

ON MEASURE IN ABSTRACT SETS

BY

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INTRODUCTION

In their paper⁽¹⁾ *Sur les décompositions dénombrables* Banach and Tarski obtained a result which can be restated as follows:

A necessary and sufficient condition that two Lebesgue measurable, euclidean sets a and a' , shall have equal measure is that there exist null sets $n < a$ and $n' < a'$, such that $a - n$ and $a' - n'$ are the respective unions of sequences of disjoint measurable sets, of which corresponding sets are congruent.

If we then identify sets differing on only a null set, there exists a class of transformations on the measurable sets such that two sets are of equal measure if and only if they correspond under some transformation of that class. Then just as the elementary notion of the volume of an n -dimensional interval is generalized to that of any measure function on a Borel field, so the equally elementary notion of equality of volume, defined for these figures by the relation congruence, can be generalized to the notion of equality of measure defined for some field of sets by a suitable class of transformations. This is done as follows:

In Part I, we consider a complemented, distributive σ -lattice M with a zero element, and a class Φ of σ -isomorphisms on the principal ideals of M . The lattice M is to be taken to correspond to a family of measurable sets modulo the null sets, and Φ as a semi-group of measure-preserving transformations. Then Φ generates an equivalence relation $a \sim b$ between elements of M which is countably additive and hereditary in the sense that $a \sim b$, $a' < a$ imply that there exists a $b' < b$ such that $a' \sim b'$. It is then shown for the bounded elements of M , that is, those which are not equivalent to any proper subelement, that the relation $a \sim b$ is also preserved by subtraction and by taking limits of monotonic sequences. It may be remarked that these latter results yield an independent proof⁽²⁾ of the theorem of Banach and Tarski. A construction is also given leading to a definition of a complete measure. In Part II, the measure of an element of M is defined as the totality of its equiva-

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(1) *Fundamenta Mathematicae*, vol. 6 (1924), pp. 244-277. The theorem quoted is on page 277. This result has been extended to the case of the Haar measure on a locally compact group by Professor John von Neumann in a series of lectures at the Institute for Advanced Study.

(2) See Example 15.3 for a proof in a generalized form.

lent elements, that is, of images under Φ ; and a partial ordering and an operation of addition are defined in the set of measures. A necessary and sufficient condition is then given that the set of measure values be order and addition isomorphic to a set of positive numbers. Such an isomorphism is necessarily unique, up to a multiplicative constant. These results will be analyzed in detail in another paper. In Part III, a sufficient condition is given that, for a family of sets upon which a numerical measure is defined, two sets of equal measure shall correspond under some measure-preserving transformation, and so that the given measure shall be derivable by the procedure of Part II. It turns out that this is the case for a measure on a separable Borel field, provided there are no minimum sets of positive measure. However, a trivial generalization of the theory of Part II can be seen to be applicable to any measure function.

PART I

1. **Ideals and isomorphisms in the lattice M .** Let $M \equiv \{a, b, c, \dots\}$ be a complemented distributive σ -lattice with a zero element; i.e., satisfying the system (α):

- i. M has a partial ordering $a < b$ (b contains a) such that $a < b$ excludes $a = b$.
- ii. Every sequence $(a_n) \subseteq M$ has a supremum and an infimum $\sum a_n$ and $\prod a_n$, respectively. If (a_n) is monotone increasing (decreasing), we denote $\sum a_n$ ($\prod a_n$) by $\lim a_n$. We also write $a + b$ and ab for the supremum and infimum of (a, b) .
- iii. M has a zero element θ . Any elements a, b such that $ab = \theta$ are called disjoint.
- iv. If $a < b$, then there exists a unique element, which will be written $b - a$, such that $(b - a)a = \theta$, and $a + (b - a) = b$.
- v. For any sequence (a_n) and any a , $a(\sum a_n) = \sum aa_n$.

1.1. DEFINITION. A class $M' \subseteq M$ is called a σ -ideal if $a' < a \in M'$ implies that $a' \in M'$, and if $(a_n) \subseteq M'$ implies that $\sum a_n \in M'$. The ideal M' is called principal if there exists an element a such that $a' \in M'$ if and only if $a' \leq a$. The principal ideal corresponding to an element a will be written $I(a)$.

1.2. DEFINITION. A univocal correspondence ϕ defined on a σ -ideal $M' \subseteq M$ onto a σ -ideal $M'' \subseteq M$ is called a σ -isomorphism if for every sequence $(a_n) \subseteq M$, $\phi(\sum a_n) = \sum \phi(a_n)$, and $\phi(\prod a_n) = \prod \phi(a_n)$. The correspondence inverse to ϕ is also a σ -isomorphism which will be written ϕ^{-1} .

1.21. For any σ -isomorphism ϕ , $\phi(a) = \theta \equiv a = \theta$; $a' < a \equiv \phi(a') < \phi(a)$, and $\phi(a - a') = \phi(a) - \phi(a')$.

1.22. Let m be a σ -ring of subsets of a set s ; n , a σ -ideal in m ; and M , the residue class ring m/n . Then any univocal point correspondence between the subsets \mathbf{a} and \mathbf{b} in m which leaves the rings m and n invariant leads to a σ -isomorphism of the principal ideals in M corresponding to \mathbf{a} and \mathbf{b} .

1.23. DEFINITION. Let (ϕ_n) be a sequence of σ -isomorphisms between the principal ideals $I(a_n)$ and $I(b_n)$. If the sequences (a_n) and (b_n) are each disjoint, then (ϕ_n) is said to have a sum $\sum \phi_n$; namely, the isomorphism $\phi(a')$ defined for $a' \leq \sum a_n$ by $\phi(a') = \sum \phi_n(a'a_n)$.

2. The class Φ of σ -isomorphisms. Equivalence. Boundedness. Let Φ be a class of σ -isomorphisms each defined between principal ideals of M , and satisfying (β) :

- i. The inverse of any isomorphism in Φ is in Φ .
- ii. If ϕ, ϕ' are in Φ , and are defined on $I(a), I(a')$, respectively, then the resultants $\phi\phi'$ and $\phi'\phi$, defined on $a\phi'(a')$ and $a'\phi(a)$, are in Φ .
- iii. If $(\phi_n) \subset \Phi$, and if $\sum \phi_n$ exists, then $\sum \phi_n \in \Phi$.
- iv. At least one isomorphism of Φ is defined on any principal ideal of M .

2.1. DEFINITION. Two elements of M are called Φ -equivalent if they correspond under some isomorphism of Φ . This relation will be written $a \sim b$.

