ON A THEOREM OF SCHUR AND ON FRACTIONAL INTEGRALS OF PURELY IMAGINARY ORDER

BY

H. KOBER

1. Let $L_p(a, b)$ be the space of all functions $f(y)$ whose pth power is integrable over (a, b) or which are measurable and essentially bounded over (a, b) for $1 \leqq p < \infty$ or $p = \infty$ respectively, with the norm

$$|f|_p = \left\{ \int_a^b |f(y)|^p dy \right\}^{1/p} \qquad [1 \leqq p < \infty],$$

$$|f|_p = \operatorname*{ess.\ u.b.}_{a<y<b} |f(y)| \qquad [p = \infty];$$

let $p' = p/(p-1)$ and

$$L_p = L_p(0, \infty).$$

The following theorem is in substance due to I. Schur(1):

Let $K(x, y)$ be homogeneous of degree -1 and $K(x, y) \geqq 0$ for $0 < x < \infty$, $0 < y < \infty$, let $K(x, y)x^{-1/2} \subset L_1$, and let $f(x) \subset L_2$; then

$$\left| \int_0^\infty K(x, y) f(x) dx \right|_2 \leqq \kappa |f(y)|_2,$$

where

$$\kappa = \int_0^\infty K(x, 1) x^{-1/2} dx = \int_0^\infty K(1, y) y^{-1/2} dx.$$

The constant κ is the best possible.

Of course the inequality is true when $K(x, y)$ takes negative or even complex values also, if we replace κ by

$$\tilde{} = \int_0^\infty |K(x, 1)| x^{-1/2} dx = \int_0^\infty |K(1, y)| y^{-1/2} dy.$$

However $\tilde{\kappa}$ is not the best possible constant any more. We shall give a better

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(1) Journal für die reine und angewandte Mathematik, vol. 140 (1911), pp. 1–28. The corresponding theorem for $f \subset L_p$, $1 < p < \infty$, was proved by G. H. Hardy, J. E. Littlewood, and G. Pólya, vide *Inequalities*, Cambridge, 1934, Theorem 319.

theorem for this case and we shall use it to deal with fractional integrals the order of which is an imaginary number, thus filling a gap in the literature.

Throughout this paper we denote constants depending on the given parameters by the single symbol C; α and β denote finite numbers such that $\Re(\alpha)>0$, $\Re(\beta)=0$.

2. THEOREM I. *Let* (i) $K(x, y)$ *be homogeneous of degree* -1, (ii) $K(x, 1)x^{-1/2} \subset L_1$, (iii) $f(x) \subset L_2$; *then the function*

$$Wf = \int_1^\infty K(x, y)f(x)dx$$

exists for almost all values of y in $(0, \infty)$, *and*

$$|Wf|_2 \leqq \kappa_0 |f(y)|_2 = \max_{-\infty<\tau<\infty} |\omega(\tau)| \cdot |f(y)|_2,$$

where

$$\omega(\tau) = \int_0^\infty K(x, 1)x^{-1/2-i\tau}dx = \int_0^\infty K(1, y)y^{-1/2+i\tau}dy, \qquad \kappa_0 = \max_{-\infty<\tau<\infty} |\omega(\tau)|.$$

The constant is the best possible.

Obviously $\kappa_0 \leqq \tilde{\kappa}$; when $K(x, y) \geqq 0$ then $\kappa_0 = \kappa$, as we may see taking $\tau = 0$.

Proof. Without loss of generality we may suppose $K(x, y)$ to be no null-function; then $\kappa_0 > 0$. Let $1 < a < \infty$, $f(x, a) = f(x)$ in (a^{-1}, a), $f(x, a) = 0$ otherwise, and let

$$M\phi = \underset{N\to\infty}{\text{l.i.m. sq.}} \int_{1/N}^N \phi(x)x^{-1/2+i\tau}dx \qquad [\phi \subset L_2].$$

In consequence of Schur's theorem W is a bounded linear transformation in L_2; the Mellin transform M is a bounded linear transformation from $L_2(0, \infty)$ into $L_2(-\infty, \infty)$. We have

$$\begin{aligned} \int_0^\infty y^{-1/2+i\tau} W\{f(x, a)\}dy &= \int_0^\infty y^{-1/2+i\tau}dy \int_{1/a}^a K(x, y)f(x)dx \\ &= \int_{1/a}^a f(x)x^{-1/2+i\tau}dx \int_0^\infty K(1, v)v^{-1/2+i\tau}dv \end{aligned} \tag{2.1}$$

when we put $y = vx$ and make use of the homogeneousness of $K(x, y)$; the interchanging of the integrations is justified by absolute convergence of the right-hand repeated integral. Since the left-hand integral exists, it must be equal to $MW\{f(x, a)\}$, therefore we have

$$MW\{f(x, a)\} = \omega(\tau)M\{f(x, a)\}.$$

Since $|f(x, a)-f(x)|_2 \to 0$ $[a \to \infty]$, and $|\omega(\tau)| \leqq \kappa_0 < \infty$, by the continuity of the operations M and W we get

$$MWf = \omega(\tau)Mf; \tag{2.2}$$

therefore $|MWf|_2 \leqq \kappa_0 |Mf|_2$ in $L_2(-\infty, \infty)$. Now the operator $(2\pi)^{-1/2}M$ is isometric, and so we obtain the first assertion of the theorem.

The function $\omega(\tau)$ is continuous in consequence of (ii) and attains its maximum value at a finite point τ, since, by the Riemann-Lebesgue theorem, $\omega(\tau) \to 0$ $[\tau \to \pm\infty]$. Now let λ be any positive number smaller than κ_0. Then we can easily show the existence of functions $f(x) \subset L_2$ such that $|Wf|_2 > \lambda |f|_2$.

Let E be a set of measure $m(E) > 0$ such that $|\omega(\tau)| > \lambda$ in E and E is included in some finite interval. Take $\phi(\tau) = 1$ in E and $\phi(\tau) = 0$ otherwise, and let $f = M^{-1}\phi$. Then from (2.2) we have

$$\int_{-\infty}^{\infty} |MWf|^2 d\tau = \int_E |\omega(\tau)|^2 d\tau > \lambda^2 m(E)$$

$$= \lambda^2 \int_{-\infty}^{\infty} |\phi(\tau)|^2 d\tau = 2\pi\lambda^2 \int_0^{\infty} |f(x)|^2 dx,$$

$$\int_0^{\infty} |Wf|^2 dy > \lambda^2 \int_0^{\infty} |f(x)|^2 dx.$$

Hence the theorem is proved.

