THE PARADOX OF KLEENE AND ROSSER[1]

BY
HASKELL B. CURRY

PART I. FORMULATION AND PRELIMINARY DISCUSSION

1. **Introduction.** In 1935 Kleene and Rosser published a proof that certain systems of formal logic are inconsistent, in the sense that every formula which can be expressed in their notation is also demonstrable[2]. There are, it so happens, only two systems in the literature to which this inconsistency proof applies; viz., the Church system of 1932–1933[3], and what I called an $\mathfrak{L}$-system in 1934[4]. But in spite of this limited application, the argument of Kleene and Rosser represents a theorem of great importance for the guidance of future research. It is a theorem of the same general character as the famous incompleteness theorems of Löwenheim, Skolem, and Gödel, which have played so prominent a role in recent research in mathematical foundations.

The proof of Kleene and Rosser is long and intricate[5], and contains complications which tend to obscure the essential meaning of their theorem. Accordingly there is some interest in the problem of making this paradox more accessible, and of presenting it in such a way as to make this essential meaning stand out more clearly. This is what the present paper attempts to do. The paradox is here presented and derived by a method which has many advantages, from the point of view indicated, over that of the original discoverers.

Before we enter into a detailed discussion, it will be expedient to examine the paradox in a vague preliminary way, and to explain in intuitive terms the central idea in its derivation.

One of the goals toward which mathematicians strive in setting up formal

Presented to the Society, March 27, 1937; received by the editors July 21, 1940. Minor additions were made December 7, 1940.

(1) This paper was first worked through in 1937. Since then it has been considerably revised. Part of this revision was done while I was in residence at the Institute for Advanced Study, Princeton, N. J. A general account of this subject was also presented to the Harvard Mathematical Colloquium on May 4, 1939. In the final preparation of the paper the author is greatly indebted to the referee, who checked the paper very carefully and conscientiously and made helpful suggestions.

(2) C 545.1. (For explanation of references, see end of the introduction.)

(3) C 359.4 and 6.

(4) C 396.6, p. 854.

(5) Although the paper C 545.1 is relatively short; yet the proof depends on results in a whole series of previous papers (viz., Church's C 359.4 and 6, Kleene's C 497.1 and 2, and Rosser's 546.1), totalling 162 pages. Of course some of the space in these papers is taken up with matters not considered here (cf. later parts of this introduction and the footnote to 3.15) and also with matters not bearing on the paradox.

systems is completeness—by which I mean not completeness in the technical sense, but simply the adequacy of the system for some purpose or other. There are two kinds of such completeness which especially concern us in this paper; both of them are desirable properties of formal systems of mathematical logic. I shall call these *combinatorial completeness* and *deductive completeness* respectively. They may be roughly explained as follows. A theory is combinatorially complete if and only if every expression $\mathfrak{M}$ formed from the terms of the system and an auxiliary indeterminate or variable x, can be represented within the system as a function of x (i.e., we can form in the system a function whose value for any argument is the same as the result of substituting that argument for x in $\mathfrak{M}$). A theory is deductively complete if whenever we can derive a proposition $\mathbf{B}$ on the hypothesis that another proposition $\mathbf{A}$ holds then we can derive without hypothesis a third proposition (such as $\mathbf{A} \supset \mathbf{B}$)(6) expressing this deducibility. Combinatorial completeness is thus a property relating to the possible constructions of terms (or formulas) within the system; deductive completeness relates to the possible derivations. Deductive completeness is a well known property of certain systems(7); whereas combinatorial completeness has only been achieved in recent years(8).

The essence of the Kleene-Rosser theorem is that it shows that these two kinds of completeness are incompatible—i.e., that any system which possesses both of them is inconsistent. The argument is essentially a refinement of the Richard paradox; it shows, in fact, that the Richard paradox can be set up formally within the system(9).

(6) Strictly speaking deductive completeness requires that when A and B are properties expressible by functions in the system such that B is formally derivable from A, then a formula expressing that fact, such as $(x).\ \mathsf{A}(x) \supset \mathsf{B}(x)$ is also derivable without hypothesis. However, the discussion in the text is sufficient for introductory purposes. For the precise formulation see 3.65 below.

(7) The name "deductive completeness" was suggested by the "Deduktionstheorem" of Hilbert-Bernays (C 507.1, p. 155), which establishes the deductive completeness of the ordinary logical calculus. The deductive completeness of the Church system is shown (essentially) by his Theorem I (C 359.4, p. 358); that of an $\mathfrak{L}$-system, by Theorem 14 of C 396.6.

(8) The possibility of achieving combinatorial completeness is due to Schönfinkel (C 304.1), who introduced certain operators, called "combinators" in C 396.2. An $\mathfrak{L}$-system, by definition, includes the theory of combinators, and so is combinatorially complete (in the strong sense—cf. below). The combinatorial completeness, according to the above rough definition, of the Church system was postulated in its formulation; that it can be so formulated as to be combinatorially complete (in the weak sense) according to the more precise definition of 3.3 (below) was shown by Rosser in C 546.1.

(9) Added in proof, August 21, 1941: Since this was written, I have discovered a much simpler way of deriving the contradiction for a system which is combinatorially complete in the strong sense (see 3.3 below). This new derivation which is based on the Russell paradox is contained in a paper, *The inconsistency of certain formal logics*, now in preparation. The new proof uses only the formulation of the present paper, viz. through 3.4.

In view of this circumstance the significance of most of the developments of this paper is quite different from what it seemed to be when this introduction was written. The uses which these developments have for other purposes is now where the chief interest lies. For these other

In order to see this in a preliminary way let us set up the Richard paradox(10), as follows. In any formal system of arithmetic the number of definable numerical functions of natural numbers is enumerable; let them, therefore be enumerated, say in a sequence

$$\phi_1(x),\ \phi_2(x),\ \phi_3(x),\ \cdots . \tag{1}$$

Consider now the function

$$f(x) = \phi_x(x) + 1. \tag{2}$$

If this is a definable function, then it must be in the sequence (1), say it is ϕ_n; then putting ϕ_n for f and n for x in (2), we have

$$\phi_n(n) = \phi_n(n) + 1,$$

which is a contradiction.

Leaving aside the explanation of this paradox from an intuitive point of view, let us consider what happens in the case of a system which is both combinatorially and deductively complete. In such a system if a function is a numerical function—i.e., if it gives numerical values for all numerical arguments(11)—then a formal statement of that fact can be demonstrated within the system, since it is deductively complete; by means of a recursive enumeration of all the theorems the set of all numerical functions can then be effectively enumerated in a sequence (1). Since the theory is combinatorially complete, we can then define within the system the function f; this is demonstrably a numerical function, and the demonstration of this fact will tell us effectively the value of n such that the above contradiction will certainly arise.

This shows in a rough way the nature of the paradox. Before we proceed to the formal developments I shall interpolate a few remarks concerning the present proof and its relation to that of Kleene and Rosser.

purposes naturally certain modifications of the methods here given will eventually be necessary; what these modifications will be is for the future to determine. However, the following may be noted as significant now. First, wherever in this paper we have a theorem of the form

$$\vdash \mathfrak{N}X_1 \& \cdots \& \mathfrak{N}X_n \to \mathfrak{G}X_1 \cdots X_n$$

where $\mathfrak{G}$ does not contain F or $\mathfrak{N}$, then on the basis of combinatory completeness alone it can be proved that

$$\vdash \mathfrak{G}\mathfrak{Z}_{p_1} \cdots \mathfrak{Z}_{p_n},$$

where $p_1, \cdots, p_n$ are any integers. Moreover where the second of these theorems is established here it follows from combinatorial completeness alone. This is true, in particular for 9.6; in fact under the hypothesis of 9.6 $T\mathfrak{Z}_a$ reduces to $\mathfrak{A}$. Second, we can go a certain, as yet undetermined, distance with a system, similar to that described in 5.6 for which consistency proof has now been obtained.

(10) See also Church's discussion of this paradox in C 359.7.

(11) More precisely, if from the assumption that the argument is numerical it follows formally that the functional value is numerical.

In the investigations of Church and his students the combinatorial completeness postulated is weakened in that it is required that the $\mathfrak{M}$ actually contain x—so that there need not be an apparatus for representing a constant as a function; there is also a weakening on the score of deductive completeness. These complications do not avoid the paradox, as Kleene and Rosser have shown; but they do increase considerably the length and intricacy of the derivation. If the object is to lay bare the central nerve of the paradox, the logical approach is to carry through the proof for the simpler case, that of combinatorial (and deductive) completeness in the strong sense, and then to show what modifications are necessary to carry through the proof for the more complicated case. Originally I had intended to do just this. But since the problems which led to this investigation, and the applications which I intend to make of it, are concerned with systems which are combinatorially complete in the strong sense, I have decided to leave this part of the investigation open. The present proof is, therefore, carried through in detail for the simpler case only. I believe that essentially the same method can be carried through with suitable modifications in the Church case, and that the result will have certain advantages over the original proof of Kleene and Rosser; but I leave the verification of this hunch to the readers who are more at home with such systems than I am.

Again, the present paper is essentially self-contained. I assume no acquaintance on the part of the reader with any previous papers either by Church or any of his students or myself. Reference is made to these papers for motivation or historical purposes only. The only information required of the reader which is not contained in the paper itself concerns the theory of recursive functions in ordinary arithmetic; in this case reference is made to papers by R. Péter, and such familiarity with the technique as is probably common knowledge among mathematicians is assumed. Except for these considerations regarding recursive functions (especially §6)[12] all proofs are given in full.

The following system is used for the making of references. The paper is arranged according to the decimal system. Reference to different parts of the paper are made by these decimal numbers. References to other papers are made as follows: The letter "J" refers to the Journal of Symbolic Logic; it is followed by volume and page numbers where the paper referred to (or a review of it) may be found. The letter "C" refers to Church's *A bibliography of symbolic logic* in J 1, 121–218 and J 3, 178–212; the individual items are referred to according to the system explained in J 3, 193.

2. **Preliminary explanations.** In this section some general conventions regarding notation, and some considerations regarding formal systems in general, will be stated. Certain technical terms which are italicized are to be regarded as defined by the content in which they so appear.

([12]) On the proofs of these refer to the remarks at the beginning of 6.8.

2.1. *Formal systems.* The notion of formal system is the central concept in any formalistic view of mathematics. Such a system is characterized by the fact that the process of proof is specified by explicitly stated rules. The initial conventions specifying such a formal system I shall call collectively its *primitive frame.* This will consist of three kinds of conventions, as in 2.11–2.13.

2.11. First we shall have conventions stating what the objects of the theory, which I shall call its *terms*, shall be. These conventions will consist of a list of *primitive terms*, a list of *operations* for the formation of further terms and a set of *rules of formation* describing how new terms are to be formed from the primitive ones by use of the operations.

2.12. Next we have conventions specifying a set of propositions, which I shall call *elementary propositions*, concerning the terms. Ordinarily these conventions will consist simply of a list of *predicates*, representing categories of terms, relations between terms, etc.; and an elementary proposition is then formed by applying a predicate to the appropriate number of terms as arguments.

2.13. Finally we have specifications determining which of the elementary propositions are true. A certain set of these elementary propositions, called *axioms*, are stated to be true outright; and specific *rules of procedure* are given which are to determine how the derivation of new elementary theorems(13) is to proceed. This process amounts to a recursive definition of the elementary theorems.

2.2. *Relations to symbolism.* This is not the place to go into an extended discussion of the nature of such a system; but it is necessary to make a few remarks, and, in particular, to compare it with the related notion of syntax of a language.

Any statement of the primitive frame must, of course, make use of symbols. However, these symbols are to be thought of as designating the term (or other notion)(14) and not as a specimen of it. In other words, these symbols are not the objects of discourse, but refer to them; they are not constituents of some "language" whose "syntax" we are studying, but are technical terms to be adjoined to ordinary language as designating some objects or other having the properties set down for them in the primitive frame. The primitive frame does not specify what these objects are; nor is it necessary that it should do so. The essence of formal reasoning is that we use these symbols as nouns, verbs, etc., in our ordinary language without saying exactly what their meanings are.

Of course, the reader who wishes may interpret the theory as talking about symbols. In that case the symbols used in the primitive frame and our dis-

(13) 'Theorem' is used here and elsewhere as synonymous with 'true proposition.'

(14) The discussion here has primary reference to the symbols which are nouns. The modifications necessary for the other kinds of symbols are left to the reader.

cussions refer to these symbols being talked about. To avoid confusion it is desirable to use symbols of the second class distinct from those of the first. So interpreted the theory is essentially the syntax of an "object language." But this interpretation is not the only one possible, and the reader who insists on it must invent his own "object language." The symbols we are using become, in this interpretation, syntactical([15]).

In order to have a systematic notation for names of symbols, Carnap (following Frege) uses a symbol enclosed in quotation marks as a name for that symbol (or expression). In this paper I shall, on occasion, use single quotes for this purpose, reserving double quotes for their ordinary uses (which sometimes conflict with the above technical use). But in some cases where there is no confusion, I shall use an expression to designate itself; in these cases the distinction is made clear by the context, as in the sentences 'Paris is a city' and 'Paris is dissyllabic'([16]).

2.3. *Propositions.* 'Proposition' is here used in an intuitive sense, i.e., as designating the meaning of a sentence.

The elementary propositions are specified by 2.12; and 2.13 gives a recursive determination of all of them which are true. This, incidentally, defines the fundamental predicates uniquely; so that once we know what the terms are, no further indeterminateness in regard to the predicates exists.

Further propositions may be formed from the elementary propositions by using the connectives and processes of ordinary discourse. These will be called *metapropositions*, and the study of the theory by means of them metatheoretic([17]). Examples of them are given below.

([15]) For a further discussion of the nature of a formal system, see my address *Some aspects of the problem of mathematical rigor*, Bulletin of the American Mathematical Society, vol. 47 (1941), pp. 221–241.

([16]) This example is Carnap's (J 4,82—p. 153). Most of the cases of this "autonymous" mode of speech in the present paper will be found to be concerned with symbols (or expressions) which are not nouns—e.g. sentences or verbs (such as '$\vdash$'). When such symbols are used as nouns I omit the quotes systematically when the reference is to the meaning of the symbol, rather than its shape (or sound). Whether this is an instance of the "autonymous" mode of speech is a matter which I leave for the reader; cf. the preceding footnote.

([17]) This is not the usual sense of 'metatheoretic.' In the ordinary usage, in fact, 'metatheoretic' and 'theoretic' are applied not to propositions but to expressions—i.e., rows of symbols; the distinction between them corresponds to the distinction between syntax language and object language. Cf. 2.2 and the paper cited in Footnote 15.

The connectives of ordinary discourse are, of course, to be understood as including processes which, as they occur in ordinary discourse, are only vaguely formulated. The precise definition of certain metatheoretic processes may require passing to a higher level of formalization. Examples of such processes are the deducibility connectives of 2.4 (cf. the next footnote) and substitution. (On the formulation of the latter, however, compare my paper cited in Footnote 15.) Consequently there is a certain amount of arbitrariness in the distinction between elementary and metatheoretic propositions. Whether we regard a proposition as metatheoretic for a system $\mathfrak{S}$, or as elementary for a more extended system $\mathfrak{S}'$, is a matter of the point of view. Although the latter may be more exact, yet often we prefer the former for the same reason that

2.4. *Metatheoretic connectives.* The following are used as symbols having a significance independent of the primitive frame of the system([18]).

2.41. '$\equiv$' (for definitional identity).

2.42. '$\rightarrow$' (for deducibility; the proposition on the right follows from that on the left by the rules of procedure).

2.43. '&' (for intuitive conjunction).

2.44. '$\rightleftarrows$' (for equivalence).

2.45. Dots will be used for avoiding parentheses in the customary manner (cf. J 2, 26–28). The order of seniority of the connectives for this purpose is '$\equiv$,' '$\rightleftarrows$,' '$\rightarrow$,' '&,' '$=$' (defined in §3.2).

2.5. *Term extensions.* One method of forming metapropositions concerning a system $\mathfrak{S}$ is to consider the adjunction to it of indeterminates, precisely as in algebra we adjoin indeterminates to a field to form its transcendental extension. If 'X' is not defined with reference to a system $\mathfrak{S}$, then we can adjoin to the primitive frame for $\mathfrak{S}$ the specification 'X is a term.' If nothing else is specified concerning X we call such an extension a *term extension* of $\mathfrak{S}$ with respect to the indeterminate X, and denote it by $\mathfrak{S}(X)$. Similar notations will be used for extensions with respect to several indeterminates.

The following are some special conventions regarding indeterminates.

2.51. 'Term' will refer to any term of $\mathfrak{S}$ or its term extensions; a *constant*, however, is a term belonging to the underlying system.

2.52. By an *X-term* we shall mean a term of $\mathfrak{S}(X)$; similarly an $(X_1, \cdots, X_m)$-term is a term of $\mathfrak{S}(X_1, \cdots, X_m)$.

---

we do not rush to Paris when we wish to measure a meter. Thus, the elementary propositions of the system $\mathfrak{r}$ of §6 formalize what are ordinarily satisfactorily thought of as metatheorems of a more elementary system of arithmetic. It is even possible to conceive of a system in which there are no elementary propositions, all propositions being metatheoretic; certain constructionist theories amount, essentially, to just this.

([18]) The exact definition of these metatheoretic connectives is discussed in my paper *Some properties of formal deducibility* (not yet published). For the purposes of the present paper it will, in general, be sufficient to take them as intuitively given. However, in regard to the connective $\rightarrow$ it seems to be necessary to make the following remarks.

As between elementary propositions we may distinguish two sorts of implication of a metatheoretic nature. Let A and B be two such propositions. Then it may happen that if A is adjoined to the primitive frame as additional axiom, then B is a theorem of the resulting system. This is the relation symbolized by $\mathsf{A}\rightarrow\mathsf{B}$; and it is generally what is referred to in connection with such phrases as '(metatheoretic) implication,' 'implies,' 'follows from,' etc. Moreover, it holds between A and B whenever B follows directly from A by a rule of procedure; hence, it is legitimate to state the rules of procedure in terms of $\rightarrow$. On the other hand $\mathsf{A}\rightarrow\mathsf{B}$ does not exhaust the possibilities of implication relations between A and B; we may, for example, have a process for converting a proof of A into a proof of B. The most general implication relation is expressed here as "if A, then B;" this is to be regarded as including the possibility $\mathsf{A}\rightarrow\mathsf{B}$ as a special case.

The difference between these two may be illustrated by an example. Consider for instance the classical algebra of propositions; and let p be a variable and $\mathfrak{A}$ an arbitrary formula therein. Then if the algebra is formulated with a substitution rule, we have $\vdash p \rightarrow \vdash \mathfrak{A}$, otherwise not. On the other hand the proposition

2.53. If a proposition holds when $X_1, \cdots, X_m$, which may appear in it, are indeterminate, that fact may be emphasized by attaching a subscript 'X' to the principal connective or predicate in it.

2.6. *Substitution.* If $\mathfrak{X}$ is an $(X_1, \cdots, X_m)$-term, and if $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ are terms, then the notation

$$\begin{bmatrix} \mathfrak{A}_1, \cdots, \mathfrak{A}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}$$

denotes the term $\mathfrak{Y}$ got by replacing X_i wherever it occurs in $\mathfrak{X}$ by $\mathfrak{A}_i$—i.e., the term which is constructed from $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ by the process that leads from $X_1, \cdots, X_m$ to $\mathfrak{X}$. We say that $\mathfrak{Y}$ is obtained from $\mathfrak{X}$ by substitution of $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ for $X_1, \cdots, X_m$. We evidently have the following property.

2.61. Any relation between the $(X_1, \cdots, X_m)$ terms $\mathfrak{X}_1, \mathfrak{X}_2, \cdots, \mathfrak{X}_n$ which is valid in $\mathfrak{S}(X_1, \cdots, X_m)$ will also hold, in the appropriate term extension, between the terms obtained by substituting any terms $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ for $X_1, \cdots, X_m$ throughout.

2.7. *Use of letters*(19).

$$\text{if} \quad \vdash p \quad \text{then} \quad \vdash \mathfrak{A}$$

is vacuous because the hypothesis is false; but we may note that it is a special case of a general substitution theorem which is demonstrable even when the algebra is formulated in terms of axiom schemes without a substitution rule.

This example shows that implications of the if-then type are ambiguous when the hypothesis is not verified. Usually we are only interested in such theorems when it is. In this paper theorems concerning $\mathfrak{S}$ stated in the if-then form are to be understood as making no particular assertion unless the hypotheses are verified; and theorems are occasionally stated in that form when that is all that interests us, even though the relation may actually be one of deducibility. Exceptions to this rule, made mostly for typographical reasons, are indicated by an asterisk prefixed to the number; such theorems are deducibility theorems. For the system $\mathfrak{r}$ all theorems and rules are stated in the if-then terminology.

In the paper above cited I have considered deducibility relations between propositions which are themselves compounded out of simpler ones by the above connectives and some others; and have advanced reasons for assuming these connectives subject to the rules of the Heyting calculus. If this is done, then I have shown that, for systems like $\mathfrak{S}$, the relation of deducibility holds between two elementary propositions if and only if the conclusion is deducible from the premise by the axioms and rules of procedure of the system. However, these general results are not needed here; we only need $\rightarrow$ between propositions which are conjunctions of elementary ones.

It should be noted that theorems in terms of deducibility which are true for $\mathfrak{S}$ are true for any extension of it.

The notations adopted above for distinguishing between deducibility and other forms of implication are not always rigidly adhered to. This is because when the paper was first written the distinction was left to be determined from the context. The changes made since will assist the reader at the more important points; but it is difficult to be sure that every instance of ambiguity has been caught in the revision.

(19) These conventions do not apply to the introduction, nor to certain intuitive remarks not strictly necessary for logical completeness.

2.71. Capital letters are used for notations connected with the formal system $\mathfrak{S}$ and its various term extensions, as follows:

2.711. 'U,' 'V,' 'X,' 'Y,' 'Z' are for indeterminates.

2.712. Other Latin, and also Greek or script, capitals are for specific constants. Their meaning is fixed throughout the paper.

2.713. '$\mathfrak{S}$' designates the system.

2.714. '$\mathfrak{F}$' and '$\mathfrak{T}$' designate specific functions from numerical expressions to terms. Their meaning is also fixed throughout the paper.

2.715. Other German letters[20] are for unspecified terms, or for terms which are specified only for a particular context. They are the variables of the intuitive discussion.

2.72. Lower case letters and Arabic numerals are for natural numbers and notions related to them, as follows:

2.721. Arabic numerals are for natural numbers in the ordinary sense.

2.722. 'u,' 'v,' 'x,' 'y,' 'z' are for numerical variables.

2.723. Other Latin letters are for unspecified numbers either taken intuitively or as constituents of the system of §6.

2.724. Greek letters[21] are for specific numerical functions. They have a fixed meaning throughout the paper.

2.725. $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ are for unspecified numerical expressions (cf. §6).

2.726. $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ are for unspecified functions.

2.727. $\mathfrak{m}$, $\mathfrak{p}$, $\mathfrak{q}$ are for unspecified expressions whether functional or numerical.

2.728. $\mathfrak{e}$, $\mathfrak{f}$, $\mathfrak{n}$, $\mathfrak{o}$, $\mathfrak{r}$, $\mathfrak{v}$, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ are for miscellaneous special purposes. The first six of these have a fixed meaning; the others are for unspecified classes of numerical variables.

2.729. Gothic letters are for operations of the system $\mathfrak{r}$ in §6.

3. **Formulation of the system** $\mathfrak{S}$. The assumptions concerning the system $\mathfrak{S}$ are those stated in 3.1–3.6. It will be noted that, aside from the very general assumptions in 3.1, the essential restrictions in the system are those of 3.3 and 3.6, which represent combinatorial and deductive completeness respectively. The other assumptions are essentially either definitions of particular terms of $\mathfrak{S}$ (and so are primarily notational) or consequences of the other assumptions.

3.1. *General restrictions on the primitive frame.*

3.11. There are a finite number of primitive terms, which we shall designate $E_0, E_1, \cdots, E_t$. (That this is not an essential restriction is shown in 3.123.)

3.12. The only operation is a single binary one of universal applicability.

---

(20) '$\mathfrak{L}$' is used in connection with 'system' to refer to the system so defined in C 396.6, p. 854 and some of its modifications.

(21) Except 'ϵ' which is used in the sense of Gödel (C 418.3, p. 180) in 6 and also (in the form '$\in$') in its ordinary set-theoretic meaning, and 'λ' which is given a specific meaning in 3.3.

We shall call this *application* and denote the term formed by application of $\mathfrak{A}$ to $\mathfrak{B}$ simply by $\mathfrak{A}\mathfrak{B}$. Then we have as sole rule of formation:

If $\mathfrak{A}$ and $\mathfrak{B}$ are terms, $\mathfrak{A}\mathfrak{B}$ is a term.