2.2. The relation Φ -equivalence has the following properties:

- i. It is an equivalence relation, and so partitions M into a set of exhaustive and mutually exclusive classes M_α . The unique class containing an element a may be denoted by $M(a)$. Obviously, $M(a) = M(a')$ if and only if there is an isomorphism ϕ of Φ such that $a' = \phi(a)$.
- ii. If $(a_n), (b_n)$ are sequences of disjoint elements of M such that $a_n \sim b_n$, then $\sum a_n \sim \sum b_n$; i.e., Φ -equivalence is countably additive.
- iii. It is hereditary in the sense that if $a \sim b$, and $a' < a$, then there exists a $b' < b$ such that $a' \sim b'$.

2.3. DEFINITION. An element a of M is unbounded (with respect to Φ) if there exists an element $a' < a$ such that $a \sim a'$, and $a - a' \neq \theta$. Otherwise a is bounded.

2.31. i. If a is bounded and $a' < a$, then a' is bounded. ii. If $a \sim a + b$, and $b - ab \neq \theta$, then a is unbounded. iii. If $a \sim a'$, then a and a' are bounded or unbounded together.

2.32. DEFINITION. A class M_α of equivalent elements of M is bounded or unbounded according as its elements are either all bounded or all unbounded.

2.33. THEOREM. A necessary and sufficient condition that an element a be unbounded is that a contain a sequence of disjoint equivalent elements $a_n \neq \theta$.

Proof. Suppose such a sequence exists. Then by 2.2, ii, $\sum a_n \sim \sum a_{2n}$; and since $\sum a_{2n-1} \neq \theta$, $\sum a_n$ is unbounded. Conversely, if a is unbounded, there exists an element $a' < a$ and an isomorphism ϕ of Φ such that a' and $a - a'$ are different from θ and $\phi(a) = a - a'$. Then an infinite sequence of elements satisfying the required conditions can be defined inductively by the formula $a_1 = a$, and for $n > 1$, $a_n = \phi(a_{n-1})$.

3. Equivalence of complements of bounded equivalent elements. The results of this section can be summarized as follows:

If $a \sim b$, and if either (i) there exists a σ -isomorphism ϕ of Φ under which ab is invariant, or (ii) the elements a and b are bounded, then for any $c > (a+b)$, $c - a \sim c - b$.

The assertions leading to this statement are the following:

3.1. A necessary and sufficient condition that for any element $c \geq (a+b)$, $c - a \sim c - b$, is that $a - ab \sim b - ab$. The necessity is obvious, and the sufficiency follows from the fact that $c - (a+b) \sim c - (a+b)$ (identity), and $a - ab \sim b - ab$ (by hypothesis). Since the terms on either side are disjoint, their respective sums are equivalent and $c - a \sim c - b$.

3.2. **THEOREM.** *If $a \sim b$, and a and b are bounded, then $a - ab \sim b - ab$.*

The general argument of the proof is as follows: First it is shown that there exist sequences (a_n) and (b_n) of elements of M contained in $a - ab$ and $b - ab$, respectively, such that $\phi^n(a_n) < ab$, and $\phi^{n+1}(a_n) = b_n$. From this it is verified that both families are disjoint. Finally it is shown that $\sum a_n = a - ab$ and $\sum b_n = b - ab$, which proves the theorem.

Let us first restate this in the more convenient form:

If $a+x \sim b+x$, where $a+x$, $b+x$ are bounded, and $ax = bx = \theta$, then $a \sim b$.

Proof. We have given that there exists an isomorphism ϕ of Φ such that $\phi(a+x) = b+x$. We shall write this as

$$(1) \quad a + x \sim b + x (\phi).$$

Then

$$(2) \quad a\phi^{-1}(b) \sim b\phi(a).$$

By taking differences on both sides we get

$$(3) \quad a\phi^{-1}(x) + x \sim b\phi(x) + x.$$

If we can show that

$$(4) \quad a\phi^{-1}(x) \sim b\phi(x),$$

then by summing corresponding sides of (2) and (4), we get the required result. Now let

$$(5, i) \quad a_0 \equiv a\phi^{-1}(x) \equiv a - a\phi^{-1}(b),$$

$$(5, ii) \quad b_0 \equiv b\phi(x) \equiv b - b\phi(a),$$

$$(5, iii) \quad x_0 \equiv \phi(a_0) \leq x.$$

Since ϕ is defined on $x \leq a+b$, we can define inductively for $n > 0$

$$(6) \quad x_n \equiv x\phi(x_{n-1}).$$

It can be shown by induction that the elements a , x_0 , x_1 , \dots , are dis

joint. For since $a_0 \leq a$, and $x_0 \leq x$, a_0 and x_0 are disjoint. Suppose that a_0, x_0, \dots, x_{n-1} are disjoint. But for $0 < k \leq n$, $\phi^{-1}(x_k) \leq x_{k-1}$, and $\phi^{-1}(x_0) = a_0$. Hence for $0 \leq k \leq n$, the elements $\phi^{-1}(x_k)$, and so also the elements x_k , are disjoint by the hypothesis of the induction, which is now complete. We define

$$(7) \quad b_n = b\phi(x_{n-1}), \quad n = 1, 2, 3, \dots$$

Since $\phi^{-1}(b_n) \leq x_{n-1}$, it follows that the elements b_n are disjoint. Further, since $\phi(x_n) \leq \phi(x) \leq b + x$,

$$(8) \quad \phi(x_{n-1}) = x\phi(x_{n-1}) + b\phi(x_{n-1}) = x_n + b_n,$$

that is,

$$(9) \quad x_{n-1} \sim x_n + b_n (\phi) \quad \text{for } n = 1, 2, 3, \dots,$$

and we also have

$$(10) \quad a_0 \sim x_0 (\phi).$$

Since the terms on the right (the left) side of (9) and (10) are disjoint, their respective sums are equivalent and

$$(11) \quad \left(a_0 + \sum_0^\infty x_n \right) \sim \left(\sum_0^\infty x_n + \sum_1^\infty b_n \right) (\phi).$$

But from (3) and (5)

$$(12) \quad a_0 + x \sim b_0 + x (\phi),$$

and for $n \geq 1$, $b_n \leq b\phi(x) = b_0$, whence the respective differences of corresponding sides of (11) and (12) are equivalent; that is,

$$(13) \quad x - \sum_0^\infty x_n \sim \left(b_0 - \sum_1^\infty b_n \right) + \left(x - \sum_0^\infty x_n \right).$$

