ON A THEOREM OF SCHUR AND ON FRACTIONAL INTEGRALS OF PURELY IMAGINARY ORDER

BY

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1. Let $L_p(a, b)$ be the space of all functions $f(y)$ whose $p$th power is integrable over $(a, b)$ or which are measurable and essentially bounded over $(a, b)$ for $1 \leq p < \infty$ or $p = \infty$ respectively, with the norm

$$
\|f\|_p = \left\{ \int_a^b |f(y)|^p \, dy \right\}^{1/p}
$$

for $1 \leq p < \infty$, and

$$
\|f\|_\infty = \text{ess. u.b. } |f(y)|
$$

for $p = \infty$; let $p' = p/(p-1)$ and

$$
L_p = L_p(0, \infty).
$$

The following theorem is in substance due to I. Schur(1):

Let $K(x, y)$ be homogeneous of degree $-1$ and $K(x, y) \geq 0$ for $0 < x < \infty$, $0 < y < \infty$, let $K(x, y)x^{-1/2} \in L_1$, and let $f(x) \in L_2$; then

$$
\left| \int_0^\infty K(x, y)f(x) \, dx \right| \leq \kappa \|f(y)\|_2,
$$

where

$$
\kappa = \int_0^\infty K(x, 1)x^{-1/2} \, dx = \int_0^\infty K(1, y)y^{-1/2} \, dy.
$$

The constant $\kappa$ is the best possible.

Of course the inequality is true when $K(x, y)$ takes negative or even complex values also, if we replace $\kappa$ by

$$
\widetilde{\kappa} = \int_0^\infty |K(x, 1)|x^{-1/2} \, dx = \int_0^\infty |K(1, y)|y^{-1/2} \, dy.
$$

However $\widetilde{\kappa}$ is not the best possible constant any more. We shall give a better

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theorem for this case and we shall use it to deal with fractional integrals the order of which is an imaginary number, thus filling a gap in the literature.

Throughout this paper we denote constants depending on the given parameters by the single symbol $C$; $\alpha$ and $\beta$ denote finite numbers such that $\Re(\alpha) > 0$, $\Re(\beta) = 0$.

2. Theorem I. Let (i) $K(x, y)$ be homogeneous of degree $-1$, (ii) $K(x, 1)x^{-1/2} \subset L_1$, (iii) $f(x) \subset L_2$; then the function

$$Wf = \int_{1}^{\infty} K(x, y)f(x)\,dx$$

exists for almost all values of $y$ in $(0, \infty)$, and

$$\|Wf\|_{2} \leq k_{0} \|f(y)\|_{2} = \max_{-\infty < r < \infty} \left| \omega(\tau) \right| \cdot \|f(y)\|_{2},$$

where

$$\omega(\tau) = \int_{0}^{\infty} K(x, 1)x^{-1/2+i\tau}dx = \int_{0}^{\infty} K(1, y)y^{-1/2+i\tau}dy, \quad k_{0} = \max_{-\infty < r < \infty} \left| \omega(\tau) \right|.$$  

The constant is the best possible.

Obviously $k_{0} \leq \tilde{\kappa}$; when $K(x, y) \geq 0$ then $k_{0} = \kappa$, as we may see taking $\tau = 0$.

**Proof.** Without loss of generality we may suppose $K(x, y)$ to be no null-function; then $k_{0} > 0$. Let $1 < a < \infty$, $f(x, a) = f(x)$ in $(a^{-1}, a)$, $f(x, a) = 0$ otherwise, and let

$$M\phi = \text{l.i.m. sq.} \int_{1/a}^{N} \phi(x)x^{-1/2+i\tau}dx \quad [\phi \subset L_2].$$

In consequence of Schur's theorem $W$ is a bounded linear transformation in $L_2$; the Mellin transform $M$ is a bounded linear transformation from $L_2(0, \infty)$ into $L_2(-\infty, \infty)$. We have

$$\int_{0}^{\infty} y^{-1/2+i\tau}W\{f(x, a)\}dy = \int_{0}^{\infty} y^{-1/2+i\tau}dy \int_{1/a}^{\infty} K(x, y)f(x)\,dx$$

$$= \int_{1/a}^{\infty} f(x)x^{-1/2+i\tau}dx \int_{0}^{\infty} K(1, y)y^{-1/2+i\tau}dy$$

(2.1)

when we put $y = vx$ and make use of the homogeneousness of $K(x, y)$; the interchange of the integrations is justified by absolute convergence of the right-hand repeated integral. Since the left-hand integral exists, it must be equal to $MW\{f(x, a)\}$, therefore we have

$$MW\{f(x, a)\} = \omega(\tau)M\{f(x, a)\}.$$
Since \(|f(x, a) - f(x)| \to 0 \ [a \to \infty]\), and \(\omega(\tau) \leq \kappa_0 < \infty\), by the continuity of the operations \(M\) and \(W\) we get

\[(2.2) \quad MWf = \omega(\tau) Mf;\]

therefore \(\|MWf\|_2 \leq \kappa_0 \|Mf\|_2\) in \(L_2(-\infty, \infty)\). Now the operator \((2\pi)^{-1/2}M\) is isometric, and so we obtain the first assertion of the theorem.

The function \(\omega(\tau)\) is continuous in consequence of (ii) and attains its maximum value at a finite point \(\tau\), since, by the Riemann-Lebesgue theorem, \(\omega(\tau) \to 0 \ [\tau \to \pm \infty]\). Now let \(\lambda\) be any positive number smaller than \(\kappa_0\). Then we can easily show the existence of functions \(f(x) \subset L_2\) such that \(\|WF\|_2 > \lambda\|f\|_2\).

Let \(E\) be a set of measure \(m(E) > 0\) such that \(\|w(t)\| > \lambda\) in \(E\) and \(E\) is included in some finite interval. Take \(\phi(\tau) = 1\) in \(E\) and \(\phi(\tau) = 0\) otherwise, and let \(f = M^{-1} \phi\). Then from (2.2) we have

\[
\lambda^2 \int_{-\infty}^{\infty} |\phi(\tau)|^2 d\tau = 2\pi \lambda^2 \int_0^{\infty} |f(x)|^2 dx,
\]

\[
\int_0^{\infty} |Wf|^2 dy > \lambda^2 \int_0^{\infty} |f(x)|^2 dx.
\]

Hence the theorem is proved.