3. We could give an alternative proof by the theory of "general transforms," without making use of Schur's theorem. Let V be a transformation of the form

$$Vf = \int_0^{\infty} L(x, y) f(x) dx,$$

the infinite integral being defined in some sense. Then it turns out that, roughly speaking, the class of all transformations which are representable in the form $V_1 V_2$ is identical with the class of the transformations

$$Wf = \int_0^{\infty} K(x, y) f(x) dx,$$

where $K(x, y)$ is homogeneous of degree -1. We leave that proof of I to the reader(2).

(2) W belongs to the so-called "product-class." We need the lemmas:

A. *Let* $y^{-1}\chi(y) \subset L_2$, *let* $\omega(\tau) = (\frac{1}{2} - i\tau) M\{y^{-1}\chi(y)\}$ *be essentially bounded in* $(-\infty, \infty)$, *and let* $\chi(y)$ *have the form* $\chi(y) = \int_1^y H(\xi) d\xi + c$, *where* $c = \chi(1)$ *is an arbitrary constant. Then, for any* $f \subset L_2$, *the function*

$$g(y) = Wf = \underset{N\to\infty}{\text{l.i.m. sq.}} \int_{1/N}^{N} H\left(\frac{y}{x}\right) f(x) \frac{dx}{x}$$

4. We replace the customary operators([3])

$$f_\alpha^{+}(y) = \frac{1}{\Gamma(\alpha)}\int_0^y (y-x)^{\alpha-1} f(x)dx, \tag{4.1}$$

$$f_\alpha^{-}(y) = \frac{1}{\Gamma(\alpha)}\int_y^\infty (x-y)^{\alpha-1} f(x)dx \tag{4.2}$$

by the more general ones

$$f_{\eta,\alpha}^{+}(y) = I_{\eta,\alpha}^{+}f = \frac{y^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^y (y-x)^{\alpha-1} x^{\eta} f(x)dx, \tag{4.3}$$

$$f_{\eta,\alpha}^{-}(y) = J_{\eta,\alpha}^{-}f = \frac{y^{\eta}}{\Gamma(\alpha)}\int_y^\infty (x-y)^{\alpha-1} x^{-\eta-\alpha} f(x)dx, \tag{4.4}$$

where η is a given parameter. Obviously

$$f_\alpha^{+}(y) = y^{\alpha} I_{0,\alpha}^{+}f, \qquad f_\alpha^{-}(y) = y^{\alpha} J_{-\alpha,\alpha}^{-}f. \tag{4.5}$$

In another paper we have proved that $I_{\eta,\alpha}^{+}f$ and $J_{\eta,\alpha}^{-}f$ are bounded linear transformations in L_p for $1 \leqq p \leqq \infty$ when $\Re(\alpha) > 0$ and when $\Re(\eta) > -1/p'$ or $\Re(\eta) > -1/p$ respectively([4]). Obviously the definitions above have no meaning at all when we replace α by an imaginary number β, but we shall show that the operators $I_{\eta,\beta}^{+}f$ and $J_{\eta,\beta}^{-}f$ exist in some sense for any $f \subset L_2$. Those definitions are of importance in the theory of Hankel transforms, as will be shown in a joint paper of A. Erdélyi and myself.

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exists, and $Mg = \omega(\tau)Mf$.

Vide H. Kober, Quarterly Journal of Mathematics, Oxford, vol. 8 (1937), pp. 172–185, §6 and Theorem 2A.

B. *Let* $\phi(x) \subset L_1$ *and* $\psi(y) = y^{-1}\int_0^y \phi(x)x^{1/2}dx$; *then* $y^{1/2}\psi(y) \to 0$ *for* $y \to 0$ *and* $y \to \infty$, *and* $|\psi|_2 \leqq |\phi|_1$.

C. *Let* $y^{-1}\chi(y) \subset L_2$, *let* $y^{-1/2}\chi(y) \to 0$ *for* $y \to 0$ *and* $y \to \infty$, *and let* $\chi(y)$ *have the form as in Lemma* A; *then*

$$(\tfrac{1}{2} - i\tau)M\{y^{-1}\chi(y)\} = \lim_{N\to\infty}\int_{1/N}^{N} H(y)y^{-1/2+i\tau}dy,$$

if the right-hand limit exists.

Cf. H. Kober, loc. cit., Theorem 3(i).

([3]) Cf. H. Weyl, Vierteljahrsschrift der Naturforschende Gesellschaft, Zurich, vol. 62 (1917), pp. 296–302; G. H. Hardy and J. E. Littlewood, Mathematische Zeitschrift, vol. 27 (1928), pp. 565–606; E. R. Love and L. C. Young, Proceedings of the London Mathematical Society, (2), vol. 44 (1938), pp. 1–28. The operator $y^{-\alpha} f_\alpha^{-}(y)$ exists in L_p with domain L_p and is bounded when $1 \leqq p < \{\Re(\alpha)\}^{-1}$. Also cf. J. D. Tamarkin, Annals of Mathematics, (2), vol. 31 (1930), pp. 219–228.

([4]) Cf. *Inequalities*, Theorem 329. H. Kober, Quarterly Journal of Mathematics, Oxford, vol. 11 (1940), pp. 193–211.

Let $\Re(\eta) > -1/2$ and[4]

$$K(x, y) = \begin{cases} \{\Gamma(\alpha)\}^{-1}(y - x)^{\alpha-1}x^{\eta}y^{-\eta-\alpha} & [0 < x < y], \\ 0 \quad [x > y]; \end{cases}$$

then $K(x, y)$ satisfies the hypotheses of Theorem I, and we have

$$\omega(\tau) = \int_0^\infty K(x, 1)x^{-1/2-i\tau}dx = \frac{1}{\Gamma(\alpha)}\int_0^1 (1 - x)^{\alpha-1}x^{\eta-1/2-i\tau}dx$$

$$= \frac{\Gamma(\eta + \frac{1}{2} - i\tau)}{\Gamma(\eta + \alpha + \frac{1}{2} - i\tau)},$$

$$\overset{+}{I}_{\eta,\alpha}f = \int_0^\infty K(x, y)f(x)dx \qquad [f \subset L_2],$$

therefore by the theorem

$$| \overset{+}{I}_{\eta,\alpha}f |_2 \leqq \max_{-\infty<\tau<\infty} \left| \frac{\Gamma(\eta + \frac{1}{2} - i\tau)}{\Gamma(\eta + \alpha + \frac{1}{2} - i\tau)} \right| | f |_2 = \kappa_0 | f |_2.$$

When we take $|\alpha| < C$, then, in consequence of a well known property of the gamma function, κ_0 is uniformly bounded for $\Re(\alpha) > 0$; therefore

$$| \overset{+}{I}_{\eta,\alpha}f |_2 \leqq C | f |_2,$$

where C depends on η only. Let β be any fixed imaginary number or zero; then, by a well known theorem on weak convergence, a sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ and a function $\phi(y) \subset L_2$ exist such that $I^+_{\eta,\alpha_n}f$ converges weakly to $\phi(y)$ when α_n tends to β $[n \to \infty]$.