Since $a + x$ is bounded, so is $x - \sum_0^\infty x_n$, and hence

$$(14) \quad b_0 - \sum_1^\infty b_n = \theta.$$

For all n , ϕ^{-1} is defined on x_n , and

$$(15) \quad \phi^{-1}(x_n) \leq x_{n-1}.$$

From this it follows that ϕ^{-n} is defined on x_n and satisfies

$$(16) \quad \phi^{-n}(x_n) \leq \phi^{-n+1}(x_{n-1}) \leq \dots \leq \phi^{-1}(x_1) \leq x_0,$$

and hence that

$$(17) \quad \phi^{-(n+1)}(x_n) \leq \phi^{-(n-k+1)}(x_{n-k}) \leq a_0 \quad \text{for any } k < n.$$

But from (8) we have

$$(18) \quad \phi^{-1}(b_n) = x_{n-1} - \phi^{-1}(x_n),$$

so we can define for $n \geq 1$

$$(19) \quad a_n \equiv \phi^{-(n+1)}(b_n) = \phi^{-n}(x_{n-1}) - \phi^{-(n+1)}(x_n) \leq a_0.$$

It follows from (19) and (16) that for $k > 0$

$$(20) \quad a_n a_{n+k} \leq a_n \phi^{-(n+k)}(x_{n+k-1}) \leq a_n \phi^{-(n+1)}(x_n) = \theta,$$

and hence that the elements a_n , $n = 1, 2, 3, \dots$, are disjoint to each other. From this and from the fact that by definition $a_n \sim b_n$ (ϕ^{n+1}), we have

$$(21) \quad \sum_1^{\infty} a_n \sim \sum_1^{\infty} b_n = b_0.$$

Since the element x is disjoint to both sides of (21), this yields

$$(22) \quad \sum_1^{\infty} a_n + x_n \sim b_0 + x,$$

which with (12) gives

$$(23) \quad \sum_1^{\infty} a_n + x \sim a_0 + x.$$

But by hypothesis $a + x$ is bounded, and so

$$(24) \quad a_0 - \sum_1^{\infty} a_n = \theta.$$

We now have, by combining (21), and (24),

$$(25) \quad a_0 \sim b_0,$$

which completes the proof of (4) and so of the required theorem.

4. **Consequences of Theorem 3.2.** We prove first the following statement.

4.1. **THEOREM.** *If a, b are bounded and if $a' \leq a$ and $b' \leq b$, then $a \sim b$ and $a' \sim b'$ implies that $a - a' \sim b - b'$; i.e., for bounded elements, Φ -equivalence is subtractive.*

Proof. Let $a \sim b$ (ϕ) and $a' \sim b'$ (ϕ'); then $a' \sim b' \sim \phi^{-1}(b') \leq a$. Then by (3.2) $a - a' \sim a - \phi^{-1}(b') \sim b - b'$ (ϕ). Therefore $a - a' \sim b - b'$.

4.11. **THEOREM.** *If (a_n) and (b_n) are monotonic sequences of bounded elements of M and if $a_n \sim b_n$ for all n , then $\lim a_n \sim \lim b_n$; i.e., the respective limits of monotonic sequences of bounded equivalent elements are equivalent.*

Proof. Let us call these limits a and b , respectively. Suppose (a_n) and there-

fore also (b_n) are non-decreasing. Then $a = a_1 + (a_2 - a_1) + (a_3 - a_2) + \dots$, and $b = b_1 + (b_2 - b_1) + (b_3 - b_2) + \dots$. Then since a_n and b_n are bounded and $a_n \sim b_n$ for all n , by 4.1 we may sum corresponding sides and so get $a \sim b$. If the sequences are non-increasing, then from 4.1 $a_n - a_{n-1} \sim b_n - b_{n-1}$. Therefore $a_1 - a \sim b_1 - b$. Since further $a_1 \sim b_1$, by again applying 4.1 we get $a \sim b$.

4.12. THEOREM. *The sum of any finite collection of bounded elements is bounded.*

Proof. We shall prove 4.12 for the case of two bounded, disjoint elements a_1 and a_2 , since the general case is readily reducible to this. Suppose then that under these conditions $a_1 + a_2$ is unbounded. Then there exists an element $a' \neq \theta$ contained in $a \equiv a_1 + a_2$ such that $a - a' \neq \theta$, and $a \sim a'$. Then $a_1 + a_2 \sim a'_1 + a'_2$, where $a'_1 = \phi(a_1)$ and $a'_2 = \phi(a_2)$. From 3.2 we have

$$(1) \quad a_1 - a_1 a'_1 \sim a'_1 - a_1 a'_1,$$

$$(2) \quad a_2 - a_2 a'_2 \sim a'_2 - a_2 a'_2.$$

But since $a'_1 - a_1 a'_1 = a'_1 a_2$ and $a_1 - a_1 a'_1 = (a_1 - a_1 a') + a_1 a'_2$ we have

$$(3) \quad (a_1 - a_1 a') + a_1 a'_2 \sim a'_1 a_2,$$

and similarly

$$(4) \quad (a_2 - a_2 a') + a_2 a'_1 \sim a'_2 a_1.$$

Now since the left-hand members of (3) and (4) are disjoint, we may substitute in (4) for $a'_1 a'_2$ and we get

$$(5) \quad (a_2 - a_2 a') + (a_1 - a_1 a') + a_1 a'_2 \sim a_1 a'_2.$$

But $a_1 a'_2$ is bounded and so

$$(6) \quad (a_2 - a_2 a') + (a_1 - a_1 a') = a - a' = \theta,$$

contrary to hypothesis. Therefore a is bounded.

It may be noted that the class M of all bounded elements of M is a finitely additive ideal (and a restricted σ -ideal) in M , and that the set of all limits of (sequences of) bounded elements is a σ -ideal in M .

5. **Lattices M derived from rings of sets.** Let m be a σ -ring of subsets of a fixed set s ; n , a σ -ideal in m ; and Π a class of univocal point transformations Π_α , defined between pairs of sets in m and leaving m and n invariant. Let M be the residue class ring m/n , and suppose the isomorphisms on M derived from the transformations of Π (see 1.22) satisfy (β) . Suppose further that conditions (γ) hold:

5.1. i. *If x is any subset of s and if there exists a set $n \in n$ such that $x \subseteq n$, then $x \in n$.*

ii. *If $x \subseteq s$, then there exists a set $x^* \in m$ such that (a) $x \subset x^*$ and (b) if $x \subset a \in m$, then $x^* - x^* a \in n$.*

Let us denote the residue class of a set $x \in m$ by $m(x)$ or by x' . Then

5.11. If x_1^* and x_2^* satisfy ii, then $x_1^* = x_2^*$.

5.12. If $x \subseteq x'$, then $x^* \subseteq x'^*$.

5.13. For any $x \subseteq s$, there exists a set x_* of m , namely, $x^* - (x^* - x)^*$, such that i. $x_* \subseteq x$; ii. if $x_*' \subseteq x$, then $x_*' \subseteq x_*$, and iii. if x_* and y_* satisfy i and ii, then $x_* = y_*$.