3.121. Parentheses will be used in the customary manner to indicate the construction of complex terms. To avoid superfluous parentheses, however, it shall be understood that in a row of terms without parentheses the operation shall be performed from left to right; so that $\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$, for instance, means the same as $((\mathfrak{A}\mathfrak{B})\mathfrak{C})\mathfrak{D}$.

3.122. The intuitive meaning of this operation is that of application of a function to an argument; thus if $\mathfrak{A}$ is a function $\mathfrak{g}$ and $\mathfrak{B}$ is a value b of its argument, $\mathfrak{A}\mathfrak{B}$ is what we ordinarily write as $\mathfrak{g}(b)$; if $\mathfrak{A}$ is a function of two variables, such as the addition function $x+y$, and $\mathfrak{B}$ is a value b, then $\mathfrak{A}\mathfrak{B}$ is the function of one variable got by giving the first argument in $\mathfrak{A}$ the value b—i.e., the function $b+x$—; this can be applied again to an argument $\mathfrak{C}$ representing a value c, and the result, $\mathfrak{A}\mathfrak{B}\mathfrak{C}$, is $b+c$, etc. This notion, due to Schönfinkel(22), eliminates the necessity of separate consideration of functions of different numbers of variables. However it is an important assumption concerning $\mathfrak{S}$ that $\mathfrak{A}\mathfrak{B}$ is a term whenever $\mathfrak{A}$ and $\mathfrak{B}$ are, so that there must be terms which are intuitively meaningless.

3.123. Next it is fitting to point out, by a heuristic argument, that this involves no essential restriction on $\mathfrak{S}$. For suppose $\mathfrak{S}$ contains an enumerable (possibly infinite) set of operations $\mathfrak{O}_1, \mathfrak{O}_2, \cdots$ where $\mathfrak{O}_i$ has n_i arguments. With each $\mathfrak{O}_i$ let us associate a term $\mathfrak{O}_i^*$ (possibly a new primitive term); then if we replace $\mathfrak{O}_i(\mathfrak{A}_1, \cdots, \mathfrak{A}_{n_i})$ everywhere by $\mathfrak{O}_i^*\mathfrak{A}_1 \cdots \mathfrak{A}_{n_i}$ we have a new system $\mathfrak{S}^*$ in which the only operation is application. Of course the notion of term in $\mathfrak{S}^*$ is more general than in $\mathfrak{S}$; but a limitation to terms corresponding to terms in $\mathfrak{S}$ can be incorporated in the rules of procedure, and this limitation can presumably be expressed by elementary propositions. Suppose, now, that $\mathfrak{S}$ has the primitive terms $E_1, E_2, E_3, \cdots$. Let $\mathfrak{C}$ be a new primitive term and define

$$E_1 \equiv \mathfrak{C}\mathfrak{C},\ E_2 \equiv \mathfrak{C}E_1,\ E_3 \equiv \mathfrak{C}E_2, \cdots;$$

then the resulting system has essentially the same content as the old, and has only one binary operation and one primitive term(23).

3.13. There is a single unary predicate denoted by the Frege sign '$\vdash$.' Elementary propositions are thus of the form $\vdash \mathfrak{A}$, where $\mathfrak{A}$ is a term.

3.131. As to the intuitive meaning of this predicate, note that the sentence '$\vdash \mathfrak{A}$' expresses the fact that $\mathfrak{A}$ belongs to a certain category of terms, which category is defined recursively by the axioms and rules of procedure. We shall

(22) C 304.1.

(23) The possibility of reducing to one primitive term in the above manner is mentioned in Schönfinkel's paper cited above. Somewhat the same idea appears in various papers by Chwistek, who uses a single primitive term '*c*' and a single binary operation denoted by a prefixed '*.'

call the members of this category *assertions* and, at times, read '$\vdash \mathfrak{A}$' as '$\mathfrak{A}$ is an assertion' or '$\mathfrak{A}$ is asserted.' Thus '$\vdash$' plays the same role as Hilbert's 'ist beweisbar' or Church's 'is provable'(24).

3.132. As in 3.123 we can show that this involves no essential restriction on $\mathfrak{S}$. For additional predicates can be expressed by means of $\vdash$ and terms (cf. the definition of equality in 3.2).

3.133. As to the meaning of '$\vdash_X$,' see 2.53.

3.14. There is a finite number of axioms. The terms asserted in these I shall denote by $A_0, A_1, \cdots, A_s$, so that the axioms are

$$\vdash A_i \qquad (i = 0, 1, \cdots, s).$$

Although in 2.13 'axiom' was defined as designating a proposition, it will be convenient to apply this name also to the A_i themselves; this is in accordance with current usage among logicians, who almost universally use 'axiom' to designate "formulas," i.e., terms.

3.15. There are a finite number of rules of procedure of the form

$$\vdash \mathfrak{X}_1 \,\&\, \cdots \,\&\, \vdash \mathfrak{X}_n \rightarrow_X \vdash \mathfrak{Y},$$

where $\mathfrak{X}_1, \cdots, \mathfrak{X}_n, \mathfrak{Y}$ are $(X_1, \cdots, X_m)$-terms such that every X_i which occurs in $\mathfrak{Y}$ also occurs in some $\mathfrak{X}_j$. (The restriction to rules of this character is not quite trivial, inasmuch as it implies that all conditions for significance are expressed by elementary propositions(25).)

3.16. *Remark.* It is sufficient to assume that $\mathfrak{S}$ contains a subsystem satisfying 3.14 and 3.15(26), i.e., that $\mathfrak{S}$ contains a finite number of elementary theorems of kind 3.14 and metatheorems of kind 3.15. For then, the proof of 15.4 will go through in the subsystem, and from this the inconsistency of the whole system follows as in 15.5.

(24) This usage of '$\vdash$' differs somewhat from that of Frege and the *Principia Mathematica*. There the symbol is placed before expressions which are already thought of as sentences, or at least as noun clauses, to indicate that the proposition concerned is true. Here '$\vdash$' is an intransitive verb; when placed before a noun the two together form a sentence; the proposition so denoted may or may not be taken as true according to the context.

(25) That the rules should be of the form given is a suggestion of Rosser (see Church C 359.7, footnote on p. 358); in his paper C 546.1 he showed how the Church system could be formulated as a system with rules of that character. The Rosser technique is also applicable to an $\mathfrak{K}$-system; this I plan to show in detail in a forthcoming paper. [This has since appeared; see J 6, pp. 41–53 (1941).]

(26) And, for that matter, 3.11. Naturally the other assumptions 3.2–3.6 must be satisfied by the subsystem. As to which of these will be satisfied automatically when they are satisfied for the system, that is a question which is not here gone into.

Kleene has pointed out to me that similar finiteness restrictions do not enter into the proof given by him and Rosser; because they first proved the inconsistency of a special system and then, in Theorem C, showed that more general systems include this as a subsystem. This part of the Kleene-Rosser argument can be superimposed on that given here to show that the finiteness restrictions of 3.14 and 3.15 are unnecessary. This argument can, presumably, be consider-

3.2. *Equality*. There exists a constant Q such that the relation of equality defined by

$$\mathfrak{X} = \mathfrak{Y} \equiv \vdash Q\mathfrak{X}\mathfrak{Y}$$

has the properties 3.21–3.27[27]. In consequence of these, 3.28 and 3.29 make convenient abbreviations.

3.21. $\mathfrak{A} = \mathfrak{A}$.

3.22. $\mathfrak{A} = \mathfrak{B} \rightarrow \mathfrak{B} = \mathfrak{A}$.

3.23. $\mathfrak{A} = \mathfrak{B} \;\&\; \mathfrak{B} = \mathfrak{C} \rightarrow \mathfrak{A} = \mathfrak{C}$.

3.24. $\mathfrak{A} = \mathfrak{B} \rightarrow \mathfrak{C}\mathfrak{A} = \mathfrak{C}\mathfrak{B}$.

3.25. $\mathfrak{A} = \mathfrak{B} \rightarrow \mathfrak{A}\mathfrak{C} = \mathfrak{B}\mathfrak{C}$.

3.26. $\mathfrak{A} = \mathfrak{B} \;\&\vdash \mathfrak{A} \rightarrow \vdash \mathfrak{B}$.

3.27. (General consequence of the foregoing.) If certain terms $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ occurring in the construction of a term $\mathfrak{X}$ are replaced by terms $\mathfrak{B}_1, \cdots, \mathfrak{B}_m$, thus forming a term $\mathfrak{Y}$, then

$$\mathfrak{A}_1 = \mathfrak{B}_1 \;\&\; \cdots \;\&\; \mathfrak{A}_m = \mathfrak{B}_m \rightarrow \mathfrak{X} = \mathfrak{Y}.$$

3.28. Definition. $\mathfrak{X} = \mathfrak{Y} = \mathfrak{Z}. \equiv . \mathfrak{X} = \mathfrak{Y} \;\&\; \mathfrak{Y} = \mathfrak{Z}$. (It then follows, by 3.23, that $\mathfrak{X} = \mathfrak{Z}$.)

3.29. Definition. $\vdash \mathfrak{X} = \vdash \mathfrak{Y}. \equiv . \vdash \mathfrak{X} \;\&\; \mathfrak{X} = \mathfrak{Y}$. (It then follows, by 3.26, that $\vdash \mathfrak{Y}$.)

3.3. *Combinatorial completeness*. Let $\mathfrak{S}'$ be an extension of $\mathfrak{S}$, and let $X_1, X_2, \cdots, X_n$ be indeterminates for $\mathfrak{S}'$. Let $\mathfrak{M}$ be any term of $\mathfrak{S}'(X_1, X_2, \cdots, X_n)$. Then there exists a term $\mathfrak{M}^*$ of $\mathfrak{S}'$ such that

$$\mathfrak{M}^* X_1 X_2 \cdots X_n = \mathfrak{M}.$$

This $\mathfrak{M}^*$ I shall denote, following Church, by the notation

$$\lambda^n X_1 X_2 \cdots X_n . \mathfrak{M}\text{(28)}.$$

The above I shall call combinatorial completeness in the strong sense. Church

---

ably simplified. All that it it necessary to do is show that any system $\mathfrak{S}$ satisfying all the hypotheses of this paper except the finiteness restrictions contains a subsystem $\mathfrak{S}'$ which satisfies all, and the particular $\mathfrak{S}'$ used by Kleene and Rosser is not the only one which will serve the purpose. It is, for instance, almost immediately evident that any such $\mathfrak{S}$ will contain a subsystem which is an $\mathfrak{L}$-system, and that an $\mathfrak{L}$-system satisfies all the hypotheses of this paper has been shown in my papers cited elsewhere. The point will not be discussed further here.

(27) In terms of the notions introduced in 3.3, 3.4, and 3.6 such a Q can be defined by

$$Q \equiv \lambda^2 XY.F(\lambda Z.ZX)(\lambda Z.ZY)I$$

(cf. Rosser C 546.1); but this is a detail which will be passed over.

(28) The period is to be interpreted as a bracket separating the complex operation '$\lambda^n X_1 \cdots X_n$' from its operand. The latter extends as far to the right as is consistent with indicated parentheses and other symbols of separation (cf. 2.45).

imposes the additional condition that $\mathfrak{M}$ must actually contain all of the $X_1, X_2, \cdots, X_n$; when this is imposed I shall speak of combinatorial completeness in the weak sense. In either case we have the following properties 3.31–3.35.

3.31. $(\lambda^n X_1 \cdots X_n . \mathfrak{M}) X_1 \cdots X_n = \mathfrak{M}$. (This follows immediately by the definition.)

3.32. If $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n$ are any terms, then

$$(\lambda^n X_1 \cdots X_n . \mathfrak{M}) \mathfrak{A}_1 \cdots \mathfrak{A}_n = \begin{bmatrix} \mathfrak{A}_1, \cdots, \mathfrak{A}_n \\ X_1, \cdots, X_n \end{bmatrix} \mathfrak{M}.$$

(This follows by 2.61.)

3.33. If $Y_1, Y_2, \cdots, Y_n$ are also indeterminates for $\mathfrak{S}'$, then

$$\lambda^n Y_1 \cdots Y_n . \begin{bmatrix} Y_1, \cdots, Y_n \\ X_1, \cdots, X_n \end{bmatrix} \mathfrak{M} = \lambda^n X_1 \cdots X_n . \mathfrak{M}.$$

(This is a separate assumption, but its plausibility is suggested by the meaning attached to indeterminates in 2.5.)

3.34. If $\mathfrak{M} =_X \mathfrak{N}$, then

$$\lambda^n X_1 \cdots X_n . \mathfrak{M} = \lambda^n X_1 \cdots X_n . \mathfrak{N}\text{(29)}.$$

3.35. $\lambda^{n+1} X_0 X_1 \cdots X_n . \mathfrak{M} = \lambda X_0 (\lambda^n X_1 X_2 \cdots X_n . \mathfrak{M})$. (This shows that it is sufficient to assume combinatorial completeness for $n=1$; but it is convenient to have the above properties to refer to for general n.)

3.4. *Some special constants.*

3.41. $I \equiv \lambda X . X$. Then I is the identity combinator. It has (by 3.32) the property that for any term $\mathfrak{A}$, $I\mathfrak{A} = \mathfrak{A}$.

3.42. $\mathsf{S} \equiv \lambda^3 XYZ . Y(XYZ)$. Then S is the successor function for natural numbers (cf. 3.44).

3.43. $K \equiv \lambda^2 XY . X$. This K is called the cancellation combinator. Its definition (unlike those of I and S) requires combinatorial completeness in the strong sense. It has the property that for any terms $\mathfrak{A}$ and $\mathfrak{B}$

$$K\mathfrak{A}\mathfrak{B} = \mathfrak{A}.$$

$K\mathfrak{A}$ is thus, intuitively, the function having $\mathfrak{A}$ as its constant value. In the case of combinatorial completeness in the weak sense such constancy functions cannot be defined over the range of all terms (i.e., for $\mathfrak{B}$ unrestricted), but constancy functions over restricted ranges can.

3.44. The natural numbers are represented in the system by a sequence of terms $\mathsf{Z}_0, \mathsf{Z}_1, \mathsf{Z}_2, \cdots$, such that

(29) This is assumed for the system $\mathfrak{S}$ and its term extensions, but not for systems obtained by adjoining axioms to such term extensions. With reference to such axiom extensions this property is invoked only when the premise is derivable without using the extraneous axioms.

$$\mathfrak{Z}_{n+1} \equiv \mathfrak{S}\, \mathfrak{Z}_n.$$

3.45. If the system is combinatorially complete in the strong sense, then we define

$$\mathfrak{Z}_0 \equiv KI,$$

and in that case we have, for any term $\mathfrak{G}$,

$$\mathfrak{Z}_0\mathfrak{G} = I, \qquad \mathfrak{Z}_1\mathfrak{G} = \lambda X.\mathfrak{G}X, \qquad \mathfrak{Z}_2\mathfrak{G} = \lambda X.\mathfrak{G}(\mathfrak{G}X),$$

and generally

$$\mathfrak{Z}_{n+1}\mathfrak{G} = \lambda X.\mathfrak{G}(\mathfrak{Z}_n\mathfrak{G}X);$$

so that, if $\mathfrak{G}$ is a function, $\mathfrak{Z}_n\mathfrak{G}$ is $\mathfrak{G}^n$, the nth iterate of $\mathfrak{G}$(30).

3.5. *The Gödel representation.* With each constant $\mathfrak{A}$ there is associated a positive integer $\mathfrak{n}(\mathfrak{A})$ such that:

3.51. A number is assigned to only one term, i.e., if $\mathfrak{n}(\mathfrak{A})=\mathfrak{n}(\mathfrak{B})$, then $\mathfrak{A}$ and $\mathfrak{B}$ are identical, and this means that they are not merely equal but are the same combination of the same primitive terms.

3.52. There exists a recursive function $\mu(x, y)$(31) such that:

3.521. $\mathfrak{n}(\mathfrak{A}\mathfrak{B})=\mu(\mathfrak{n}(\mathfrak{A}), \mathfrak{n}(\mathfrak{B}))$;

3.522. $x<\mu(x, y)$; $y<\mu(x, y)$;

3.523. If $\mu(x, y)=\mu(x', y')$, then $x=x'$ & $y=y'$(32);

3.524. $\mu(x, y)\neq e_k$ where e_k is the number of a primitive term.

3.6. *Deducibility assumptions.* There exist two terms $\mathcal{N}$ and F whose intuitive meanings are as follows: $\mathcal{N}$ represents the property of being an integer, i.e., the proposition $\vdash \mathcal{N}\mathfrak{A}$ is to mean the same as '$\mathfrak{A}$ is a natural number;' on the other hand F represents the notion of being a function, in the sense

(30) This way of introducing the natural numbers is due to Church (C 359.6, p. 863).

(31) A μ satisfying all conditions would be given by

$$\mu(x\cdot y) \equiv 2^x\cdot 3^y$$

provided the primitive terms are assigned numbers not of that form. Another such μ can be obtained from a recursive enumeration of number pairs (such as the $\sigma(x, y)$ of Hilbert-Bernays (C 507.1, p. 321) by skipping enough numbers to get in the e_k. Still another possibility is the following. Let application be denoted, for the time being, by a prefixed '*' (à la Chwistek), so that no parentheses are necessary. Assign to E_i the number $i+1$ and to * the number 0; then every term is assigned a unique number in the $(t+1)$-adic system; and conversely every such number with not too many 0's is assigned to a unique term (the number of prefixed 0's is uniquely determined). From this a function μ can be determined.

(32) We must understand this to mean that for all x, y, x', y' either $\mu(x, y)\neq\mu(x', y')$, or $x=x'$ and $y=y'$. In the notation of §6.9 this is expressed by

$$|\ \delta_2(\mu(x, y), \mu(x', y'))\ |\cdot(\delta_2(x, x') + \delta_2(y, y')) = 0.$$

A similar remark applies to 3.524. Note that a part of 3.523 and 3.524 is a consequence of 3.51; thus, for 3.524, if $a=\mathfrak{n}(\mathfrak{A})$ and $b=\mathfrak{n}(\mathfrak{B})$, $\mu(a, b)\neq e_k$ follows by 3.51; but if a and b are not assigned to terms, 3.524 is an additional restriction on μ. These additional restrictions facilitate the proofs in §9, but are not essential.

that '$\vdash F\mathfrak{A}\mathfrak{B}\mathfrak{G}$' is to mean the same as '$\mathfrak{G}$ is a function on $\mathfrak{A}$ to $\mathfrak{B}$'(33). These intuitive meanings are expressed by the following formal assumptions concerning them(34):

3.61. $\vdash \mathfrak{N} Z_0$, i.e., Z_0 is a natural number.

3.62. $\vdash F\mathfrak{N}\mathfrak{N}S$; i.e., S is a function on $\mathfrak{N}$ to $\mathfrak{N}$, i.e., a numerical function.

3.63. $\vdash F\mathfrak{A}\mathfrak{B}\mathfrak{G}$ & $\vdash \mathfrak{A}X \rightarrow_X \vdash \mathfrak{B}(\mathfrak{G}X)$, i.e., if $\mathfrak{G}$ is a function on $\mathfrak{A}$ to $\mathfrak{B}$ and X is on the range $\mathfrak{A}$, then $\mathfrak{G}X$ is in $\mathfrak{B}$. This is the characteristic property of F according to its intuitive meaning. This rule will be sometimes referred to as "*Principle F.*"

*3.64(35). (Mathematical induction.) If

$$\vdash \mathfrak{M} Z_0 \;\&\; \vdash F\mathfrak{M}\mathfrak{M}S,$$

then

$$\vdash \mathfrak{N}X \rightarrow_X \vdash \mathfrak{M}X.$$

3.65. (Property of deductive completeness.) If $\mathfrak{M}_1, \cdots, \mathfrak{M}_n, \mathfrak{A}, \mathfrak{B}, \mathfrak{G}$ are terms of an extension $\mathfrak{S}'$ of $\mathfrak{S}$ with respect to $Y_1, \cdots, Y_m$, and X is an indeterminate for $\mathfrak{S}'$, and if

$$\vdash \mathfrak{M}_1 \;\& \cdots \&\vdash \mathfrak{M}_n \;\&\; \vdash \mathfrak{A}X \rightarrow_{X,Y} \vdash \mathfrak{B}(\mathfrak{G}X);$$

then

$$\vdash \mathfrak{M}_1 \;\& \cdots \&\; \mathfrak{M}_n \rightarrow_Y \vdash F\mathfrak{A}\mathfrak{B}\mathfrak{G}(36).$$

---

(33) The postulation of F is equivalent to postulating a term representing formal implication. For if P^* represents formal implication, P^* and F are interdefinable, thus

$$F \equiv \lambda^3 XYZ.P^*X(\lambda U.Y(ZU)),$$
$$P^* \equiv \lambda^2 XY.FXYI.$$

(It is understood, of course, that in place of 3.63 the appropriate rule to go with P^* is

$$\vdash P^*\mathfrak{A}\mathfrak{B} \;\&\vdash \mathfrak{A}U \rightarrow_U \vdash \mathfrak{B}U.)$$

Many formal relations are simpler in terms of F.

(34) Of the "assumptions" here stated it is evident that the really important ones are 3.63 and 3.65, expressing the intuitive meaning of F and deductive completeness respectively. The other properties 3.61, 3.62, and 3.64 will automatically hold for any system which is adequate for arithmetic at all. Incidentally it is not necessary to postulate them as independent assumptions. For if, using the Λ defined in §5.4, we adopted the definition

$$\mathfrak{N} \equiv \lambda X.F(\lambda Y.\Lambda(YZ_0)(FYYS))(\lambda Y.YX)I,$$

they would become deducible as theorems. (The proof of this uses the full strength of 3.65, but is no more difficult than the theorems in §5.) When 3.64 is deduced in this way, it holds in the sense of deducibility (cf. the next footnote).

(35) On the significance of the * in 3.64 see the footnote to 2.42. That there is no loss of generality in assuming 3.64 in this way is evident from the preceding footnote, since the properties of conjunction (5.42) are deducible without the use of 3.64.

(36) The referee has called attention to the fact that, since we do not have conjunction in the system, we need this assumption for $n \geqq 2$ in order to derive 5.42 according to the method

4. **General discussion of the proof.** The derivation of the paradox depends on three principal theorems which may be roughly described as follows:

First, the system $\mathfrak{S}$ is strong enough to include all of recursive arithmetic, in the sense that every recursive arithmetic function has a representative in $\mathfrak{S}$, called its formalization, such that the properties of the recursive functions are reflected in corresponding theorems about their formalizations.

Second, we can construct a constant, here called T, which formalizes the Gödel representation; this means that if $a = \mathfrak{n}(\mathfrak{A})$ the proposition

$$\mathfrak{A} = TZ_a$$

is formally demonstrable as an elementary theorem in $\mathfrak{S}$.

Third, we can construct within the system a formal enumeration of the assertions [3.131]. Indeed by the aid of the above described T such an enumeration can be obtained from a formalization Θ of a primitive recursive function $\theta(x)$, which enumerates the Gödel numbers of those assertions. For this Θ we can further prove that

$$\vdash \; \mathcal{N}X \rightarrow_X T(\Theta X).$$

These three theorems are established in Parts II, III, and IV respectively. On the basis of them it is shown in Part V that the Richard paradox can be set up in the system on much the same lines as in the introduction.

One or two general remarks may be interpolated here before we proceed.

The reader should note the difference between proving that

$$\vdash \; \mathfrak{M}Z_n \qquad\qquad (n = 0, 1, 2, \cdots)$$

and that

$$\vdash \; \mathcal{N}X \rightarrow_X \vdash \; \mathfrak{M}X.$$

The first of these requires simply that we know that $\mathfrak{M}Z_n$ is an assertion for every non-negative integral value of n; the second requires the exhibition of a process for passing from the premise to the conclusion by the use of the rules of procedure for $\mathfrak{S}(X)$ only, X being treated as a constant. The second of these theorems is stronger than the first; if we have the second, the first follows by 5.27 and 2.61.

Again the reader should avoid being confused by the different senses of

---

given below. Of course, as soon as conjunction is introduced there is no increased generality in assuming a general n. If 5.42 were deduced by the method sketched in 5.6, then it would be sufficient to assume 3.65 for $n=1$.

With this form of 3.65 it is evident that it holds not only for $\mathfrak{S}$ but also for any system obtained from $\mathfrak{S}$ by adjoining additional axioms (even if they contain additional terms). This is also true for 3.61–3.64; for 3.61 and 3.62 are elementary propositions, and 3.63 and 3.64 are deducibility properties in the sense of the footnote to 2.42. The same invariance with respect to extension also holds for various theorems in the sequel; this is not always explicitly indicated.

'induction,' 'recursion' and the like. These are sometimes used in an intuitive sense, sometimes as referring to a formal process related to $\mathfrak{S}$ (e.g., a use of 3.64) or the system $\mathfrak{r}$ of §6.

Part II. Formalization of recursive arithmetic

We now turn to the proof of the first of the three major theorems mentioned in §4. Some preliminary lemmas concerning special constants are proved in §5. Then in §6 ordinary recursive arithmetic is formulated precisely as a formal system $\mathfrak{r}$. This leads up to the proof of the final theorems in §7.