3. We could give an alternative proof by the theory of "general transforms," without making use of Schur's theorem. Let \(V\) be a transformation of the form

\[Vf = \int_0^\infty L(x, y)f(x)dx,\]

the infinite integral being defined in some sense. Then it turns out that, roughly speaking, the class of all transformations which are representable in the form \(V_1V_2\) is identical with the class of the transformations

\[Wf = \int_0^\infty K(x, y)f(x)dx,\]

where \(K(x, y)\) is homogeneous of degree \(-1\). We leave that proof of I to the reader(\cite{footnote}).

(\cite{footnote}) \(W\) belongs to the so-called "product-class." We need the lemmas:

A. Let \(y \in L_2\), let \(\omega(\tau) = (1 - i\tau)M\{y \times \chi(y)\}\) be essentially bounded in \((-\infty, \infty)\), and let \(\chi(y)\) have the form \(\chi(y) = \int_0^\infty H(\xi) d\xi + c\), where \(c = \chi(1)\) is an arbitrary constant. Then, for any \(f \subset L_2\), the function

\[g(y) = Wf = \lim_{N \to \infty} \int_{1/N}^N H(\xi) f(x) \frac{dx}{x}\]
4. We replace the customary operators\(^{(3)}\)

\[
\int_{y}^{v} f(x) \, dx, \\
\int_{y}^{\infty} f(x) \, dx,
\]

by the more general ones

\[
\int_{y}^{v} x^{-\alpha} f(x) \, dx, \\
\int_{y}^{\infty} x^{-\alpha} f(x) \, dx,
\]

where \(\alpha\) is a given parameter. Obviously

\[
\int_{y}^{v} x^{-\alpha} f(x) \, dx = y^{-\alpha} \int_{y}^{v} x^{-\alpha} f(x) \, dx, \\
\int_{y}^{\infty} x^{-\alpha} f(x) \, dx = y^{-\alpha} \int_{y}^{\infty} x^{-\alpha} f(x) \, dx,
\]

In another paper we have proved that \(\int_{y}^{v} x^{-\alpha} f(x) \, dx\) and \(\int_{y}^{\infty} x^{-\alpha} f(x) \, dx\) are bounded linear transformations in \(L_p\) for \(1 \leq p \leq \infty\) when \(\Re(\alpha) > 0\) and when \(\Re(\eta) > -1/p\) or \(\Re(\eta) > -1/p\) respectively\(^{(4)}\). Obviously the definitions above have no meaning at all when we replace \(\alpha\) by an imaginary number \(\beta\), but we shall show that the operators \(\int_{y}^{v} x^{-\alpha} f(x) \, dx\) and \(\int_{y}^{\infty} x^{-\alpha} f(x) \, dx\) exist in some sense for any \(f \in L_2\). Those definitions are of importance in the theory of Hankel transforms, as will be shown in a joint paper of A. Erdélyi and myself.

\(\int_{y}^{v} x^{-\alpha} f(x) \, dx\) and \(\int_{y}^{\infty} x^{-\alpha} f(x) \, dx\).

exits, and \(Mg = \omega(\tau)Mf\).


B. Let \(\phi(x) \in L_1\) and \(\psi(y) = \int_{y}^{\infty} \phi(x) x^{1/2} \, dx\); then \(\psi^{1/2}(y) \to 0\) for \(y \to 0\) and \(y \to \infty\), and

\[
|\phi|_{1/2} = \lim \psi^{1/2}(y).
\]

C. Let \(y^{-1/2} \in L_1\), let \(y^{-1} x(y) \to 0\) for \(y \to 0\) and \(y \to \infty\), and let \(x(y)\) have the form as in Lemma A; then

\[
(1 - it) M y^{-1} x(y) = \lim_{N \to \infty} \int_{1/N}^{N} H(y) y^{-1+ir} \, dy,
\]

if the right-hand limit exists.

Cf. H. Kober, loc. cit., Theorem 3(i).

\(\int_{y}^{v} x^{-\alpha} f(x) \, dx\) exists in \(L_p\) with domain \(L_p\) and is bounded when \(1 \leq p < \{\Re(\alpha)\}^{-1}\). Also cf. J. D. Tamarkin, Annals of Mathematics, (2), vol. 31 (1930), pp. 219–228.

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Let $\Re(\eta) > -1/2$ and (4)

$$K(x, y) = \begin{cases} \{\Gamma(\alpha)\}^{-1}(y - x)^{\alpha - 1}y^{\alpha - \alpha} & [0 < x < y], \\ 0 & [x > y]; \end{cases}$$

then $K(x, y)$ satisfies the hypotheses of Theorem I, and we have

$$\omega(\tau) = \int_0^\infty K(x, 1)x^{-1/2-ir}dx = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - x)^{\alpha - 1}x^{\eta - 1/2 - ir}dx$$

$$= \frac{\Gamma(\eta + \frac{1}{2} - ir)}{\Gamma(\eta + \alpha + \frac{1}{2} - ir)},$$

$$I_{\eta, \alpha f}^+ = \int_0^\infty K(x, y)f(x)dx \quad [f \subset L_2],$$

therefore by the theorem

$$I_{\eta, \alpha f}^+ = \max_{-\infty < \kappa < \infty} \left| \frac{\Gamma(\eta + \frac{1}{2} - ir)}{\Gamma(\eta + \alpha + \frac{1}{2} - ir)} \right| |f|_2 = \kappa_0 |f|_2.$$

When we take $|\alpha| < C$, then, in consequence of a well known property of the gamma function, $\kappa_0$ is uniformly bounded for $\Re(\alpha) > 0$; therefore

$$|I_{\eta, \alpha f}^+|_2 \leq C |f|_2,$$

where $C$ depends on $\eta$ only. Let $\beta$ be any fixed imaginary number or zero; then, by a well known theorem on weak convergence, a sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ and a function $\phi(y) \subset L_2$ exist such that $I_{\eta, \alpha_n f}^+$ converges weakly to $\phi(y)$ when $\alpha_n$ tends to $\beta \ [n \to \infty]$. A similar argument applies to $J_{\alpha, \alpha f}^-$, and we now define

\begin{equation}
I_{\eta, \beta f}^+ = \text{weak limit}_{\alpha_n \to \beta} I_{\eta, \alpha_n f}^+ \quad [\Re(\eta) > -\frac{1}{2}, f \subset L_2]
\end{equation}

\begin{equation}
J_{\eta, \beta f}^- = \text{weak limit}_{\alpha_m \to \beta} J_{\eta, \alpha_m f}^-
\end{equation}

for some sequences $\{\alpha_n\}, \{\alpha_m\}$.