A similar argument applies to $J^-_{\eta,\alpha}f$, and we now define

$$(4.6) \qquad \overset{+}{I}_{\eta,\beta}f = \operatorname*{weak\ limit}_{\alpha_n \to \beta} \overset{+}{I}_{\eta,\alpha_n}f$$

$$[\Re(\eta) > -\tfrac{1}{2}, f \subset L_2]$$

$$(4.7) \qquad \overset{-}{J}_{\eta,\beta}f = \operatorname*{weak\ limit}_{\alpha_m \to \beta} \overset{-}{J}_{\eta,\alpha_m}f$$

for some sequences $\{\alpha_n\}$, $\{\alpha_m\}$.

5. **Strong convergence.** Starting from $I^+_{\eta,\beta}\psi$ and $J^-_{\eta,\beta}\psi$ for step-functions ψ we can show that $I^+_{\eta,\alpha}f$ or $J^-_{\eta,\alpha}f$ converges to $I^+_{\eta,\beta}f$ or $J^-_{\eta,\beta}f$ in the strong sense also for any $f \subset L_2$ when α tends to β. We can also proceed in a shorter way. By (2.2) we have

$$(5.1) \qquad M\overset{+}{I}_{\eta,\alpha}f = \Gamma(\eta + \tfrac{1}{2} - i\tau)\{\Gamma(\eta + \alpha + \tfrac{1}{2} - i\tau)\}^{-1}Mf$$

and, taking

$$K(x, y) = \begin{cases} 0 & [0 < x < y] \\ \{\Gamma(\alpha)\}^{-1}(x - y)^{\alpha-1}x^{-\eta-\alpha}y^{\eta} & [x > y], \end{cases}$$

we get

$$MJ^{-}_{\eta,\alpha}f = \Gamma(\eta + \tfrac{1}{2} + i\tau)\{\Gamma(\eta + \alpha + \tfrac{1}{2} + i\tau)\}^{-1}Mf. \tag{5.2}$$

Let

$$Mf = g(\tau);\ \omega(\tau; \alpha) = \Gamma(\eta + \tfrac{1}{2} - i\tau)/\Gamma(\eta + \alpha + \tfrac{1}{2} - i\tau).$$

Then

$$\int_{-\infty}^{\infty} | MI^{+}_{\eta,\alpha}f - \omega(\tau; \beta)g(\tau) |^2 d\tau = \int_{-\infty}^{\infty} | g(\tau) |^2 | \omega(\tau; \alpha) - \omega(\tau; \beta) |^2 d\tau$$
$$= \int_{-\infty}^{-N} + \int_{N}^{\infty} + \int_{-N}^{N} = Z_1 + Z_2 + Z_3.$$

Since $\omega(\tau; \alpha)$ is bounded in $(-\infty, \infty)$ uniformly when $\Re(\alpha) > 0$ and $|\alpha| < C$, and since $g(\tau) \subset L_2(-\infty, \infty)$, we can fix N sufficiently large such that $Z_1 < \epsilon/3$, $Z_2 < \epsilon/3$ uniformly in α for any given $\epsilon > 0$. Now it is easy to show that $Z_3 < \epsilon/3$ when $|\alpha - \beta|$ is sufficiently small. Hence $MI^{+}_{\eta,\alpha}f$ converges strongly to the function $\omega(\tau; \beta)g(\tau)$ and, by the property of the Mellin transformation mentioned above, $I^{+}_{\eta,\alpha}f$ to $M^{-1}\{\omega(\tau; \beta)g(\tau)\}$. By the same argument we get the corresponding result for $J^{-}_{\eta,\alpha}f$, and so we have

THEOREM II. *Let $\Re(\eta) > -1/2$, let $\Re(\alpha) > 0$ and $\Re(\beta) = 0$ and $\alpha \to \beta$, and let $f \subset L_2$; then the functions $I^{+}_{\eta,\alpha}f$ and $J^{-}_{\eta,\alpha}f$ converge strongly to $I^{+}_{\eta,\beta}f$ and $J^{-}_{\eta,\beta}f$ respectively, where*

$$I^{+}_{\eta,\beta}f = M^{-1}\left\{\frac{\Gamma(\eta + \frac{1}{2} - i\tau)}{\Gamma(\eta + \beta + \frac{1}{2} - i\tau)} Mf\right\}, \tag{5.3}$$

$$J^{-}_{\eta,\beta}f = M^{-1}\left\{\frac{\Gamma(\eta + \frac{1}{2} + i\tau)}{\Gamma(\eta + \beta + \frac{1}{2} + i\tau)} Mf\right\}. \tag{5.4}$$

Evidently $I^{+}_{\eta,0}f = f$, $J^{-}_{\eta,0}f = f$.

6. **The inversions of the operators** $I^{+}_{\eta,\beta}f$, $J^{-}_{\eta,\beta}f$. The operators $(I^{+}_{\eta,\beta})^{-1}$ and $(J^{-}_{\eta,\beta})^{-1}$ are also bounded linear transformations in L_2. We have

THEOREM III. *Let $\Re(\eta) > -1/2$, $\Re(\beta) = 0$, let $f(x) \subset L_2$, and let*

$$I^{+}_{\eta,\beta}f = g(y); \qquad J^{-}_{\eta,\beta}f = h(y). \tag{6.1}$$

Then

$$f = I^{+}_{\eta+\beta,-\beta}g; \qquad f = J^{-}_{\eta+\beta,-\beta}h. \tag{6.2}$$

The proof follows from (5.3) and (5.4), immediately; for instance,

$$MI^{+}_{\eta+\beta,-\beta}g = \frac{\Gamma(\eta+\beta+\frac{1}{2}-i\tau)}{\Gamma(\eta+\beta-\beta+\frac{1}{2}-i\tau)} Mg = Mf.$$

From (5.3) and (5.4) we may also see that both the domain and the range of $(I^{+}_{\eta,\beta})^{-1}$ and $(J^{-}_{\eta,\beta})^{-1}$ are L_2, since $\{\omega(\tau;\beta)\}^{-1}$ is bounded in $(-\infty,\infty)$.

Of course the operators $I^{+}_{\eta,\alpha}$ and $J^{-}_{\eta,\alpha}$ do not possess this simply property.