5. **Some special constants and their properties.** In this section we study some special functions useful in setting up recursive arithmetic within the system. Besides some properties of the elementary constants already postulated, we study the ordered pair D_2 and the recursion combinator R. It is necessary to include some generalizations of the deducibility postulates of 3.6.

5.1. *The F sequence.* Although functions of several variables can be regarded as special cases of functions of one variable, yet it saves cumbersome formulas to introduce an abbreviation for the term, definable in terms of F, which plays the same role for functions of n variables that F does for functions of one variable. This term is defined by induction thus:

$$F_0 \equiv I, \qquad F_1 \equiv F,$$
$$F_{n+1} \equiv \lambda^{n+2}XY_1 \cdots Y_nZ.FX(F_nY_1 \cdots Y_nZ).$$

It will be seen that for instance $\vdash F_2\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{G}$ says that $\mathfrak{G}$ is on $\mathfrak{A}$ to $F\mathfrak{B}\mathfrak{C}$, and hence is a function on the composite range $\mathfrak{A}\mathfrak{B}$ to $\mathfrak{C}$.

5.11. Theorem. *For $m \geqq 1$, $n \geqq 0$,*

$$F_{m+n} \equiv \lambda^{m+n+1}X_1 \cdots X_mY_1 \cdots Y_nZ.F_mX_1 \cdots X_m(F_nY_1 \cdots Y_nZ).$$

Proof. For $m=1$, this is true by definition. Suppose the theorem is true for $m=k$. Then [3.31]

$$F_{n+k}X_1 \cdots X_kY_1 \cdots Y_nZ = F_kX_1 \cdots X_k(F_nY_1 \cdots Y_nZ). \tag{1}$$

Therefore [by definition of F_{k+1}]

$$\begin{aligned} FU(F_{n+k}X_1 \cdots X_kY_1 \cdots Y_nZ) &= FU(F_kX_1 \cdots X_k(F_nY_1 \cdots Y_nZ)) \\ &= F_{k+1}UX_1 \cdots X_k(F_nY_1 \cdots Y_nZ). \end{aligned}$$

The theorem for $m=k+1$ now follows by 3.34.

*5.12. Theorem. (Generalization of Principle F.) *If, for $m \geqq 1$, $n \geqq 0$,*

(a) $\vdash F_{m+n}\mathfrak{K}_1\mathfrak{K}_2 \cdots \mathfrak{K}_m\mathfrak{L}_1\mathfrak{L}_2 \cdots \mathfrak{L}_n\mathfrak{M}\mathfrak{G}$, *and*

(b) $\vdash \mathfrak{K}_i\mathfrak{A}_i$ $(i=1, 2, \cdots, m)$;

then

$$\vdash F_n\mathfrak{L}_1\mathfrak{L}_2 \cdots \mathfrak{L}_n\mathfrak{M}(\mathfrak{G}\mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_m).$$

Proof. Suppose that for a given $k<m$ we have established

(1) $$\vdash F_{m+n-k}\mathfrak{K}_{k+1} \cdots \mathfrak{K}_m\mathfrak{L}_1 \cdots \mathfrak{L}_n\mathfrak{M}(\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_k).$$

Then [5.11] $\vdash F\mathfrak{K}_{k+1}(F_{m+n-k-1}\mathfrak{K}_{k+2} \cdots \mathfrak{K}_m\mathfrak{L}_1 \cdots \mathfrak{L}_n\mathfrak{M})(\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_k)$. Then by Principle F and the $k+1$st hypothesis (b) we have (1) for $k+1$ in the place of k. On the other hand (1) is true for $k=0$ by (a). Hence by induction (1) is true for all $k \leqq m$, hence for $k=m$, which gives the theorem.

5.13. Theorem. (Generalization of 3.65.) *If*

$$\vdash \mathfrak{A}_1X_1 \ \& \cdots \& \vdash \mathfrak{A}_mX_m \rightarrow_X \vdash \mathfrak{B}(\mathfrak{G}X_1 \cdots X_m);$$

then

$$\vdash F_m\mathfrak{A}_1 \cdots \mathfrak{A}_m\mathfrak{B}\mathfrak{G}.$$

Proof. By induction from 3.65.

5.14. Theorem. *If* (a) $\vdash F_n\mathfrak{A}_1 \cdots \mathfrak{A}_n\mathfrak{B}\mathfrak{G}$, (b) $\vdash F\mathfrak{A}_i'\mathfrak{A}_iI$ $(i=1, \cdots, n)$, (c) $\vdash F\mathfrak{B}\mathfrak{B}'I$; *then*

$$\vdash F_n\mathfrak{A}_1' \cdots \mathfrak{A}_n'\mathfrak{B}'\mathfrak{G}.$$

Proof. The proof follows by 5.13, since from the hypothesis

$$\vdash \mathfrak{A}_i'X_i \qquad (i = 1, 2, \cdots, n),$$

we conclude $\vdash \mathfrak{B}'(\mathfrak{G}X_1 \cdots X_n)$ by the hypotheses, 5.12, and 3.41.

5.2. *Properties of the elementary constants I, K, S, Z_n.*

5.21. Theorem. $\vdash_U FUUI$.

Proof. By 3.41,

$$\vdash UX \rightarrow_{U,X} \vdash U(IX),$$

whence the theorem follows by 3.65.

5.22. Theorem. $\vdash_U F_2U_1U_2U_1K$.

Proof. By 3.43,

$$\vdash U_1X_1 \ \& \vdash U_2X_2 \rightarrow_{U,X} \vdash U_1(KX_1X_2).$$

The theorem follows by 5.13.

5.23. Theorem.

$$\vdash_U F(F(FU_1U_3)(FU_2U_1))(F(FU_1U_3)(FU_2U_3))\mathsf{S}.$$

Proof. Suppose

(1) $$\vdash U_2Z,$$

(2) $\vdash FU_1U_3Y,$

(3) $\vdash F_2(FU_1U_3)U_2U_1X.$

Then [5.12]

$$\vdash U_1(XYZ).$$

Therefore

$$\vdash U_3(Y(XYZ)) \qquad \text{by [5.12],}$$

(4) $= \vdash U_3(\mathsf{S}XYZ)$ [3.42; 3.2].

Thus (1), (2), (3) imply (4). Hence, by 5.13,

$$\vdash_U F_3(F_2(FU_1U_3)U_2U_1)(FU_1U_3)U_2U_3\mathsf{S}.$$

By 5.11 this gives the theorem.

5.231. Corollary. $\vdash_U F(F(FUU)(FUU))(F(FUU)(FUU))\mathsf{S}.$

5.24. Theorem. $\vdash_U F_2U_1U_2U_2\mathsf{Z}_0.$

Proof. Similar to that of 5.22; or from 5.22 (substituting FU_2U_2 for the U_1 and U_1 for the U_2 of 5.22) and 5.21 by Principle F.

5.25. Theorem. $\vdash_U F(FUU)(FUU)\mathsf{Z}_n$ $(n=0, 1, \cdots)$.

Proof. For $n=0$, this follows from 5.24 (take $U_1 \equiv FUU$, $U_2 \equiv U$). Assuming it for Z_n, it follows for Z_{n+1} by 5.231, 3.63 (in view of 3.44). Hence the theorem follows by intuitive induction.

5.26. Theorem. $\vdash_U F\mathcal{N}(F(FUU)(FUU))I.$

Proof. Similar to that of 5.25; only we replace the intuitive induction by a formal induction using 3.64.

5.27. Theorem. $\vdash \mathcal{N}\mathsf{Z}_n$ [3.61, 3.62, 3.63, 3.44].

5.3. *The ordered pair D_2.* The definition of the ordered pair here used, viz.,

$$D_2 \equiv \lambda^3 XYZ.Z(KY)X,$$

is due to Bernays(37).

5.31. Theorem. $D_2\mathfrak{A}\mathfrak{B}\mathsf{Z}_0 = \mathfrak{A}.$

Proof. By 3.32

$$\begin{aligned} D_2\mathfrak{A}\mathfrak{B}\mathsf{Z}_0 &= \mathsf{Z}_0(K\mathfrak{B})\mathfrak{A} \\ &= \mathfrak{A} \qquad [3.45, 3.41]. \end{aligned}$$

(37) Oral communication, May 2, 1936.

5.32. THEOREM. $D_2\mathfrak{A}\mathfrak{B}(\mathsf{S}X)=\mathfrak{B}$.

Proof. By 3.32

$$\begin{aligned} D_2\mathfrak{A}\mathfrak{B}(\mathsf{S}X) &= \mathsf{S}X(K\mathfrak{B})\mathfrak{A} \\ &= K\mathfrak{B}(X(K\mathfrak{B})\mathfrak{A}) && [3.42;\ 3.32], \\ &= \mathfrak{B} && [3.43]. \end{aligned}$$

5.33. THEOREM. $\vdash_U F_3U_1U_2(F(FU_3U_2)(FU_1U_4))U_4D_2$.

Proof. If we assume $\vdash U_1X$, $\vdash U_2Y$, and $\vdash F(FU_3U_2)(FU_1U_4)Z$, then [5.22; 5.12] $\vdash FU_3U_2(KY)$. Therefore [5.12]

$$\vdash U_4(D_2XYZ).$$

The theorem follows by 5.13.

5.34. THEOREM. $\vdash_U F_3UU\mathfrak{N}UD_2$.

Proof. In 5.33 put $U_1\equiv U_2\equiv U_3\equiv U_4\equiv U$. The theorem follows by 5.26, 5.21, and 5.14.

5.4. *Definition of conjunction.*

5.41. DEFINITION. $\Lambda\equiv\lambda^2XY.F(F(FII)(FII))I(D_2XY)$.

5.42. THEOREM. $\vdash\Lambda\mathfrak{A}\mathfrak{B}\rightleftarrows\vdash\mathfrak{A}\ \&\ \vdash\mathfrak{B}$.

Proof. Suppose $\vdash\Lambda\mathfrak{A}\mathfrak{B}$. Then, since by 5.25 $\vdash F(FII)(FII)Z_0$ and $\vdash F(FII)(FII)Z_1$, we have by Principle F

$$\vdash D_2\mathfrak{A}\mathfrak{B}Z_0\ \&\ \vdash D_2\mathfrak{A}\mathfrak{B}Z_1.$$

Therefore [5.31, 5.32]

(1) $$\vdash\mathfrak{A}\ \&\ \vdash\mathfrak{B}.$$

Conversely suppose (1) holds. Then by 3.41

(2) $$\vdash I\mathfrak{A}\ \&\ \vdash I\mathfrak{B}.$$

But, by 5.33,

(3) $$\vdash F_3II(F(FII)(FII))ID_2.$$

Therefore [(2); (3); 5.12]

$$\vdash F(F(FII)(FII))I(D_2\mathfrak{A}\mathfrak{B}) = \vdash\Lambda\mathfrak{A}\mathfrak{B},$$

q.e.d.

5.43. The more general conjunction Λ_n is definable recursively by

$$\Lambda_2\equiv\Lambda,$$
$$\Lambda_{n+1}\equiv\lambda^{n+1}XY_1\cdots Y_n.\Lambda X(\Lambda_nY_1\cdots Y_n).$$

It can then be shown, by intuitive induction, that

$$\vdash \Lambda_n \mathfrak{A}_1 \cdots \mathfrak{A}_n$$

is equivalent to

$$\vdash \mathfrak{A}_1 \,\&\, \cdots \,\&\, \vdash \mathfrak{A}_n.$$

5.44. THEOREM. (Generalization of mathematical induction.) *If*

(a) $\vdash \mathfrak{M} Z_0$,

(b) $\vdash \mathcal{N} X \,\&\, \vdash \mathfrak{M} X \rightarrow {}_X\vdash \mathfrak{M}(\mathsf{S} X)$,

then

(c) $\vdash \mathcal{N} X \rightarrow {}_X\vdash \mathfrak{M} X$.

Proof. Let $\mathfrak{M}' \equiv \lambda X . \Lambda(\mathcal{N} X)(\mathfrak{M} X)$. Then [(a), 3.61]

(1) $$\vdash \mathfrak{M}' Z_0.$$

Assume

(2) $$\vdash \mathfrak{M}' X.$$

Then [5.42] $\vdash \mathcal{N} X \,\&\, \vdash \mathfrak{M} X$. Therefore [(b)] $\vdash \mathfrak{M}(\mathsf{S} X)$. But [3.62, 3.63] $\vdash \mathcal{N}(\mathsf{S} X)$. Therefore [5.42] $\vdash \mathfrak{M}'(\mathsf{S} X)$. Since this follows from (2), we have by 3.65

$$\vdash F \mathfrak{M}' \mathfrak{M}' \mathsf{S}.$$

Therefore [3.64]

$$\vdash F \mathcal{N} \mathfrak{M}' I.$$

From this the theorem follows by 3.63, 3.41, and 5.42.

5.5. *The recursion combinator R.* We seek now to find a function $\mathfrak{G}$ which satisfies the following "definition" in terms of a given function $\mathfrak{H}$ and constant $\mathfrak{A}$

$$\mathfrak{G} Z_0 = \mathfrak{A},$$

$$\mathcal{N} X \rightarrow_X \mathfrak{G}(\mathsf{S} X) = \mathfrak{H} X(\mathfrak{G} X).$$

This $\mathfrak{G}$ is defined by

$$\mathfrak{G} \equiv R \mathfrak{H} \mathfrak{A},$$

where R is the combinator defined in 5.52. 5.51 is preliminary; 5.52 contains the definition; while the theorems are stated in 5.53 and 5.54. The method of treatment is due to Bernays([38]).

5.51. In order to find such a $\mathfrak{G}$ we consider first that if $\mathfrak{G}$ satisfied the condition

([38]) See note to 5.3.

$$\mathfrak{G}(\mathsf{S}\,X) = \mathfrak{H}(\mathfrak{G}X),$$

its definition would be very simple; viz.,

$$\mathfrak{G} \equiv \lambda X.X\mathfrak{H}\mathfrak{A}.$$

In the more difficult case suppose we had such a $\mathfrak{G}$. Then the function

$$\mathfrak{L} \equiv \lambda X.D_2X(\mathfrak{G}X)$$

satisfies a recursion of the simpler type. For, assuming the above condition for $\mathfrak{G}$, we obtain

$$\begin{aligned} \mathfrak{L}(\mathsf{S}\,X) &= D_2(\mathsf{S}\,X)(\mathfrak{G}(\mathsf{S}\,X)) \\ &= D_2(\mathsf{S}\,X)(\mathfrak{H}X(\mathfrak{G}X)) \\ &= D_2(\mathsf{S}\,(\mathfrak{L}X\mathsf{Z}_0))(\mathfrak{H}(\mathfrak{L}X\mathsf{Z}_0)(\mathfrak{L}X\mathsf{Z}_1)) \qquad [5.31, 5.32], \\ &= \mathfrak{K}(\mathfrak{L}X), \end{aligned}$$

where $\mathfrak{K} \equiv \lambda X.D_2(\mathsf{S}\,(X\mathsf{Z}_0))(\mathfrak{H}(X\mathsf{Z}_0)(X\mathsf{Z}_1))$. This enables us to define $\mathfrak{L}$ and hence $\mathfrak{G}$.

These $\mathfrak{G}$, $\mathfrak{L}$ and $\mathfrak{K}$ are functions of $\mathfrak{H}$ and $\mathfrak{A}$. If we express them as such we have the R, $\mathfrak{R}_3$ and $\mathfrak{R}_2$ of the following definitions. It is convenient to define $\mathfrak{R}_2$ in two stages, of which $\mathfrak{R}_1$ is the first stage.

5.52. Definitions.

$$\mathfrak{R}_1 \equiv \lambda^3 UXY.D_2(\mathsf{S}\,X)(UXY), \qquad \mathfrak{R}_2 \equiv \lambda^2 UX.\mathfrak{R}_1U(X\mathsf{Z}_0)(X\mathsf{Z}_1),$$

$$\mathfrak{R}_3 \equiv \lambda^3 UVX.X(\mathfrak{R}_2U)(D_2\mathsf{Z}_0V), \qquad R \equiv \lambda^3 UVX.\mathfrak{R}_3UVX\mathsf{Z}_1.$$

5.53. Theorem. *If we define* $\mathfrak{G} \equiv RUV$; *then*

(a) $\mathfrak{G}\mathsf{Z}_0 =_{UV} V$,

(b) $\vdash \mathcal{N}X \rightarrow_{UVX} \mathfrak{G}(\mathsf{S}\,X) = UX(\mathfrak{G}X)$.

Proof. Let us define

(1) $$\mathfrak{K}_1 \equiv \mathfrak{R}_1U,$$

(2) $$\mathfrak{K}_2 \equiv \mathfrak{R}_2U,$$

(3) $$\mathfrak{L} \equiv \mathfrak{R}_3UV,$$

(4) $$\mathfrak{B} \equiv D_2\mathsf{Z}_0V.$$

Then the above definitions give

(5) $$\mathfrak{K}_1 = \lambda^2 XY.D_2(\mathsf{S}\,X)(UXY),$$

(6) $$\mathfrak{K}_2 = \lambda X.\mathfrak{K}_1(X\mathsf{Z}_0)(X\mathsf{Z}_1),$$

(7) $$\mathfrak{L} = \lambda X.X\mathfrak{K}_2\mathfrak{B},$$

(8) $$\mathfrak{G} = \lambda X.\mathfrak{L}X\mathsf{Z}_1.$$

Then [(7)]

$$\mathfrak{L} Z_0 = Z_0 \mathfrak{K}_2 \mathfrak{B}$$
$$= \mathfrak{B} \qquad [3.45].$$

Therefore

(9) $$\mathfrak{L} Z_0 Z_0 = Z_0 \qquad [5.31],$$

(10) $$\mathfrak{G} Z_0 = \mathfrak{L} Z_0 Z_1 = V \qquad [5.32].$$

This proves the first part of the theorem. To prove the second part—

$$\begin{aligned} \mathfrak{L}(\mathsf{S} X) &= \mathsf{S} X \mathfrak{K}_2 \mathfrak{B} && [(7)], \\ &= \mathfrak{K}_2(X \mathfrak{K}_2 \mathfrak{B}) && [3.42], \\ &= \mathfrak{K}_2(\mathfrak{L} X) && [(7)], \\ &= \mathfrak{K}_1(\mathfrak{L} X Z_0)(\mathfrak{G} X) && [(6), (8)], \end{aligned}$$

(11) $$= D_2(\mathsf{S}(\mathfrak{L} X Z_0))(U(\mathfrak{L} X Z_0)(\mathfrak{G} X)) \qquad [(5)].$$

Therefore [5.31]

$$\mathfrak{L}(\mathsf{S} X) Z_0 = \mathsf{S}(\mathfrak{L} X Z_0).$$

This with (9) gives a proof by mathematical induction [5.44 with $\mathfrak{M} \equiv \lambda X . Q(\mathfrak{L} X Z_0) X$] that

$$\vdash \mathcal{N} X \rightarrow_X \mathfrak{L} X Z_0 = X.$$

If we put this in (11), then

$$\begin{aligned} \vdash \mathcal{N} X \rightarrow_X \mathfrak{G}(\mathsf{S} X) &= \mathfrak{L}(\mathsf{S} X) Z_1 \\ &= U X (\mathfrak{G} X) \qquad [5.32], \end{aligned}$$

q.e.d.

5.54. Theorem. $\vdash F_2(F_2 \mathcal{N} \mathfrak{Q} \mathfrak{Q}) \mathfrak{Q} (F \mathcal{N} \mathfrak{Q}) R$.

Proof. Let $\mathfrak{G}$ be as in 5.53, and suppose that

(1) $$\vdash F_2 \mathcal{N} \mathfrak{Q} \mathfrak{Q} U,$$

(2) $$\vdash \mathfrak{Q} V.$$

Then [5.53 (a)] $\vdash \mathfrak{Q}(\mathfrak{G} Z_0)$.

Assume

(3) $$\vdash \mathcal{N} X \;\&\; \vdash \mathfrak{Q}(\mathfrak{G} X).$$

Then [(1), 5.12] $\vdash \mathfrak{Q}(U X (\mathfrak{G} X))$; that is [5.53 (b)],

(4) $$\vdash \mathfrak{Q}(\mathfrak{G}(\mathsf{S} X)).$$

Therefore [(3), (4), 5.44, 3.65]

$$\vdash F\mathcal{N}\mathfrak{Q}\mathfrak{G}.$$

This means that, assuming (1) and (2), it follows that

$$\vdash F\mathcal{N}\mathfrak{Q}(RUV).$$

The theorem follows by 5.13.

5.6. *Discussion.* Throughout the foregoing repeated use has been made of 3.65. Some interest attaches to the question of whether these theorems can be derived without using 3.65 in its full power, inasmuch as systems with weakened forms of 3.65 may conceivably be consistent. This section will be closed with some informal discussion of results bearing on this question.

Suppose we assume the following postulates concerning F:

$$\vdash F_2(FYZ)(FXY)(FXZ)B,$$
$$\vdash F(F_2YXZ)(F_2XYZ)C,$$
$$\vdash F(F_2XXY)(FXY)W,$$
$$\vdash FX(FYX)K,$$

where B, C, W are the "primitive combinators" which may be defined, in our present notation, as follows:

$$B \equiv \lambda^3 XYZ.X(YZ),$$
$$C \equiv \lambda^3 XYZ.XZY,$$
$$W \equiv \lambda^2 XY.XYY.$$

These postulates may, of course, be expressed as axioms; they are easy consequences of 3.65. Moreover they are highly plausible assumptions concerning F (cf. the intuitive discussions in my paper C 396.7). With these postulates in the place of 3.65 all the above theorems except 5.13 can be derived, provided that mathematical induction in the form 5.44, and also 3.61, 3.62, and 5.26 be assumed(39). This, however, I have accomplished not for the above definition of R, but for R defined in a way more akin to that used by Kleene. The sequence of definitions for this purpose is the following:

(39) These derivations are contained in papers now in preparation which employed the methods of my paper J 1, 65. The essential point is the following lemma: If $\mathfrak{X}$ is an $(X_1, \cdots, X_n)$-term and $\mathfrak{A}_1, \cdots, \mathfrak{A}_n, \mathfrak{B}$ are constants such that from the assumptions

$$\vdash \mathfrak{A}_i X_i \qquad (i = 1, 2, \cdots, n),$$

it follows, by following through the construction of $\mathfrak{X}$ and applying Principle F only, that

$$\vdash \mathfrak{B}\mathfrak{X};$$

then

$$\vdash F_n \mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{B}(\lambda^n X_1 \cdots X_n.\mathfrak{X}).$$

This lemma can then replace 5.13. Presumably then the inductive assumptions can be derived by adjoining appropriate postulates for Λ, and defining $\mathcal{N}$ as in the second footnote to 3.6.

$$\mathfrak{D}_1 \equiv \lambda X.D_2X(\mathsf{S}X), \qquad \mathfrak{D}_2 \equiv \lambda X.\mathfrak{D}_1(XZ_1),$$

$$\mathfrak{D}_3 \equiv \lambda X.X\mathfrak{D}_2(D_2Z_0Z_0), \qquad \nabla \equiv \lambda X.\mathfrak{D}_3XZ_0,$$

$$\mathfrak{R}_1 \equiv \lambda^3 UYX.UX(Y(\nabla X)), \qquad \mathfrak{R}_2 \equiv \lambda^3 UVX.X(\mathfrak{R}_1U)(KV),$$

$$R \equiv \lambda^3 UVX.\mathfrak{R}_2UVX(\nabla X).$$

The ∇ so defined is the predecessor function; it has the properties

$$\nabla Z_0 = Z_0, \qquad \vdash \ \mathcal{N}X \rightarrow_X \nabla(\mathsf{S}X) = X,$$

while R has the same properties as in 5.53 and 5.54. The new definitions are easier to handle because the ordered pairs involved are homogeneous, whereas in 5.52 they are heterogeneous. On the present basis, however, the Bernays definition is easier.

Note, too, that the definition of conjunction in §5.4 is such as not to require any use of mathematical induction (even in the above weakened system). If the definition

$$\lambda^2 XY.F\mathcal{N}I(D_2XY)$$

were used, then we should require 5.34, and hence 5.26, to establish 5.42.

Many of the theorems in the later sections can also be derived in the so-modified system. How far we can go with it is, as yet, undetermined.

6. **Recursive arithmetic as a formal system.** We are now in possession of enough formal machinery to show that recursive arithmetic can be incorporated in the system $\mathfrak{S}$. In order to make this result explicit, it is desirable to formulate recursive arithmetic with all possible precision. In the literature recursive arithmetic is usually treated in a semi-intuitive manner. However, greater exactness and definiteness are attained by treating it as itself a formal system. The present section will be devoted to formulating recursive arithmetic in that manner. After some preliminary explanations in 6.1, the formal statement of the primitive frame of the system is given in 6.2–6.7; in 6.8 there are some general theorems about the system $\mathfrak{r}$, and in 6.9 there are stated, for future reference, the definitions of some special recursive functions[40].