5. **Strong convergence.** Starting from $I_{\eta, \beta f}^+$ and $J_{\eta, \beta f}^-$ for step-functions $\psi$ we can show that $I_{\eta, \alpha f}^+$ or $J_{\eta, \alpha f}^-$ converges to $I_{\eta, \beta f}^+$ or $J_{\eta, \beta f}^-$ in the strong sense also for any $f \subset L_2$ when $\alpha$ tends to $\beta$. We can also proceed in a shorter way. By (2.2) we have

\begin{equation}
MI_{\eta, \alpha f}^+ = \Gamma(\eta + \frac{1}{2} - ir)\{\Gamma(\eta + \alpha + \frac{1}{2} - ir)\}^{-1}Mf
\end{equation}

and, taking
\[ K(x, y) = \begin{cases} 0 & [0 < x < y] \\ \{\Gamma(\alpha)\}^{-1}(x - y)^{\alpha - 1}x^{-\alpha}y^{\alpha} & [x > y], \end{cases} \]

we get

\[ (5.2) \quad M^+_{\eta, \alpha}f = \Gamma(\eta + \frac{1}{2} + ir)\{\Gamma(\eta + \alpha + \frac{1}{2} + ir)\}^{-1}Mf. \]

Let

\[ Mf = g(r); \quad \omega(\tau; \alpha) = \Gamma(\eta + \frac{1}{2} - ir)/\Gamma(\eta + \alpha + \frac{1}{2} - ir). \]

Then

\[ \int_{-\infty}^{\infty} | M^+_{\eta, \alpha}f - \omega(\tau; \beta)g(\tau) |^2 d\tau = \int_{-\infty}^{\infty} | g(\tau) |^2 | \omega(\tau; \alpha) - \omega(\tau; \beta) |^2 d\tau = \int_{-\infty}^{-N} + \int_{N}^{\infty} + \int_{-N}^{N} = Z_1 + Z_2 + Z_3. \]

Since \( \omega(\tau; \alpha) \) is bounded in \((-\infty, \infty)\) uniformly when \( \Re(\alpha) > 0 \) and \( |\alpha| < C \), and since \( g(\tau) \in L^2(-\infty, \infty) \), we can fix \( N \) sufficiently large such that \( Z_1 < \epsilon/3 \), \( Z_2 < \epsilon/3 \) uniformly in \( \alpha \) for any given \( \epsilon > 0 \). Now it is easy to show that \( Z_3 < \epsilon/3 \) when \( |\alpha - \beta| \) is sufficiently small. Hence \( M^+_{\eta, \alpha}f \) converges strongly to the function \( \omega(\tau; \beta)g(\tau) \) and, by the property of the Mellin transformation mentioned above, \( I^+_{\eta, \alpha}f \) to \( M^{-1}\{\omega(\tau; \beta)g(\tau)\} \). By the same argument we get the corresponding result for \( J^-_{\eta, \alpha}f \), and so we have

**Theorem II.** Let \( \Re(\eta) > -1/2 \), let \( \Re(\alpha) > 0 \) and \( \Re(\beta) = 0 \) and \( \alpha \to \beta \), and let \( f \in L^2 \); then the functions \( I^+_{\eta, \alpha}f \) and \( J^-_{\eta, \alpha}f \) converge strongly to \( I^+_{\eta, \beta}f \) and \( J^-_{\eta, \beta}f \) respectively, where

\[ (5.3) \quad I^+_{\eta, \alpha}f = M^{-1}\{\frac{\Gamma(\eta + \frac{1}{2} - ir)}{\Gamma(\eta + \alpha + \frac{1}{2} - ir)}MF\}, \]

\[ (5.4) \quad J^-_{\eta, \alpha}f = M^{-1}\{\frac{\Gamma(\eta + \frac{1}{2} + ir)}{\Gamma(\eta + \beta + \frac{1}{2} + ir)}MF\}. \]

Evidently \( I^+_{\eta, \alpha}f = f, J^-_{\eta, \alpha}f = f \).

**6. The inversions of the operators \( I^-_{\eta, \alpha}f, J^+_{\eta, \alpha}f \).** The operators \( (I^+_{\eta, \alpha})^{-1} \) and \( (J^-_{\eta, \alpha})^{-1} \) are also bounded linear transformations in \( L^2 \). We have

**Theorem III.** Let \( \Re(\eta) > -1/2, \Re(\beta) = 0, \) let \( f(x) \subset L^2 \), and let

\[ (6.1) \quad I^+_{\eta, \alpha}f = g(y); \quad J^-_{\eta, \alpha}f = h(y). \]

Then

\[ (6.2) \quad f = I^+_{\eta, \alpha - \beta}g; \quad f = J^-_{\eta, \alpha - \beta}h. \]
The proof follows from (5.3) and (5.4), immediately; for instance,

\[ M I_{\eta+\beta,-g}^+ = \frac{\Gamma(\eta + \beta + \frac{1}{2} - it)}{\Gamma(\eta + \beta - \beta + \frac{1}{2} - it)} Mg = Mf. \]

From (5.3) and (5.4) we may also see that both the domain and the range of \((I_{\eta}^+)^{-1}\) and \((J_{\eta}^-)^{-1}\) are \(L_2\), since \(\{\omega(\tau; \beta)\}^{-1}\) is bounded in \((-\infty, \infty)\).

Of course the operators \(I_{\eta}^+\) and \(J_{\eta}^-\) do not possess this simply property.