7. **Application to the customary fractional integrals, to that of Riemann-Liouville and to that of Weyl.** Let the operators $X_\alpha f = f^{+}_{\alpha}(y)$ and $Y_\alpha f = f^{-}_{\alpha}(y)$ be defined by

$$X_\beta f = y^{\beta} I^{+}_{0,\beta} f, \qquad Y_\beta f = y^{\beta} J^{-}_{-\beta,\beta} f \qquad [f \subset L_2] \tag{7.1}$$

when they are of imaginary order, in accordance with (4.5). Since $|y^\beta| = 1$, X_β and Y_β are bounded linear transformations in L_2 with domain L_2, and it is not difficult to show that, for $\alpha \to \beta$,

$$| y^{\beta-\alpha} f^{+}_{\alpha} - X_\beta |_2 \to 0, \qquad | y^{\beta-\alpha} f^{-}_{\alpha} - Y_\beta |_2 \to 0.$$

The semi-group property of f^{+}_{α} in $L_p(0, a)$ for $1 \leq p \leq \infty$, $0 < a < \infty$ is well known(5). Here we shall prove

THEOREM IV. *The transformations X_β or Y_β form a group in L_2.*

Since $X_0 f = I^{+}_{0,0} f = f$ and $Y_0 f = J^{-}_{0,0} f = f$, we have only to prove that, for any imaginary numbers β, γ

$$X_\beta X_\gamma = X_{\beta+\gamma}; \qquad Y_\beta Y_\gamma = Y_{\beta+\gamma}. \tag{7.2}$$

We need the following lemmas:

LEMMA 1. *When $f \subset L_2$, $\Re(\eta) > -1/2$, $\Re(\lambda) \geq 0$, $\Re(\mu) \geq 0$,*

$$I^{+}_{\eta+\lambda,\mu} I^{+}_{\eta,\lambda} = I^{+}_{\eta,\lambda+\mu} = I^{+}_{\eta,\lambda} I^{+}_{\eta+\lambda,\mu}; \qquad J^{-}_{\eta+\lambda,\mu} J^{-}_{\eta,\lambda} = J^{-}_{\eta,\lambda+\mu} = J^{-}_{\eta,\lambda} J^{-}_{\eta+\lambda,\mu}. \tag{7.3}$$

LEMMA 2. *When $f \subset L_2$, $\Re(\lambda) \geq 0$, $\Re(\nu) = 0$,*

$$I^{+}_{\eta,\lambda} f = y^{\nu} I^{+}_{\eta+\nu,\lambda} \{x^{-\nu} f(x)\}; \qquad J^{-}_{\eta,\lambda} f = y^{-\nu} J^{-}_{\eta+\nu,\lambda} \{x^{\nu} f(x)\}. \tag{7.4}$$

We can easily prove Lemma 1 by taking the Mellin transforms of both sides and employing (5.1)–(5.4).

The proof of Lemma 2 follows from the definitions (4.3) and (4.4) immediately when $\Re(\lambda) > 0$, since $x^{\nu} f(x) \subset L_2$, $x^{-\nu} f(x) \subset L_2$. Taking $\Re(\lambda) > 0$, $\lambda \to \beta$, we have

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(5) E. Hille, Annals of Mathematics, (2), vol. 40 (1939), 4.4. In this paper the theory of semi-groups is developed. Cf. E. Hille, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 159–161.

$$| I^{+}_{\eta,\lambda} - I^{+}_{\eta,\beta} |_2 \to 0;$$

$$| y^{\nu} I^{+}_{\eta+\nu,\lambda}\{x^{-\nu} f(x)\} - y^{\nu} I^{+}_{\eta+\nu,\beta}\{x^{-\nu} f(x)\} |_2$$
$$= | I^{+}_{\eta+\nu,\lambda}\{x^{-\nu} f(x)\} - I^{+}_{\eta+\nu,\beta}\{x^{-\nu} f(x)\} |_2 \to 0,$$

and so (7.3) is true for $\lambda = \beta$ also.

Now by (7.1)

$$X_\beta X_\gamma f = y^{\beta} I^{+}_{0,\beta}\{x^{\gamma} I^{+}_{0,\gamma} f\},$$

and $I^{+}_{0,\beta}\{x^{\gamma}\phi\} = y^{\gamma} I^{+}_{\gamma,\beta}\phi$ by Lemma 2. Hence, by Lemma 1,

$$X_\beta X_\gamma f = y^{\beta+\gamma} I^{+}_{\gamma,\beta} I^{+}_{0,\gamma} f = y^{\beta+\gamma} I^{+}_{0,\beta+\gamma} f = X_{\beta+\gamma} f.$$

Similarly we have

$$Y_\beta Y_\gamma f = y^{\beta} J^{-}_{-\beta,\beta}\{x^{\gamma} J^{-}_{-\gamma,\gamma} f\} = y^{\beta+\gamma} J^{-}_{-\beta-\gamma,\beta} J^{-}_{-\gamma,\gamma} f = y^{\beta+\gamma} J^{-}_{-\beta-\gamma,\beta+\gamma} f = Y_{\beta+\gamma} f.$$

COROLLARY. *The transformations* $(X_\beta)^{-1}$ *and* $(Y_\beta)^{-1}$ *are linear and bounded in* L_2 *with domain* L_2, *and* $(X_\beta)^{-1} = X_{-\beta}$, $(Y_\beta)^{-1} = Y_{-\beta}$.

8. We shall now deal with the corresponding problems in L_p for $p \geqq 1$. We do not know if Theorem I can be generalized in some way for $p \neq 2$. Therefore we cannot extend the results of §§4–7 to the general case $f \subset L_p$ $[1 \leqq p \leqq \infty]$. We have to restrict ourselves to certain subspaces of L_p or, as in Theorem VI, to the case when α tends to zero under certain conditions. Moreover we shall discuss the characteristic values (§10).