6.1. *Preliminary explanations.* The system of recursive arithmetic will be called the system $\mathfrak{r}$. Its terms will be called *expressions* to distinguish them from the terms of the system $\mathfrak{S}$. These expressions include not only numbers and numerical variables, but also functions of every degree of multiplicity. Accordingly we consider an infinite series of categories of expressions $\mathfrak{e}_0, \mathfrak{e}_1, \mathfrak{e}_2, \cdots$. The expressions of $\mathfrak{e}_0$ are the numerical variables and expressions capable of being substituted for them—we shall call these *numerical expressions*—those of $\mathfrak{e}_n$ for $n>0$ are *functions* of n arguments. It is also

(40) See also my paper *A formalization of recursive arithmetic*, American Journal of Mathematics, vol. 63 (1941), pp. 263–282. (This paper was written after the presentation.)

desirable to make explicit the variables on which a numerical expression depends or may depend; thus if $\mathfrak{x}$ is any set of numerical variables $\mathfrak{e}_0(\mathfrak{x})$ is the set of all numerical expressions whose variables, if any, are in $\mathfrak{x}$. In this connection the null set of variables is denoted by $\mathfrak{o}$ and the set of all variables by $\mathfrak{v}$, so that $\mathfrak{e}_0$ is the same as $\mathfrak{e}_0(\mathfrak{v})$. On the use of letters for expressions see 2.72. '$\in$' and '$\subset$' are used with their ordinary set-theoretic meanings. In the statement of the primitive frame parts of the text enclosed in parentheses refer to the intuitive interpretation or other extraneous consideration; these are inserted to increase intelligibility, but are not a part of the formal specifications.

6.2. *Primitive expressions.* There is an infinite set of these, constituted as follows:

6.21. (A constant numeral) 0.

6.22. An infinite sequence $\mathfrak{v}$ of *numerical variables*: $v_1, v_2, v_3, \cdots$. The letters 'x,' 'y,' 'z,' etc., stand for unspecified members of this sequence.

6.23. (A function) σ (this is the successor function, converting any numerical expression $\mathfrak{a}$ into $\mathfrak{a}+1$).

6.24. (A function) κ_0 (this is the function having 0 as its constant value; cf. 6.62 below).

6.25. A set (of functions) κ_{nm} where $n=1, 2, 3, \cdots$ and $m=1, 2, 3, \cdots, n$. (κ_{nm} picks out the mth argument from a sequence of n; cf. 6.63 below.)

6.3. *Operations.* These also form an infinite set, but may be grouped into three kinds, as in 6.31–6.33. Operations are denoted by Gothic letters, which are prefixed to their arguments; since the multiplicity of each is definite, parentheses are unnecessary (as in the logical notation of the Poles).

6.31. For each $n>0$, there is an $(n+1)$-ary operation $\mathbf{a}_n$. (This represents the application of an n-ary function to n arguments. In accordance with custom) let us define

$$\mathfrak{g}(\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n) \equiv \mathbf{a}_n\mathfrak{g}\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n,$$

when $\mathfrak{g}\in\mathfrak{e}_n$ and $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n\in\mathfrak{e}_0$.

6.32. For each $n>0$, and $m>0$ there is an $(m+1)$-ary operation $\mathbf{s}_{mn}$ (representing the substitution of m n-ary functions in an m-ary one, cf. 6.64 below).

6.33. For each $n\geqq 0$ a binary $\mathbf{r}_n$ (representing the recursive definition of an $(n+1)$-ary function by an n-ary one and an $(n+2)$-ary one, cf. 6.65 below).

6.4. *Rules of formation.* We turn now to the rules for the formation of derived expressions, and for the classification of expressions into the categories mentioned in 6.1.

6.41. $0\in\mathfrak{e}_0(\mathfrak{x})$ for any $\mathfrak{x}$.

6.42. $\mathfrak{x}\subset\mathfrak{e}_0(\mathfrak{x})$.

6.43. $\sigma\in\mathfrak{e}_1$.

6.44. $\kappa_0\in\mathfrak{e}_1$.

6.45. $\kappa_{nm}\in\mathfrak{e}_n$.

6.46. If $\mathfrak{g} \in \mathfrak{e}_n$ and $\mathfrak{a}_i \in \mathfrak{e}_0(\mathfrak{x})$ for $i=1, 2, \cdots, n$, then $\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n) \in \mathfrak{e}_0(\mathfrak{x})$.

6.47. If $\mathfrak{h} \in \mathfrak{e}_m$ and $\mathfrak{k}_i \in \mathfrak{e}_n$ for $i=1, 2, \cdots, m$, then $\mathbf{s}_{mn}\mathfrak{h}\mathfrak{k}_1 \cdots \mathfrak{k}_m \in \mathfrak{e}_n$.

6.48. If $\mathfrak{h} \in \mathfrak{e}_n$ and $\mathfrak{k} \in \mathfrak{e}_{n+2}$, then $\mathbf{r}_n\mathfrak{h}\mathfrak{k} \in \mathfrak{e}_{n+1}$([41]).

6.5. *Elementary propositions.* The only primitive predicate of the system is the binary relation of equality between numerical expressions. The elementary propositions are those of the form

$$\mathfrak{a} = \mathfrak{b},$$

where $\mathfrak{a} \in \mathfrak{e}_0(\mathfrak{v})$, $\mathfrak{b} \in \mathfrak{e}_0(\mathfrak{v})$.

6.6. *Axioms.* In stating the axioms we have two alternatives: either we may state specific axioms with a substitution rule, or we may state axiom-schemes in the sense of von Neumann. The latter course is followed here. It is, then, to be understood that $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are any expression of $\mathfrak{e}_0$ and $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ are functions.

6.61. $\mathfrak{a} = \mathfrak{a}$([42]).

6.62. $\kappa_0(\mathfrak{a}) = 0$.

6.63. $\kappa_{nm}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n) = \mathfrak{a}_m$.

6.64. If $\mathfrak{h} \in \mathfrak{e}_m$ and $\mathfrak{k}_1, \cdots, \mathfrak{k}_m \in \mathfrak{e}_n$ and $\mathfrak{g} \equiv \mathbf{s}_{mn}\mathfrak{h}\mathfrak{k}_1 \cdots \mathfrak{k}_m$, then

$$\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n) = \mathfrak{h}(\mathfrak{k}_1(\mathfrak{a}_1, \cdots, \mathfrak{a}_n), \cdots, \mathfrak{k}_m(\mathfrak{a}_1, \cdots, \mathfrak{a}_n)).$$

6.65. If $\mathfrak{h} \in \mathfrak{e}_n$([43]), $\mathfrak{k} \in \mathfrak{e}_{n+2}$, $\mathfrak{g} \equiv \mathbf{r}_n\mathfrak{h}\mathfrak{k}$, then

$$\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, 0) = \mathfrak{h}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n),$$

$$\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \sigma(\mathfrak{b})) = \mathfrak{k}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \mathfrak{b}, \mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \mathfrak{b})).$$

6.7. *Rules of procedure.* As before $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are arbitrary expressions of $\mathfrak{e}_0$.

6.71. If $\mathfrak{a} = \mathfrak{b}$, then $\mathfrak{b} = \mathfrak{a}$.

6.72. If $\mathfrak{a} = \mathfrak{b}$, and $\mathfrak{b} = \mathfrak{c}$, then $\mathfrak{a} = \mathfrak{c}$.

6.73. If $\mathfrak{g} \in \mathfrak{e}_n$, and if

$$\mathfrak{a}_i = \mathfrak{b}_i \qquad (i = 1, 2, \cdots, n);$$

then $\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n) = \mathfrak{g}(\mathfrak{b}_1, \cdots, \mathfrak{b}_n)$.

6.74. (Mathematical induction([44]).) If $\mathfrak{g}$, $\mathfrak{h} \in \mathfrak{e}_{n+1}$, $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathfrak{e}_0(\mathfrak{x})$ are such that

(a) $\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, 0) = \mathfrak{h}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, 0)$,

(b) on the hypothesis that

$$\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, y) = \mathfrak{h}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, y), \tag{1}$$

([41]) If $n=0$, then $\mathfrak{e}_n$ is to be understood as meaning $\mathfrak{e}_0(\mathfrak{o})$.

([42]) This is, of course, redundant.

([43]) If $n=0$, $\mathfrak{h}$ must be in $\mathfrak{e}_0(\mathfrak{o})$.

([44]) It will be noticed that this rule has a metatheoretic premise, and therefore does not have the simple character prescribed for the rules of $\mathfrak{S}$ in 3.14.

where y is a variable not in $\mathfrak{x}$, it follows by 6.71–6.73 (and known theorems) that

$$\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \sigma(y)) = \mathfrak{h}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \sigma(y)); \tag{2}$$

then $\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \mathfrak{b}) = \mathfrak{h}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \mathfrak{b})$.

6.8. *Some general theorems.* The following theorems are obvious enough from the intuitive point of view; but they may be proved to follow formally from the above definitions. The proofs are omitted here; it is planned to publish them in a separate paper(45).

6.81. THEOREM. *If* $\mathfrak{a} \in \mathfrak{e}_0(x_1, \cdots, x_m)$ *and* $\mathfrak{a}'$ *is obtained from* $\mathfrak{a}$ *by substituting for* $x_1, \cdots, x_m$ *expressions* $\mathfrak{b}_1, \cdots, \mathfrak{b}_m \in \mathfrak{e}_0(y_1, \cdots, y_n)$, *then* $\mathfrak{a}' \in \mathfrak{e}_0(y_1, \cdots, y_n)$.

6.82. THEOREM. *If* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{e}_0(x_1, \cdots, x_n)$ *such that*

$$\mathfrak{a} = \mathfrak{b};$$

and if $\mathfrak{a}', \mathfrak{b}'$ *are obtained by substituting* $\mathfrak{c}_1, \cdots, \mathfrak{c}_n \in \mathfrak{e}_0$ *for* $x_1, \cdots, x_n$ *respectively in* $\mathfrak{a}$ *and* $\mathfrak{b}$; *then*

$$\mathfrak{a}' = \mathfrak{b}'.$$

6.83. THEOREM. *If* $\mathfrak{a} \in \mathfrak{e}_0(x_1, \cdots, x_n)$, *then there exists a* $\mathfrak{g} \in \mathfrak{e}_n$ *such that*

$$\mathfrak{a} = \mathfrak{g}(x_1, \cdots, x_n).$$

This $\mathfrak{g}$ *we denote by* $\mathfrak{f}_{x_1 \cdots x_n}(\mathfrak{a})$; *it is defined by intuitive recursion as follows*:

$$\begin{aligned} \mathfrak{f}_{x_1 \cdots x_n}(x_i) &\equiv \kappa_{ni}, \\ \mathfrak{f}_{x_1 \cdots x_n}(0) &\equiv \mathbf{s}_{1n}\kappa_0\kappa_{n1}, \\ \mathfrak{f}_{x_1 \cdots x_n}(\mathfrak{h}(\mathfrak{a}_1, \cdots, \mathfrak{a}_m)) &\equiv \mathbf{s}_{mn}\mathfrak{h}\mathfrak{g}_1 \cdots \mathfrak{g}_m, \end{aligned}$$

where $\mathfrak{g}_i \equiv \mathfrak{f}_{x_1 \cdots x_n}(\mathfrak{a}_i)$.

6.84. THEOREM. *If* $\mathfrak{a} \in \mathfrak{e}_0(x_1, \cdots, x_n)$, $\mathfrak{b} \in \mathfrak{e}_0(x_1, \cdots, x_n, y, z)$; *then there exists a* $\mathfrak{g} \in \mathfrak{e}_{n+1}$ *such that*

$$\mathfrak{g}(x_1, \cdots, x_n, 0) = \mathfrak{a}, \qquad \mathfrak{g}(x_1, \cdots, x_n, \sigma(y)) = \mathfrak{b}',$$

where $\mathfrak{b}'$ *is the result of substituting* $\mathfrak{g}(x_1, \cdots, x_n, y)$ *for* z *in* $\mathfrak{b}$.

6.85. THEOREM. (Generalization of mathematical induction.) *If* $\mathfrak{a}, \mathfrak{b}$ *are in* $\mathfrak{e}_0(x_1, \cdots, x_n, y)$ *such that*

(a) *we have*

$$\begin{bmatrix} 0 \\ y \end{bmatrix}\mathfrak{a} = \begin{bmatrix} 0 \\ y \end{bmatrix}\mathfrak{b},$$

(45) See the paper cited in the footnote to the beginning of §6.

(b) *from the assumption*

$$\mathfrak{a} = \mathfrak{b}$$

it follows by rules 6.71–6.73 *only that*

$$\begin{bmatrix} \sigma(y) \\ y \end{bmatrix} \mathfrak{a} = \begin{bmatrix} \sigma(y) \\ y \end{bmatrix} \mathfrak{b};$$

then

$$\mathfrak{a} = \mathfrak{b}.$$

6.86. The formulation above given is sufficient for recursive arithmetic as defined by Hilbert and Bernays[46], in the sense that, if

$$\mathfrak{a} = \mathfrak{b}$$

is deducible in the formalism of Hilbert-Bernays, then it is deducible also in the system $\mathfrak{r}$.

6.9. *Special functions*. The object of this section is to give explicitly, primarily for the purpose of fixing the notation, the definitions of some special recursive functions which we need in the sequel. For some of these functions we use an operational, rather than a Greek letter notation. The functions are defined within the system $\mathfrak{r}$ by virtue of 6.83 or 6.84. Some of the properties of these functions are also listed. On the proofs of the statements made see 6.8 (especially 6.86).

For explanation of references to this number in the later portions of the paper see 7.6.

6.91. The sum $x+y$ is defined by

$$x + 0 = x, \qquad x + \sigma(y) = \sigma(x + y),$$

and obeys the commutative and associative laws.

6.92. The product $x \cdot y$ is defined by

$$x \cdot 0 = 0, \qquad x \cdot \sigma(y) = x \cdot y + x.$$

This product is commutative and associative and distributive with respect to addition.

6.93. Functions $\delta_1(x)$ and $x \dot{-} y$, related to subtraction, are defined as follows:

$$\delta_1(0) = 0, \qquad \delta_1(\sigma(x)) = x;$$
$$x \dot{-} 0 = x, \qquad x \dot{-} \sigma(y) = \delta_1(x \dot{-} y).$$

(Intuitively $x \dot{-} y$ is $x - y$ if $x \geqq y$, otherwise it is 0.) Some of the properties of these functions are as follows:

---

[46] C 507.1, p. 307.

6.931. $\delta_1(\mathfrak{a}) = \mathfrak{a} \dot{-} 1$.
6.932. $\mathfrak{a} \dot{-} \mathfrak{a} = 0 \dot{-} \mathfrak{a} = 0$.
6.933. $(\mathfrak{a}+\mathfrak{c}) \dot{-} (\mathfrak{b}+\mathfrak{c}) = \mathfrak{a} \dot{-} \mathfrak{b}$.
6.934. $(\mathfrak{a}+\mathfrak{b}) \dot{-} \mathfrak{b} = \mathfrak{a}$.
6.935. $\mathfrak{a} \dot{-} (\mathfrak{b}+\mathfrak{c}) = (\mathfrak{a} \dot{-} \mathfrak{b}) \dot{-} \mathfrak{c} = (\mathfrak{a} \dot{-} \mathfrak{c}) \dot{-} \mathfrak{b}$.
6.936. $\mathfrak{a} \cdot (\mathfrak{b} \dot{-} \mathfrak{c}) = (\mathfrak{a} \cdot \mathfrak{b}) \dot{-} (\mathfrak{a} \cdot \mathfrak{c})$.
6.937. $(\mathfrak{a} \dot{-} \mathfrak{b}) \cdot (\mathfrak{b} \dot{-} \mathfrak{a}) = 0$.
6.938. $\mathfrak{a} + (\mathfrak{b} \dot{-} \mathfrak{a}) = \mathfrak{b} + (\mathfrak{a} \dot{-} \mathfrak{b})$.
6.94. In terms of subtraction we define the negation function $|x|$ thus:

$$|x| \equiv 1 \dot{-} x.$$

Then we have the following properties:
6.941. $|0| = 1$; $|\sigma(\mathfrak{a})| = 0$.
6.942. $||0|| = 0$; $||\sigma(\mathfrak{a})|| = 1$.
6.943. If $|\mathfrak{a}| \cdot \mathfrak{b} = 0$, and $\mathfrak{a} = 0$, then $\mathfrak{b} = 0$.
6.944. $\mathfrak{a} = ||\mathfrak{a}|| + \delta_1(\mathfrak{a})$.
6.95. The equality function δ_2 is defined by

$$\delta_2(x, y) = (x \dot{-} y) + (y \dot{-} x).$$

6.96. General sums and products are defined thus:
(a) $\sum_{y=0}^{x} \mathfrak{g}(y)$ is the function $\mathfrak{h}(x)$ such that

$$\mathfrak{h}(0) = \mathfrak{g}(0), \qquad \mathfrak{h}(\sigma(x)) = \mathfrak{h}(x) + \mathfrak{g}(\sigma(x)).$$

(b) $\prod_{y=0}^{x} \mathfrak{g}(y)$ is the function $\mathfrak{k}(x)$ such that

$$\mathfrak{k}(0) = \mathfrak{g}(0), \qquad \mathfrak{k}(\sigma(x)) = \mathfrak{k}(x) \cdot \mathfrak{g}(\sigma(x)).$$

The first of these functions is 0 if and only if $\mathfrak{g}(y) = 0$ for every $y \leqq x$; the second is 0 if and only if $\mathfrak{g}(y) = 0$ for some $y \leqq x$. It is also convenient to define

$$(\exists y \leqq x)\mathfrak{g}(y) = \left|\left| \prod_{y=0}^{x} \mathfrak{g}(y) \right|\right|.$$

6.97. Given $\mathfrak{g}(x)$, $(\epsilon y \leqq x)\mathfrak{g}(y)$ is defined as that function $\mathfrak{h}(x)$ such that

$$\mathfrak{h}(0) = 0,$$

$$\mathfrak{h}(\sigma(x)) = \mathfrak{h}(x) \cdot \left| \prod_{y=0}^{x} \mathfrak{g}(y) \right| + \sigma(x) \cdot |\, \mathfrak{g}(\sigma(x))| \cdot (\exists y \leqq x)\mathfrak{g}(y).$$

Intuitively $(\epsilon y \leqq x)\mathfrak{g}(y)$ is the least $z \leqq x$ such that $\mathfrak{g}(z) = 0$, if there is any such z; otherwise it is 0. Thus we have the following properties:
6.971. $(\epsilon y \leqq x)\mathfrak{g}(y) \dot{-} x = 0$.
6.972. $|(\exists y \leqq x)\mathfrak{g}(y)| \cdot \mathfrak{g}((\epsilon y \leqq x)\mathfrak{g}(y)) = 0$.
6.98. Given $\mathfrak{a}$, $\mathfrak{b}_0, \cdots, \mathfrak{b}_p$, $\mathfrak{c} \in \mathfrak{e}_0(x_1, \cdots, x_n)$, a function $\mathfrak{g}$ such that

$$\mathfrak{g}(x_1, \cdots, x_n) = \begin{cases} \mathfrak{b}_k & \text{if} \quad \mathfrak{a} = k \leqq p, \\ \mathfrak{c} & \text{if} \quad |\, \mathfrak{a} \dot{-} p \,| = 0, \end{cases}$$

is given by

$$\mathfrak{g}(x_1, \cdots, x_n) \equiv \sum_{k=0}^{p} |\, \delta_2(\mathfrak{a}, k) \,| \cdot \mathfrak{b}_k + \|\mathfrak{a} \dot{-} p\| \cdot \mathfrak{c}.$$

6.99. The function

$$\left[\frac{x}{y}\right],$$

representing as usual the integral part of x/y, may be defined thus:

$$\left[\frac{x}{y}\right] \equiv (\epsilon z \leqq x)(\sigma(x) \dot{-} y \cdot \sigma(z)).$$

7. **The representation of recursive functions.** In this section we study a method of associating with every expression $\mathfrak{m}$, numerical or functional, of the system $\mathfrak{r}$, a term called its formalization and denoted by '$\mathfrak{F}(\mathfrak{m})$.' This association is such that if $\mathfrak{m}$ is a function, or is an $\mathfrak{e}_0(\mathfrak{o})$, then $\mathfrak{F}(\mathfrak{m})$ is a constant in $\mathfrak{S}$; while if $\mathfrak{m}$ contains variables, $\mathfrak{F}(\mathfrak{m})$ contains corresponding indeterminates. Furthermore, with each elementary proposition of $\mathfrak{r}$ we associate a proposition concerning $\mathfrak{S}$ which we call the formalization of the former proposition; if this elementary proposition is $\mathfrak{a} = \mathfrak{b}$, and $X_1, X_2, \cdots, X_n$ are the indeterminates in $\mathfrak{F}(\mathfrak{a})$ and $\mathfrak{F}(\mathfrak{b})$, this formalization is

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_n \rightarrow_X \mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{b}).$$

The object is to show that this formalization is true whenever the original proposition of $\mathfrak{r}$ is true.

The definition of $\mathfrak{F}$ is given in 7.1, and some preliminary theorems are found in 7.2; in 7.3 it is shown that the formalizations of all the $\mathfrak{r}$-axioms are true, while in 7.4 we establish the validity of the formalized $\mathfrak{r}$-rules as principles of inference between such formalized propositions; this leads to the general result and some related general theorems in 7.5. In 7.6 are listed some special cases which are useful in what follows.

7.1. *Definition of* $\mathfrak{F}$. $\mathfrak{F}(\mathfrak{m})$ is defined by recursion on the construction of $\mathfrak{m}$ as follows:

7.11. $\mathfrak{F}(0) \equiv Z_0$.

7.12. $\mathfrak{F}(v_i) \equiv V_i$—that is, to each variable v_i of $\mathfrak{r}$ we associate a particular indeterminate V_i. When we use 'x,' 'x_k,' 'y_j,' etc., to stand for unspecified variables of $\mathfrak{r}$, then it is to be understood that 'X,' 'X_k,' 'Y_j,' etc., stand for the corresponding indeterminates of $\mathfrak{S}$.

7.13. $\mathfrak{F}(\sigma) \equiv \mathsf{S}$.

7.14. $\mathfrak{F}(\kappa_0) \equiv KZ_0$.

7.15. $\mathfrak{F}(\kappa_{nm}) \equiv \lambda^n X_1 \cdots X_n . X_m$.

7.16. If $\mathfrak{F}(\mathfrak{g}) \equiv \mathfrak{G}$, and $\mathfrak{F}(\mathfrak{a}_i) \equiv \mathfrak{A}_i$, then

$$\mathfrak{F}(\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n)) \equiv \mathfrak{G}\mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_n.$$

7.17. If $\mathfrak{g} \equiv \mathbf{s}_{mn}\mathfrak{h}\mathfrak{k}_1 \cdots \mathfrak{k}_m$, and if $\mathfrak{F}(\mathfrak{h}) \equiv \mathfrak{H}$, $\mathfrak{F}(\mathfrak{k}_i) \equiv \mathfrak{K}_i$, then

$$\mathfrak{F}(\mathfrak{g}) \equiv \lambda^n X_1 \cdots X_n . \mathfrak{H}(\mathfrak{K}_1 X_1 \cdots X_n) \cdots (\mathfrak{K}_m X_1 \cdots X_n).$$

7.18. If $\mathfrak{h} \equiv \mathbf{r}_n \mathfrak{h}\mathfrak{k}$, and $\mathfrak{F}(\mathfrak{h}) \equiv \mathfrak{H}$, $\mathfrak{F}(\mathfrak{k}) \equiv \mathfrak{K}$, then

$$\mathfrak{F}(\mathfrak{g}) \equiv \lambda^{n+1} X_1 \cdots X_n Y . R\mathfrak{K}^*\mathfrak{H} Y X_1 \cdots X_n,$$

where

$$\mathfrak{K}^* \equiv \lambda^{n+2} Y Z X_1 \cdots X_n . \mathfrak{K} X_1 \cdots X_n Y(Z X_1 \cdots X_n).$$

7.2. *Functional character of the formalization of a numerical function.* It is convenient to introduce $\mathcal{N}_k$ as an abbreviation for $F_k\mathcal{N} \cdots \mathcal{N}$ (where the '$\mathcal{N}$' is repeated $k+1$ times). The precise definition is a recursive one, thus (cf. 5.1):

$$\mathcal{N}_0 \equiv \mathcal{N}, \qquad \mathcal{N}_{k+1} \equiv F\mathcal{N}\mathcal{N}_k.$$

7.21. Theorem. *If* $\mathfrak{g} \in \mathfrak{e}_n$, *and* $\mathfrak{G} \equiv \mathfrak{F}(\mathfrak{g})$, *then* $\vdash \mathcal{N}_n\mathfrak{G}$.

Proof. By induction on the construction of $\mathfrak{g}$ in 7.211–7.215.

