ON THE NUMBER OF PARTITIONS OF A NUMBER INTO UNEQUAL PARTS[1]

BY

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1. **Introduction.** Let $q(n)$ be the number of partitions of an integer n into unequal parts, or into odd parts[2]. Then

$$
(1.1)\qquad
\begin{aligned}
f(x) &= 1 + \sum_{n=1}^{\infty} q(n)x^n = (1+x)(1+x^2)(1+x^3)\cdots \\
&= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.
\end{aligned}
$$

Hardy and Ramanujan[3] indicated that by their fundamental analytic method one can obtain the following result:

$$
\begin{aligned}
q(n) = {} & \frac{1}{2^{1/2}} \frac{d}{dn} J_0\left[i\pi\left\{\tfrac{1}{3}\left(n+\tfrac{1}{24}\right)\right\}^{1/2}\right] \\
& + 2^{1/2}\cos\left(\tfrac{2}{3}\pi n - \tfrac{1}{9}\pi\right)\frac{d}{dn} J_2\left[\tfrac{1}{3} i\pi\left\{\tfrac{1}{3}\left(n+\tfrac{1}{24}\right)\right\}^{1/2}\right] + \cdots \\
& + \text{to } \left[\alpha n^{1/2}\right] \text{ terms} + O(1)
\end{aligned}
$$

where α is an arbitrary constant. This result is less satisfactory than that concerning the number $p(n)$ of partitions (unrestricted) of n, since in the latter case the error term approaches zero with increasing n. Recently Rademacher[4] obtained an equality for $p(n)$. The object of the present paper is to find an equality for $q(n)$. The work of this paper is a straightforward application of Hardy-Ramanujan's method with two modifications. These modifications are Kloosterman's sum and Rademacher's "Farey dissection of infinite order."

The present method may also be applied to find the explicit formula for

$$\sum_{x=1}^{[n^{1/2}]} p(n-x^2)$$

where $p(n)$ is the number of unrestricted partitions of n.

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(1) This paper was accepted by Acta Arithmetica before the war.

(2) Cf. MacMahon, *Combinatory Analysis*, vol. 2, 1916, p. 11.

(3) Proceedings of the London Mathematical Society, (2), vol. 17 (1918), pp. 75–115.

(4) Proceedings of the London Mathematical Society, (2), vol. 43 (1937), pp. 241–254.

2. **Statement of the result.** Let

$$\epsilon_{h,k} = \begin{cases} \exp\left(-\pi i\left(\frac{(h'^2-1)}{8}\left(\frac{1-hh'}{k}-1\right)+\frac{h'(1-hh')}{8k}\right.\right. \\ \qquad\left.\left.+\frac{1}{24}\left(k+\frac{1-hh'}{k}\right)(hh'^2-h'-h)\right)\right), & \text{for } 2\mid k, \\ \exp\left(\frac{\pi i}{24}\left(k+\frac{1-hh'}{k}\right)(h+h'-h^2h')\right), & \text{for } 2\nmid k,\ 2\nmid h, \\ \exp\left(-\frac{\pi i}{8}\left(k^2-1-hk+\frac{1}{3}(h+h')\left(hh'k-\frac{hh'-1}{k}\right)\right)\right), & \\ & \text{for } 2\nmid k,\ 2\mid h, \end{cases}$$

and

$$\omega_{h,k} = \begin{cases} \epsilon_{h,k}\exp\left(-\frac{\pi i}{12k}(h+h')\right), & \text{for } 2\mid k, \\ \epsilon_{h,k}\exp\left(-\frac{\pi i}{24k}(2h-h')\right), & \text{for } 2\nmid k, \end{cases}$$

where $hh'\equiv 1 \pmod{k}$, $h\equiv h' \pmod{2}$.

THEOREM. *The number of partitions of an integer n into unequal parts is given by*

$$q(n) = \frac{1}{2^{1/2}}\sum_{k=1,k\text{ odd}}^{\infty}\ \sum_{(h,k)=1,0<h\leq k}\omega_{h,k}e^{-2\pi ihn/k}\frac{d}{dn}J_0\left(\frac{i\pi}{k}\left\{\tfrac{2}{3}\left(n+\tfrac{1}{24}\right)\right\}^{1/2}\right),$$

where $J_0(x)$ is the Bessel function of the 0*th order.*

3. **Farey dissection.** By means of Cauchy's integral formula we obtain for (1.1)

