) and converges strongly to a function \( g_{\beta}(y, n) \) as \( \alpha \rightarrow \beta \),

\[
|g_{\alpha}(y, n) - g_{\beta}(y, n)|_2 \rightarrow 0.
\]

We now define \( I_{\sigma f} \) in \((0, \infty)\) by putting

\[
I_{\sigma f} = g_{\beta}(y, n) \quad \text{for} \quad n - 1 < y \leq n \quad [n = 1, 2, \cdots].
\]

This function then has the desired properties. For \( 0 < y \leq n \) obviously

\[
g_{\alpha}(y, n) = I_{\sigma f},
\]

and so

\[
\int_0^n |I_{\sigma f} - g_{\beta}(y, n)|^2 dy \leq \int_0^\infty |g_{\alpha}(y, n) - g_{\beta}(y, n)|^2 dy \rightarrow 0
\]

as \( \alpha \) tends to \( \beta \). Therefore, for \( 1 \leq m < n \),
\[
\left( \int_0^n |g_\beta(y, n) - g_\beta(y, m)|^2 \, dy \right)^{1/2} \leq \left( \int_0^n |I_{r, \alpha f} - g_\beta(y, m)|^2 \, dy \right)^{1/2} \\
+ \left( \int_0^n |g_\beta(y, n) - I_{r, \alpha f}|^2 \, dy \right)^{1/2} \rightarrow 0 \quad \alpha \rightarrow \beta.
\]

Hence \( g_\beta(y, m) \equiv g_\beta(y, n) \) in \((0, m)\), and

\[
I_{r, \alpha f} = g_\beta(y, n)
\]

in \((0, n)\) for \(n = 1, 2, \ldots\).

Therefore \( I_{r, \alpha f} \subset L_2(0, n) \) for any \(n\), and, by (11.3) and (11.4),

\[
\int_0^n |I_{r, \alpha f} - I_{r, \alpha f}|^2 \, dy = \int_0^n |I_{r, \alpha f} - g_\beta(y, n)|^2 \, dy \rightarrow 0
\]
as \(\alpha \rightarrow \beta\); which proves the first part of the theorem.

Let us take (cf. 4.4) \(0 < \gamma \leq n\),

\[
J_{r, \alpha f} = \frac{\gamma^n}{\Gamma(\alpha)} \left\{ \int_0^n + \int_0^\infty \right\} (x - y)^{\alpha-1} x^{-\gamma-x} f(x) \, dx \\
= \phi_\alpha(y, n) + \psi_\alpha(y, n).
\]

Plainly \( \phi_\alpha(y, n) = J_{r, \alpha f} \) in \((0, n)\). Now \( f_n \subset L_2(0, \infty) \) and \( \Re(\eta) > -1/\rho > -1/2 \), and so, by Theorem II, \( J_{r, \alpha f} \) exists, belongs to \( L_2 \), and

\[
\int_0^n |\phi_\alpha(y, n) - J_{r, \alpha f}|^2 \, dy \leq \int_0^n |J_{r, \alpha f} - J_{r, \alpha f}|^2 \, dy \rightarrow 0 \quad \alpha \rightarrow \beta.
\]

Let \( b \) be any positive number smaller than \( n \). Then there exists a function \( \psi_\beta(y, n) \), depending on \( n \) but not on \( b \), such that

\[
\int_0^n |\psi_\alpha(y, n) - \psi_\beta(y, n)|^2 \, dy \rightarrow 0 \quad \alpha \rightarrow \beta,
\]
as we shall now show. For \(0 < \gamma < n\) and \( n \leq x < \infty\), the function

\[
\chi(x, y, \alpha) = \left\{ \Gamma(\alpha) \right\}^{-1} y^{-\gamma} (x - y)^{\alpha-1} x^{-\gamma-x} f(x)
\]
tends to \( \chi(x, y, \beta) \) as \(\alpha \rightarrow \beta\), and

\[
\chi(x, y, \alpha) < Kx^{-\Re(\gamma)-1} |f(x)| \subset L_1(n, \infty)
\]
when \(\gamma\) is fixed and \(|\alpha| < K\). Hence, by Lebesgue's convergence theorem, \( \psi_\beta(y, n) \) exists, and \( \psi_\alpha(y, n) - \psi_\beta(y, n) \rightarrow 0 \) when \(0 < \gamma < n\). For \(0 < \gamma \leq b < n\), we have \( x - y \equiv (1 - b/n)x = cx\),

\[
|\psi_\alpha(y, n)| \leq |\Gamma(\alpha)|^{-1} y^{\Re(\gamma)} \int_n^\infty \left| f(x) \right| \, dx \leq Ky^{\Re(\gamma)}
\]
where $K$ does not depend on $\alpha$ when $|\alpha - \beta| < 1$. Hence also
\[ |\psi_{n}(y, n) - \psi_{n}(y, n)| \leq K \gamma^{y}_{\beta(n)}. \]

Since $y^{\gamma}_{\beta(n)} \subseteq L_{2}(0, 1)$, applying Lebesgue’s theorem again, we obtain (11.7). In consequence of (11.5)–(11.7), for $h_{\beta}(y, n) = \phi_{\beta}(y, n) + \psi_{\beta}(y, n)$ we have
\[
(11.8) \quad \int_{0}^{b} |\tilde{J}_{\gamma, \alpha} - h_{\beta}(y, n)|^{2} dy \to 0 \quad [\alpha \to \beta].
\]

When we define $\tilde{J}_{\gamma, \beta} f$ in $(0, \infty)$ by
\[
\tilde{J}_{\gamma, \beta} f = h_{\beta}(y, n) \quad \text{for } n - 1 \leq y < n \quad [n = 1, 2, \cdots],
\]
then, by the argument applied in (11.4), it easily follows that $\tilde{J}_{\gamma, \beta} f = h_{\beta}(y, n)$ for $0 < y < n$, which completes the proof.