7. Application to the customary fractional integrals, to that of Riemann-Liouville and to that of Weyl. Let the operators \(X_\alpha f = f^{\alpha+}(y)\) and \(Y_\alpha f = f^{\alpha-}(y)\) be defined by

\[
(7.1) \quad X_\alpha f = y^{\alpha} I_{0}^{\alpha} f, \quad Y_\alpha f = y^{\alpha} J_{0}^{\alpha} f \quad \quad [f \in L_2]
\]

when they are of imaginary order, in accordance with (4.5). Since \(|y^{\beta}| = 1\), \(X_\beta\) and \(Y_\beta\) are bounded linear transformations in \(L_2\) with domain \(L_2\), and it is not difficult to show that, for \(\alpha \to \beta,\)

\[
|y^{\beta-a} f_n - X_\beta |_2 \to 0, \quad |y^{\beta-a} f_n - Y_\beta |_2 \to 0.
\]

The semi-group property of \(f^\alpha\) in \(L_\alpha(0, a)\) for \(1 \leq p \leq \infty, 0 < a < \infty\) is well known (h). Here we shall prove

**Theorem IV.** The transformations \(X_\beta\) or \(Y_\beta\) form a group in \(L_2\).

Since \(X_\alpha f = I_{0}^{\alpha} f = f\) and \(Y_\alpha f = J_{0}^{\alpha} f = f\), we have only to prove that, for any imaginary numbers \(\beta, \gamma\)

\[
(7.2) \quad X_\beta X_\gamma = X_{\beta+\gamma}; \quad Y_\beta Y_\gamma = Y_{\beta+\gamma}.
\]

We need the following lemmas:

**Lemma 1.** When \(f \in L_2\), \(\Re(\eta) > -1/2, \Re(\lambda) \geq 0, \Re(\mu) \geq 0,\)

\[
(7.3) \quad I_{\eta+\lambda,\mu}^+ I_{\eta,\lambda+\mu}^+ = I_{\eta+\lambda,\mu+\eta}^+; \quad J_{\eta+\lambda,\mu}^- J_{\eta,\lambda+\mu}^- = J_{\eta+\lambda,\mu+\eta}^- J_{\eta,\lambda+\mu}^-.
\]

**Lemma 2.** When \(f \in L_2\), \(\Re(\lambda) \geq 0, \Re(\nu) = 0,\)

\[
(7.4) \quad I_{\eta,\lambda}^+ f = y^{r} I_{\eta,\lambda+\nu}^{+}\{x^r f(x)\}; \quad J_{\eta,\lambda}^- f = y^{-r} J_{\eta+\nu,\lambda}^{-}\{x^r f(x)\}.
\]

We can easily prove Lemma 1 by taking the Mellin transforms of both sides and employing (5.1)-(5.4).

The proof of Lemma 2 follows from the definitions \((4.3)\) and \((4.4)\) immediately when \(\Re(\lambda) > 0\), since \(x^r f(x) \in L_2, x^{-r} f(x) \in L_2\). Taking \(\Re(\lambda) > 0, \lambda \to \beta,\) we have

\[ (*) \quad E. \text{Hille, Annals of Mathematics, (2), vol. 40 (1939), 4.4. In this paper the theory of semi-groups is developed. Cf. E. Hille, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 159-161.\]
\[ | I_{\gamma, \lambda}^+ - I_{\gamma, \beta}^+ |_2 \to 0; \]
\[ | y^{\gamma} I_{\gamma, \lambda}^+ \{ x^{-\gamma} f(x) \} - y^{\gamma} I_{\gamma, \beta}^+ \{ x^{-\gamma} f(x) \} |_2 \]
\[ = | I_{\gamma, \lambda}^+ \{ x^{-\gamma} f(x) \} - I_{\gamma, \beta}^+ \{ x^{-\gamma} f(x) \} |_2 \to 0, \]
and so (7.3) is true for \( \lambda = \beta \) also.

Now by (7.1)
\[ X_{\gamma} X_{\gamma} f = y^{\gamma} I_{\gamma, \beta}^+ \{ x^{\gamma} I_{\beta, \gamma}^+ \}, \]
and \( I_{\beta, \gamma}^+ \{ x^{\gamma} \phi \} = y^{\gamma} I_{\beta, \gamma}^+ \phi \) by Lemma 2. Hence, by Lemma 1,
\[ X_{\beta} X_{\gamma} f = y^{\beta + \gamma} I_{\beta, \gamma}^+ I_{\beta, \gamma}^+ = y^{\beta + \gamma} I_{\beta, \gamma}^+ = X_{\beta + \gamma} f. \]

Similarly we have
\[ Y_{\gamma} Y_{\gamma} f = y^{\beta + \gamma} J_{\gamma, \beta}^- J_{\gamma, \beta}^- = y^{\beta + \gamma} J_{\gamma, \beta}^- J_{\gamma, \beta}^- = Y_{\beta + \gamma} f. \]

**Corollary.** The transformations \( (X_{\beta})^{-1} \) and \( (Y_{\beta})^{-1} \) are linear and bounded in \( L_2 \) with domain \( L_2 \), and \( (X_{\beta})^{-1} = X_{-\beta}, (Y_{\beta})^{-1} = Y_{-\beta} \).

8. We shall now deal with the corresponding problems in \( L_p \) for \( p > 1 \). We do not know if Theorem I can be generalized in some way for \( p \neq 2 \). Therefore we cannot extend the results of §§4–7 to the general case \( f \in L_p \) \([1 \leq p \leq \infty]\). We have to restrict ourselves to certain subspaces of \( L_p \) or, as in Theorem VI, to the case when \( \alpha \) tends to zero under certain conditions. Moreover we shall discuss the characteristic values (§10).