$$q(n) = \frac{1}{2\pi i}\int_C \frac{f(x)}{x^{n+1}}\,dx.$$

The path of integration may be the circle defined as $|x| = e^{-2\pi N^{-2}}$ where N is a certain positive integer at our disposal. In the usual way we divide the circle into Farey arcs $\xi_{h,k}$ of order N. The Farey arc $\xi_{h,k}$ is defined by

$$x = \exp(2\pi ih/k - 2\pi N^{-2} + 2\pi i\vartheta), \qquad (h, k) = 1, \tag{3.1}$$

and

$$-\vartheta_1(h, k) = \frac{h+h_1}{k+k_1} - \frac{h}{k} \leq \vartheta \leq \frac{h+h_2}{k+k_2} - \frac{h}{k} = \vartheta_2(h, k) \tag{3.2}$$

where h_1/k_1, h/k, h_2/k_2 are three consecutive fractions in the Farey sequence of order N. It is well known that

$$(3.3)\qquad \frac{1}{k(N+k)} \leqq \vartheta_1(h, k) < \frac{1}{k(N+1)}, \frac{1}{k(N+k)} \leqq \vartheta_2(h, k) < \frac{1}{k(N+1)}.$$

We obtain then

$$(3.4)\qquad q(n) = \frac{1}{2\pi i} \sum_{(h,k)=1, 0<h\leqq k\leqq N} \int_{\xi_{h,k}} \frac{f(x)}{x^{n+1}}\, dx.$$

Let I_1 and I_2 denote the sums of those terms satisfying $2|k$, and $2\nmid k$, respectively. Then, by (3.4), we have

$$(3.5)\qquad q(n) = I_1 + I_2.$$

4. Lemmas on Kloosterman's sums.

LEMMA 4.1([5]). *Let*

$$g(N, \vartheta, h, k) = \begin{cases} 1 & \text{for } -\vartheta_1(h, k) \leqq \vartheta \leqq \vartheta_2(h, k), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$g = \sum_{r=1}^{k} b_r e^{2\pi i r h'/k}$$

where h' iz an integer satisfying

$$hh' \equiv 1 \pmod{k},$$

and b_r is independent of h and

$$\sum_{r=1}^{k} |b_r| < \log 4k.$$

LEMMA 4.2. *Let a be an absolute constant. Then*

$$\sum_{0<h\leqq ak, (h,ak)=1, h\equiv l(a)} \exp\left(\frac{2\pi i}{ak}(nk + mh')\right) = O(k^{2/3+\epsilon}(n, k)^{1/3}).$$

LEMMA 4.3. *If k is even and $\omega_{h,k}$ as defined in* §2, *then*

$$S_k = \sum_{1\leqq h\leqq k, (h,k)=1, hh'\equiv 1\,(k)} \omega_{h,k} e^{2\pi i(nh+mh')/k} = O(k^{2/3+\epsilon}(n, k)^{1/3}).$$

Proof. For the sake of simplicity I give here only the proof of the case $24|k$.

([5]) T. Estermann, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 7 (1929), pp. 93, 94.

Then

$$S_k = \sum_{1 \leqq l \leqq 24, (l,24)=1} \sum_{1 \leqq h \leqq k, (h,k)=1, hh' \equiv 1, h \equiv l \ (24)} \omega_{h,k} e^{2\pi i(nh+mh')/k}.$$

The inner sum becomes a Kloosterman's sum as in Lemma 4.2. Therefore we have

$$S_k = O(k^{2/3+\epsilon}(n, k)^{1/3}).$$

As to the proof of the other cases, nothing is difficult but a little complicated, and the following fact is used: let

$$F(h, k) = \omega_{h,k} e^{2\pi i(nh+mh')/k};$$

then $F(h+k, k) = F(h, k)$.

LEMMA 4.4. *Let $2 \nmid k$ and $\omega_{h,k}$ be as defined in* §2, *then*

$$S = \sum_{1 \leqq h \leqq k, (h,k)=1, hh' \equiv 1 \ (k), h' \text{ odd}} \omega_{h,k} e^{\pi i(2nh+mh')/k} = O(k^{2/3+\epsilon}(h, k)^{1/3}).$$

The proof is similar to that of Lemma 4.3, only notice that

$$S = \sum_{1 \leqq h < 2k, (h,2k)=1, hh' \equiv 1 \ (2k)}$$

5. **Lemmas from the theory of the linear transformation of the elliptic modular functions.**