Let $0 < a < \infty$ and let the step-function $\phi_a(x)$ be defined by $\phi_a(x) = 1$ for $0 \leqq x \leqq a$, $\phi_a(x) = 0$ otherwise. When $\Re(\zeta) > -1$, we easily find

$$\text{(8.11)} \qquad I^{+}_{\zeta,\alpha}\phi_a = \frac{\Gamma(\xi + 1)}{\Gamma(\zeta + \alpha + 1)} \quad \text{or} \quad \frac{1}{\Gamma(\alpha)} \int_0^{a/y} (1 - t)^{\alpha-1} t^{\zeta} dt,$$

$$\text{(8.12)} \qquad J^{-}_{\eta,\alpha}\phi_a = \frac{1}{\Gamma(\alpha + 1)} \left\{ \left(1 - \frac{y}{a}\right)^{\alpha} \left(\frac{y}{a}\right)^{\eta-1} + (\eta - 1) \int_{y/a}^{1} (1 - t)^{\alpha} t^{\eta-2} dt \right\} \quad \text{or} \quad 0$$

for $0 < y < a$ or $a < y < \infty$ respectively; therefore, in these open intervals, $I^{+}_{\zeta,\alpha}\phi_a$ and $J^{-}_{\eta,\alpha}\phi_a$ certainly exist and are continuous, also when we replace α by a purely imaginary number β, and $I^{+}_{\zeta,\alpha}\phi_a \to I^{+}_{\zeta,\beta}\phi_a$, $J^{-}_{\eta,\alpha}\phi_a \to J^{-}_{\eta,\beta}\phi_a$ as $\alpha \to \beta$. Hence, for any step-function f, $I^{+}_{\zeta,\beta} f$ and $J^{-}_{\eta,\beta} f$ exist and are continuous almost everywhere in $(0, \infty)$, and $I^{+}_{\zeta,0} f \equiv J^{-}_{\eta,0} f \equiv f$, as we can easily deduce from (8.11) and (8.12). The following theorem holds:

THEOREM V. *Let* $1 \leqq p < \infty$ *and* $\Re(\zeta) > -1/p' = 1 - 1/p$, $\Re(\eta) > -1/p$, *let*

$\Re(\lambda) \geqq 0$, *and let* $f(x)$ *be any step-function. Then, as* $\alpha \to \lambda$,

$$\left| \overset{+}{I}_{\zeta,\alpha} f - \overset{+}{I}_{\zeta,\lambda} f \right|_p \to 0, \tag{8.21}$$

$$\left| \overset{-}{J}_{\eta,\alpha} f - \overset{-}{J}_{\eta,\lambda} f \right|_p \to 0. \tag{8.22}$$

To prove (8.21), we simply take $f = \phi_a$ and $|\alpha - \lambda| < 1$. Then

$$\int_0^\infty \left| \overset{+}{I}_{\zeta,\alpha}\phi_a - \overset{+}{I}_{\zeta,\lambda}\phi_a \right|^p dy = \int_0^a \left| \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+\alpha+1)} - \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+\lambda+1)} \right|^p dy$$
$$+ \int_a^\infty |\psi(\alpha, y) - \psi(\lambda, y)|^p dy = V_1 + V_2,$$

say, where

$$\psi(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^{a/y} (1-t)^{\alpha-1} t^\zeta dt. \tag{8.3}$$

Obviously $V_1 \to 0$ as $\alpha \to \lambda$. When $a < y < \infty$ and y is fixed, then $\psi(\alpha, y) \to \psi(\lambda, y)$ by the Lebesgue convergence theorem, since $|(1-t)^{\alpha-1}t^\zeta| \leqq (1-a/y)^{-1} t^{\Re(\zeta)}$ and $t^{\Re(\zeta)} \subset L_1(0, 1)$. To prove $V_2 \to 0$ we need only show that $|\psi(\alpha, y)| \leqq U(y)$, where $U(y)$ does not depend on α and belongs to $L^p(a, \infty)$.

For $2a < y < \infty$, we have $1 - a/y > 1/2$,

$$|\psi(\alpha, y)| \leqq \frac{2}{|\Gamma(\alpha)|} \int_0^{a/y} t^{\Re(\zeta)} dt \leqq K y^{-\Re(\zeta)-1} = U_0(y) \subset L_p(2a, \infty).$$

For $a < y < 2a$, we have

$$\psi(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^{1/2} + \frac{1}{\Gamma(\alpha)} \int_{1/2}^{a/y} = \psi_1(\alpha, y) + \psi_2(\alpha, y),$$

$$\begin{aligned} |\psi_1(\alpha, y)| &\leqq \frac{2}{|\Gamma(\alpha)|} \int_0^{1/2} t^{\Re(\zeta)} dt = C_1 = U_1(y) \subset L_p(a, 2a). \\ |\psi_2(\alpha, y)| &= \frac{1}{|\Gamma(\alpha+1)|} \left| 2^{-\alpha-\zeta} - \left(1 - \frac{a}{y}\right)^\alpha \left(\frac{a}{y}\right)^\zeta \right. \\ &\quad \left. + \zeta \int_{1/2}^{a/y} (1-t)^\alpha t^{\zeta-1} dt \right| < C_2 = U_2(y) \subset L_p(a, 2a). \end{aligned} \tag{8.3.0}$$

Applying Lebesgue's theorem again, we have $V_2 \to 0$, which completes the proof.

The proof of (8.22) is similar. Let $|\alpha - \lambda| < 1$ again. We have to take into consideration that $\overset{-}{J}_{\eta,\alpha}\phi_a$ is bounded uniformly in α and y for $a/2 < y < a$. Furthermore, for $0 < y < a/2$ we have to replace (8.12) by

$$\text{(8.12.0)}\qquad \begin{aligned} \bar{J}_{\eta,\alpha}\phi_a &= \frac{2^{1-\eta-\alpha}}{\Gamma(\alpha+1)} + \frac{\eta-1}{\Gamma(\alpha+1)}\int_{1/2}^{1}(1-t)^{\alpha}t^{\eta-2}dt \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{y/a}^{1}(1-t)^{\alpha-1}t^{\eta-1}dt = W_1 + W_2 + W_3, \end{aligned}$$

where $|W_1| \leqq K$, $|W_2| \leqq K$, $|W_3| \leqq K y^{\eta_1} \subset L_p(0, a/2)$, when $-1/p < \eta_1 < \min(0, \Re(\eta))$. For $2 \leqq p < \infty$, the theorem also follows from Theorem IIb (§9). Theorem V holds also when we suppose that ζ and η are not fixed, but that $\zeta \to \zeta_0$ and $\eta \to \eta_0$ as $\alpha \to \lambda$, where $\Re(\zeta_0) > -1/p'$ and $\Re(\eta_0) > -1/p$.