Let \( 0 < a < \infty \) and let the step-function \( \phi_a(x) \) be defined by \( \phi_a(x) = 1 \) for \( 0 \leq x \leq a \), \( \phi_a(x) = 0 \) otherwise. When \( \Re(\zeta) > -1 \), we easily find
\[ I_{\gamma, \alpha}^a = \frac{\Gamma(\zeta + 1)}{\Gamma(\alpha + 1)} \left[ \frac{1}{\Gamma(a)} \int_0^{a/y} (1 - t)^{a-1} dt \right], \]
(8.11)
\[ J_{\eta, \alpha}^a = \frac{1}{\Gamma(\alpha + 1)} \left\{ \left( 1 - \frac{y}{a} \right)^{a} \left( \frac{y}{a} \right)^{-1} \right\} \]
\[ + (\eta - 1) \int_{y/a}^{1} (1 - t)^{a-2} dt \] or 0
(8.12)
for \( 0 < y < a \) or \( a = y < \infty \) respectively; therefore, in these open intervals, \( I_{\gamma, \alpha}^a \) and \( J_{\eta, \alpha}^a \) exist and are continuous, also when we replace \( \alpha \) by a purely imaginary number \( \beta \), and \( I_{\gamma, \alpha}^a \to I_{\gamma, \beta}^a \alpha \to J_{\eta, \alpha}^a \to J_{\eta, \beta}^a \) as \( \alpha \to \beta \). Hence, for any step-function \( f \), \( I_{\gamma, \alpha}^a f \) and \( J_{\eta, \alpha}^a f \) exist and are continuous almost everywhere in \( (0, \infty) \), and \( I_{\gamma, \alpha}^a f = J_{\eta, \alpha}^a f = f \), as we can easily deduce from (8.11) and (8.12). The following theorem holds:

**Theorem V.** Let \( 1 \leq p < \infty \) and \( \Re(\zeta) > -1/p' = 1 - 1/p, \Re(\eta) > -1/p \), let
\( \Re(\lambda) \geq 0, \) and let \( f(x) \) be any step-function. Then, as \( \alpha \to \lambda, \)

\begin{align*}
(8.21) & \quad |I_{t,\alpha}f - I_{t,\lambda}f|_p \to 0, \\
(8.22) & \quad |J_{s,\alpha}f - J_{s,\lambda}f|_p \to 0.
\end{align*}

To prove (8.21), we simply take \( f = \phi_0 \) and \( |\alpha - \lambda| < 1. \) Then

\[
\left| \int_0^\infty |\phi_0| \frac{e^{-\alpha x} - e^{-\lambda x}}{\Gamma(\xi + \alpha + 1)} \right|^p dy = V_1 + V_2,
\]

say, where

\begin{align*}
(8.3) & \quad \psi(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^{\alpha/y} (1 - t) e^{-t \xi} dt.
\end{align*}

Obviously \( V_1 \to 0 \) as \( \alpha \to \lambda. \) When \( a < y < \infty \) and \( y \) is fixed, then \( \psi(\alpha, y) \to \psi(\lambda, y) \) by the Lebesgue convergence theorem, since \( |(1 - t)^{-1/2}| \leq (1 - a/y)^{-1/2} \) and \( e^{\beta(t)} \leq C_1 \) for \( 0 < y < \infty. \) To prove \( V_2 \to 0 \) we need only show that \( \psi(\alpha, y) \) is bounded uniformly in \( \alpha \) and belongs to \( L^p(a, \infty). \)

For \( 2a < y < \infty, \) we have

\[
|\psi(\alpha, y)| \leq \frac{2}{\Gamma(\alpha)} \int_0^{\alpha/y} e^{\beta(t)} dt \leq K y^{-\beta(\xi)-1} = U_0(y) \subset L_p(2a, \infty).
\]

For \( a < y < 2a, \) we have

\[
\psi(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^{1/2} + \frac{1}{\Gamma(\alpha)} \int_{1/2}^{\alpha/y} = \psi_1(\alpha, y) + \psi_2(\alpha, y),
\]

\[
|\psi_1(\alpha, y)| \leq \frac{2}{\Gamma(\alpha)} \int_0^{1/2} e^{\beta(t)} dt = C_1 \subset U_1(y) \subset L_p(a, 2a).
\]

\[
(8.3.0) & \quad |\psi_2(\alpha, y)| = \frac{1}{\Gamma(\alpha + 1)} \left| 2^{-a-t} - \left( 1 - \frac{a}{y} \right)^{\alpha} \left( \frac{a}{y} \right)^t \right|
\]

\[
+ t \int_{1/2}^{\alpha/y} (1 - t)^{-1/2} dt \leq C_2 = U_2(y) \subset L_p(a, 2a).
\]

Applying Lebesgue's theorem again, we have \( V_2 \to 0, \) which completes the proof.

The proof of (8.22) is similar. Let \( |\alpha - \lambda| < 1 \) again. We have to take into consideration that \( J_{s,\alpha} \phi_0 \) is bounded uniformly in \( \alpha \) and \( y \) for \( a/2 < y < a. \) Furthermore, for \( 0 < y < a/2 \) we have to replace (8.12) by
\[ J_{\eta, \alpha} \phi_a = \frac{2^{1-\nu-a}}{\Gamma(\alpha + 1)} + \frac{\eta - 1}{\Gamma(\alpha + 1)} \int_{1/2}^{1} (1 - t)^{\alpha-3} dt \]

\[ + \frac{1}{\Gamma(\alpha)} \int_{y/a}^{1} (1 - \xi)^{\alpha-3} d\xi = W_1 + W_2 + W_3, \]

where \( |W_1| \leq K, |W_2| \leq K, |W_3| \leq Ky_n \subset L_p(0, a/2) \), when \(-1/p < \eta_1 < \min (0, \Re(\eta))\). For \(2 \leq p < \infty\), the theorem also follows from Theorem IIb (§9).

Theorem V holds also when we suppose that \( \xi \) and \( \eta \) are not fixed, but that \( \xi \to \xi_0 \) and \( \eta \to \eta_0 \) as \( \alpha \to \lambda \), where \( \Re(\xi_0) > -1/p' \) and \( \Re(\eta_0) > -1/p \).