LEMMA 5.1. *Suppose that $2 \nmid h$, $2 | k$; that h' is a positive integer satisfying $hh' \equiv 1$ (mod k); that $\omega_{h,k}$ is defined in* §2; *and that*

$$x = \exp\left(-\frac{2\pi z}{k} + \frac{2h\pi i}{k}\right), \qquad x' = \exp\left(-\frac{2\pi}{kz} - \frac{2h'\pi i}{k}\right),$$

where the real part of z is positive. Then

$$f(x) = \omega_{h,k} \exp\left(-\frac{\pi}{12kz} + \frac{\pi z}{12k}\right) f(x').$$

Proof. If we take $a = h$, $b = -k$, $c = (1-hh')/k$, $d = h'$, so that $ad - bc = 1$, and write

$$x = q^2 = e^{2\pi i\tau}, \qquad x' = Q^2 = e^{2\pi iT},$$
$$\tau = (h + iz)/k, \qquad T = (-h' + i/z)/k,$$

then we can easily verify that

$$T = \frac{c + d\tau}{a + b\tau}.$$

Also, in the notation of Tannery and Molk, we obtain

$$f(x) = \frac{1}{2^{1/3}} q^{-1/12} \frac{\phi(\tau)}{\chi(\tau)}, \qquad f(x') = \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(T)}{\chi(T)}.$$

Then

$$\begin{aligned} f(x') &= \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(T)}{\chi(T)} = \exp\left(\pi i\left(\tfrac{1}{8}(d^2-1)(c-1) + \frac{cd}{8}\right.\right. \\ &\qquad \left.\left. - \frac{(b-c)(bcd-a)}{24}\right)\right) \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(\tau)}{\chi(\tau)} \\ &= \exp\left(\pi i\left(\tfrac{1}{8}(d^2-1)(c-1) + \frac{cd}{8} - \frac{(b-c)(bcd-a)}{24}\right)\right) q^{1/12} Q^{-1/12} f(x) \\ &= \exp\left(\pi i\left(\tfrac{1}{8}(d^2-1)(c-1) + \frac{cd}{8} - \frac{(b-c)(bcd-a)}{24}\right)\right) \\ &\qquad \cdot \exp\left(\frac{\pi}{12k}\left(\frac{1}{z} - z\right)\right) \exp\left(\frac{\pi i}{12k}(h+h')\right) f(x). \end{aligned}$$

LEMMA 5.2. *Suppose that* $2 \nmid hk$ *and* $hh' \equiv 1 \pmod{2k}$, *that*

$$f_1(x) = \prod_{1}^{\infty} (1 + x^{n-1/2}) = 1 + \sum_{n=1}^{\infty} q_1(n) x^{n/2}.$$

Then

$$f(x) = \frac{\omega_{h,k}}{2^{1/2}} \exp\left(\frac{\pi}{12k}\left(z + \frac{1}{2z}\right)\right) f_1(x').$$

Proof. As in Lemma 5.1, we have

$$f_1(x) = f_1(q^2) = \prod(1 + q^{2n-1}) = 2^{1/6} q^{1/24} \frac{1}{\chi(\tau)},$$

$$\begin{aligned} f_1(x') &= 2^{1/6} Q^{1/24} \frac{1}{\chi(T)} = 2^{1/6} Q^{1/24} \exp\left(-\frac{(b-c)(abc-d)}{24}\pi i\right) \frac{\phi(\tau)}{\chi(\tau)} \\ &= 2^{1/6} Q^{1/24} \exp\left(-\frac{(b-c)(abc-d)}{24}\pi i\right) 2^{1/3} q^{1/12} f(x) \\ &= \exp\left(-\frac{(b-c)(abc-d)}{24}\pi i\right) 2^{1/2} \\ &\qquad \cdot \exp\left(\frac{\pi i}{24}\left(-\frac{h'}{k} + \frac{i}{kz} + \frac{2h}{k} + \frac{2iz}{k}\right)\right) f(x). \end{aligned}$$

LEMMA 5.3. *Suppose that* $2|h$, $2\nmid k$, $hh'\equiv 1 \pmod{k}$, $2|h'$ *and suppose that*

$$f_2(x) = \prod_1^\infty (1 - x^{n-1/2}) = 1 + \sum q_2(n) x^{n/2}.$$

Then

$$f(x) = \frac{\omega_{h,k}}{2^{1/2}} \exp\left(\frac{\pi}{12k}\left(z + \frac{1}{2z}\right)\right) f_2(x').$$