5.14. A set $x \subset s$ is in m if and only if $x^* = x_*$, in which case $x^* = x_* = x$.

5.15. If $x_n \subset s$, then for any x_n^* , x_{n*} , $m(\sum x_n^*) = \sum m(x_n^*)$; $m[(\sum x_n)_*] \geq m(\sum x_{n*})$; $m[(\prod x_n)^*] \leq m(\prod x_n^*)$; $m[(\prod x_n)_*] = m(\prod x_{n*})$.

Since the proofs of these relations are all alike, we shall prove only the first. From the fact that $(\sum x_n)^* \geq \sum x_n > x$ for all n , $m(x_n^*) \leq m[(\sum x_n)^*]$, and so $m[(\sum x_n)^*] \geq m[\sum x_n^*]$. Similarly, $x_n^* \geq x_n$ for all n , and so $\sum x_n^* \geq \sum x_n$. Therefore $m(\sum x_n^*) > m[(\sum x_n)^*]$, and the relation is proved.

5.16. LEMMA. A necessary and sufficient condition that $x \in m$ is that for any $y \subseteq s$, $(xy)^*(y - xy)^* \in n$.

Proof. Suppose $x \notin m$. Then for any x^* , $x^* - x$ is not in m and so not in n . Then from i of (γ) , $(x^* - x)^*$ is not in n . Now let $y = x^*$. Then $(xy)^*(x - xy)^* = (x^* - x)^* \notin n$. If $x \in m$, then for any $y \subseteq s$, $(xy)^* \leq x$, and so $m[(xy)^*] \leq m(x)$. Also $y - xy < s - x < (s - x)^* - (s - x)^*(xy)^*$, which with ii of (γ) completes the proof.

5.17. If π is any transformation of Π defined on a set $a \in m$, then for any $x \subset a$, we define $\pi(x)$ as the image of x under π . Then since π leaves m and n invariant, $\pi(a) \in m$, and $\pi(a) \supseteq \pi(x^*) \supseteq \pi(x)$, so $m[\pi(x^*)] > m[\pi^*(x)]$. But since $\pi^*(x) \supseteq \pi(x)$, it follows that $\pi^{-1}[\pi^*(x)] \supseteq x$. Therefore $\pi^{-1}[\pi^*(x)] \supseteq x^*$, whence $m[\pi(x^*)] = m[\pi^*(x)]$. By a similar argument with reference to x , we also have $m[\pi(x_*)] = m[\pi_*(x)]$. That is, $m(x^*) \sim m[\pi(x)^*]$, and $m(x)_* \sim m[\pi(x)_*]$.

PART II

6. **Measuring systems in abstract sets.** A set s is said to have a *measuring system* (m, n, Φ) if m is a σ -ring of subsets of s ; n , a σ -ideal in m (notice that under these conditions $M \equiv m/n$ satisfies (α) of §1); and Φ , a class of σ -isomorphisms defined on M and satisfying (β) (§2). If (γ) of §5 also holds, the system (m, n, Φ) is *complete*. Any set has a trivial measuring system, namely, (the totality of its subsets, the void set, the identity transformations on every subset). In euclidean space, the Lebesgue measurable sets, the null sets, and the transformations derived from direct sums of rigid motions form such a system.

Now let us consider a set s with a given system (m, n, Φ) . An isomorphism of Φ is said to be *measure-preserving*, and the elements a, b, c, \dots , of M to be *measurable* and of *measure* $M(a), M(b), M(c), \dots$, (2.2, i), respectively. Henceforth these measures will be written a or $m(a)$, b or $m(b)$, c or $m(c), \dots$. In particular, the element n is said to have null measure $m(n) = 0$. The ele-

ments a and b are of equal measure if and only if they correspond under some measure-preserving transformation, i.e., if and only if $a \sim b$. If the system (m, n, Φ) is complete, a set $a \in m$ is said to be *measurable* and of *measure* $m(m(a))$, where $m(a)$ is the residue class of a . For any $x \subseteq s$, the elements x^* and x_* (§5) are called a *hull* and a *kernel* of x , respectively. Then $m(x^*)$ and $m(x_*)$, which will now be written ξ^* and ξ_* , will be called the *outer measure* and the *inner measure* of x .

7. Partial ordering of the measure-values. Addition. Let $\mathfrak{M}^* = (a, b, c, \dots)$ be the totality of measures, and let \mathfrak{M} be the subset of bounded measures (2.32). Henceforth $a < \infty$ shall mean a is bounded.

7.1. DEFINITION. If $a < \infty$, we define $a < b$ to mean that there exist corresponding elements a and b in M such that $a < b$. It follows that $a < b$ is a (proper) partial ordering of \mathfrak{M} . The null element 0 of \mathfrak{M}^* is the zero of the ordering.

7.11. Any monotonic increasing (decreasing) sequence (a_n) has a supremum (an infimum) written $\lim a_n$. For suppose a_n increasing, and let a_n be a sequence of corresponding elements from M . For each n , there exists an $a'_n < a_{n+1}$ such that $a'_n \sim a_n$. Then $m[\sum(a_{n+1} - a'_n)]$ can be verified to be the desired supremum.

7.2. DEFINITION. If $(a_n) \subset \mathfrak{M}^*$, and if there exists a sequence (a_n) of disjoint corresponding elements in M , then (a_n) is said to have a sum $\sum a_n$, namely, $m(\sum a_n)$. A binary sum will be written $a + b$.

It follows from the considerations of Part I that:

7.21. Whenever it exists, the sum is unique, and the operation of addition is commutative and associative, and $\sum a_n = \lim \sum_1^n a_i$.

7.22. If $a_i = a$ for all i , then we write $\sum_1^n a_i$ as na . If either side of the following exists, so does the other; and $na + ma = (n + m)a$; $n(a + b) = na + nb$; $n(ma) = m(na) = nma$.

7.23. For any $a \in \mathfrak{M}^*$, $a + 0$ exists and is equal to a ; if $a < \infty$ then $a + b = a$ if and only if $b = 0$.