7.211. The theorem is true for $\mathfrak{g} \equiv \sigma$ by 7.13, and 3.62.

7.212. For $\mathfrak{g} \equiv \kappa_0$ the theorem follows by 7.14, 5.22 (with $U_1 \equiv U_2 \equiv \mathcal{N}$), 3.61, and 5.12.

7.213. The theorem is true for $\mathfrak{g} \equiv \kappa_{nm}$, for

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_n \rightarrow_X \vdash \mathcal{N}X_m \qquad (m \leqq n),$$
$$\rightarrow_X \vdash \mathcal{N}(\mathfrak{F}(\kappa_{nm}))X_1 \cdots X_n \qquad [7.15, 3.31, 3.26],$$

whence the theorem follows by 5.13.

7.214. Suppose $\mathfrak{g} \equiv \mathbf{s}_{mn}\mathfrak{h}\mathfrak{k}_1, \cdots, \mathfrak{k}_m$, and the theorem is true for $\mathfrak{h}, \mathfrak{k}_1, \cdots, \mathfrak{k}_m$. Let $\mathfrak{G} \equiv \mathfrak{F}(\mathfrak{g})$, $\mathfrak{H} \equiv F(\mathfrak{h})$, $\mathfrak{K}_j \equiv \mathfrak{F}(\mathfrak{k}_j)$ $(j = 1, 2, \cdots, m)$. Then by the hypothesis of the induction and 5.12

$$(1) \quad \vdash \mathcal{N}Y_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}Y_m \rightarrow_Y \vdash \mathcal{N}(\mathfrak{H}Y_1 \cdots Y_m),$$
$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_n \rightarrow_X \vdash \mathcal{N}(\mathfrak{K}_j X_1 \cdots X_n) \qquad (j = 1, \cdots, m),$$
$$\rightarrow \vdash \mathcal{N}(\mathfrak{H}(\mathfrak{K}_1 X_1 \cdots X_n) \cdots (\mathfrak{K}_m X_1 \cdots X_n)) \qquad [(1), 2.61],$$
$$\rightarrow \vdash \mathcal{N}(\mathfrak{G}X_1 \cdots X_n) \qquad [7.17, 3.31, 3.26].$$

The theorem then follows by 5.13.

7.215. Suppose $\mathfrak{g} \equiv \mathbf{r}_n\mathfrak{h}\mathfrak{k}$, and that the hypothesis of the induction is true for $\mathfrak{h}$ and $\mathfrak{k}$. Let $\mathfrak{H} \equiv \mathfrak{F}(\mathfrak{h})$, $\mathfrak{K} \equiv \mathfrak{F}(\mathfrak{k})$, and $\mathfrak{K}^*$ be as in 7.18. Then by the hypothesis of the induction,

(1) $$\vdash \mathcal{N}_n\mathfrak{H},$$

and [5.11]

(2) $$\vdash F_2\mathcal{N}\mathcal{N}\mathcal{N}_n\mathfrak{K}.$$

From (2) and 5.12, 5.13 (much as in 7.214) we can then conclude

(3) $$\vdash F_2\mathcal{N}\mathcal{N}_n\mathcal{N}_n\mathfrak{K}^*.$$

From (1), (3), 5.54, and 5.12 we have

(4) $$\vdash F\mathcal{N}\mathcal{N}_n(R\mathfrak{K}^*\mathfrak{H}).$$

But, by 7.18,

$$\mathfrak{G} \equiv \lambda^{n+1}X_1 \cdots X_nY.R\mathfrak{K}^*\mathfrak{H}YX_1 \cdots X_n.$$

Hence from (4), using first 5.12 then 5.13, we have the theorem for $\mathfrak{G}$.

7.22. THEOREM. *If* $\mathfrak{a}\in\mathfrak{e}_0(x_1, \cdots, x_m)$, *then*

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m \rightarrow \vdash \mathcal{N}(\mathfrak{F}(\mathfrak{a})).$$

Proof. We can prove the theorem by induction on the construction of $\mathfrak{a}$, bearing in mind that if $\mathfrak{a}$ is primitive then it must be 0 or some v_i, and if it is not primitive it must be constructed from primitives by 6.46 only. If $\mathfrak{a}\equiv 0$, the result follows by 7.11 and 3.61. If $\mathfrak{a}\equiv v_i$, the theorem to be proved becomes $\vdash\mathcal{N}V_i\rightarrow\vdash\mathcal{N}V_i$ (by 7.12) which is a tautology. Suppose $\mathfrak{a}\equiv\mathfrak{g}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n)$ and that the theorem is true for $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$. Let $\mathfrak{A}_i\equiv\mathfrak{F}(\mathfrak{a}_i)$, $\mathfrak{A}\equiv\mathfrak{F}(\mathfrak{a})$, $\mathfrak{G}\equiv\mathfrak{F}(\mathfrak{g})$. Then, by the hypothesis of the induction,

$$\begin{aligned} \vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m \rightarrow{} & \vdash \mathcal{N}\mathfrak{A}_j && (j = 1, 2, \cdots, n), \\ \rightarrow{} & \vdash \mathcal{N}(\mathfrak{G}\mathfrak{A}_1, \cdots, \mathfrak{A}_n) && [7.21, 5.12], \\ \rightarrow{} & \vdash \mathcal{N}\mathfrak{A} && [7.16], \end{aligned}$$

q.e.d.

7.3. *Verification of the axioms.* We are now in a position to show that if $\mathfrak{a}=\mathfrak{b}$ is an axiom, and $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{e}_0(x_1, \cdots, x_m)$, then $\vdash\mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash\mathcal{N}X_m \rightarrow_X\mathfrak{F}(\mathfrak{a})=\mathfrak{F}(\mathfrak{b})$.

7.31. In the case of the axiom-schemes 6.61–6.64 we have

$$\vdash \mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{b})$$

without any hypothesis [by 3.21; 3.43; 3.32; respectively]; hence, what is to be proved follows a fortiori.

7.32. Let us now turn to the axiom-schemes of 6.65. Suppose that $\mathfrak{a}_1, \cdots, \mathfrak{a}_n, \mathfrak{b}\in\mathfrak{e}_0(x_1, \cdots, x_m)$, and let $\mathfrak{A}_j\equiv\mathfrak{F}(\mathfrak{a}_j)$, $\mathfrak{B}\equiv\mathfrak{F}(\mathfrak{b})$, $\mathfrak{G}\equiv\mathfrak{F}(\mathfrak{g})$, $\mathfrak{H}\equiv\mathfrak{F}(\mathfrak{h})$, $\mathfrak{K}\equiv\mathfrak{F}(\mathfrak{k})$. Then by 7.13, 7.16, what we have to show is that if

(1) $$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m,$$

then

$$\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_n Z_0 = \mathfrak{H}\mathfrak{A}_1 \cdots \mathfrak{A}_n, \tag{2}$$

$$\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_n(\mathsf{S}\,\mathfrak{B}) \equiv \mathfrak{K}\mathfrak{A}_1 \cdots \mathfrak{A}_n\mathfrak{B}(\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_n\mathfrak{B}). \tag{3}$$

Let $\mathfrak{K}^*$ be defined as in 7.18, and let $\mathfrak{G}^* \equiv R\mathfrak{K}^*\mathfrak{H}$. Then [7.18, 3.32]

$$\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_n Y =_Y \mathfrak{G}^* Y \mathfrak{A}_1 \cdots \mathfrak{A}_n. \tag{4}$$

By 5.53, $\mathfrak{G}^* Z_0 = \mathfrak{H}$. From this and (4) we have (2).

Again, by 5.53,

$$\vdash \mathcal{N} Y \rightarrow_Y \mathfrak{G}^*(\mathsf{S}\, Y) = \mathfrak{K}^* Y(\mathfrak{G}^* Y). \tag{5}$$

By 7.2, on the hypothesis (1) we have

$$\vdash \mathcal{N}\mathfrak{B}.$$

Therefore [(5), 2.61]

$$\mathfrak{G}^*(\mathsf{S}\,\mathfrak{B}) = \mathfrak{K}^*\mathfrak{B}(\mathfrak{G}^*\mathfrak{B}); \tag{6}$$

hence

$$\begin{aligned} \mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_n(\mathsf{S}\,\mathfrak{B}) &= \mathfrak{G}^*(\mathsf{S}\,\mathfrak{B})\mathfrak{A}_1 \cdots \mathfrak{A}_n && [(4)], \\ &= \mathfrak{K}^*\mathfrak{B}(\mathfrak{G}^*\mathfrak{B})\mathfrak{A}_1 \cdots \mathfrak{A}_n && [(6)], \\ &= \mathfrak{K}\mathfrak{A}_1 \cdots \mathfrak{A}_n\mathfrak{B}(\mathfrak{G}^*\mathfrak{B}\mathfrak{A}_1 \cdots \mathfrak{A}_n) && [7.18, 3.32], \\ &= \mathfrak{K}\mathfrak{A}_1 \cdots \mathfrak{A}_n\mathfrak{B}(\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_n\mathfrak{B}) && [(4)]. \end{aligned}$$

Then (3) follows by 3.23.

7.4. *Verification of the rules.* Next we have to show that the rules of procedure of $\mathfrak{r}$ are valid principles of inference between the formalizations of the elementary propositions.

7.41. As a preliminary to this let us make a remark concerning these formalizations. The formalization of $\mathfrak{a} = \mathfrak{b}$ is defined as

$$\vdash \mathcal{N} X_1 \ \& \cdots \& \vdash \mathcal{N} X_m \rightarrow_X \mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{b}), \tag{1}$$

where $X_1, \cdots, X_m$ are the indeterminates, if any, corresponding to variables appearing in $\mathfrak{a}$ or $\mathfrak{b}$ (or both), and hence are all the indeterminates in $\mathfrak{F}(\mathfrak{a})$ or $\mathfrak{F}(\mathfrak{b})$ (or both). But this formalization is equivalent to

$$\vdash \mathcal{N} Y_1 \ \& \cdots \& \vdash \mathcal{N} Y_n \rightarrow_Y \mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{b}), \tag{2}$$

provided $Y_1, \cdots, Y_n$ are any set of indeterminates which include the $X_1, \cdots, X_m$. For if (1) is true, then (2) is true a fortiori. On the other hand, if (2) is true, then for any Y_j not occurring in either $\mathfrak{F}(\mathfrak{a})$ or $\mathfrak{F}(\mathfrak{b})$ substitute Z_0 for Y_j; the corresponding premise is redundant by 3.61, and the conclusion is unaltered; the premises can then be rearranged (the relation designated by '&' is intuitive conjunction) to give (1).

7.42. This established, we may show that the formalized 6.71–6.73 follow from the rules of 3.2. Let us take 6.72 as a typical case; the others are proved similarly. In accordance with 7.41 what we have to show is that, if

(1) $\vdash \mathcal{N}X_1 \& \cdots \& \vdash \mathcal{N}X_m \rightarrow_X \mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{b}),$

(2) $\vdash \mathcal{N}X_1 \& \cdots \& \vdash \mathcal{N}X_m \rightarrow_X \mathfrak{F}(\mathfrak{b}) = \mathfrak{F}(\mathfrak{c}),$

then

(3) $\vdash \mathcal{N}X_1 \& \cdots \& \vdash \mathcal{N}X_m \rightarrow_X \mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{c}),$

where $X_1, \cdots, X_m$ is a set of indeterminates including all which occur in $\mathfrak{F}(\mathfrak{a})$, $\mathfrak{F}(\mathfrak{b})$, or $\mathfrak{F}(\mathfrak{c})$. Very well, let (1) and (2) be assumed. Then on the assumption that the hypothesis of (3) holds, we have $\mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{b})$ and $\mathfrak{F}(\mathfrak{b}) = \mathfrak{F}(\mathfrak{c})$, hence $\mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{c})$ by 3.23.

7.43. Let us turn now to the rule of mathematical induction (6.74). Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ be as in the hypotheses of 6.74, and let $\mathfrak{G} \equiv \mathfrak{F}(\mathfrak{g})$, $\mathfrak{H} \equiv \mathfrak{F}(\mathfrak{h})$, $\mathfrak{A}_i \equiv \mathfrak{F}(\mathfrak{a}_i)$ $(i = 1, 2, \cdots, n)$, $\mathfrak{B} \equiv \mathfrak{F}(\mathfrak{b})$. Let $X_1, \cdots, X_m$ be a sufficiently inclusive set of indeterminates, and let

$$\mathfrak{K} \equiv Q(\mathfrak{G}\mathfrak{A}_1 \cdots \mathfrak{A}_n Y)(\mathfrak{H}\mathfrak{A}_1 \cdots \mathfrak{A}_n Y),$$
$$\mathfrak{L} \equiv \lambda^{m+1} Y X_1 \cdots X_m . \mathfrak{K}.$$

Then the formalization of hypothesis (a), which we assume to hold, is

$$\vdash \mathcal{N}X_1 \& \cdots \& \vdash \mathcal{N}X_m \rightarrow_X \vdash \mathfrak{L} Z_0 X_1 \cdots X_m,$$

which is equivalent [3.41] to

(1) $\vdash \mathcal{N}X_1 \& \cdots \& \vdash \mathcal{N}X_m \rightarrow_X \vdash I(\mathfrak{L} Z_0 X_1 \cdots X_m).$

If, now, we define

$$\mathfrak{M} \equiv \lambda Y . F_m \mathcal{N} \cdots \mathcal{N} I(\mathfrak{L} Y),$$

then from (1), 5.13 we have

(2) $\vdash \mathfrak{M} Z_0.$

Suppose, now, that

(3) $\vdash \mathcal{N}Y \& \vdash \mathfrak{M} Y.$

Then [5.12, definition of $\mathfrak{M}$, 3.41]

$$\vdash \mathcal{N}X_1 \& \cdots \& \vdash \mathcal{N}X_m \rightarrow_X \vdash \mathfrak{L} Y X_1 \cdots X_m.$$

Therefore, a fortiori,

(4) $\vdash \mathcal{N}X_1 \& \cdots \& \vdash \mathcal{N}X_m \& \vdash \mathcal{N}Y \rightarrow_{X,Y} \vdash \mathfrak{L} Y X_1 \cdots X_m.$

This is the formalization of 6.74 (1). Now the passage 6.74 (1) to 6.74 (2) was by rules 6.71–6.73 only; hence, by 7.42 from (4) we can pass([47]) to

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m \,\&\, \vdash \mathcal{N}Y \rightarrow_{X,Y} \vdash \mathfrak{L}(\mathsf{S}\, Y)X_1 \cdots X_m,$$

and thence [(3)] to

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m \rightarrow_X \vdash \mathfrak{L}(\mathsf{S}\, Y)X_1 \cdots X_m.$$

Therefore [5.13]

$$\vdash \mathfrak{M}(\mathsf{S}\, Y). \tag{5}$$

We have deduced (5) on the hypothesis (3); hence by (2) and 5.44

$$\vdash \mathcal{N}Y \rightarrow_Y \vdash \mathfrak{M}Y.$$

Therefore [5.12]

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m \,\&\, \vdash \mathcal{N}Y \rightarrow \vdash \mathfrak{K}.$$

The formalization of the conclusion of 6.74 now follows by substitution of $\mathfrak{B}$ for Y, which is permissible by 2.61 and 7.22.

7.5. *General theorems.* As mentioned above, the following may be established.

7.51. THEOREM. *If*

$$\mathfrak{a} = \mathfrak{b} \tag{1}$$

is valid in $\mathfrak{r}$, *and if* $X_1, \cdots, X_n$ *are all the indeterminates occurring in* $\mathfrak{F}(\mathfrak{a})$ *or* $\mathfrak{F}(\mathfrak{b})$, *then*

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_n \rightarrow_X \mathfrak{F}(\mathfrak{a}) = \mathfrak{F}(\mathfrak{b}). \tag{2}$$

Proof. By induction on the proof of (1) in 7.3 and 7.4.

7.52. THEOREM. *If* $\mathfrak{a} \in \mathfrak{e}_0(x_1, \cdots, x_n)$, *and if we define*

$$\mathfrak{F}_{x_1 x_2 \cdots x_n}(\mathfrak{a}) \equiv \mathfrak{F}(\mathfrak{f}_{x_1 \cdots x_n}(\mathfrak{a})),$$

where $\mathfrak{f}_{x_1 \cdots x_n}(\mathfrak{a})$ *is defined as in* 6.83, *then*

([47]) In case known theorems are used in 6.74 we must imagine that we are dealing with a particular proof of a theorem in $\mathfrak{r}$, and showing that this can be converted into a proof in $\mathfrak{S}$ of the corresponding formalization. If we suppose the theorems in the original proof are numbered consecutively from 1 to n, then we have to show by induction on k that the formalization of the first k steps are theorems in $\mathfrak{S}$. Supposing this done for a given k, we are interested here in the case where the $k+1$st step is by 6.74. Then the known theorems used will be those of index $\leq k$. For these the formalizations in $\mathfrak{S}$ are true by the hypothesis of the induction. The argument in 7.42 shows that the formalization of 6.74 (2) is deducible from (4) and these formalizations, and hence, since the latter are true, from (4) only. This deducibility depends, of course, on properties of $\rightarrow$ not explicitly formulated in this paper; but we may suppose, by 7.41, that all the formalizations in this subsidiary proof have the same premises, and then the deducibility relations hold between the conclusions—as in 7.42—by virtue of the rules of procedure of $\mathfrak{S}$ alone.

$$\mathfrak{F}_{x_1\cdots x_n}(\mathfrak{a}) = \lambda^n X_1 \cdots X_n . \mathfrak{F}(\mathfrak{a}).$$

Proof. We proceed by intuitive induction in the construction of $\mathfrak{a}$. We adopt the abbreviation $\mathfrak{g} \equiv \mathfrak{f}_{x_1\cdots x_n}(\mathfrak{a})$.

If $\mathfrak{a} \equiv x_i$, then $\mathfrak{F}(\mathfrak{a}) \equiv X_i$, $\mathfrak{g} \equiv \kappa_{ni}$ and the theorem holds by 7.15. If $\mathfrak{a} \equiv 0$, then $\mathfrak{g} \equiv \mathbf{s}_{1n}\kappa_0\kappa_{n1}$, and if we set

$$\mathfrak{K}_{nm} \equiv \mathfrak{F}(\kappa_{nm}),$$

we have [7.17, 7.14]

$$\begin{aligned} \mathfrak{F}(\mathfrak{g}) &\equiv \lambda^n X_1 \cdots X_n . K Z_0(\mathfrak{K}_{n1} X_1 \cdots X_n), \\ &= \lambda^n X_1 \cdots X_n . Z_0 && [3.34, 3.43], \\ &= \lambda^n X_1 \cdots X_n . \mathfrak{F}(\mathfrak{a}) && [7.11]. \end{aligned}$$

The theorem is therefore true if $\mathfrak{a}$ is primitive.

Suppose $\mathfrak{a} \equiv \mathfrak{h}(\mathfrak{a}_1, \cdots, \mathfrak{a}_m)$, and the theorem is true for $\mathfrak{a}_1, \cdots, \mathfrak{a}_m$. Then, if we let $\mathfrak{g}_1, \cdots, \mathfrak{g}_m$ be as in 6.83, and set

$$\mathfrak{G}_i \equiv \mathfrak{F}(\mathfrak{g}_i), \qquad \mathfrak{A}_i \equiv \mathfrak{F}(\mathfrak{a}_i), \qquad \mathfrak{H} \equiv \mathfrak{F}(\mathfrak{h}),$$

we have

$$\mathfrak{G}_i \equiv \lambda^n X_1 \cdots X_n . \mathfrak{A}_i.$$

Therefore [3.31]

$$\mathfrak{G}_i X_1 \cdots X_n = \mathfrak{A}_i. \tag{1}$$

Again [6.83], $\mathfrak{g} \equiv \mathbf{s}_{mn}\mathfrak{h}\mathfrak{g}_1 \cdots \mathfrak{g}_m$. Therefore

$$\begin{aligned} \mathfrak{F}(\mathfrak{g}) &\equiv \lambda^n X_1 \cdots X_n . \mathfrak{H}(\mathfrak{G}_1 X_1 \cdots X_n) \cdots (\mathfrak{G}_m X_1 \cdots X_n) && [7.17], \\ &= \lambda^n X_1 \cdots X_n . \mathfrak{H}\mathfrak{A}_1 \cdots \mathfrak{A}_m && [(1), 3.34], \\ &= \lambda^n X_1 \cdots X_n . \mathfrak{F}(\mathfrak{a}) && [7.16]. \end{aligned}$$

This completes the proof by intuitive induction.

7.53. THEOREM. *If a is a numeral*[48], *then* $\mathfrak{F}(a) = Z_a$.

Proof. Clear by 7.11, 7.13, 3.44.

7.54. THEOREM. *If $\mathfrak{g} \in \mathfrak{e}_n$, and $a_1, a_2, \cdots, a_n$, b are numerals such that*

$$b = \mathfrak{g}(a_1, \cdots, a_n),$$

then

$$Z_b = \mathfrak{G} Z_{a_1} Z_{a_2} \cdots Z_{a_n},$$

where $\mathfrak{G} \equiv \mathfrak{F}(\mathfrak{g})$.

---

(48) A numeral is an expression of the form $\sigma(\sigma(\cdots \sigma(0) \cdots))$.

Proof. Clear by 7.51, 7.53, 7.16.

7.6. *Special functions.* The formalizations of the special functions of 6.9 will be symbolized as explained below. To substantiate statements of properties of these functions which follow by 6.9 and 7.5 I shall often refer simply to 6.9. In interpreting the operational definitions it is to be understood that the definitions hold when $\mathfrak{A}$, $\mathfrak{B}$, etc., are respectively the formalizations of $\mathfrak{a}$, $\mathfrak{b}$, etc.

$$\begin{aligned} \mathfrak{A} + \mathfrak{B} &\equiv \mathfrak{F}(\mathfrak{a} + \mathfrak{b}), \\ \mathfrak{A} \cdot \mathfrak{B} &\equiv \mathfrak{F}(\mathfrak{a} \cdot \mathfrak{b}), \\ \Delta_1 &\equiv \mathfrak{F}(\delta_1), \\ \mathfrak{A} \dot{-} \mathfrak{B} &\equiv \mathfrak{F}(\mathfrak{a} \dot{-} \mathfrak{b}), \\ |\mathfrak{A}| &\equiv \mathfrak{F}(|\mathfrak{a}|), \\ \Delta_2 &\equiv \mathfrak{F}(\delta_2), \\ \sum_{k=0}^{p} \mathfrak{A}_k &\equiv \mathfrak{F}\left(\sum_{k=0}^{p} \mathfrak{a}_k \right) \text{(}^{49}\text{)}. \end{aligned}$$

Certain properties of these functions, useful as lemmas, are given in 7.61–7.63. In these $\mathfrak{A}$, $\mathfrak{B}$ stand for terms subject to the hypothesis $\vdash \mathcal{N}\mathfrak{A}$ & $\vdash \mathcal{N}\mathfrak{B}$.

7.61. THEOREM. $\mathfrak{A}+\mathfrak{B}=Z_0 \rightarrow \mathfrak{A}=Z_0$ & $\mathfrak{B}=Z_0$.

Proof. By 6.932

$$\begin{aligned} \mathfrak{A} + \mathfrak{B} = Z_0 &\rightarrow (\mathfrak{A} + \mathfrak{B}) \dot{-} \mathfrak{B} = Z_0; \\ (1) \qquad &\rightarrow \mathfrak{A} = Z_0 && [6.934]. \\ \mathfrak{A} + \mathfrak{B} = Z_0 &\rightarrow \mathfrak{B} + \mathfrak{A} = Z_0 && [6.91], \\ &\rightarrow \mathfrak{B} = Z_0 && [(1)]. \end{aligned}$$

7.62. THEOREM. $\Delta_2\mathfrak{A}\mathfrak{B}=Z_0 \rightarrow \mathfrak{A}=\mathfrak{B}$.

Proof. We have

$$\begin{aligned} \Delta_2\mathfrak{A}\mathfrak{B} = Z_0 &\rightarrow (\mathfrak{A} \dot{-} \mathfrak{B}) + (\mathfrak{B} \dot{-} \mathfrak{A}) = Z_0 && [\text{definition}] \\ &\rightarrow \mathfrak{A} \dot{-} \mathfrak{B} = Z_0 \;\&\; \mathfrak{B} \dot{-} \mathfrak{A} = Z_0 && [7.61], \\ &\rightarrow \mathfrak{A} = \mathfrak{A} + (\mathfrak{B} \dot{-} \mathfrak{A}) && [6.91], \\ &= \mathfrak{B} + (\mathfrak{A} \dot{-} \mathfrak{B}) && [6.938], \\ &= \mathfrak{B} && [6.91], \end{aligned}$$

q.e.d.

*7.63. THEOREM. *If* $|\mathfrak{A}| \cdot \mathfrak{B}=Z_0$, *and* $\mathfrak{A}=Z_0$, *then* $\mathfrak{B}=Z_0$ [6.941, 6.92, 6.91].

(49) Note that p and k are here taken in an intuitive sense, and are not formalized.