Proof. We take

$$a = -h, \quad b = k, \quad c = (hh' - 1)/k, \quad d = -h'.$$

Then

$$\begin{aligned} f_2(x') = f_2(Q^2) &= 2^{1/6} Q^{1/24} \frac{\psi(T)}{\chi(T)} \\ &= 2^{1/6} Q^{1/24} \exp\left(\frac{\pi i}{2}\left(\frac{b^2-1}{4} + \frac{ab}{4} - \frac{(a+d)(abd-c)}{12}\right)\right) \frac{\phi(\tau)}{\chi(\tau)} \\ &= 2^{1/2} \exp\left(\frac{\pi i}{2}\left(\frac{b^2-1}{4} + \frac{ab}{4} - \frac{(a+d)(abd-c)}{12}\right)\right) Q^{1/24} q^{1/12} f(x). \end{aligned}$$

6. **Approximation of the integrand.** Let

$$z = k(N^{-2} - i\vartheta).$$

Then

$$\begin{aligned} I_1 &= \sum_{1\leqq k\leqq N, 2|k} \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\vartheta) f(e^{(2\pi i h - 2\pi z)/k}) e^{-2\pi i h n/k + 2\pi z n/k} d\vartheta \\ &= \sum_{1\leqq k\leqq N, 2|k} \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\vartheta) \omega_{h,k} e^{(\pi/12k)(z-1/z)} \\ &\qquad \cdot f(x') e^{-2\pi i h n/k + 2\pi z n/k} d\vartheta \\ &= \sum_{1\leqq k\leqq N, 2|k} \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\vartheta) \omega_{h,k} \\ &\qquad \cdot e^{(\pi/12k)(z-1/z) - 2\pi i h n/k + 2\pi z n/k} \sum_{\nu=0}^\infty q(\nu) e^{-(2\pi/kz + 2h'\pi i/k)\nu} d\vartheta \\ &= \sum_{1\leqq k\leqq N, 2|k} \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} \sum_{\nu=0}^\infty q(\nu) e^{-(2\pi/kz)(\nu+1/24) + (2\pi z/k)(n+1/24)} \\ &\qquad \cdot \sum_{r=1}^k b_r e^{2\pi i r h'/k} \omega_{h,k} e^{-2\pi i h n/k - 2\pi i h'\nu/k} d\vartheta. \end{aligned} \tag{6.1}$$

Since $(1/k)\Re(1/z)\geqq\frac{1}{2}$, we have

$$|I_1|\leqq\sum_{1\leqq k\leqq N,2|k}\int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}}\sum_{\nu=0}^{\infty}q(\nu)$$

$$\cdot\exp\left\{-\frac{2\pi}{k}\left(\nu+\frac{1}{24}\right)\Re\frac{1}{z}+\frac{2\pi}{k}\left(n+\frac{1}{24}\right)\Re z\right\}$$

$$\sum_{r=1}^{k}|b_r|\left|\sum_{(h,k)=1}\omega_{h,k}e^{-2\pi ihn/k+2h'(r-\nu)\pi i/k}\right|d\vartheta$$

$$=O\left(\sum_{k=1}^{N}\int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}}\sum_{\nu=0}^{\infty}q(\nu)e^{-\pi(\nu+1/24)}\sum_{r=1}^{k}|b_r|\,k^{2/3}d\vartheta\right)$$

$$=O\left(\sum_{k=1}^{N}\log k\cdot k^{2/3}\frac{1}{kN}\right)=O\left(\frac{1}{N}\sum_{k=1}^{N}k^{-1/3+\epsilon}\right)$$

$$=O(N^{-1/3+\epsilon}).$$

Let

$$J=\frac{1}{2^{1/2}}\sum_{k=1,k\text{ odd}}^{N}\sum_{(h,k)=1,0<h\leqq k}\int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}}g(\vartheta)\omega_{h,k}$$

$$\cdot e^{(\pi/24k)(2z+1/z)-2\pi ihn/k+2\pi zn/k}d\vartheta.$$

The same method will give us that $|I_2-J|=O(N^{-1/3+\epsilon})$.

7. **A contour integration.** Let $w=N^{-2}-i\vartheta$. Then

$$J=\frac{-i}{2^{1/2}}\sum_{1\leqq k\leqq N,k\text{ odd}}\sum_{(h,k)=1,0<h\leqq k}\omega_{h,k}e^{-2\pi ihn/k}\int_{N^{-2}-i\vartheta_2}^{N^{-2}+i\vartheta_1}e^{2\pi w(n+1/24)+\pi/24k^2w}dw$$

$$=\frac{i}{2^{1/2}}\sum_{1\leqq k\leqq N,k\text{ odd}}\sum_{(h,k)=1,0<h\leqq k}\omega_{h,k}e^{-2\pi ihn/k}\left(\int_{N^{-2}+i\vartheta_1}^{N^{-2}+ik^{-1}(N+1)^{-1}}\right.$$