7.24. Binary addition has a unique inverse for bounded elements; i.e., $a + \xi = b < \infty$ has a unique solution for fixed a and b , denoted by $b - a$.

7.25. A necessary and sufficient condition that $a < b$ is that for some c , $a + c = b$ holds. If $a < b < \infty$ then there exists a maximum integer n such that $na \leq b$.

7.26. If $\sum m(a_n)$ exists, then $\sum m(a_n) \geq m(\sum a_n)$, and equality holds if and only if the elements a_n are disjoint, in which case $\sum m(a_n)$ does exist.

8. Algebraic properties of \mathfrak{M}^* .

8.1. We now see that any measuring system (m, n, Φ) gives rise to a set \mathfrak{M}^* of measure-values with a commutative and associative operation of addition defined for certain denumerable sets of elements of \mathfrak{M}^* , in terms of which a partial ordering of the bounded elements can be derived. Then

- i. There exists an element 0 such that $a \neq 0$ implies that $a > 0$ and $a + 0 = a$.
- ii. If $a < \infty$, and $a < b$, then $m(a) < m(b)$.
- iii. $\sum m(a_n) \geq m(\sum a_n)$ and equality holds if and only if the elements a_n are disjoint.

If, further, the measure defined is complete, then to any $x \subseteq s$ there corresponds an outer and an inner measure ξ^* and ξ_* , respectively, and

- iv. If $x^* < \infty$, and $x < y$, then $\xi^* \leq \eta^*$.
- v. If (x_n) is a sequence of sets such that $\sum x_n^*$ exists, then $m(\sum x_n)^* \leq \sum \xi_n^*$ if $(\sum x_n)^* < \infty$.
- vi. The set x is measurable if and only if the equivalent conditions a and b (the Carathéodory condition) hold:

- a. $x^* = x_*$, in which case $x^* = x_* = x$, and
- b. for any $y \subseteq s$, $m(xy)^* + m(y - xy)^* = y^*$ [$= m((xy)^* \cdot (y - xy)^*)$].

9. **Ordered sets of measure-values.** In the remainder of Part II it is assumed that the system (m, n, Φ) satisfies condition (δ) .

(δ) For any a, b of M either there exists a $b' \leq b$ such that $a \sim b'$ or there exists an $a' < a$ such that $a' \sim b$.

9.1. A necessary and sufficient condition that (δ) hold is that for any two elements a and b of M^* either there exists an a' such that $a + a' = b$ or a b' such that $b + b' = a$.

We shall show that (δ) is a sufficient as well as a trivially necessary condition that M be order and addition isomorphic to a set of positive numbers.

9.2. For any bounded measures a and b , $a \prec b$ implies that $a \geq b$; i.e., \mathfrak{M} is linearly ordered, and either $b - a$ or $a - b$ exists.

9.2.1. From 9.2, one and only one of the following occurs: a. \mathfrak{M} contains the single element 0; b. there exists an element a of \mathfrak{M} such that $a \neq 0$, and if $a' < a$, then $a' = 0$; c. for any $a > 0$ there exists an a' such that $0 < a' < a$. In the last case, we may assume that $2a' \leq a$, for we can take the lesser of a' and $a - a'$.

9.2.2. If a_n, a are bounded, and $a_n \leq a_{n+1}$ ($a_n \geq a_{n+1}$), then $a < \lim a_n$ ($a > \lim a_n$) implies that there exists a k such that $a \leq a_k$ ($a \geq a_k$).

9.3. i. For any a, b in \mathfrak{M} , there is a unique integer n such that $b = na + c$, $c < a$. ii. If a and b are bounded, then for every integer n such that na and nb exist, na and nb stand in the same relative order as do a and b .

9.4. There exists an element s of M^* such that $a < \infty$ implies that $a \leq s$. If $s \prec \infty$, the sum of any two bounded elements always exists; if $s < \infty$, $a + b$ exists if and only if the equivalent conditions $a < s - b$ and $b < s - a$ hold. In this latter case, the element s is unique; and for any a of M , $a \leq s$.

9.5. If \mathfrak{M} satisfies b of 9.21, that is, if there is a least element $a_1 > 0$, then it follows from 9.3 and 9.4 that the elements of \mathfrak{M} are of the form na_1 , and that according as s is unbounded or bounded and equal to ka_1 , n will assume all positive integral values or all those not greater than k .

9.6. LEMMA. If \mathfrak{M} satisfies c of 9.21, i.e., \mathfrak{M} has no minimal non-null ele-

ments, and, if each element of \mathfrak{M} is contained in one or the other, but not both of two non-void subclasses \mathfrak{M}' and \mathfrak{M}'' and if every element of \mathfrak{M}'' is greater than every element of \mathfrak{M}' , then either there is a least element of \mathfrak{M}'' and no greatest in \mathfrak{M}' , or there is a greatest element in \mathfrak{M}' and no least in \mathfrak{M}'' .

Proof. Since neither \mathfrak{M}' nor \mathfrak{M}'' is void there exist elements a_1' in \mathfrak{M}' and a_1'' in \mathfrak{M}'' with $a_1' < a_1''$. From c , there is a third element between the two, so both cannot be the extreme elements of their respective classes. If either one is, the lemma is proved; if not, since there is some $c_1 > 0$ such that $2c_1 \leq a_1'' - a_1'$, there is a maximum integer n_1 such that $a_1' + n_1c_1 < \mathfrak{M}'$, and a maximum integer $n_1' \geq n_1$ such that $a_1' + n_1'c_1 \leq a_1''$. Let $a_2' = a_1' + n_1c_1$ and let a_2'' be either $a_1' + (n_1' + 1)c_1$ or a_1'' , according as $n_1' > n_1$ or $n_1' = n_1$. Then $a_2' \in \mathfrak{M}'$, $a_2'' \in \mathfrak{M}''$, $a_1' \leq a_2' < a_2'' \leq a_1''$, and $2(a_2'' - a_2') \leq (a_1'' - a_1')$. Let c_2 be the minimum of $a_1'' - n_1'c_1$ and $c_1 - (a_1'' - n_1'c_1)$. Then $2c_2 \leq c_1$ and hence $4c_2 \leq a_1'' - a_1'$. Either if a_2' is the maximum of its class, or if a_2'' is the minimum of its class, the assertion holds. If not, we define a_3' and a_3'' just as in the preceding case. By proceeding in this way, we arrive at a pair of sequences (a_n') and (a_n'') of elements of \mathfrak{M}' and \mathfrak{M}'' , respectively, such that $a_n' \leq a_{n+1}' \leq a_{n+1}'' \leq a_n''$ and $2^{n-1}(a_n'' - a_n') \leq a_1'' - a_1'$. The sequences are monotonic and contained in M , so their respective limits a' and a'' exist and are bounded. Also, since for any $b' \in \mathfrak{M}'$ and for any integer m , $b' < a_m''$, it follows from 7.11 that $b' \leq a'' = \lim a_n''$. In particular this relation holds for all elements of (a_n') , and so $a' = \lim a_n' \leq a''$. Further since $a_n' \leq a' \leq a'' \leq a_n''$ for all n , it follows that $a'' - a' \leq a_n'' - a_n' \leq 2^{n-1}(a_1'' - a_1')$ and hence that $a'' - a' = \theta$, i.e., $a' = a'' = a$. From the fact that a is the limit of a sequence from \mathfrak{M}' and of one from \mathfrak{M}'' it follows that a is the greatest or the least element of its class according as it is in \mathfrak{M}' or in \mathfrak{M}'' , and that the class to which a does not belong has no extreme element.