Now for $1 \leqq p \leqq \infty$ and $\Re(\zeta) > -1/p'$, $\Re(\eta) > -1/p$, we have[6]

$$\text{(8.4)}\qquad \begin{aligned} |\overset{+}{I}_{\zeta,\alpha}g|_p &\leqq \frac{\Gamma\{\Re(\alpha)\}\Gamma\{\Re(\zeta)+1/p'\}}{|\Gamma(\alpha)|\cdot\Gamma\{\Re(\zeta+\alpha)+1/p'\}}|g|_p; \\ |\bar{J}_{\eta,\alpha}g|_p &\leqq \frac{\Gamma\{\Re(\alpha)\}\Gamma\{\Re(\eta)+1/p\}}{|\Gamma(\alpha)|\cdot\Gamma\{\Re(\eta+\alpha)+1/p\}}|g|_p. \end{aligned}$$

When in (8.21) and (8.22) $\Re(\lambda)$ is greater than zero, then $|\overset{+}{I}_{\zeta,\alpha}g|_p$ and $|\bar{J}_{\eta,\alpha}g|_p$ are bounded uniformly in α for $|\alpha-\lambda| < \frac{1}{2}\Re(\lambda)$; by approximating to $g(x)$ by a sequence of step-functions we see that, for $\Re(\lambda) > 0$ and $1 \leqq p < \infty$, Theorem V is valid for any $g \subset L_p$. It is an open question whether or not it holds for $\Re(\lambda) = 0$ also, when $1 \leqq p < 2$ or $2 < p < \infty$ (cf. Theorem II), but it is certainly true in the following sense for $\lambda = 0$:

THEOREM VI. *Let $1 \leqq p < \infty$ and $f(x) \subset L_p$, let $\Re(\zeta) > -1/p'$ and $\Re(\eta) > -1/p$; let Θ be any positive number smaller than $\pi/2$, and let $\alpha \to 0$ with the restriction $|\arg\alpha| \leqq \Theta$. Then*

$$|\overset{+}{I}_{\zeta,\alpha}f - f(y)|_p \to 0; \qquad |\bar{J}_{\eta,\alpha}f - f(y)|_p \to 0.$$

The proof is an immediate consequence of Theorem V, since in L_p the operations $\overset{+}{I}_{\zeta,\alpha}$ and $\bar{J}_{\eta,\alpha}$ are uniformly bounded for $|\arg\alpha| \leqq \Theta$, when $|\alpha| < K$: Let $\alpha = \alpha_1 + i\alpha_2$; then $\alpha_1 > 0$, $|\alpha_2| < \alpha_1 \operatorname{tg}\Theta$, and

$$\frac{\Gamma\{\Re(\alpha)\}}{|\Gamma(\alpha)|} = \frac{\Gamma(\alpha_1+1)}{|\Gamma(\alpha+1)|}\left|1 + i\frac{\alpha_2}{\alpha_1}\right| \leqq \frac{\Gamma(\alpha_1+1)}{|\Gamma(\alpha+1)|}(1+\operatorname{tg}\Theta) < K.$$

Therefore, by (8.4), the operations have the desired property.

COROLLARY. *Let α be restricted as in Theorem* VI *and let $f \subset L_p$; then*

$$|y^{-\alpha}\overset{+}{f}_\alpha(y) - f(y)|_p \to 0, \quad |y^{-\alpha}\bar{f}_\alpha(y) - f(y)|_p \to 0$$

for $1 < p < \infty$ and $1 \leqq p < \infty$ respectively, as α tends to zero.

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(6) Cf. our paper cited in Footnote 4.

The case $p=1$ is not included for $f_\alpha^+(y)$ by this theorem; some results for $f_\alpha^+(y)$ $[\alpha\to 0]$ under the hypothesis $f\subset L_1(0, A)$ were given by Hardy-Littlewood (loc. cit.) and by J. D. Tamarkin(7).

9. We can state some better theorems for $p>2$. Let $2<p\leqq\infty$ and let $\mathfrak{M}_p$ be the set of all functions $f\subset L_p$ which possess a Mellin transform $F(\tau)=Mf$ in the well defined sense that $f(x)$ is representable in the form

$$(9.1)\qquad f(x) = \underset{a,b\to\infty}{\overset{\text{index } p}{\text{l.i.m.}}}\ \frac{1}{2\pi}\int_{-a}^{b} F(\tau)x^{-1/p-i\tau}d\tau = M^{-1}F,$$

where $F(\tau)\subset L_{p'}(-\infty,\ \infty)$. In consequence of the well known theory of Fourier transforms, $M^{-1}F$ is a bounded linear transformation from $L_{p'}(-\infty,\ \infty)$ into $L_p(0,\ \infty)$ and $\mathfrak{M}_p$ a subspace of L_p and smaller than L_p. When $p=2$ obviously $\mathfrak{M}_p=L_2$(8).

Lemma 3. *Let $2<p\leqq\infty$ and $\Re(\zeta)>-1/p'$ and $\Re(\eta)>-1/p$, and let $f\subset\mathfrak{M}_p$. Then $I_{\zeta,\alpha}^{+}f$ and $J_{\eta,\alpha}^{-}f$ belong to $\mathfrak{M}_p$ also, and*

$$(9.21)\qquad M\overset{+}{I}_{\zeta,\alpha}f = \omega(\tau,\ \alpha)Mf;$$

$$(9.22)\qquad M\overset{-}{J}_{\eta,\alpha}f = \chi(\tau,\ \alpha)Mf,$$

where

$$(9.3)\qquad \omega(\tau,\ \alpha) = \frac{\Gamma(\zeta+1/p'-i\tau)}{\Gamma(\zeta+\alpha+1/p'-i\tau)};\quad \chi(\tau,\ \alpha) = \frac{\Gamma(\eta+1/p+i\tau)}{\Gamma(\eta+\alpha+1/p+i\tau)}.$$

We shall only outline the proof. Let $F_n(\tau)=F(\tau)$ in $(-n,\ n)$, $F_n(\tau)=0$ for $|\tau|>n$, and let $f(x,\ n)=M^{-1}F_n$. Then, for $0<y<\infty$,

$$\begin{aligned} \overset{+}{I}_{\zeta,\alpha}\{f(x,\ n)\} &= \frac{y^{-\zeta-\alpha}}{2\pi\Gamma(\alpha)}\int_0^y (y-x)^{\alpha-1}x^{\zeta}dx\int_{-n}^{n}F(\tau)x^{-1/p-i\tau}d\tau \\ &= \frac{1}{2\pi}\int_{-n}^{n}F(\tau)y^{-1/p-i\tau}d\tau\cdot\frac{1}{\Gamma(\alpha)}\int_0^1(1-\xi)^{\alpha-1}\xi^{\zeta-1/p-i\tau}d\xi \\ &= M^{-1}\{F_n(\tau)\omega(\tau,\ \alpha)\}, \end{aligned}$$

and so (9.21) follows by $\int|F-F_n|^{p'}d\tau\to 0$, $|f(x)-f(x,\ n)|_p\to 0$ $[n\to\infty]$, and by (8.4).