Now for \(1 \leq p \leq \infty\) and \( \Re(\xi) > -1/p', \Re(\eta) > -1/p\), we have\(^{(4)}\)

\[ |I_{\xi, \alpha} g|_p \leq \frac{\Gamma(\Re(\alpha)) \Gamma(\Re(\xi) + 1/p')}{\Gamma(\alpha) \cdot \Gamma(\Re(\xi) + 1/p')} |g|_p; \]

\[ |J_{\eta, \alpha} g|_p \leq \frac{\Gamma(\Re(\alpha)) \Gamma(\Re(\eta) + 1/p)}{\Gamma(\alpha) \cdot \Gamma(\Re(\eta) + 1/p')} |g|_p. \]

When in (8.21) and (8.22) \( \Re(\lambda) \) is greater than zero, then \( |I_{\xi, \alpha} g|_p \) and \( |J_{\eta, \alpha} g|_p \) are bounded uniformly in \( \alpha \) for \( |\alpha - \lambda| < \frac{1}{2} \Re(\lambda) \); by approximating to \( g(x) \) by a sequence of step-functions we see that, for \( \Re(\lambda) > 0 \) and \( 1 \leq p < \infty \), Theorem V is valid for any \( g \subset L_p \). It is an open question whether or not it holds for \( \Re(\lambda) = 0 \) also, when \( 1 \leq p < 2 \) or \( 2 < p < \infty \) (cf. Theorem II), but it is certainly true in the following sense for \( \lambda = 0 \):

**Theorem VI.** Let \( 1 \leq p < \infty \) and \( f(x) \subset L_p \), let \( \Re(\xi) > -1/p' \) and \( \Re(\eta) > -1/p \); let \( \Theta \) be any positive number smaller than \( \pi/2 \), and let \( \alpha \to 0 \) with the restriction \( |\arg \alpha| \leq \Theta \). Then

\[ |I_{\xi, \alpha} f - f(y)|_p \to 0; \quad |J_{\eta, \alpha} f - f(y)|_p \to 0. \]

The proof is an immediate consequence of Theorem V, since in \( L_p \) the operations \( I_{\xi, \alpha} \) and \( J_{\eta, \alpha} \) are uniformly bounded for \( |\arg \alpha| \leq \Theta \), when \( |\alpha| < K \):

Let \( \alpha = \alpha_1 + i\alpha_2 \); then \( \alpha_1 > 0 \), \( |\alpha_2| < \alpha_1 \tan \Theta \), and

\[ \frac{\Gamma(\Re(\alpha))}{\Gamma(\alpha)} = \frac{\Gamma(\alpha_1 + 1)}{\Gamma(\alpha + 1)} \cdot 1 + i \frac{\alpha_2}{\alpha_1} \leq \frac{\Gamma(\alpha_1 + 1)}{\Gamma(\alpha + 1)} (1 + \tan \Theta) < K. \]

Therefore, by (8.4), the operations have the desired property.

**Corollary.** Let \( \alpha \) be restricted as in Theorem VI and let \( f \subset L_p \); then

\[ |y^{-\alpha} f_a(y) - f(y)|_p \to 0; \quad |y^{-\alpha} f_a(y) - f(y)|_p \to 0 \]

for \( 1 < p < \infty \) and \( 1 \leq p < \infty \) respectively, as \( \alpha \) tends to zero.

\(^{(4)}\) Cf. our paper cited in Footnote 4.
The case \( p = 1 \) is not included for \( f_+^+ (y) \) by this theorem; some results for \( f_+^+ (y) \) \( [\alpha \to 0] \) under the hypothesis \( f \in L_1(0, A) \) were given by Hardy-Littlewood (loc. cit.) and by J. D. Tamarkin\(^7\).

9. We can state some better theorems for \( p > 2 \). Let \( 2 < p \leq \infty \) and let \( \mathcal{M}_p \) be the set of all functions \( f \in L_p \) which possess a Mellin transform \( F(\tau) = Mf \) in the well defined sense that \( f(x) \) is representable in the form

\[
(9.1) \quad f(x) = \lim_{a \to 0+} \frac{1}{2\pi} \int_{-a}^a F(\tau) x^{-1/p - i\tau} d\tau = M^{-1} F,
\]

where \( F(\tau) \in L_p(-\infty, \infty) \). In consequence of the well known theory of Fourier transforms, \( M^{-1} F \) is a bounded linear transformation from \( L_p(-\infty, \infty) \) into \( L_p(0, \infty) \) and \( \mathcal{M}_p \) a subspace of \( L_p \) and smaller than \( L_p \). When \( p = 2 \) obviously \( \mathcal{M}_2 = L_2(\mathbb{R}) \).

**Lemma 3.** Let \( 2 < p \leq \infty \) and \( \Re(\xi) > -1/p' \) and \( \Re(\eta) > -1/p \), and let \( f \in \mathcal{M}_p \). Then \( f^+ \) and \( J^+ f \) belong to \( \mathcal{M}_p \) also, and

\[
(9.21) \quad Mf^+ = \omega(\tau, \alpha) Mf; \quad Mf^+ \in \mathcal{M}_p;
\]

\[
(9.22) \quad Mf^+ = \chi(\tau, \alpha) Mf; \quad Mf^+ \in \mathcal{M}_p,
\]

where

\[
(9.3) \quad \omega(\tau, \alpha) = \frac{\Gamma(\tau + 1/p' - i\alpha)}{\Gamma(\tau + \alpha + 1/p' - i\alpha)}; \quad \chi(\tau, \alpha) = \frac{\Gamma(\eta + 1/p + i\alpha)}{\Gamma(\eta + \alpha + 1/p + i\alpha)}.
\]

We shall only outline the proof. Let \( F_n(\tau) = F(\tau) \) in \( (-n, n) \), \( F_n(\tau) = 0 \) for \| \tau \| > n \), and let \( f(x, n) = M^{-1} F_n \). Then, for \( 0 < y < \infty \),

\[
I_{f^+}(x, n) = \frac{y^{-1/n}}{2\pi \Gamma(\alpha)} \int_0^y (y-x)^{a-1} \xi dx \int_{-n}^n F(\tau) x^{-1/p - i\tau} d\tau
\]

\[
= \frac{1}{2\pi} \int_{-n}^n F(\tau) y^{-1/p - i\tau} d\tau \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\xi)^{a-1} \xi^{1/p - i\tau} d\xi
\]

\[
= M^{-1} \{ F_n(\tau) \omega(\tau, \alpha) \},
\]

and so (9.21) follows by \( \int |F - F_n| \tau d\tau \to 0 \), \( |f(x) - f(x, n)| \tau \to 0 \) \( [n \to \infty] \), and by (8.4).

\(^7\) Annals of Mathematics, (2), vol. 31 (1930), pp. 219–228.