10. Isomorphisms of \mathfrak{M} onto sets of positive numbers.

10.1. THEOREM. Under condition (δ) the set \mathfrak{M} of bounded measure-values is isomorphic to a subset Γ of the set of non-negative numbers; i.e., there exists a one-to-one mapping $f(a)$ of \mathfrak{M} onto Γ such that whenever $\sum a_n$ exists and is bounded then $\sum f(a_n)$ converges to $f(\sum a_n)$, and whenever $\sum a_n$ exists and is unbounded, then $\sum f(a_n)$ diverges to infinity. The mapping f is completely determined up to a multiplicative constant; that is, if f' is any other mapping of M onto Γ with the above property, then there exists a constant ξ such that for all $a < \infty$, $f(a) = \xi f'(a)$. According as conditions a, b, or c, of 9.21 hold, the set Γ will consist of a. the number zero, b. the set $nf(a_1)$, where $n = 1, 2, 3, \dots$ or $1, 2, \dots, k$, or c. the interval $0 \leq \xi \leq f(\mathfrak{s})$, or $0 \leq \xi \leq \infty$, according as the element \mathfrak{s} of 9.4 is bounded or unbounded.

The theorem follows immediately for cases a and b. In case c, we shall use the following lemma :

LEMMA. To every ordered pair (a, b) of bounded elements different from 0, there corresponds a unique positive finite number $\mu(a, b)$ such that

- i. if $\sum a_n$ exists, then $\mu(\sum a_n, b) = \sum \mu(a_n, b)$,
- ii. $\mu(a, a) = 1$.

Proof. For any $a \neq 0$, the set $\mathfrak{M}' \subseteq \mathfrak{M}$ of elements a' such that $2a' \leq a$, and the set $\mathfrak{M}'' \subseteq \mathfrak{M}$ of elements a'' such that $2a'' > a$, satisfy the hypotheses of 9.6. From this it follows that there exists an element a^1 such that $2a^1 = a$, and such that for every integer n an a^n exists such that $2^n a^n = a$. Then

(1)
$$a^n = 2^k a^{n+k},$$

(2)
$$\sum_{i=1}^p a^{n_i} = \left(\sum_1^p 2^{n_i-m} \right) a^m \quad \text{if } n_i > m \text{ for } i = 1, 2, \dots, p.$$

Further if $a' > 0$, there exists an n such that $a' > a^n$. For by 9.4 there exists a k such that $a = ka' + a''$, where $a'' < a'$, and we may take $n = k + 1$. More generally,

(3) if $a' > a''$, there exists an n for which $a' < a'' + a^n$.

For every pair of elements a and b there exists an integer $\mu_n(a, b)$ such that $a = \mu_n b^n + c_n$, where $c_n < b^n$. From the previous, it follows that the μ_n satisfy

(4)
$$2\mu_n \leq \mu_{n-1} \leq 2(\mu_n - 1);$$

(5) if $c > a$, there exists a k such that for all $p > 0$, $\mu_{k+p}(c, b) > \mu_{k+p}(a, b)$;

(6)
$$\mu_k \left(\sum_{i=1}^p a^{n_i}, a \right) = \sum_1^p \mu_k(a^{n_i}, a) \quad \text{for } n_i > k \text{ for } i = 1, 2, \dots, p;$$

(7)
$$\mu_r(b + ka^r, a) = \mu_r(b, a) + k/2^r, \quad \text{for any } b.$$

From (4) the sequence of numbers $\mu_n(b, a)$ is monotonically increasing and bounded and therefore approaches a unique limit $\mu(b, a)$. We have at once that

(8)
$$\mu(0, a) = 0; \quad \mu(a_n, a) = 1/2^n;$$

(9) if $c < b$, then $\mu(c, a) < \mu(b, a)$;

(10)
$$\mu(c + ka^r) = \mu(c) + k/2^r.$$

Also for any finite p , $\mu(\sum_1^p a_i, a) = \sum_1^p \mu(a_i, a)$. For $\sum_1^p a_i = \sum_1^p \mu_k(a', a) a^k + \sum_{i=1}^p a'_{k,i}$, where $a'_{k,i} < a^k$. It can be seen that $\mu_k(\sum_{i=1}^p a_i, a) < p$; then applying (10), dividing by 2^k and passing to the limit give the desired result. Now suppose that $a^* = \sum_1^\infty a_i$ exists. If $a^* < \infty$, then since for every n $\mu_n(a^*, a) a^n \leq a^*$ it follows that there is a p_n for which $\mu_n(a^*, a) a^n \leq \sum_1^{p_n} a_i$. Then $\mu_n(a^*, a)/2^n = \mu[\mu_n(a^*, a) a^n] \leq \mu(\sum_{i=1}^{p_n} a_i, a) < \mu(a^n, a)$, or $\mu(a^*, a) - \mu(\sum_1^{p_n} a_i, a) < 1/2^n$. If $\sum_1^\infty a_i \ll \infty$, then for any n , $na < \sum_1^\infty a_i$, whence $\mu(\sum a_i, a) > \mu(na, a) \geq n$.