(7) Annals of Mathematics, (2), vol. 31 (1930), pp. 219–228.

(8) When $2<p<\infty$, then (9.1) implies

$$F(\tau) = \underset{a\to\infty}{\overset{\text{index } p'}{\text{l.i.m.}}}\int_{1/a}^{a} f(x)x^{-1/p'+i\tau}dx$$

in consequence of the Hille-Tamarkin theorem, vide Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 768–774.

For Lemma 3 see also our paper cited above.

By (9.21) and (9.22) $I_{\zeta,\alpha}^{+}f$ and $J_{\eta,\alpha}^{-}f$ are defined also when we replace α by β; for $\omega(\tau,\beta)$ and $\chi(\tau,\beta)$ are bounded for $-\infty<\tau<\infty$, therefore $\omega(\tau,\beta)Mf=\omega(\tau,\beta)F(\tau)$ and $\chi(\tau,\beta)Mf=\chi(\tau,\beta)F(\tau)$ belong to $L_{p'}(-\infty,\infty)$. Also to every $g\subset\mathfrak{M}_p$ corresponds a uniquely determined function $f\subset\mathfrak{M}_p$ or $\phi\subset\mathfrak{M}_p$ such that $I_{\zeta,\beta}^{+}f=g$ or $J_{\eta,\beta}^{-}\phi=g$ respectively (cf. (6.1) and (7.2)). By the same reasoning as in §5 and §7, we have the theorems:

THEOREM IIb. *Let $2\leqq p\leqq\infty$, let $\Re(\zeta)>-1/p'$, $\Re(\eta)>-1/p$, and let $f\subset\mathfrak{M}_p$. Then*

$$\left| I_{\zeta,\alpha}^{+}f - I_{\zeta,\lambda}^{+}f\right|_p \to 0; \qquad \left| J_{\eta,\alpha}^{-}f - J_{\eta,\lambda}^{-}f\right|_p \to 0$$

as α tends to λ, where $\Re(\lambda)\geqq 0$.

LEMMA 4. *The operator $X_\beta f$, defined by* (7.1), *exists in $L_p(0,\infty)$ with both domain and range $\mathfrak{M}_p$ when $2\leqq p\leqq\infty$. So does $Y_\beta f$ when $2\leqq p<\infty$.*

THEOREM IVb. *Under the restrictions of Lemma* 4, *the transformations X_β or Y_β form a group in L_p, and $\left|X_\beta-X_{\beta_0}\right|_p\to 0$, $\left|Y_\beta-Y_{\beta_0}\right|_p\to 0$ as $\beta\to\beta_0$.*

10. **Characteristic values.** From (7.1) and (9.2) we easily have

$$MX_\beta f=\frac{\Gamma(1/p'-\beta-i\tau)}{\Gamma(1/p'-i\tau)}F(\tau-i\beta) \qquad [2\leqq p\leqq\infty], \tag{10.11}$$

$$MY_\beta f=\frac{\Gamma(1/p+i\tau)}{\Gamma(1/p+\beta+i\tau)}F(\tau-i\beta) \qquad [2\leqq p<\infty], \tag{10.12}$$

where $f=M^{-1}F\subset\mathfrak{M}_p$. We shall now deal with the equations

$$X_\beta f = lf, \tag{10.21}$$

$$Y_\beta f = lf, \tag{10.22}$$

$$X_\beta f = l\bar{f}. \tag{10.23}$$

Let $\beta=i\rho$, where ρ is real. Obviously (10.21) is equivalent to

$$h(\tau+\rho)=lh(\tau);\ \ F(\tau)=h(\tau)/\Gamma(1/p'-i\tau)\subset L_{p'}(-\infty,\infty), \tag{10.31}$$

and (10.22) or (10.23) to

$$k(\tau+\rho)=lk(\tau);\ \ F(\tau)=k(\tau)\Gamma(1/p+i\tau)\subset L_{p'}(-\infty,\infty) \tag{10.32}$$

or

$$h(\tau+\rho)=l\bar{h}(-\tau);\ \ F(\tau)=h(\tau)/\Gamma(1/p'-i\tau)\subset L_{p'}(-\infty,\infty). \tag{10.33}$$

By means of the well known asymptotic expansion of $\left|\Gamma(\sigma+i\tau)\right|$ for $t\to\pm\infty$ we arrive at the following results:

The transformation $X_\beta f$ $[\beta\neq 0]$ with domain $\mathfrak{M}_p$ $[2\leqq p\leqq\infty]$ has no characteristic values at all.

The characteristic values of $Y_\beta f$ $[\beta\neq 0$, *domain* $\mathfrak{M}_p$, $2\leqq p<\infty]$ *are the set of points l for which* $\exp(-\frac{1}{2}\pi|\beta|)<|l|<\exp(\frac{1}{2}\pi|\beta|)$. *To every characteristic value l corresponds an infinity of characteristic functions.*

We construct all these functions by (10.32), taking $k(\tau)$ in $(0, |\rho|)$ as an arbitrary function belonging to $L_{p'}(0, |\rho|)$.

Also, from the group property of Y_β, we can deduce the result:

Let $\Re(\kappa)=\Re(\lambda)=0$, *let* κ/λ *be no rational number, and let* $Y_\kappa f=f$ *and* $Y_\lambda f=f$ *and* $f\subset\mathfrak{M}_p$. *Then* $f(x)=ce^{-x}$, *and* $Y_\gamma f=f$ *for any* γ *such that* $\Re(\gamma)\geqq 0$.

Furthermore, by (10.33), we can prove the result:

The equation (10.23) *has a solution* $f\subset\mathfrak{M}_p$ *if, and only if,* $|l|=1$. *To every number l of this kind corresponds an infinity of solutions* $f\subset\mathfrak{M}_p$.

For instance, $f(x)=e^{-1/x}x^{-\beta-1}$ is a solution of (10.23) for $l=1$.