\(^8\) When \( 2 < p < \infty \), then (9.1) implies

\[
F(\tau) = \lim_{a \to \infty} \frac{1}{2\pi} \int_{1/a}^a f(x) x^{-1/p + i\tau} dx
\]


For Lemma 3 see also our paper cited above.
By (9.21) and (9.22) \( I_{\alpha}^{+}f \) and \( J_{-\alpha}^{-}f \) are defined also when we replace \( \alpha \) by \( \beta \); for \( \omega(\tau, \beta) \) and \( \chi(\tau, \beta) \) are bounded for \(-\infty < \tau < \infty\), therefore \( \omega(\tau, \beta)Mf = \omega(\tau, \beta)F(\tau) \) and \( \chi(\tau, \beta)Mf = \chi(\tau, \beta)F(\tau) \) belong to \( L_{p'}(-\infty, \infty) \). Also to every \( g \subset M_{p} \) corresponds a uniquely determined function \( f \subset M_{p} \) such that \( I_{\alpha}^{+}g = \phi \) and \( J_{-\alpha}^{-}\phi = g \) respectively (cf. (6.1) and (7.2)). By the same reasoning as in §5 and §7, we have the theorems:

**Theorem IIb.** Let \( 2 \leq p \leq \infty \), let \( \Re(\chi) > -1/p' \), \( \Re(\eta) > -1/p \), and let \( f \subset M_{p} \). Then

\[
| I_{\alpha}^{+}f - I_{\alpha}^{+}\phi |_{p} \to 0; \quad | J_{-\alpha}^{-}f - J_{-\alpha}^{-}\phi |_{p} \to 0
\]

as \( \alpha \) tends to \( \lambda \), where \( \Re(\lambda) \geq 0 \).

**Lemma 4.** The operator \( X_{\alpha}f \), defined by (7.1), exists in \( L_{p}(0, \infty) \) with both domain and range \( M_{p} \) when \( 2 \leq p \leq \infty \). So does \( Y_{\beta}f \) when \( 2 \leq p < \infty \).

**Theorem IVb.** Under the restrictions of Lemma 4, the transformations \( X_{\beta} \) or \( Y_{\beta} \) form a group in \( L_{p} \), and \( | X_{\beta} - X_{\beta_{0}} |_{p} \to 0 \), \( | Y_{\beta} - Y_{\beta_{0}} |_{p} \to 0 \) as \( \beta \to \beta_{0} \).

10. **Characteristic values.** From (7.1) and (9.2) we easily have

\[
MX_{\alpha}f = \frac{\Gamma(1/p' - \beta - ir)}{\Gamma(1/p' - ir)} F(\tau - i\beta) \quad [2 \leq p \leq \infty],
\]

\[
MY_{\alpha}f = \frac{\Gamma(1/p + \beta + ir)}{\Gamma(1/p + ir)} F(\tau - i\beta) \quad [2 \leq p < \infty],
\]

where \( f = M^{-1}F \subset M_{p} \). We shall now deal with the equations

\[
X_{\alpha}f = lf,
\]

\[
Y_{\alpha}f = lf,
\]

\[
X_{\alpha}f = l\bar{f}.
\]

Let \( \beta = i\rho \), where \( \rho \) is real. Obviously (10.21) is equivalent to

\[
h(\tau + \rho) = lk(\tau); \quad F(\tau) = h(\tau)/\Gamma(1/p' - ir) \subset L_{p'}(-\infty, \infty),
\]

and (10.22) or (10.23) to

\[
k(\tau + \rho) = lk(\tau); \quad F(\tau) = k(\tau)/\Gamma(1/p + ir) \subset L_{p'}(-\infty, \infty)
\]

or

\[
h(\tau + \rho) = l\bar{k}(-\tau); \quad F(\tau) = h(\tau)/\Gamma(1/p' - ir) \subset L_{p'}(-\infty, \infty).
\]

By means of the well known asymptotic expansion of \( |\Gamma(\sigma + ir)| \) for \( t \to \pm \infty \) we arrive at the following results:

The transformation \( X_{\alpha}f [\beta \neq 0] \) with domain \( M_{p} \) \( [2 \leq p \leq \infty] \) has no characteristic values at all.
The characteristic values of $Y_{\beta f}$ [$\beta \neq 0$, domain $\mathbb{M}_p$, $2 \leq p < \infty$] are the set of points $l$ for which $\exp (-\frac{1}{2} \pi |\beta|) < |l| < \exp (\frac{1}{2} \pi |\beta|)$. To every characteristic value $l$ corresponds an infinity of characteristic functions.

We construct all these functions by (10.32), taking $k(\tau)$ in $(0, |\rho|)$ as an arbitrary function belonging to $L_p'(0, |\rho|)$.

Also, from the group property of $Y_\beta$, we can deduce the result:

Let $\Re(\kappa) = \Re(\lambda) = 0$, let $\kappa/\lambda$ be no rational number, and let $Y_{\kappa f} = f$ and $Y_{\lambda f} = f$ and $f \subset \mathbb{M}_p$. Then $f(x) = ce^{-x}$, and $Y_{\gamma f} = f$ for any $\gamma$ such that $\Re(\gamma) \geq 0$.

Furthermore, by (10.33), we can prove the result:

The equation (10.23) has a solution $f \subset \mathbb{M}_p$ if, and only if, $|l| = 1$. To every number $l$ of this kind corresponds an infinity of solutions $f \subset \mathbb{M}_p$.

For instance, $f(x) = e^{-1/2} e^{-\beta x}$ is a solution of (10.23) for $l = 1$.

11. It is not difficult to show that $I_{\beta f}$ and $J_{\gamma f}$ certainly exist and are integrable for any $f \subset L_p$ when $\Re(\xi) > -1/2$ and $\Re(\eta) > -1/p$ and $2 < p < \infty$.