Complete additivity of $\mu(b, a)$ follows from these last two relations and the finite additivity; i.e.,

$$(11) \quad \mu(\sum a_i, a) = \sum \mu(a_i, a).$$

Relations i and ii of the lemma are now proved, and it remains to show for each ordered pair b, a that $\mu(b, a)$ is unique. But any $\mu'(b, a)$ satisfying i and ii can be seen to be such that

$$(12) \quad k\mu'(a^n, a) = k/2^n = \mu(a^n, a), \quad \text{for all } k \text{ and } n;$$

and so

$$(13) \quad \mu'(b, a) = \mu'(\lim \mu_n(b, a)a^n) = \lim \mu'(\mu_n(b, a)a^n) = \lim \mu_n/2^n = m(b, a).$$

We can also verify by a limiting process that if none of a, b, c is 0, then

$$(14) \quad \mu(a, b) = \mu(a, c)\mu(c, b).$$

Then the isomorphism required in 10.1 can be given by $f(a) = \mu(a, a_1)$ for any $a_1 > 0$. From (14), $f(a)/f(b) = \mu(a, a_1)/\mu(b, a_1) = \mu(a, b)$, which is independent of f and of a_1 . But if $g(a)$ is any mapping whatever satisfying the conditions of 10.1, then the ratio $g(a)/g(b)$ satisfies the conditions of the lemma, and so is equal to $\mu(a, b)$. Therefore $g(a) = g(a_1)\mu(a, a_1)$, and $g(a)/f(a) = g(a_1)$, a constant.

Let ξ be any positive finite number less than $\mu(\mathfrak{g}, a)$, where $a \neq 0$, and \mathfrak{g} satisfies 9.4. Then ξ can be expressed uniquely in the form $\zeta + \xi/2^k$ where ζ is an integer, $\xi_k = 0$ or 1 and an infinite set of the ξ_k are zero. It follows from 9.3 and relation (11) that all the sums $\xi_k a^k$ exist and that $\mu(\sum \xi_k)a = \xi$. Therefore the range of any isomorphism, which must be of the form $f(b) = \mu(b, a)$, is a closed finite or infinite interval with left end point zero. This completes the proof of 10.1.

10.2. Let s' and s'' be any two unbounded elements of M , if such exist, which are the respective limits of monotonic increasing sequences (s'_n) and (s''_n) of bounded elements. Then there exist subsequences $(s'_{n'_i})$ and $(s''_{n''_i})$ such that $s_{n'_i} \leq s_{n''_i} < s'_{n'_i+1}$. It follows that $s' = s''$, that is, all unbounded limits of bounded elements of M are of equal measure, which we denote by $+\infty$. If the system (m, n, Φ) is such that

(ϵ) every element of M is the limit of a sequence of bounded elements, then the range of measure-values in \mathfrak{M}^* is $0 \leq \xi \leq +\infty$.

11. Isomorphisms of \mathfrak{M} onto the infinite half-interval. Conclusion. We have shown that for any set s , any system (m, n, Φ) satisfying (α) and (β) leads to a set \mathfrak{M}^* of measures, the bounded subset \mathfrak{M} of which satisfies conditions which may be regarded as a generalization of the Lebesgue conditions on a numerical measure. If in addition (δ) holds, the set \mathfrak{M}^* of limits of bounded measures is isomorphic to either a set of successive non-negative integers or to a closed interval with left end points zero. If (ϵ) holds, i.e., if

$\mathfrak{M}_* = \mathfrak{M}^*$, the measure determined by (m, n, Φ) can be represented by a function $f(a)$ which is unique up to a multiplicative constant, assumes the values $0 \leq \xi \leq \infty$, and is such that $f(a) = 0$ if and only if $a = \theta$; $f(a) = f(b)$ if and only if a and b correspond under some measure-preserving transformation, and $f(a) < \infty$ if and only if $a < \infty$. It may be pointed out that no reference is made to any topological or metric properties of the set s .

It is not necessarily true that every numerical measure can be derived by the procedure above, for this would require the existence of a measure-preserving transformation between any two sets of equal measure. However, it is shown in Part III that this is true for many important measures. The general case can be treated by introducing into M an equivalence relation, equality of measure, which is countably disjointly additive, hereditary in the sense of 2.2, and for bounded elements is also subtractive and preserved by taking limits of monotonic sequences. It can easily be verified that the entire theory of Part II would hold in this case. Further if M has a countable basis, a family of isomorphisms can be defined which generate the given congruence, which reduces this case to the previous one.

PART III

12. Lattices satisfying (ζ) . Bases. Let M be a lattice satisfying (α) of §1 and $|a|$ be a functional defined on M and satisfying (ζ) :

- ia. $0 \leq |a| \leq \infty$, for any a of M , and $|a| = 0$ if and only if $a = \theta$;
- ib. if $|a| = \infty$, then there exists a sequence of elements a_n such that $|a_n| < \infty$, and $\sum a_n = a$;
- ii. $|\sum a_n| = \sum |a_n|$, for any sequence of disjoint a_n ;
- iii. for any $a \neq \theta$ there is an $a' < a$ such that $0 \leq |a'| \leq \infty$;
- iv. for any a of M , there is a countable set of elements a_n of M such that for any $a' < a$, and for any $\epsilon > 0$, some subsequence (a_m) covers a' and $|\sum a_m - a'| < \epsilon$.

The object of the next sections is to prove under conditions (ζ) that if $|a| = |a'|$ then there exists a σ -isomorphism ϕ of $I(a)$ onto $I(a')$ (see 1.1) such that for any $b < a$, $|\phi(b)| = |b|$, which in turn leads to the result that the $|a|$ can be determined up to a constant multiple by a set of isomorphisms satisfying (β) .

Henceforth a family of elements a_n satisfying iv of (ζ) will be called a *basis* of a , and any element of a basis a *base element*.

12.1. From the fact that if each element of a basis can be expressed as the sum of elements of a family (a_n') then (a_n') is also a basis, it follows that a has a basis of the following form:

Let (b_n) be any basis of a . Then let a_n be the n th element of the form $b_k - \sum_1^{k-1} b_i$ which is different from θ ; let $a_{n,0}$ be the first element of the form $a_n b_{k+i}$, where a_n is derived from b_k such that $\theta < a_n b_{k+i} < a_n$; and let $a_{n,1} = a_n - a_{n,0}$. Then if a_ν is defined for every sequence $\nu = (n, \epsilon_1, \dots, \epsilon_k)$, $\epsilon_i = \pm 1$, as some product $\epsilon_1 b_1 \dots \epsilon_k b_k$ where $-b = a - b$, then $a_{\nu,0}$ is given

by $\epsilon_1 b_1 \cdots \epsilon_k b_k b_m$, where m is the least value for which $0 < \epsilon_1 b_1 \cdots \epsilon_k b_k b_m < \epsilon_1 b_1 \cdots \epsilon_k b_k$, and $a_{\nu,1} = a_\nu - a_{\nu,0}$. It follows at once that every b_n is the sum of a finite set of the a_ν . Therefore (a_ν) is a basis. Let $n(\nu)$ be the number of terms in the sequence ν , and Γ_k the totality of sequences ν such that $n(\nu) = k$. We define $\gamma < \nu$, where $\gamma = (\epsilon_1, \epsilon_2, \dots, \epsilon_k)$ and $\nu = (\epsilon'_1, \epsilon'_2, \dots, \epsilon'_k)$ if $k > k'$, and $(\epsilon_1, \epsilon_2, \dots, \epsilon_k) \equiv (\epsilon'_1, \epsilon'_2, \dots, \epsilon'_k)$.