Part III. Formalization of the Gödel representation

The subject of this part is the second of the major theorems considered in §4, the construction of T. §8 is preliminary; the construction proper is given in §9.

8. Numerical functions associated with the Gödel representation.

8.1. *The functions* $\mu_1(x)$, $\mu_2(x)$, $\mu_3(x)$. We seek now primitive recursive functions μ_1 and μ_2 such that if $x=\mathfrak{n}(\mathfrak{X}_1\mathfrak{X}_2)$ then $\mu_1(x)=\mathfrak{n}(\mathfrak{X}_1)$, $\mu_2(x)=\mathfrak{n}(\mathfrak{X}_2)$. This we may do as follows:

$$\mu_1(x) \equiv (\epsilon y \leqq x)(\exists z \leqq x)\delta_2(x, \mu(y, z)), \tag{1}$$

$$\mu_2(x) \equiv (\epsilon z \leqq x)\delta_2(x, \mu(\mu_1(x), z)). \tag{2}$$

Then we have

$$\mu_1(\mu(x, y)) = x, \tag{3}$$

$$\mu_2(\mu(x, y)) = y. \tag{4}$$

If we define further

$$\mu_3(x) \equiv (\exists y \leqq x)(\exists z \leqq x)(\delta_2(x, \mu(y, z))), \tag{5}$$

then by the processes of recursive arithmetic we have

$$\mu_3(\mu(x, y)) = 0. \tag{6}$$

Also, $\mu_3(x)=0 \rightarrow x=\mu(\mu_1(x), \mu_2(x))$; in fact, we have

$$\mu_3(x) = \left\|\delta_2(x, \mu(\mu_1(x), \mu_2(x)))\right\|. \tag{7}$$

Likewise we have

$$\mu_3(x)\cdot\mu_1(x) = \mu_3(x)\cdot\mu_2(x) = 0. \tag{8}$$

8.2. *The functions* M, M_1, M_2, M_3. These are defined as the formalizations of μ, μ_1, μ_2, μ_3, respectively. Then by 8.1 and the theorems of 7.5 we have the following.

8.21. $\vdash \mathcal{N}X \;\&\; \vdash \mathcal{N}Y \rightarrow_{X,Y} M_1(MXY)=X \;\&\; M_2(MXY)=Y$.

8.22. $\vdash \mathcal{N}X \;\&\; M_3X=Z_0 \rightarrow_X M(M_1X)(M_2X)=X$.

8.23. $\vdash \mathcal{N}X \;\&\; \vdash \mathcal{N}Y \rightarrow_{X,Y} M_3(MXY)=Z_0$.

8.3. *The function* $\mathfrak{T}$(50). It is convenient to denote by '$\mathfrak{T}$' the correspondence inverse to $\mathfrak{n}$, so that

$$a = \mathfrak{n}(\mathfrak{A}) \rightleftarrows \mathfrak{T}(a) = \mathfrak{A}. \tag{1}$$

This defines $\mathfrak{T}$ uniquely for a number of the form $\mathfrak{n}(\mathfrak{A})$. For other values of x we may define

(50) This discussion of $\mathfrak{T}$ is for heuristic purposes only. The conclusions of the paper do not depend on the properties of $\mathfrak{T}$ here stated.

$$\mathfrak{T}(x) \equiv I.$$

Such a $\mathfrak{T}$ will have, then, the following property. Suppose $a = \mathfrak{n}(\mathfrak{A})$, $b = \mathfrak{n}(\mathfrak{B})$, $c = \mathfrak{n}(\mathfrak{C})$, where $\mathfrak{C} \equiv \mathfrak{A}\mathfrak{B}$. Then

$$\begin{aligned} c &= \mu(a, b), \\ \mathfrak{C} &= \mathfrak{T}(c) = \mathfrak{T}(\mu(a, b)), \\ \mathfrak{A}\mathfrak{B} &= \mathfrak{T}(a)(\mathfrak{T}(b)). \end{aligned}$$

Therefore

$$\mathfrak{T}(\mu(a, b)) = \mathfrak{T}(a)(\mathfrak{T}(b)),$$

i.e.,

$$\mathfrak{T}(\mu(x, y)) = \mathfrak{T}(x)(\mathfrak{T}(y)). \tag{2}$$

This equation represents a course-of-values recursion[51] for $\mathfrak{T}$. If we now define

$$\begin{aligned} \mathfrak{T}(e_k)^{(52)} &\equiv E_k, \\ \mathfrak{T}(a) &\equiv I, \qquad \text{if no term is associated with } a, \end{aligned} \tag{3}$$

and define $\mathfrak{T}$ recursively by (2) if $\mu_3(x) = 0$, then $\mathfrak{T}(x)$ will be defined for all x and will have the property (1).

9. **The function** T. The constant T, mentioned in §4, is a formalization, so to speak, of the $\mathfrak{T}$ considered in 8.3. Its characteristic properties are the following:

$$\begin{gathered} a = \mathfrak{n}(\mathfrak{A}) \rightarrow TZ_a = \mathfrak{A}, \\ \vdash \mathcal{N}X \;\&\; \vdash \mathcal{N}Y \rightarrow T(MXY) = TX(TY), \end{gathered}$$

which are motivated in 8.3. The preliminary analysis leading up to this function T is given in 9.1; the formal definition is given in 9.2; 9.3 and 9.4 are lemmas; while the final proof that the above two properties hold occupies 9.5 and 9.6.

9.1. *Preliminary analysis.* As pointed out in 8.3 the second property amounts to a course-of-values recursion for T, which, together with

$$TZ_0 \equiv I, \qquad TZ_{e_k} \equiv E_k, \tag{1}$$

essentially defines T. In order to take care of this recursion by the methods of §5 we must first consider a "course-of-values function," call it $\mathfrak{V}$, such that, intuitively,

(51) This expression is a translation of the German term 'Wertverlaufsrekursion' introduced by Péter (C 466.2). Cf. Hilbert-Bernays C 507.1, p. 326. The translation is used by Péter herself in Acta Szeged, vol. 9 (1940), p. 233.

(52) $e_k \equiv \mathfrak{n}(E_k)$.

$$\mathfrak{B} Z_x Z_y = \begin{cases} T Z_y & \text{if} \quad y \leqq x, \\ I & \text{if} \quad y > x. \end{cases} \tag{2}$$

If we can define such a $\mathfrak{B}$ then we can define T as $\lambda X . \mathfrak{B} X X$. This $\mathfrak{B}$ will be definable by a simple recursion of the kind considered in §5.

It is evidently in accordance with (1) to specify that

$$\mathfrak{B} Z_0 = KI = Z_0, \tag{3}$$

and with (2) to require that

$$\mathfrak{B} Z_{n+1} Z_m = \mathfrak{B} Z_n Z_m, \quad \text{if} \quad m \neq n + 1,$$

while for $m = n+1$ we set

$$\mathfrak{B} Z_{n+1} Z_m = \mathfrak{U}_1 Z_{n+1},$$

where $\mathfrak{U}_1$ is yet to be defined. These can be combined by setting

$$\mathfrak{B}(\mathfrak{S} X) = \lambda Y . D_2(\mathfrak{U}_1(\mathfrak{S} X))(\mathfrak{B} X Y)(\Delta_2(\mathfrak{S} X) Y). \tag{4}$$

Let us turn now to $\mathfrak{U}_1$. If we set

$$\mathfrak{U}_1 \equiv \lambda X . D_2(\mathfrak{U}_2 X)(\mathfrak{U}_3 X)(M_3 X), \tag{5}$$

then it is evident that

$$\mathfrak{U}_1 Z_x = \begin{cases} \mathfrak{U}_2 Z_x & \text{if} \quad \mu_3(x) = 0, \\ \mathfrak{U}_3 Z_x & \text{if} \quad \mu_3(x) \neq 0. \end{cases}$$

Turning now to $\mathfrak{U}_2$ and setting

$$\mathfrak{U}_2 \equiv \lambda X . \mathfrak{B} X (M_1 X)(\mathfrak{B} X (M_2 X)),$$

we have that if $\mu_3(n) = 0$

$$\begin{aligned} T Z_{n+1} &= \mathfrak{B} Z_{n+1} Z_{n+1} \\ &= \mathfrak{U}_1 Z_{n+1} \\ &= \mathfrak{U}_2 Z_{n+1} \\ &= \mathfrak{B} Z_{n+1}(M_1 Z_{n+1})(\mathfrak{B} Z_{n+1}(M_2 Z_{n+1})) \\ &= T(M_1 Z_{n+1})(T(M_2 Z_{n+1})) \qquad [\text{by (2)}]. \end{aligned}$$

Turn now to $\mathfrak{U}_3$, and define functions $\mathfrak{K}_0, \mathfrak{K}_1, \cdots, \mathfrak{K}_t$ by

$$\begin{aligned} \mathfrak{K}_i &\equiv \lambda X . D_2 E_i(\mathfrak{K}_{i+1} X)(\Delta_2 Z_{e_i} X) \qquad (i = 0, 1, \cdots, t-1), \\ \mathfrak{K}_j &\equiv \lambda X . D_2 E_t I(\Delta_2 Z_{e_t} X). \end{aligned}$$

Then if we take $\mathfrak{K}_0$ for $\mathfrak{U}_3$ our definition of T is complete.

If we substitute for $\mathfrak{U}_1$, $\mathfrak{U}_2$, $\mathfrak{U}_3$ in (4) we have a recursive definition for $\mathfrak{B}$. This leads to the definition given formally in 9.2.

9.2. *Definitions.*

9.21. $\mathfrak{K}_i \equiv \lambda Y . D_2 E_i(\mathfrak{K}_{i+1} Y)(\Delta_2 Z_{e_i} Y)$ $(i=0,\ 1, \cdots,\ t-1)$, and $\mathfrak{K}_t \equiv \lambda Y . D_2 E_t I(\Delta_2 Z_{e_t} Y)$.

9.22. $\mathfrak{U}_2 \equiv \lambda^2 VY . V(M_1 Y)(V(M_2 Y))$.

9.23. $\mathfrak{U}_1 \equiv \lambda^2 VY . D_2(\mathfrak{U}_2 VY)(\mathfrak{K}_0 Y)(M_3 Y)$.

9.24. $\mathfrak{G} \equiv \lambda^3 XVY . D_2(\mathfrak{U}_1 VY)(VY)(\Delta_2(\mathsf{S} X) Y)$.

9.25. $\mathfrak{B} \equiv R \mathfrak{G} Z_0$.

9.26. $T \equiv \lambda X . \mathfrak{B} XX$.

*9.3. LEMMA. *If* $\vdash \mathcal{N} X$ & $\vdash \mathcal{N} Y$, *then*

$$\mathfrak{B}(X + Y)X = TX.$$

Proof. If $Y = Z_0$, then

$$\begin{aligned} \mathfrak{B}(X + Y)X &= \mathfrak{B}(X + Z_0)X \\ &= TX && [9.26, 6.91]. \end{aligned}$$

Suppose the lemma true for a given Y[53]. Then

$$\begin{aligned} \mathfrak{B}(X + \mathsf{S} Y)X &= \mathfrak{B}(\mathsf{S}(X + Y))X && [6.91], \\ &= \mathfrak{G}(X + Y)(\mathfrak{B}(X + Y))X && [9.25; 5.53], \\ &= D_2 \mathfrak{W}(\mathfrak{B}(X + Y)X)(\Delta_2(\mathsf{S}(X + Y))X) && [9.24], \end{aligned}$$

where $\mathfrak{W}$ is a term which does not concern us for the moment. But, by 6.91,

$$\begin{aligned} \Delta_2(\mathsf{S}(X + Y))X &= \Delta_2(X + \mathsf{S} Y)X \\ &= ((X + \mathsf{S} Y) \dot{-} X) + (X \dot{-} (X + \mathsf{S} Y)) && [6.95], \\ &= \mathsf{S} Y + Z_0 && [6.934, 6.935, 6.932], \\ &= \mathsf{S} Y && [6.91]. \end{aligned}$$

Therefore

$$\begin{aligned} \mathfrak{B}(X + \mathsf{S} Y)X &= \mathfrak{B}(X + Y)X && [5.32], \\ &= TX && \text{(by hypothesis of induction)}. \end{aligned}$$

The lemma follows by 5.44.

*9.4. LEMMA. *If* $\vdash \mathcal{N} X$, *then* $T(\mathsf{S} X) = D_2(\mathfrak{U}_2(\mathfrak{B} X)(\mathsf{S} X))(\mathfrak{K}_0(\mathsf{S} X))(M_3(\mathsf{S} X))$.

Proof. We have

$$\begin{aligned} T(\mathsf{S} X) &= \mathfrak{B}(\mathsf{S} X)(\mathsf{S} X) && [9.26], \\ &= \mathfrak{G} X(\mathfrak{B} X)(\mathsf{S} X) && [9.25; 5.53], \\ &= D_2(\mathfrak{U}_1(\mathfrak{B} X)(\mathsf{S} X))\mathfrak{B}(\Delta_2(\mathsf{S} X)(\mathsf{S} X)) && [9.24], \quad (1) \end{aligned}$$

[53] I.e., suppose $\vdash \mathcal{N} Y$ & $\vdash \mathfrak{M} Y$, where

$$\mathfrak{M} \equiv \lambda Y . Q(\mathfrak{B}(X + Y)X)(TX).$$

where $\mathfrak{W}$ is a term which does not concern us. By 6.95, $\Delta_2(\mathfrak{S}X)(\mathfrak{S}X)=Z_0$. Therefore [(1); 5.31]

$$T(\mathfrak{S}X) = \mathfrak{U}_1(\mathfrak{B}X)(\mathfrak{S}X).$$

The lemma follows by 9.23.

*9.5. THEOREM. *If* $\vdash \mathcal{N}X$ & $\vdash \mathcal{N}Y$, *then* $T(MXY)=TX(TY)$.

Proof. By 3.522, $|\mu(x, y)|=0$. Therefore [6.944] $\mu(x, y) = \sigma(\delta_1(\mu(x, y)))$. Therefore [7.51, hypothesis]

$$MXY = \mathfrak{S}\mathfrak{A}, \tag{1}$$

where $\mathfrak{A}\equiv\Delta_1(MXY)$. Therefore

$$\begin{aligned} T(MXY) &= T(\mathfrak{S}\mathfrak{A}) \\ &= D_2(\mathfrak{U}_2(\mathfrak{B}\mathfrak{A})(\mathfrak{S}\mathfrak{A}))(\mathfrak{K}_0(\mathfrak{S}\mathfrak{A}))(M_3(\mathfrak{S}\mathfrak{A})) && [9.4], \\ &= \mathfrak{U}_2(\mathfrak{B}\mathfrak{A})(\mathfrak{S}\mathfrak{A}) && [(1), 8.23, 5.31], \\ &= \mathfrak{B}\mathfrak{A}(M_1(\mathfrak{S}\mathfrak{A}))(\mathfrak{B}\mathfrak{A}(M_2(\mathfrak{S}\mathfrak{A}))) && [9.22], \\ &= \mathfrak{B}\mathfrak{A}X(\mathfrak{B}\mathfrak{A}Y) && [(1), 8.21]. \end{aligned} \tag{2}$$

But, by 3.522, $\mu(x, y)>x$ & $\mu(x, y)>y$. Therefore

$$\delta_1(\mu(x, y)) \geqq x.$$

Hence

$$\delta_1(\mu(x, y)) = x + (\delta_1(\mu(x, y)) \dot{-} x).$$

It follows [7.51] that $\mathfrak{A}=X+(\mathfrak{A}\dot{-}X)$. Similarly $\mathfrak{A}=Y+(\mathfrak{A}\dot{-}Y)$. Hence, by 9.3

$$\mathfrak{B}\mathfrak{A}X = TX, \qquad \mathfrak{B}\mathfrak{A}Y = TY.$$

Substituting these in (2) we have the theorem.

9.6. THEOREM. *If* $a=\mathfrak{n}(\mathfrak{A})$, *then* $\mathfrak{A}=TZ_a$.

Proof. We proceed by induction on the construction of $\mathfrak{A}$. In 9.61 we take the case where $\mathfrak{A}$ is a primitive term, in 9.62 the induction from $\mathfrak{B}$ and $\mathfrak{C}$ to $\mathfrak{A}=\mathfrak{B}\mathfrak{C}$.

9.61. Suppose $\mathfrak{A}\equiv E_k$; then $a=\mathfrak{n}(\mathfrak{A})=e_k>0$. Let

$$b = e_k - 1.$$

Then

$$a = b + 1.$$

Therefore [3.44]

(1) $$\mathfrak{Z}_a = \mathfrak{S}\, \mathfrak{Z}_b;$$

hence [9.4]

(2) $$T\mathfrak{Z}_a = D_2\mathfrak{W}(\mathfrak{K}_0\mathfrak{Z}_a)(M_3\mathfrak{Z}_a),$$

where $\mathfrak{W}$ is a term which does not concern us. By 8.1, 3.524, $\mu_3(a)>0$. Hence there is a number c such that $\mu_3(a)=c+1$. Hence [7.54, 3.44] $M_3\mathfrak{Z}_a=\mathfrak{S}\,\mathfrak{Z}_c$. Therefore [(2); 5.32]

(3) $$T\mathfrak{Z}_a = \mathfrak{K}_0\mathfrak{Z}_a.$$

Now for $i<k$, $\delta_2(e_i, a)>0$. Therefore [7.6]

$$\Delta_2\mathfrak{Z}_{e_i}\mathfrak{Z}_a = \mathfrak{S}\,(\Delta_1(\Delta_2\mathfrak{Z}_{e_i}\mathfrak{Z}_a)).$$

Thus [9.21; 5.32]

(4) $$\mathfrak{K}_i\mathfrak{Z}_a = \mathfrak{K}_{i+1}\mathfrak{Z}_a.$$

But since $\Delta_2\mathfrak{Z}_{e_k}\mathfrak{Z}_a=\mathfrak{Z}_0$, we have [5.31]

(5) $$\mathfrak{K}_k\mathfrak{Z}_a = E_k.$$

By (3), (4), (5) we have $T\mathfrak{Z}_a=\mathfrak{A}$.

9.62. Suppose now $\mathfrak{A}\equiv\mathfrak{B}\mathfrak{C}$, and that the theorem is true for $\mathfrak{B}$ and $\mathfrak{C}$. Let $b\equiv\mathfrak{n}(\mathfrak{B})$ and $c\equiv\mathfrak{n}(\mathfrak{C})$; then [3.521] $a=\mu(b, c)$. Hence [7.54]

$$\mathfrak{Z}_a = M\mathfrak{Z}_b\mathfrak{Z}_c.$$

Therefore

$$\begin{aligned} T\mathfrak{Z}_a &= T(M\mathfrak{Z}_b\mathfrak{Z}_c) \\ &= T\mathfrak{Z}_b(T\mathfrak{Z}_c) && [9.5], \\ &= \mathfrak{B}\mathfrak{C} && [\text{hypothesis of induction}], \\ &= \mathfrak{A}, \end{aligned}$$

q.e.d.

PART IV. FORMAL ENUMERATION OF THEOREMS

The object of this part is to define a term Θ which gives a formal enumeration of the theorems of the system, in the sense that every term of the form $T(\Theta\mathfrak{Z}_n)$ is assertible, and vice versa. Θ has then the property that

$$\vdash\ \mathcal{N}X \rightarrow_X T(\Theta X).$$

Θ is the formalization of a primitive recursive function $\theta(x)$; this $\theta(x)$ and some numerical functions related to it are defined in §10. The remaining sections concern the formalizations of these various functions. The final result is established in §14.

10. **The function θ and related functions.**

10.1. *Enumeration of the axioms.* We define first a function $\alpha(x)$ which enumerates the numbers of the axioms. By hypothesis there are only a finite number of axioms; let these be $A_0, A_1, \cdots, A_s$, and let $a_0, a_1, \cdots, a_s$ respectively be their associated numbers. Then let

$$\alpha_1(x) \equiv \sum_{k=0}^{s} \left| \delta_2(x, k) \right| \cdot a_k. \tag{1}$$

We can easily define a function $\beta(x)$ whose value for every x is one of the a_i, and which takes each of the a_i at least once; for instance let $\beta(x)$ be a primitive recursive function which takes on only the values $0, 1, \cdots, s$ (e.g.,

$$x \dot{-} \left[\frac{x}{s+1} \right] \cdot (s+1)).$$

Then set

$$\begin{aligned} \alpha(x) &\equiv \alpha_1(x) + \left\| x \dot{-} s \right\| \cdot \alpha_1(\beta(x)), \\ &= \sum_{k=0}^{s} \left| \delta_2(x, k) \right| \cdot \alpha_1(k) + \left\| x \dot{-} s \right\| \cdot \alpha_1(\beta(x)). \end{aligned} \tag{2}$$

10.2. *Functions connected with the rules.* Suppose we have a rule of the form

$$\vdash \mathfrak{X}_1 \,\&\, \cdots \,\&\, \vdash \mathfrak{X}_n \rightarrow_X \mathfrak{X}_{n+1},$$

where the $\mathfrak{X}_j$ are combinations of constants and the indeterminates $X_1, \cdots, X_m$. We must furthemore assume that any indeterminate which actually occurs in $\mathfrak{X}_{n+1}$ actually occurs in some $\mathfrak{X}_j$ for $j<n$. We seek to find a recursive function $\rho(x_1, \cdots, x_n)$, such that if $\mathfrak{X}_1^*, \cdots, \mathfrak{X}_{n+1}^*$ are terms obtained by substituting constants $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ for $X_1, \cdots, X_m$ in $\mathfrak{X}_1, \cdots, \mathfrak{X}_{n+1}$ respectively, and if $b_j = \mathfrak{n}(\mathfrak{X}_j^*)$, then $\rho(b_1, \cdots, b_n) = b_{n+1}$; furthermore ρ is to have the property that if $c_1, \cdots, c_n$ are numbers associated with theorems (whether obtained by substitution from the $\mathfrak{X}_j$ or not) then $\rho(c_1, \cdots, c_n)$ is also. In order to obtain such a ρ we must first define functions $\phi(\mathfrak{X}_j; x_1, \cdots, x_m)$ expressing the above b_j in terms of $\mathfrak{n}(\mathfrak{A}_1), \cdots, \mathfrak{n}(\mathfrak{A}_m)$ (this is done in 10.21), and then (in 10.22) functions allowing, so to speak, the inversion of the ϕ's. The definition of ρ is then given in 10.23.

10.21. Let $\mathfrak{X}$ be any combination of constants and the indeterminates $X_1, \cdots, X_m$. With each such $\mathfrak{X}$ we associate a function $\phi(\mathfrak{X}; x_1, \cdots, x_m)$ as specified, by intuitive recursion on the construction of $\mathfrak{X}$, in 10.211–10.213. The properties 10.214 and 10.215 then follow by induction.

10.211. If $\mathfrak{C}$ is a constant[54], then

$$\phi(\mathfrak{C}; x_1, \cdots, x_m) \equiv \mathfrak{n}(\mathfrak{C}).$$

[54] It is sufficient to make this definition for $\mathfrak{C}$ a primitive constant; it then follows by 10.213 for any constant.

10.212. For $i=1, 2, \cdots, m$,

$$\phi(X_i; x_1, \cdots, x_m) \equiv x_i.$$

10.213. $\phi(\mathfrak{Y}\mathfrak{Z}; x_1, \cdots, x_m) \equiv \mu(\phi(\mathfrak{Y}; x_1, \cdots, x_m), \phi(\mathfrak{Z}; x_1, \cdots, x_m))$.

10.214. If $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ are any constants and if $a_i = \mathfrak{n}(\mathfrak{A}_i)$, then

$$\phi(\mathfrak{X}; a_1, \cdots, a_m) = \mathfrak{n}\left(\begin{bmatrix} \mathfrak{A}_1, \cdots, \mathfrak{A}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}\right).$$

10.215. If $X_{j_1}, X_{j_2}, \cdots, X_{j_p}$ are the indeterminates actually occurring in $\mathfrak{X}$, then, if

$$\phi(\mathfrak{X}; a_1, \cdots, a_m) = \phi(\mathfrak{X}; b_1, \cdots, b_m),$$

we have

$$a_{j_1} = b_{j_1} \;\&\; \cdots \;\&\; a_{j_p} = b_{j_p};$$

furthermore $\phi(\mathfrak{X}; a_1, \cdots, a_m)$ is not the number of any constant unless $a_{j_1}, \cdots, a_{j_p}$ are also [3.52]. The first statement may, moreover, be formally established in the form

$$(1) \qquad \begin{aligned} &\left| \, \delta_2(\phi(\mathfrak{X}; x_1, \cdots, x_m), \phi(\mathfrak{X}; y_1, \cdots, y_m)) \right| \\ &\qquad = \left| \, \delta_2(x_{j_1}, y_{j_1}) + \cdots + \delta_2(x_{j_p}, y_{j_p}) \right|. \end{aligned}$$

10.22. Consider now a rule of the form stated in the introduction to 10.2. We shall suppose that the $X_1, \cdots, X_m$ are all the indeterminates which actually occur in the $\mathfrak{X}_1, \cdots, \mathfrak{X}_n$; the rule is then such that these include all which actually occur in $\mathfrak{X}_{n+1}$ also. We define now functions ψ_i, such that, if $b_1, \cdots, b_n$ are the numbers of terms arising by substitution of $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ for $X_1, \cdots, X_m$ in the $\mathfrak{X}_j$, then $\psi_i(b_1, \cdots, b_n) = \mathfrak{n}(\mathfrak{A}_i)$.