We have

**Theorem VII.** Let $2 < p < \infty$ and $f(x) \subset L_p$; let $\Re(\xi) > -1/2$, $\Re(\eta) > -1/p$. Then there exist functions $I_{\beta f}$ and $J_{\gamma f}$, defined in $(0, \infty)$ and such that, for any positive finite number $A$, $I_{\beta f}$ and $J_{\gamma f}$ belong to $L_2(0, A)$ and that

$$
\int_0^A |I_{\gamma f} - I_{\beta f}|^2 dy \to 0; \int_0^A |J_{\gamma f} - J_{\beta f}|^2 dy \to 0 \quad [\alpha \to \beta].
$$

**Proof.** Let $f_n(x) = f(x)$ or $f_n(x) = 0$ for $0 \leq x \leq n$ or $x > n$ respectively [n = 1, 2, \ldots]. Then $f_n(x) \subset L_2(0, \infty)$, and by Theorem II, the function $g_\alpha(y, n) = I_{\alpha f_n}$ belongs to $L_2(0, \infty)$ and converges strongly to a function $g_\beta(y, n)$ as $\alpha \to \beta$,

$$
|g_\alpha(y, n) - g_\beta(y, n)| _2 \to 0.
$$

We now define $I_{\beta f}$ in $(0, \infty)$ by putting

$$
I_{\beta f} = g_\beta(y, n) \quad \text{for} \quad n - 1 < y \leq n \quad [n = 1, 2, \ldots].
$$

This function then has the desired properties. For $0 < y \leq n$ obviously

$$
g_\alpha(y, n) = I_{\alpha f},
$$

and so

$$
\int_0^n |I_{\alpha f} - g_\beta(y, n)|^2 dy \leq \int_0^\infty |g_\alpha(y, n) - g_\beta(y, n)|^2 dy \to 0
$$

as $\alpha$ tends to $\beta$. Therefore, for $1 \leq m < n$,
\[
\left( \int_0^m |g_\alpha(y, n) - g_\alpha(y, m)|^2 \, dy \right)^{1/2} \leq \left( \int_0^m |I_{1, \alpha}f - g_\alpha(y, m)|^2 \, dy \right)^{1/2} \\
+ \left( \int_0^m |g_\alpha(y, n) - I_{1, \alpha}f|^2 \, dy \right)^{1/2} \to 0 \quad [\alpha \to \beta].
\]

Hence \( g_\alpha(y, m) = g_\alpha(y, n) \) in \((0, m)\), and

\[I_{1, \alpha}f = g_\alpha(y, n)\]
in \((0, n)\) for \(n = 1, 2, \cdots \).

Therefore \( I_{1, \alpha}f \in L_2(0, n) \) for any \(n\), and, by (11.3) and (11.4),

\[
\int_0^n \left| I_{1, \alpha}f - I_{1, \alpha}g_\alpha(y, n) \right|^2 \, dy = \int_0^n \left| I_{1, \alpha}f - g_\alpha(y, n) \right|^2 \, dy \to 0
\]
as \(\alpha \to \beta\); which proves the first part of the theorem.

Let us take (cf. 4.4) \(0 < y \leq n\),

\[J_{n, \alpha}f = \frac{\gamma^n}{\Gamma(\alpha)} \left\{ \int_0^n + \int_n^\infty \right\} (x - y)^{\alpha-1}x^{-\alpha}f(x) \, dx
\]

\[= \phi_\alpha(y, n) + \psi_\alpha(y, n).
\]

Plainly \(\phi_\alpha(y, n) = J_{n, \alpha}f_n\) in \((0, n)\). Now \(f_n \in L_2(0, \infty)\) and \(\Re(\eta) > -1/p > -1/2\), and so, by Theorem II, \(J_{n, \alpha}f_n\) exists, belongs to \(L_2\), and

\[
\int_0^n \left| \phi_\alpha(y, n) - J_{n, \alpha}f_n \right|^2 \, dy \leq \int_0^\infty \left| J_{n, \alpha}f_n - J_{n, \alpha}f_n \right|^2 \, dy \to 0 \quad [\alpha \to \beta].
\]

Let \(b\) be any positive number smaller than \(n\). Then there exists a function \(\psi_\beta(y, n)\), depending on \(n\) but not on \(b\), such that

\[\int_0^b \left| \psi_\alpha(y, n) - \psi_\beta(y, n) \right|^2 \, dy \to 0 \quad [\alpha \to \beta],
\]
as we shall now show. For \(0 < y < n\) and \(n \leq x < \infty\), the function

\[\chi(x, y, \alpha) = \left\{ \Gamma(\alpha) \right\}^{-1}y^{-\alpha}(x - y)^{\alpha-1}x^{-\alpha}f(x)
\]
tends to \(\chi(x, y, \beta)\) as \(\alpha \to \beta\), and

\[|\chi(x, y, \alpha)| < Kx^{-\Re(\eta)-1}f(x) \subset L_1(n, \infty)
\]
when \(y\) is fixed and \(|\alpha| < k\). Hence, by Lebesgue’s convergence theorem, \(\psi_\beta(y, n)\) exists, and \(\psi_\alpha(y, n) - \psi_\beta(y, n) \to 0\) when \(0 < y < n\). For \(0 < y \leq b < n\), we have \(x - y \geq (1 - b/n)x = cx\),

\[|\psi_\alpha(y, n)| \leq \left| \Gamma(\alpha) \right|^{-1}y^{\Re(\eta)} \int_0^\infty e^{\Re(\alpha)-1x^{-\Re(\eta)-1}}f(x) \, dx \leq Ky^{\Re(\eta)},
\]
where $K$ does not depend on $\alpha$ when $|\alpha - \beta| < 1$. Hence also

$$|\psi_\alpha(y, n) - \psi_\beta(y, n)| \leq K_\mathbb{R}(n).$$

Since $\mathcal{Y}_\mathbb{R}(n) \subset L_2(0, b)$, applying Lebesgue’s theorem again, we obtain (11.7). In consequence of (11.5)–(11.7), for $h_\beta(y, n) = \phi_\beta(y, n) + \psi_\beta(y, n)$ we have

$$\int_0^b |F_{\alpha, \beta} - h_\beta(y, n)|^2 dy \to 0 \quad [\alpha \to \beta].$$

When we define $F_{\alpha, \beta}f$ in $(0, \infty)$ by

$$F_{\alpha, \beta}f = h_\beta(y, n) \quad \text{for } n - 1 \leq y < n \quad [n = 1, 2, \cdots],$$

then, by the argument applied in (11.4), it easily follows that $F_{\alpha, \beta}f = h_\beta(y, n)$ for $0 < y < n$, which completes the proof.

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