11. It is not difficult to show that $I^+_{\zeta,\beta}f$ and $J^-_{\eta,\alpha}f$ certainly exist and are integrable for any $f\subset L_p$ when $\Re(\zeta)>-1/2$ and $\Re(\eta)>-1/p$ and $2<p<\infty$. We have

THEOREM VII. *Let* $2<p<\infty$ *and* $f(x)\subset L_p$; *let* $\Re(\zeta)>-1/2$, $\Re(\eta)>-1/p$. *Then there exist functions* $I^+_{\zeta,\beta}f$ *and* $J^-_{\eta,\alpha}f$, *defined in* $(0,\infty)$ *and such that, for any positive finite number* A, $I^+_{\zeta,\beta}f$ *and* $J^-_{\eta,\beta}f$ *belong to* $L_2(0, A)$ *and that*

$$\int_0^A |I^+_{\zeta,\alpha}f - I^+_{\zeta,\beta}f|^2 dy\to 0;\quad \int_0^A |J^-_{\eta,\alpha}f - J^-_{\eta,\beta}f|^2 dy\to 0\quad [\alpha\to\beta].$$

Proof. Let $f_n(x)=f(x)$ or $f_n(x)=0$ for $0\leqq x\leqq n$ or $x>n$ respectively $[n=1, 2, \cdots]$. Then $f_n(x)\subset L_2(0,\infty)$, and by Theorem II, the function $g_\alpha(y, n)=I^+_{\zeta,\alpha}f_n$ belongs to $L_2(0,\infty)$ and converges strongly to a function $g_\beta(y, n)$ as $\alpha\to\beta$,

$$|g_\alpha(y, n)-g_\beta(y, n)|_2\to 0. \tag{11.1}$$

We now define $I^+_{\zeta,\beta}f$ in $(0,\infty)$ by putting

$$I^+_{\zeta,\beta}f = g_\beta(y, n)\quad\text{for}\quad n-1<y\leqq n\qquad [n=1, 2, \cdots]. \tag{11.2}$$

This function then has the desired properties. For $0<y\leqq n$ obviously

$$g_\alpha(y, n) = I^+_{\zeta,\alpha}f,$$

and so

$$\int_0^n |I^+_{\zeta,\alpha}f - g_\beta(y, n)|^2 dy \leqq \int_0^\infty |g_\alpha(y, n)-g_\beta(y, n)|^2 dy\to 0 \tag{11.3}$$

as α tends to β. Therefore, for $1\leqq m<n$,

$$\left(\int_0^m |g_\beta(y, n) - g_\beta(y, m)|^2 dy\right)^{1/2} \leqq \left(\int_0^m |I^+_{\zeta,\alpha}f - g_\beta(y, m)|^2 dy\right)^{1/2}$$
$$+ \left(\int_0^n |g_\beta(y, n) - I^+_{\zeta,\alpha}f|^2 dy\right)^{1/2} \to 0 \qquad [\alpha \to \beta].$$

Hence $g_\beta(y, m) \equiv g_\beta(y, n)$ in $(0, m)$, and

$$I^+_{\zeta,\beta}f \equiv g_\beta(y, n) \tag{11.4}$$

in $(0, n)$ for $n = 1, 2, \cdots$.

Therefore $I^+_{\zeta,\beta}f \subset L_2(0, n)$ for any n, and, by (11.3) and (11.4),

$$\int_0^n |I^+_{\zeta,\alpha}f - I^+_{\zeta,\beta}f|^2 dy = \int_0^n |I^+_{\zeta,\alpha}f - g_\beta(y, n)|^2 dy \to 0$$

as $\alpha \to \beta$; which proves the first part of the theorem.

Let us take (cf. 4.4) $0 < y \leqq n$,

$$\begin{aligned} J^-_{\alpha,\eta}f &= \frac{y^\eta}{\Gamma(\alpha)}\left\{\int_y^n + \int_n^\infty\right\}(x - y)^{\alpha-1}x^{-\eta-\alpha}f(x)dx \\ &= \phi_\alpha(y, n) + \psi_\alpha(y, n). \end{aligned} \tag{11.5}$$

Plainly $\phi_\alpha(y, n) = J^-_{\alpha,\eta}f_n$ in $(0, n)$. Now $f_n \subset L_2(0, \infty)$ and $\Re(\eta) > -1/p > -1/2$, and so, by Theorem II, $J^-_{\beta,\eta}f_n$ exists, belongs to L_2, and

$$\int_0^n |\phi_\alpha(y, n) - J^-_{\eta,\beta}f_n|^2 dy \leqq \int_0^\infty |J^-_{\eta,\alpha}f_n - J^-_{\eta,\beta}f_n|^2 dy \to 0 \quad [\alpha \to \beta]. \tag{11.6}$$

Let b be any positive number smaller than n. Then there exists a function $\psi_\beta(y, n)$, depending on n but not on b, such that

$$\int_0^b |\psi_\alpha(y, n) - \psi_\beta(y, n)|^2 dy \to 0 \qquad [\alpha \to \beta], \tag{11.7}$$

as we shall now show. For $0 < y < n$ and $n \leqq x < \infty$, the function

$$\chi(x, y, \alpha) = \{\Gamma(\alpha)\}^{-1}y^{-\eta}(x - y)^{\alpha-1}x^{-\eta-\alpha}f(x)$$

tends to $\chi(x, y, \beta)$ as $\alpha \to \beta$, and

$$|\chi(x, y, \alpha)| < Kx^{-\Re(\eta)-1}|f(x)| \subset L_1(n, \infty)$$

when y is fixed and $|\alpha| < K$. Hence, by Lebesgue's convergence theorem, $\psi_\beta(y, n)$ exists, and $\psi_\alpha(y, n) - \psi_\beta(y, n) \to 0$ when $0 < y < n$. For $0 < y \leqq b < n$, we have $x - y \geqq (1 - b/n)x = cx$,

$$|\psi_\alpha(y, n)| \leqq |\Gamma(\alpha)|^{-1}y^{\Re(\eta)}\int_n^\infty c^{\Re(\alpha)-1}x^{-\Re(\eta)-1}|f(x)|\,dx \leqq Ky^{\Re(\eta)},$$

where K does not depend on α when $|\alpha-\beta|<1$. Hence also

$$|\psi_\alpha(y, n) - \psi_\beta(y, n)| \leqq Ky^{\Re(\eta)}.$$

Since $y^{\Re(\eta)} \subset L_2(0, b)$, applying Lebesgue's theorem again, we obtain (11.7). In consequence of (11.5)–(11.7), for $h_\beta(y, n) = \phi_\beta(y, n) + \psi_\beta(y, n)$ we have

$$\int_0^b |J_{\eta,\alpha}^- f - h_\beta(y, n)|^2 dy \to 0 \qquad [\alpha \to \beta]. \tag{11.8}$$

When we define $J_{\eta,\beta}^- f$ in $(0, \infty)$ by

$$J_{\eta,\beta}^- f = h_\beta(y, n) \qquad \text{for } n-1 \leqq y < n \quad [n = 1, 2, \cdots],$$

then, by the argument applied in (11.4), it easily follows that $J_{\eta,\beta}^- f \equiv h_\beta(y, n)$ for $0 < y < n$, which completes the proof.

THE UNIVERSITY,
EDGBASTON, BIRMINGHAM, ENGLAND.