In making these definitions it is convenient to abbreviate as follows:

$$\begin{aligned} \phi_j(x_1, \cdots, x_m) &\equiv \phi(\mathfrak{X}_j; x_1, \cdots, x_m), \\ \Sigma y &\equiv y_1 + y_2 + \cdots + y_n, \\ \psi_i^* &\equiv \psi_i(y_1, \cdots, y_n), \\ \mathfrak{m}_i &= \sum_{j=1}^{n} \delta_2(y_j, \phi_j(\psi_1^*, \cdots, \psi_{i-1}^*, x_i, \cdots, x_m)). \end{aligned}$$

Then the definitions are

$$(1) \qquad \begin{aligned} \psi_0^* &\equiv (\exists x_1 \leqq \Sigma y)(\exists x_2 \leqq \Sigma y) \cdots (\exists x_m \leqq \Sigma y)\mathfrak{m}_1, \\ \psi_i^* &\equiv (\epsilon x_i \leqq \Sigma y)(\exists x_{i+1} \leqq \Sigma y) \cdots (\exists x_m \leqq \Sigma y)\mathfrak{m}_i \\ &\qquad\qquad (i = 1, 2, \cdots, m). \end{aligned}$$

From these definitions it follows that if $\psi_0(y_1, \cdots, y_n)=0$, then $\phi_j(\psi_1^*, \cdots, \psi_m^*)=y_j$; i.e.,

$$(2) \qquad \left| \psi_0(y_1, \cdots, y_n)\right| \cdot \delta_2(y_j, \phi_j(\overset{*}{\psi_1}, \cdots, \overset{*}{\psi_m})) = 0.$$

10.23. In terms of the function defined in 10.21 and 10.22 the definition of ρ is the following:

$$\rho(y_1, \cdots, y_n) \equiv \left| \overset{*}{\psi_0} \right| \cdot \phi_{n+1}(\overset{*}{\psi_1}, \cdots, \overset{*}{\psi_m}) + \left\|\overset{*}{\psi_0}\right\| \cdot \mathfrak{n}(A_0).$$

This ρ then has the properties stated in the introduction to 10.2.

10.3 *Definition of θ.* The definition of $\theta(x)$, which enumerates with repetitions the numbers associated with the theorems, is essentially due to Kleene([55]). The idea of this enumeration is as follows: First we divide the theorems into stages. In the 0th stage we put only the axiom A_0; and in the $(k+1)$st stage we put the axiom A_k (i.e., $\mathfrak{T}(\alpha(k))$) together with the theorems obtained by applying the rules in all possible ways to the theorems in all the preceding stages. We may suppose that all the rules have the same number of premises; for if any rule has less than the maximum number, we increase the number by repeating premises. Then within each stage we can arrange the theorems, first according to the rule used, then according to the first premise, then according to the second premise, etc. In such a scheme every theorem will have at least one number, and this number can be effectively ascertained when once a proof of the theorem is known.

This numbering of the theorems is determined by a set of recursive functions as follows([56]):

$\zeta(k)$ is the number of theorems in the stages preceding the kth.

$\gamma(x)=$the stage of the theorem numbered x.

$\nu(x)=$the number of theorem x in its stage.

$\pi_0(x)=$the number of the rule used([57]).

$\pi_k(x)=$the number of the kth premise $(k=1, 2, \cdots, n)$.

$\xi_k(x)=$the number of the theorem in the group in the same stage having the same rule and (for $k>0$) the same k first premises.

Let us suppose there are r rules of n premises each. Let the rules be numbered $0, 1, 2, \cdots, r-1$; and let ρ_k be the function associated with the kth rule as in 10.2. Then the above functions have the following recursive definitions:

$$(1) \qquad \begin{aligned} \zeta(0) &= 0, \\ \zeta(x+1) &= \zeta(x) + 1 + r\cdot(\zeta(x))^n, \end{aligned}$$

([55]) C 497.2.

([56]) In the enumeration defined below these theorems are placed first and A_k last, and the numbering in each of the substages begins with 0.

([57]) If the nth theorem is an axiom, $\pi_0(n)=r$, where r (as below) is the number of rules.

(2) $$\gamma(x) \equiv (\epsilon z \leqq x)\left[(\zeta(z) \dot{-} x) + ((x+1) \dot{-} \zeta(z+1))\right].$$

(3) $$\nu(x) \equiv x \dot{-} \zeta(\gamma(x)),$$

(4) $$\pi_0(x) \equiv \left[\frac{\nu(x)}{(\zeta(\gamma(x)))^n}\right],$$

(5) $$\xi_0(x) \equiv \nu(x) \dot{-} \pi_0(x) \cdot (\zeta(\gamma(x)))^n,$$

(6) $$\pi_{k+1}(x) \equiv \left[\frac{\xi_k(x)}{(\zeta(\gamma(x)))^{n-k-1}}\right] \qquad (k = 0, 1, \cdots, n-1),$$

(7) $$\xi_{k+1}(x) \equiv \xi_k(x) \dot{-} \pi_{k+1}(x) \cdot (\zeta(\gamma(x)))^{n-k-1} \qquad (k = 0, 1, \cdots, n-2).$$

Then the recursive equations for $\theta(x)$ are

(8) $$\theta(0) = \alpha(0);$$

and, for $x>0$,

(9) $$\theta(x) = \sum_{k=0}^{r-1} \left| \delta_2(\pi_0(x), k)\right| \cdot \rho_k(\theta(\pi_1(x)), \cdots, \theta(\pi_n(x))) + \left\|\pi_0(x) \dot{-} (r-1)\right\| \cdot \alpha(\gamma(x)).$$

Since for $x>0$, $i>0$,

(10) $$\pi_i(x) \leqq \nu(x) < x,$$

this is a course-of-values recursion in the sense of R. Péter([58]) for θ. According to her work it can be replaced by a primitive recursion; i.e., there exists a primitive recursive function having the properties (8) and (9).

11. **Proof by cases.** In the foregoing we have frequently had to deal with functions of the form of 6.98. In proving theorems concerning the formalizations of such functions we shall need a technique for obtaining proofs by cases—i.e., proof of a theorem by considering separately the case where some numerical term is Z_0 or is not, etc. The necessary theorems in this connection are developed in the present section.

11.1. THEOREM. *If* $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}$ *are any terms such that*
(a) $\vdash \mathfrak{M} \rightarrow \vdash \mathrm{N}\mathfrak{A}$,
(b) $\vdash \mathfrak{M} \,\&\, \mathfrak{A} = Z_0 \rightarrow \vdash \mathfrak{B}$,
(c) $\vdash \mathfrak{M} \,\&\, |\mathfrak{A}| = Z_0 \rightarrow \vdash \mathfrak{B}$,
then

$$\vdash \mathfrak{M} \rightarrow \vdash \mathfrak{B}.$$

Proof. Let $\mathfrak{L} \equiv \lambda X . \Lambda \mathfrak{M}(Q\mathfrak{A}X)$. Then [5.42]

([58]) See second footnote to 8.3.

(1) $$\vdash \mathfrak{L}Y \rightleftarrows_Y \vdash \mathfrak{M} \,\&\, \mathfrak{A} = Y.$$

Therefore

$$\begin{aligned} &\vdash \mathfrak{L}Z_0 \rightarrow \vdash \mathfrak{M} \,\&\, \mathfrak{A} = Z_0 \\ &\qquad \rightarrow \vdash \mathfrak{B} \qquad [(b)]. \end{aligned}$$

Hence [3.43]

$$\vdash K(\mathfrak{L}Z_0)X \rightarrow_X \vdash K\mathfrak{B}(IX).$$

Thus [3.65]

(2) $$\vdash F(K(\mathfrak{L}Z_0))(K\mathfrak{B})I.$$

Again, by (1),

$$\begin{aligned} \vdash \mathcal{N}Y \,\&\, \vdash \mathfrak{L}(\mathsf{S}\,Y) &\rightarrow_Y \vdash \mathfrak{M} \,\&\, \mathfrak{A} = \mathsf{S}\,Y \,\&\, \vdash \mathcal{N}Y \\ &\rightarrow_Y \vdash \mathfrak{M} \,\&\, |\mathfrak{A}| = Z_0 \qquad [6.941], \\ &\rightarrow_Y \vdash \mathfrak{B} \qquad [(c)]. \end{aligned}$$

Hence [3.43, 3.41]

$$\vdash \mathcal{N}Y \,\&\, \vdash K(\mathfrak{L}(\mathsf{S}\,Y))X \rightarrow_{X,Y} \vdash K\mathfrak{B}(IX).$$

Therefore [3.65]

(3) $$\vdash \mathcal{N}Y \rightarrow_Y \vdash F(K(\mathfrak{L}(\mathsf{S}\,Y)))(K\mathfrak{B})I.$$

Then the following can be successively deduced:

$$\begin{aligned} &\vdash \mathcal{N}Y \rightarrow_Y \vdash F(K(\mathfrak{L}Y))(K\mathfrak{B})I && [(2), (3), 5.44], \\ &\vdash \mathcal{N}Y \,\&\, \vdash K(\mathfrak{L}Y)X \rightarrow_{Y,X} \vdash K\mathfrak{B}(IX) && [3.63], \\ &\vdash \mathcal{N}Y \,\&\, \vdash \mathfrak{L}Y \rightarrow_Y \vdash \mathfrak{B} && [3.43], \\ &\vdash \mathcal{N}Y \,\&\, \vdash \mathfrak{M} \,\&\, \mathfrak{A} = Y \rightarrow_Y \vdash \mathfrak{B} && [(1)], \\ &\vdash \mathcal{N}\mathfrak{A} \,\&\, \vdash \mathfrak{M} \rightarrow \vdash \mathfrak{B} && [3.2]. \end{aligned}$$

The hypothesis $\vdash \mathcal{N}\mathfrak{A}$ is superfluous by (a), and the theorem is proved.

11.2. THEOREM. *If* $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}$ *are terms such that*

(a) $\vdash \mathfrak{M} \rightarrow \vdash \mathcal{N}\mathfrak{A}$,

and if for $k = 0, 1, 2, \cdots, p$ *we have*

(b) $\vdash \mathfrak{M} \,\&\, \mathfrak{A} = Z_k \rightarrow \vdash \mathfrak{B}$,

then

$$\vdash \mathfrak{M} \,\&\, \mathfrak{A} \dot{-} Z_p = Z_0 \rightarrow \vdash \mathfrak{B}.$$

Proof. For $p=0$ the conclusion is the same as the hypothesis [by 6.93]. Suppose the theorem true for $p=q$; I shall show that it then follows for $p=q+1$.

Suppose then $p=q+1$ and the theorem holds for q. Then the first $q+1$ hypotheses (b) (viz., for $k=0, 1, \cdots, q$) imply that

(1) $$\vdash \mathfrak{M} \;\&\; \mathfrak{A} \dot{-} Z_q = Z_0 \rightarrow \vdash \mathfrak{B}.$$

Let $\mathfrak{M}_1 \equiv \Lambda\mathfrak{M}(Q(\mathfrak{A} \dot{-} Z_p)Z_0)$. Then [5.42]

(2) $$\vdash \mathfrak{M}_1 \rightleftarrows \vdash \mathfrak{M} \;\&\; \mathfrak{A} \dot{-} Z_p = Z_0.$$

Hence

(3) $$\begin{aligned}\vdash \mathfrak{M}_1 \;\&\; \mathfrak{A} \dot{-} Z_q = Z_0 &\rightarrow \vdash \mathfrak{M} \;\&\; \mathfrak{A} \dot{-} Z_q = Z_0\\ &\rightarrow \vdash \mathfrak{B} \qquad [(1)].\end{aligned}$$

On the other hand, by 6.86, the equation

$$|x \dot{-} \sigma(y)| \cdot \|x \dot{-} y\| \cdot \delta_2(x, \sigma(y)) = 0$$

is valid in $\mathfrak{r}$; hence, by 7.51, we know that

$$\vdash \mathcal{N}X \;\&\; \vdash \mathcal{N}Y \rightarrow_{X,Y} |X \dot{-} \mathfrak{S}Y| \cdot \|X \dot{-} Y\| \cdot \Delta_2 X(\mathfrak{S}Y) = Z_0.$$

In this we can substitute [2.61] $\mathfrak{A}$ for X and Z_q for Y; the result, by (a), 5.27, 3.44, shows that

$$|\mathfrak{A} \dot{-} Z_p| \cdot \|\mathfrak{A} \dot{-} Z_q\| \cdot \Delta_2\mathfrak{A}Z_p = Z_0.$$

Therefore [(2), 7.63]

$$\vdash \mathfrak{M}_1 \rightarrow \|\mathfrak{A} \dot{-} Z_q\| \cdot \Delta_2\mathfrak{A}Z_p = Z_0.$$

Hence

(4) $$\begin{aligned}\vdash \mathfrak{M}_1 \;\&\; |\mathfrak{A} \dot{-} Z_q| = Z_0 &\rightarrow \Delta_2\mathfrak{A}Z_p = Z_0 \qquad [7.63],\\ &\rightarrow \mathfrak{A} = Z_p \qquad [7.62].\end{aligned}$$

Thus

(5) $$\begin{aligned}\vdash \mathfrak{M}_1 \;\&\; |\mathfrak{A} \dot{-} Z_q| = Z_0 &\rightarrow \vdash \mathfrak{M} \;\&\; \mathfrak{A} = Z_p \qquad [(2), (4)],\\ &\rightarrow \vdash \mathfrak{B} \qquad [(b)],\end{aligned}$$

and thus [(3), (5), 11.1]

$$\vdash \mathfrak{M}_1 \rightarrow \vdash \mathfrak{B}.$$

The conclusion of the theorem follows by (2).

11.3. THEOREM. *If* $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}$, *are terms such that*

(a) $\vdash \mathfrak{M} \rightarrow \vdash \mathcal{N}\mathfrak{A}$,

(b) *for* $k=0, 1, \cdots, p$, $\vdash \mathfrak{M} \;\&\; \mathfrak{A} = Z_k \rightarrow \vdash \mathfrak{B}$,

(c) $\vdash \mathfrak{M} \;\&\; |\mathfrak{A} \dot{-} Z_p| = Z_0 \rightarrow \vdash \mathfrak{B}$,

then

$$\vdash \mathfrak{M} \rightarrow \vdash \mathfrak{B}.$$

Proof. This follows by 11.2 and 11.1.

11.4. THEOREM. *If* $\mathfrak{A}, \mathfrak{B}_0, \cdots, \mathfrak{B}_p, \mathfrak{C}, \mathfrak{M}$ *are* $(X_1, \cdots, X_m)$*-terms, of which* $\mathfrak{A}, \mathfrak{B}_0, \cdots, \mathfrak{B}_p, \mathfrak{C}$ *are formalizations of numerical expressions, and* $\mathfrak{Q}$ *is a constant such that*

(a) $\vdash \mathfrak{M} \rightarrow_X \vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m$,

(b) $\vdash \mathfrak{M} \,\&\, \mathfrak{A} = Z_k \rightarrow_X \vdash \mathfrak{Q}\mathfrak{B}_k$ $(k = 0, 1, 2, \cdots, p)$,

(c) $\vdash \mathfrak{M} \,\&\, |\mathfrak{A} \dot{-} Z_p| = Z_0 \rightarrow_X \vdash \mathfrak{Q}\mathfrak{C}$;

and if $\mathfrak{G}$ *is defined by*

$$\mathfrak{G} \equiv \lambda^m X_1 \cdots X_m.\left[\sum_{k=0}^{p} |\, \Delta_2 \mathfrak{A} Z_k| \cdot \mathfrak{B}_k + ||\mathfrak{A} \dot{-} Z_p|| \cdot \mathfrak{C}\right];$$

then

$$\vdash \mathfrak{M} \rightarrow_X \vdash \mathfrak{Q}(\mathfrak{G}X_1 \cdots X_m).$$

Proof(59). Since $\mathfrak{G}$ is the formalization of the $\mathfrak{g}$ of 6.98 [by 7.52], we have [by 7.51, 7.22]

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m \rightarrow |\Delta_2 \mathfrak{A} Z_k| \cdot \Delta_2(\mathfrak{G}X_1 \cdots X_m)\mathfrak{B}_k = Z_0.$$

Therefore [(a); 7.62; 7.63]

(1) $$\vdash \mathfrak{M} \,\&\, \mathfrak{A} = Z_k \rightarrow \mathfrak{G}X_1 \cdots X_m = \mathfrak{B}_k.$$

Similarly,

$$\vdash \mathcal{N}X_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}X_m \rightarrow ||\mathfrak{A} \dot{-} Z_p|| \cdot \Delta_2(\mathfrak{G}X_1 \cdots X_m)\mathfrak{C} = Z_0.$$

Hence [(a); 7.63]

(2) $$\vdash \mathfrak{M} \,\&\, |\,\mathfrak{A} \dot{-} Z_p| = Z_0 \rightarrow \mathfrak{G}X_1 \cdots X_m = \mathfrak{C}.$$

The theorem now follows by 11.3, 7.22.

12. Formalization of the axiom enumeration.

THEOREM. *If* $\mathcal{A} \equiv \mathfrak{F}(\alpha)$, *then* $\vdash \mathcal{N}X \rightarrow_X T(\mathcal{A}X)$.

Proof. Let $\mathcal{A}_1 \equiv \mathfrak{F}(\alpha_1)$. Then, since for $k = 0, 1, \cdots, s$,

$$T(\mathcal{A}_1 Z_k) = T Z_{\alpha_1(k)} \qquad [7.54],$$

(59) The referee has pointed out that the proof as given is not valid for the case that $\mathfrak{A}, \mathfrak{B}_0, \cdots, \mathfrak{B}_p, \mathfrak{C}$ are unrestricted $(X_1, \cdots, X_m)$-terms such that

$$\vdash \mathfrak{M} \rightarrow_X \vdash \mathcal{N}\mathfrak{A} \,\&\, \vdash \mathcal{N}\mathfrak{B}_0 \cdots \vdash \mathcal{N}\mathfrak{B}_p \,\&\, \vdash \mathcal{N}\mathfrak{C};$$

but that the theorem may easily be extended to cover that case—in fact, the theorem may first be established for the case that $\mathfrak{A}, \mathfrak{B}_0, \cdots, \mathfrak{B}_p, \mathfrak{C}$ are indeterminates and the general case derived by substitution. However, this more general theorem is not needed in this paper.

$$T(\mathrm{Z}_{\alpha_1(k)}) = \mathfrak{T}(\alpha_1(k)) \qquad [9.6],$$
$$= A_k \qquad [\text{definition of } \alpha_1],$$

we have

(1) $$\vdash T(\mathcal{A}_1 \mathrm{Z}_k) \qquad (k = 0, 1, \cdots, s).$$

Therefore [11.2]

(2) $$\vdash \mathcal{N}X \,\&\, X \dot{-} \mathrm{Z}_s = \mathrm{Z}_0 \rightarrow \vdash T(\mathcal{A}_1 X).$$

Next, let $\mathfrak{B} \equiv \mathfrak{F}(\beta)$. Then, since $\beta(x) \dot{-} s = 0$, we have [7.51] $\vdash \mathcal{N}X \rightarrow \mathfrak{B}X \dot{-} \mathrm{Z}_s = \mathrm{Z}_0$; also [7.21, 3.63] $\vdash \mathcal{N}X \rightarrow \vdash \mathcal{N}(\mathfrak{B}X)$. Hence by substitution in (2) we have

(3) $$\vdash \mathcal{N}X \rightarrow_X \vdash T(\mathcal{A}_1(\mathfrak{B}X)).$$

Now by (2) of 10.1 and 7.52,

$$\mathcal{A} \equiv \lambda X.\, \sum_{k=0}^{s} \left| \Delta_2 X \mathrm{Z}_k \right| \cdot \mathcal{A}_1 \mathrm{Z}_k + \left\| X \dot{-} \mathrm{Z}_s \right\| \cdot \mathcal{A}_1(\mathfrak{B}X).$$

Hence the theorem follows by 11.4 from (1) and (3).

Remark. This theorem is the only place in the paper where the finiteness of the axiom system plays an essential role.

13. **Formalization of ρ and related functions.** Let $\mathfrak{R}$ be the formalization of the ρ defined in 10.2. We are concerned in this section with the proof that

$$\vdash \mathcal{N}Y_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}Y_n \,\&\, \vdash TY_1 \,\&\, \cdots \,\&\, \vdash TY_n \rightarrow \vdash T(\mathfrak{R}Y_1 \cdots Y_n),$$

under the hypothesis that the rule with which ρ is associated in 10.2 is one of the rules of procedure.

In this section the formalizations of the various functions considered in 10.2 are symbolized as follows:

$$\Phi(\mathfrak{X}) \equiv \lambda^m X_1 \cdots X_m . \mathfrak{F}(\phi(\mathfrak{X}; x_1, \cdots, x_m)),$$
$$\Phi_j \equiv \mathfrak{F}(\phi_j) \qquad (j = 1, 2, \cdots, n),$$
$$\Psi_i \equiv \mathfrak{F}(\psi_i) \qquad (i = 0, 1, 2, \cdots, m).$$

Let us also define

$$\mathfrak{M} \equiv \Lambda_{2n}(\mathcal{N}Y_1) \cdots (\mathcal{N}Y_n)(TY_1) \cdots (TY_n),$$
$$\mathfrak{Y}_i \equiv \Psi_i Y_1 \cdots Y_n \qquad (i = 0, 1, 2, \cdots, m).$$

The course of the proof is as follows: In 13.1 a lemma concerning $\Phi(\mathfrak{X})$ is established. Then in 13.2 it is shown that

$$\vdash \mathfrak{M} \,\&\, \mathfrak{Y}_0 = \mathrm{Z}_0 \rightarrow_Y \vdash T(\mathfrak{R}Y_1 \cdots Y_n),$$

while in 13.3 we shall see that

$$\vdash \mathfrak{M} \;\&\; |\mathfrak{Y}_0| = Z_0 \to_Y \vdash T(\mathfrak{R}Y_1 \cdots Y_n).$$

The theorem concerning $\mathfrak{R}$ then follows by 11.1 and 5.43.

*13.1. LEMMA. *On the assumption that*

$$\vdash \mathcal{N}\mathfrak{U}_1 \;\&\; \cdots \;\&\; \vdash \mathcal{N}\mathfrak{U}_m,$$

it follows that

$$T(\Phi(\mathfrak{X})\mathfrak{U}_1 \cdots \mathfrak{U}_m) = \begin{bmatrix} T\mathfrak{U}_1, \cdots, T\mathfrak{U}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}.$$

Proof. Let

$$\mathfrak{X}' \equiv \Phi(\mathfrak{X})\mathfrak{U}_1 \cdots \mathfrak{U}_m,$$

$$\mathfrak{X}^* \equiv \begin{bmatrix} T\mathfrak{U}_1, \cdots, T\mathfrak{U}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}.$$

Then, by 3.32, it follows that

(1) $$\mathfrak{X}' = \begin{bmatrix} \mathfrak{U}_1, \cdots, \mathfrak{U}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{F}(\phi(\mathfrak{X}; x_1, \cdots, x_m)).$$

To prove the lemma we proceed by intuitive induction on the construction of $\mathfrak{X}$.

1. Suppose $\mathfrak{X} \equiv \mathfrak{C}$, $c \equiv \mathfrak{n}(\mathfrak{C})$, where $\mathfrak{C}$ is a constant. Then [(1), 10.211, 7.53]

$$\mathfrak{X}' = Z_c.$$

Therefore [9.6]

$$T\mathfrak{X}' = \mathfrak{C}$$

$$= \mathfrak{X}^* \qquad [2.6].$$

2. Suppose $\mathfrak{X} \equiv X_i$. Then [(1), 10.212, 7.12]

$$\mathfrak{X}' = \mathfrak{U}_i.$$

Therefore [2.6]

$$T\mathfrak{X}' = T\mathfrak{U}_i = \mathfrak{X}^*.$$

3. Suppose, finally, that $\mathfrak{X} \equiv \mathfrak{X}_1\mathfrak{X}_2$ and that the theorem holds for $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then [(1), 7.52, 10.213]

(2) $$\mathfrak{X}' = M\mathfrak{X}_1'\mathfrak{X}_2'.$$

But [(1), 7.22, hypothesis]

(3) $$\vdash \mathcal{N}\mathfrak{X}_1' \;\&\; \vdash \mathcal{N}\mathfrak{X}_2'.$$

Therefore

$$T\mathfrak{X}' = T\mathfrak{X}_1'(T\mathfrak{X}_2') \qquad [(2), (3), 9.5],$$
$$= \mathfrak{X}_1^*\mathfrak{X}_2^* \qquad [\text{hypothesis of induction}],$$
$$= \mathfrak{X}^* \qquad [2.6],$$

q.e.d.