$$+\int_{N^{-2}+ik^{-1}(N+1)^{-1}}^{-N^{-2}+ik^{-1}(N+1)^{-1}}+\int_{-N^{-2}+ik^{-1}(N+1)^{-1}}^{-N^{-2}-ik^{-1}(N+1)^{-1}}+\int_{-N^{-2}-ik^{-1}(N+1)^{-1}}^{N^{-2}+ik^{-1}(N+1)^{-}}$$

$$\left.+\int_{N^{-2}-ik^{-1}(N+1)}^{N^{-2}-i\vartheta_2}-2\pi i\text{ Residue at }0\right)$$

$$=K_1+K_2+K_3+K_4+K_5+L\text{ (say)}.$$

We have

$$K_1=\frac{i}{2^{1/2}}\sum_{1\leqq k\leqq N,k\text{ odd}}\sum_{(h,k)=1,0<h\leqq k}\omega_{h,k}e^{-2\pi ihn/k}$$

$$\cdot\int_{N^{-2}+ik^{-1}(N+k)^{-1}}^{N^{-2}+ik^{-1}(N+1)^{-1}}g(\vartheta)e^{2\pi w(n+1/24)+\pi/24k^2w}dw.$$

By Lemma 3.1, we have

$$K_1 = O\left(\sum_{1 \leq k \leq N, k \text{ odd}} k^{2/3+\epsilon} \int_{k^{-1}(N+k)^{-1}}^{k^{-1}(N+1)^{-}} \exp \left\{ 2\pi\left(n + \frac{1}{24}\right)\Re w + \frac{\pi}{24k^2} \Re \frac{1}{w} \right\} dw \right.$$

$$= O\left(\sum_{k=1}^{N} k^{2/3+\epsilon} e^{-2\pi n N^{-2}} \int_{k^{-1}(N+k)^{-1}}^{k^{-1}(N+1)^{-1}} d\vartheta \right)$$

$$= O(N^{-1/3+\epsilon}).$$

Similar result holds for K_5.

We have

$$\Re \frac{1}{k^2 w} = \frac{N^{-2}}{k^2 N^{-2} + N^2}, \qquad K_2 = O\left(\sum_{k=1}^{N} N^{-2} k^{2/3+\epsilon} \right) = O(N^{-1/3+\epsilon}).$$

Similar result holds for K_4.

Applying again Kloosterman's argument to K_3, we have also $K_3 = O(N^{-1/3})$.

Finally we find the residue of $\exp(2\pi w(n+1/24)+\pi/24k^2 w)$ at $w=0$. We have the expansion

$$e^{2\pi w(n+1/24)} = \sum_{\nu=1}^{\infty} \frac{(2\pi w(n + 1/24))^{\nu}}{\nu!},$$

$$e^{\pi/24k^2 w} = \sum_{\mu=1}^{\infty} \frac{1}{\mu!} \left(\frac{\pi}{24k^2 w} \right)^{\mu}.$$

The residue is, therefore,

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu!} \left(\frac{\pi}{24k^2} \right)^{\mu} \frac{1}{(\mu - 1)!} (2\pi(n + \tfrac{1}{24}))^{\mu-1}$$

$$= \frac{1}{2\pi} \frac{d}{dn} \sum_{\mu=1}^{\infty} \frac{1}{(\mu!)^2} \left(\frac{\pi}{24k^2} \right)^{\mu} (2\pi(n + \tfrac{1}{24}))^{\mu}$$

$$= \frac{1}{2\pi} \frac{d}{dn} \sum_{\mu=1}^{\infty} \frac{1}{2^{2\mu}(\mu!)^2} \left(\frac{\pi}{k} \left\{ \tfrac{1}{3}(n + \tfrac{1}{24}) \right\}^{1/2} \right)^{2\mu}$$

$$= \frac{1}{2\pi} \frac{d}{dn} J_0\left(\frac{i\pi}{k} \left\{ \tfrac{1}{3}(n + \tfrac{1}{24}) \right\}^{1/2} \right).$$

Therefore

$$q(n) = \frac{1}{2^{1/2}} \sum_{k=1, k \text{ odd}}^{N} \sum_{(h,k)=1, 0<h\leq k} \omega_{h,k} e^{-2\pi i h n/k} \frac{d}{dn} J_0\left(\frac{i\pi}{k} \left\{ \tfrac{1}{3}(n + \tfrac{1}{24}) \right\}^{1/2} \right)$$

$$+ O(N^{-1/3+\epsilon}).$$

Let $N \to \infty$; we obtain the theorem.

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