*13.2. LEMMA. *If* (a) $\vdash \mathfrak{M}$, *and* (b) $\mathfrak{Y}_0 = Z_0$; *then*

$$\vdash T(\mathfrak{R}Y_1 \cdots Y_n).$$

Proof. By (a) and 5.43 it follows that

$$(1) \qquad \vdash \mathcal{N}Y_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}Y_n;$$

hence, by 10.22 (2), 7.51, we can conclude (for $j=1, 2, \cdots, n$) that

$$|\,\mathfrak{Y}_0| \cdot \Delta_2 Y_j(\Phi_j\mathfrak{Y}_1 \cdots \mathfrak{Y}_m) = Z_0.$$

Therefore [(b), 7.63]

$$\Delta_2 Y_j(\Phi_j\mathfrak{Y}_1 \cdots \mathfrak{Y}_m) = Z_0.$$

Hence [7.62]

$$(2) \qquad Y_j = \Phi_j\mathfrak{Y}_1 \cdots \mathfrak{Y}_m.$$

Thus

$$TY_j = T(\Phi_j\mathfrak{Y}_1 \cdots \mathfrak{Y}_m) \qquad [3.2],$$

$$(3) \qquad = \begin{bmatrix} T\mathfrak{Y}_1, \cdots, T\mathfrak{Y}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}_j \qquad [13.1, 7.22](^{60}).$$

But [(a), 5.43] $\vdash TY_j$. Therefore [(3), 3.2]

$$(4) \qquad \vdash \begin{bmatrix} T\mathfrak{Y}_1, \cdots, T\mathfrak{Y}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}_j,$$

for $j=1, 2, \cdots, n$. But by the hypothesis of this section the rule

$$\vdash \mathfrak{X}_1 \,\&\, \cdots \,\&\, \vdash \mathfrak{X}_n \rightarrow_X \vdash \mathfrak{X}_{n+1}$$

is a valid rule of procedure; hence by application of this rule to (4), it follows that

$$(5) \qquad \vdash \begin{bmatrix} T\mathfrak{Y}_1, \cdots, T\mathfrak{Y}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}_{n+1}.$$

---

(60) The hypothesis of 13.1, which in this case is

$$\vdash \mathcal{N}\mathfrak{Y}_1 \,\&\, \cdots \,\&\, \vdash \mathcal{N}\mathfrak{Y}_m,$$

is true by 7.22.

On the other hand, by the definition of ρ (in 10.23) and 7.52,

$$\mathfrak{R}Y_1 \cdots Y_n = |\mathfrak{Y}_0| \cdot (\Phi_{n+1}\mathfrak{Y}_1 \cdots \mathfrak{Y}_m) + ||\mathfrak{Y}_0|| \cdot Z_a,$$

where $a \equiv \mathfrak{n}(A_0)$. However, by (b) it follows [6.941] that

$$|\mathfrak{Y}_0| = Z_1 \; \& \; ||\mathfrak{Y}_0|| = Z_0;$$

and hence [6.91, 6.92] that

(6) $$\mathfrak{R}Y_1 \cdots Y_n = \Phi_{n+1}\mathfrak{Y}_1 \cdots \mathfrak{Y}_m.$$

Therefore

$$\begin{aligned} T(\mathfrak{R}Y_1 \cdots Y_n) &= T(\Phi_{n+1}\mathfrak{Y}_1 \cdots \mathfrak{Y}_m) \\ &= \begin{bmatrix} T\mathfrak{Y}_1, \cdots, T\mathfrak{Y}_m \\ X_1, \cdots, X_m \end{bmatrix} \mathfrak{X}_{n+1} \qquad [13.1]. \end{aligned}$$

From this and (5) the lemma follows by 3.2.

*13.3. Lemma. *If* (a) $\vdash \mathfrak{M}$, *and* (b) $|\mathfrak{Y}_0| = Z_0$, *then*

$$\vdash T(\mathfrak{R}Y_1 \cdots Y_n).$$

Proof. From the definition of $\mathfrak{R}$ and (b) it follows by 6.94, 6.92, 6.91 (cf. derivation of 13.2 (6)) that

$$\mathfrak{R}Y_1 \cdots Y_n = Z_a,$$

where $a \equiv \mathfrak{n}(A_0)$. Therefore [9.6]

$$T(\mathfrak{R}Y_1 \cdots Y_n) = A_0.$$

The lemma follows from

$$\vdash A_0,$$

which holds by definition of A_0.

13.4. Theorem. *If* $\mathfrak{R} \equiv \mathfrak{F}(\rho)$ *where* ρ *is defined as in* 10.2 *from the valid rule of procedure*

$$\vdash \mathfrak{X}_1 \; \& \cdots \& \vdash \mathfrak{X}_n \rightarrow_X \vdash \mathfrak{X}_{n+1},$$

then

$$\vdash \mathsf{N}Y_1 \; \& \cdots \& \vdash \mathsf{N}Y_n \; \& \vdash TY_1 \; \& \cdots \& \vdash TY_n \rightarrow_Y \vdash T(\mathfrak{R}Y_1 \cdots Y_n).$$

Proof. See the introduction to the section.

14. **Formalization of θ; the function Θ.** We turn now to the study of the constant Θ defined by

$$\Theta \equiv \mathfrak{F}(\theta).$$

The principal theorem to be established is that

$$\vdash \mathcal{N}X \rightarrow_X \vdash T(\Theta X),$$

which is proved in 14.6.

Since Θ satisfies a course-of-values recursion, it is natural to approach our theorems by way of a "course-of-values function" $\mathfrak{M}$, such that $\vdash \mathfrak{M}X$ means intuitively "for all $Y \leqq X$, $\vdash T(\Theta X)$." Such an $\mathfrak{M}$ can be formally defined by means of an auxiliary term $\mathfrak{L}$, thus:

$$\mathfrak{L} \equiv \lambda^2 XY . \Lambda(\mathcal{N}Y)(Q(Y \dot{-} X)Z_0),$$
$$\mathfrak{M} \equiv \lambda X . F(\mathfrak{L}X)T\Theta.$$

Other functions used in this section are symbolized thus:

$$\mathfrak{P}_j \equiv \mathfrak{F}(\pi_j) \qquad (j = 0, 1, \cdots, n),$$
$$\mathfrak{R}_k \equiv \mathfrak{F}(\rho_k) \qquad (k = 0, 1, \cdots, r-1),$$

where ρ_k is the recursive function associated with the kth rule as in 10.3, and

$$\Gamma \equiv \mathfrak{F}(\gamma).$$

The development of the proof is as follows. The idea is to prove by induction that

$$\vdash \mathcal{N}X \rightarrow_X \vdash \mathfrak{M}X.$$

The two parts of this induction are in 14.1 and 14.5. The second part splits into two cases of which the first splits again into two subcases; these cases are taken care of in 14.2–14.3.

An additional, but obvious, property of Θ is proved in 14.7.

14.1. Lemma. $\vdash \mathfrak{M}Z_0$.

Proof. Suppose that

(1) $$\vdash \mathfrak{L}Z_0 Y.$$

Then [5.42] $\vdash \mathcal{N}Y$, and $Y \dot{-} Z_0 = Z_0$. Therefore [6.93]

(2) $$Y = Z_0.$$

On the other hand, by 10.3 (8) and 7.54

$$\Theta Z_0 = Z_a,$$

where $a \equiv \mathfrak{n}(A_0)$. Therefore [9.6] $T(\Theta Z_0) = A_0$. Hence [definition of A_0, 3.2] q.e.d.

$$\vdash T(\Theta Z_0),$$

and therefore [(2)]

(3) $$\vdash T(\Theta Y).$$

Since (3) has been deduced from the assumption (1), it follows by 3.65 that

$$\begin{aligned} &\vdash F(\mathfrak{L}\mathfrak{Z}_0)T\Theta \\ &= \vdash \mathfrak{M}\mathfrak{Z}_0 \qquad \text{[definition]}, \end{aligned}$$

q.e.d.

*14.2. LEMMA. *If* (a) $\vdash \mathcal{N}X$, (b) $\vdash \mathfrak{M}X$, (c) $\mathfrak{P}_0(\mathfrak{S}X) = \mathfrak{Z}_h$ $(0 \leqq h < r)$; *then*

$$\vdash T(\Theta(\mathfrak{S}X)).$$

Proof. Let $\mathfrak{Y} \equiv \mathfrak{S}X$. Then, by 10.3 (9) and 7.51 we have

(1) $$\begin{aligned} \Theta\mathfrak{Y} = \sum_{k=0}^{r-1} \left| \Delta_2(\mathfrak{P}_0\mathfrak{Y})\mathfrak{Z}_k \right| \cdot \mathfrak{R}_k(\Theta(\mathfrak{P}_1\mathfrak{Y})) \cdots (\Theta(\mathfrak{P}_n\mathfrak{Y})) \\ + \left\| \mathfrak{P}_0\mathfrak{Y} \dot{-} \mathfrak{Z}_{r-1} \right\| \cdot \mathcal{A}(\Gamma\mathfrak{Y}). \end{aligned}$$

To this now apply (c), using the fact that [6.95]

$$\begin{aligned} \left| \Delta_2\mathfrak{Z}_h\mathfrak{Z}_k \right| &= \mathfrak{Z}_0 \qquad \text{for } h \neq k, \\ \left| \Delta_2\mathfrak{Z}_h\mathfrak{Z}_h \right| &= \mathfrak{Z}_1, \end{aligned}$$

and [6.93]

$$\left\| \mathfrak{Z}_h \dot{-} \mathfrak{Z}_{r-1} \right\| = \mathfrak{Z}_0.$$

Then [6.91, 6.92]

(2) $$\Theta\mathfrak{Y} = \mathfrak{R}_h(\Theta(\mathfrak{P}_1\mathfrak{Y})) \cdots (\Theta(\mathfrak{P}_n\mathfrak{Y})).$$

Now by 10.3 (10) and 7.51 it follows that

$$\mathfrak{P}_j\mathfrak{Y} \dot{-} X = \mathfrak{Z}_0 \qquad (j = 1, 2, \cdots, n),$$

also [7.22, (a), 3.62, 3.63] $\vdash \mathcal{N}(\mathfrak{P}_j\mathfrak{Y})$. Therefore [5.42, definition of $\mathfrak{L}$]

$$\vdash \mathfrak{L}X(\mathfrak{P}_j\mathfrak{Y}).$$

Hence [(b), 3.63]

(3) $$\vdash T(\Theta(\mathfrak{P}_j\mathfrak{Y})).$$

But [7.22, (a), 3.62, 3.63]

(4) $$\vdash \mathcal{N}(\Theta(\mathfrak{P}_j\mathfrak{Y})).$$

Therefore [(2), (3), (4), 13.4]

$$\vdash T(\Theta\mathfrak{Y}),$$

q.e.d.

*14.3. LEMMA. *If* (a) $\vdash \mathcal{N}X$, (b) $\vdash \mathfrak{M}X$, (c) $\left| \mathfrak{P}_0(\mathfrak{S}X) \dot{-} \mathfrak{Z}_{r-1} \right| = \mathfrak{Z}_0$, *then* $\vdash T(\Theta(\mathfrak{S}X))$.

Proof. As in 14.2 we conclude from 10.3 (10) and (c), using theorems of 6.9, 7.5 and 7.6, that

$$\Theta(\mathfrak{S}X) = \mathcal{A}(\Gamma(\mathfrak{S}X)).$$

By (a), 3.62, 3.63, 7.22 it follows that

$$\vdash \mathcal{N}(\Gamma(\mathfrak{S}X)).$$

Therefore [§12]

$$T(\mathcal{A}(\Gamma(\mathfrak{S}X))).$$

The lemma follows by 3.2.

*14.4. Lemma. *If* (a) $\vdash \mathcal{N}X$, (b) $\vdash \mathfrak{M}X$; *then*

$$\vdash T(\Theta(\mathfrak{S}X)).$$

Proof. [14.2, 14.3, 7.22, 11.3].

*14.5. Lemma. *If* (a) $\vdash \mathcal{N}X$, *and* (b) $\vdash \mathfrak{M}X$, *then* $\vdash \mathfrak{M}(\mathfrak{S}X)$.

Proof. Suppose that

(1) $$\vdash \mathfrak{L}(\mathfrak{S}X)Y.$$

Then [5.42]

(2) $$\vdash \mathcal{N}Y,$$

(3) $$Y \dot{-} \mathfrak{S}X = Z_0.$$

Now in 11.2 it has been shown that from (a) and (2) it follows that

$$|\, Y \dot{-} \mathfrak{S}X| \cdot \|Y \dot{-} X\| \cdot \Delta_2 Y(\mathfrak{S}X) = Z_0.$$

Therefore [(3), 7.63]

(4) $$\|Y \dot{-} X\| \cdot \Delta_2 Y(\mathfrak{S}X) = Z_0.$$

Suppose then that

(5) $$Y \dot{-} X = Z_0.$$

Then [(2), 5.42] $\vdash \mathfrak{L}XY$. Therefore [(b), 3.63]

(6) $$\vdash T(\Theta Y),$$

on the assumption that (1) and (5) hold. On the other hand, suppose that

(7) $$|\, Y \dot{-} X| = Z_0.$$

Then [(4), 7.63, 7.62] $Y = \mathfrak{S}X$. Therefore [14.4]

$$\vdash T(\Theta Y).$$

This has been shown to follow from (1) and (5) and also from (1) and (7). Hence, by 11.1, it follows from (1) only. The lemma now follows by definition of $\mathfrak{M}$ and 3.65.

14.6. THEOREM. $\vdash \mathcal{N}X \rightarrow_X \vdash T(\Theta X)$.

Proof. From 14.1, 14.5 it follows by 5.44 that

$$\vdash \mathcal{N}X \rightarrow_X \vdash \mathfrak{M}X.$$

Therefore [3.63, 5.42] $\vdash \mathcal{N}X$ & $\vdash \mathcal{N}Y$ & $Y \dot{-} X = Z_0 \rightarrow_{XY} \vdash T(\Theta Y)$. In this substitute X for Y and we have the theorem.

14.7. THEOREM. *If* $\vdash \mathfrak{A}$, *then there exists an* n *such that* $\mathfrak{A} = T(\Theta Z_n)$.

Proof. Let $a \equiv \mathfrak{n}(\mathfrak{A})$. From 10.3 it is evident that if a proof of $\vdash \mathfrak{A}$ is at hand we can find an n such that $a = \theta(n)$. Then [7.54] $Z_a = \Theta Z_n$. Therefore [9.6]

$$\mathfrak{A} = TZ_a = T(\Theta Z_n).$$

PART V. FINAL DEDUCTION

15. **Proof of inconsistency.** We are now in a position to set up the Richard paradox within the system. The following explanations may help in the understanding of the derivation. The term $\mathfrak{H}$, which in the first two lemmas is taken as unspecified, is defined to be $F\mathcal{N}\mathcal{N}$ in 15.3; so that $\mathfrak{H}$ represents the property of being a numerical function. Then the function $\omega(x)$ introduced in 15.1 is a recursive enumeration (with repetitions, of course) of the numbers associated with all the numerical functions. On the other hand $\mathfrak{U}$, introduced in 15.2, is an enumeration within the system of the numerical functions themselves; thus $\mathfrak{U}$ represents the sequence $\phi_1(x)$, $\phi_2(x)$, $\cdots$ of §1. The diagonal function, $f(x)$ in §1, is the $\mathfrak{G}$ of §15.3; and the number of $\mathfrak{G}$ in the enumeration $\mathfrak{U}$ is Z_g.

15.1. LEMMA. *If* $\mathfrak{H}$, $\mathfrak{A}$ *are terms such that*

$$\vdash \mathfrak{H}\mathfrak{A}, \tag{1}$$

then there exist recursive functions $\omega(x)$, $\sigma'(x)$ *such that*:

(i) *if* $\mathfrak{X}$ *is any term such that we have a proof that*

$$\vdash \mathfrak{H}\mathfrak{X}, \tag{2}$$

then we can find an integer n *such that*

$$\mathfrak{n}(\mathfrak{X}) = \omega(n);$$

(ii) *we have further*

$$\mu(\mathfrak{n}(\mathfrak{H}), \omega(x)) = \theta(\sigma'(x)). \tag{3}$$

Remark. Property (ii) is a kind of converse of (i); for if $\mathfrak{n}(\mathfrak{X})=\omega(n)$, then by (ii) and 3.521, $\mathfrak{n}(\mathfrak{H}\mathfrak{X})=\theta(\sigma'(n))$, and so, by the construction of θ [10.3], $\vdash \mathfrak{H}\mathfrak{X}$. Thus ω enumerates all and only those integers which are numbers of terms $\mathfrak{X}$ for which (2) holds.

Proof. Let $h\equiv\mathfrak{n}(\mathfrak{H})$,

$$a \equiv \mathfrak{n}(\mathfrak{A}), \tag{4}$$

and let b be the number, obtained from a proof of (1) as in 10.3, such that [3.521]

$$\mathfrak{n}(\mathfrak{H}\mathfrak{A}) = \mu(h, a) = \theta(b). \tag{5}$$

Let $\sigma'(x)$ be the recursive function such that[61]

$$\sigma'(x) = \begin{cases} x & \text{if} \quad \mu_1(\theta(x)) = h, \\ b & \text{if} \quad \mu_1(\theta(x)) \neq h; \end{cases} \tag{6}$$

and then let

$$\omega(x) \equiv \mu_2(\theta(\sigma'(x))). \tag{7}$$

Now suppose that (2) holds and let

$$m \equiv \mathfrak{n}\,(\mathfrak{X}).$$

Let n be such that

$$\mathfrak{n}(\mathfrak{H}\mathfrak{X}) = \mu(h, m) = \theta(n). \tag{8}$$

Then [8.1 (3)] $\mu_1(\theta(n))=h$. Therefore [(6)]

$$\sigma'(n) = n. \tag{9}$$

Therefore

$$\begin{aligned} \omega(n) &= \mu_2(\theta(n)) && [(7)], \\ &= m && [(8),\ 8.1,\ (4)], \end{aligned}$$

which proves (i).

To prove (ii), we first show that

$$\mu_1(\theta(\sigma'(x))) = h. \tag{10}$$

If $\mu_1(\theta(x))=h$, this follows at once by (6); if not, then it follows from (6) and (5) by 8.1 (3). By 6.86 this establishes (10) as a theorem in $\mathfrak{r}$. But since [(4), 3.5] $h>0$, it then follows that [8.1, (8)]

(61) An explicit definition of σ' is

$$\sigma'(x) \equiv \mathfrak{B}\,|\,\mathfrak{a}\,|\cdot x + ||\mathfrak{a}||\cdot b$$

where $\mathfrak{a}\equiv\delta_2(h,\ \mu_1(\theta(x)))$.

$$\mu_3(\theta(\sigma'(x))) = 0.$$

Therefore [8.1, (7)]

$$\theta(\sigma'(x)) = \mu(h, \omega(x)),$$

q.e.d.

15.2. LEMMA. *If* $\mathfrak{H}$, $\mathfrak{A}$, σ', ω *are as in* 15.1, *and if*

(1) $$\Omega \equiv \mathfrak{F}(\omega),$$

(2) $$\mathfrak{U} \equiv \lambda X . T(\Omega X);$$

then

(i) *if* $\mathfrak{X}$ *is any term such that*

(3) $$\vdash \mathfrak{H}\mathfrak{X},$$

then from a proof of (3) *we can find an* n *such that*

$$\mathfrak{X} = \mathfrak{U} Z_n;$$

(ii) $\vdash F\mathcal{N}\mathfrak{H}\mathfrak{U}$.

Remark. Here again (ii) implies the converse of (i) [3.63, 5.27]. $\mathfrak{U}$ gives a formal enumeration, not of numbers of the terms satisfying (3), but of the terms themselves.

Proof. Suppose that (3) holds and set

$$m \equiv \mathfrak{n}(\mathfrak{X}).$$

Then by 15.1 we can find an n such that

$$m = \omega(n).$$

Therefore [7.53]

$$Z_m = \Omega Z_n.$$

Hence

$$\begin{aligned} \mathfrak{X} &= T Z_m = T(\Omega Z_n) && [9.6], \\ &= \mathfrak{U} Z_n && [(2)]. \end{aligned}$$

This proves (i).

Let us turn to (ii). As in 15.1 let

$$h \equiv \mathfrak{n}(\mathfrak{H}), \qquad \Sigma \equiv \mathfrak{F}(\sigma').$$

Then by 15.1 (ii) and 7.51

$$\vdash \mathcal{N}X \rightarrow_X \Theta(\Sigma X) = M Z_h(\Omega X).$$

Therefore

$$\begin{aligned}(4)\qquad \vdash \mathcal{N}X \to_X T(\Theta(\Sigma X)) &= T(M Z_h(\Omega X)) \\ &= T Z_h(T(\Omega X)) \quad [9.5, 5.27, 7.22], \\ &= \mathfrak{H}(\mathfrak{U}X) \quad [9.6, (2)].\end{aligned}$$

But

$$\begin{aligned}(5)\qquad \vdash \mathcal{N}X &\to_X \vdash \mathcal{N}(\Sigma X) \quad [7.22], \\ &\to_X \vdash T(\Theta(\Sigma X)) \quad [14.6].\end{aligned}$$

Therefore [(4), (5)]

$$\vdash \mathcal{N}X \to_X \vdash \mathfrak{H}(\mathfrak{U}X).$$

(ii) now follows by 3.65.

15.3. Lemma. *Let* $\mathfrak{U}$ *be defined as in* 15.2 *with*

$$\mathfrak{H} \equiv F\mathcal{N}\mathcal{N}, \qquad \mathfrak{A} \equiv \mathsf{S},$$

and let $\mathfrak{G} \equiv \lambda X.\mathsf{S}(\mathfrak{U}XX)$; *then we may find an integer g such that*

(i) $\mathfrak{G} = \mathfrak{U}Z_g$, *and*

(ii) $\mathfrak{U}Z_gZ_g = \mathsf{S}(\mathfrak{U}Z_gZ_g)$.

Proof. The only hypothesis on $\mathfrak{H}$ and $\mathfrak{A}$ in 15.1 and 15.2 was 15.1 (1); and this is fulfilled for the present specialized $\mathfrak{H}$ and $\mathfrak{A}$ by 3.62. Hence, by 15.2 (ii),

$$\vdash F\mathcal{N}(F\mathcal{N}\mathcal{N})\mathfrak{U}.$$

Therefore

$$\begin{aligned}(1)\qquad \vdash \mathcal{N}X &\to_X \vdash F\mathcal{N}\mathcal{N}(\mathfrak{U}X) \quad [3.63], \\ &\to_X \vdash \mathcal{N}(\mathfrak{U}XX) \quad [3.63], \\ &\to_X \vdash \mathcal{N}(\mathsf{S}(\mathfrak{U}XX)) \quad [3.62, 3.63], \\ &\to_X \vdash \mathcal{N}(\mathfrak{G}X) \quad [\text{definition of } \mathfrak{G}].\end{aligned}$$

Therefore [3.65]

$$\vdash F\mathcal{N}\mathcal{N}\mathfrak{G}.$$

From this and 15.2 (i) it follows that there exists an integer g such that (i) holds. From (i) we obtain (ii) by applying both sides of (i) to Z_g and using the definition of $\mathfrak{G}$ in connection with 3.22.

15.4. Theorem. $Z_0 = Z_1$.

Proof. Let $\mathfrak{U}$, $\mathfrak{G}$ and g be as in 15.3 and let

$$\mathfrak{B} \equiv \mathfrak{U}Z_gZ_g.$$

Then [15.3 (ii)] $\mathfrak{B} = \mathsf{S}\mathfrak{B}$. Therefore

(1) $$\mathfrak{B} \dot{-} \mathfrak{B} = \mathsf{S}\mathfrak{B} \dot{-} \mathfrak{B}.$$

But [15.3 (1)] $\vdash \mathcal{N}\mathfrak{B}$. Therefore [6.93]

$$\mathfrak{B} \dot{-} \mathfrak{B} = Z_0,$$
$$\mathsf{S}\mathfrak{B} \dot{-} \mathfrak{B} = Z_1.$$

The theorem now follows by (1).

15.5. THEOREM. *If* $\mathfrak{A}$ *is a term, then* $\vdash \mathfrak{A}$.

Proof. By 15.4 $Z_0 = Z_1$. Therefore [3.2]

$$D_2 A_0 \mathfrak{A} Z_0 = D_2 A_0 \mathfrak{A} Z_1.$$

Therefore [5.31, 5.32]

$$A_0 = \mathfrak{A}.$$

But [3.14] $\vdash A_0$, hence [3.2]

$$\vdash \mathfrak{A},$$

q.e.d.

PENNSYLVANIA STATE COLLEGE,
STATE COLLEGE, PA.