12.2. Then

- i. $\sum a_\gamma = a, \gamma \in \Gamma_n$, for any fixed n ;
- ii. $a_\nu a_\gamma = \theta$ if $n(\nu) = n(\gamma)$;
- iii. $a_\nu < a_\gamma$ if and only if $\nu < \gamma$;
- iv. $a_\gamma = \sum a_\nu, \nu < \gamma, n(\nu) = n_0 < n(\gamma)$.

Now let (a_ν) be any basis satisfying 12.2. It follows that:

12.3. If (a_{ν_i}) is any finite set of distinct base elements, then (a) $\prod a_{\nu_i}$ is equal either to θ or to the (necessarily unique) element of least subscript ν_k , according as the sequence is or is not nested; (b) if $\nu_i < \nu$ then $a_\nu - \sum a_{\nu_i}$ is the sum of a finite set of disjoint base elements of subscripts ν'_k such that $n(\nu'_i) = n(\nu_k)$; (c) any countable sum of base elements can be expressed as the sum of disjoint base elements. (For $\sum a_{\nu_i} = \sum (a_{\nu_i} - \sum_{j < i} a_{\nu_j})$, where the terms of the right side are disjoint, and from (b) each consists of a finite sum of disjoint elements.)

Further, if (a_ν) and (b_ν) are bases of a and b , respectively, which satisfy 12.2:

12.4. For any finite sequence (ν_i) of which we suppose ν_k the least (a) either both $\prod a_{\nu_i} = \prod b_{\nu_i} = \theta$, or $\prod a_{\nu_i} = a_{\nu_k}$ and $\prod b_{\nu_i} = b_{\nu_k}$; (b) if $\nu > \nu_i$, then either $a_\nu = \sum a_{\nu_i}$ and $b_\nu = \sum b_{\nu_i}$, or there exists a sequence (ν'_j) , such that $a_\nu - \sum a_{\nu_i} = \sum a_{\nu'_j}$ and $b_\nu - \sum b_{\nu_i} = \sum b_{\nu'_j}$. (c) For corresponding subsequences $(a_{\nu_i}), (b_{\nu_i})$ of (a_ν) and (b_ν) there exists a sequence (ν'_j) of subscripts such that $(a_{\nu'_j}), (b_{\nu'_j})$ are disjoint and $\sum a_{\nu_i} = \sum a_{\nu'_j}, \sum b_{\nu_i} = \sum b_{\nu'_j}$.

12.5. DEFINITION. Any (finite) union of base elements of any a of M is called a finite σ -element of the basis (a_ν) . The σ -element $\sum a_{\nu_i}$ will be denoted by $s_{[\nu_i]}(a)$.

From the previous results it follows that the σ -elements of any basis (a_ν) of an element a have the properties listed below:

12.6. All finite products, all countable sums, and the relative complement of a finite σ -element with respect to any σ -element are all either σ -elements or equal to θ . All σ -elements can be expressed as the sum of a sequence of disjoint base elements.

12.7. Any element $b < a$ is the product of a nested sequence of σ -elements $s_k(a)$ such that $\lim |s_k(a)| = |b|$.

For from the definition of a basis, for any sequence of positive numbers $\epsilon_n \rightarrow 0$, there exists a sequence of σ -elements $s_n(a) > b$, and such that $|s_n - b| < \epsilon_n$. But from 12.6, the elements $\prod_{i=1}^k s_n$ are σ -elements and satisfy the required conditions.

12.8. If (a_ν) , and (a'_ν) are bases of a and a' , respectively, and if the corresponding σ -elements $s_\alpha(a) \equiv \sum a_{\nu_i(\alpha)}$ and $s_\alpha(a') \equiv \sum a'_{\nu_i(\alpha)}$ are denoted by s_α and s'_α , then, for any sets of corresponding σ -elements s_i and s'_i either both members of any one of the pairs $\prod_i^k s_i$ and $\prod_i^k s'_i$ (where k is any finite number), $\sum s_i$ and $\sum s'_i$, and the pair of symmetric differences $s_1 - s_2$ and $s'_1 - s'_2$, are equal to θ , or in each case there exists a pair of corresponding σ -elements s_{α_1} and $s_{\alpha'_1}$, or s_{α_2} and $s_{\alpha'_2}$, or s_{α_3} and $s_{\alpha'_3}$, such that $\prod_i^k s_i = s_{\alpha_1}$ and $\prod_i^k s'_i = s'_{\alpha_1}$, or $\sum s_i = s_{\alpha_2}$ and $\sum s'_i = s_{\alpha'_2}$, or $s_1 - s_2 = s_{\alpha_3}$ and $s'_1 - s'_2 = s_{\alpha_3}$.

13. Infinite products. Regular bases. We prove first

13.1. LEMMA. If (a_{ν_i}) is any infinite sequence of base elements with distinct subscripts, then $\prod a_{\nu_i} = \theta$.

Proof. We may assume that the sequence can be arranged so that $\nu_i < \nu_{i-1}$, and so that $a_{\nu_i} < a_{\nu_{i-1}}$, for otherwise from (a) of 12.3 the conclusion that $\prod a_{\nu_i} = \theta$ already holds. If under this restriction $\prod a_{\nu_i} = a' \neq \theta$, there exists a ν_0 such that $|a_{\nu_0} a'| \neq 0$. Then since for all i , $a_{\nu_0} a_{\nu_i} \neq \theta$, it follows for all but the finite set of the a_{ν_i} for which $n(\nu_i) < n(\nu_0)$, that $a_{\nu_0} < a_{\nu_i}$, and so further that $a_{\nu_0} > a'$. By iii of (ζ), since $a' \neq \theta$, there exists an $a'' < a'$ such that $\theta < a'' < a'$. But every σ -element s covering a'' must also contain a' , and hence $|s - a''| > |s - a'| > 0$, which contradicts iv of (ζ). Therefore $a' = \theta$.

13.2. LEMMA. If (a_ν) is a basis of a satisfying 12.2, for any finite positive $\xi < |a|$ there exists a σ -element s such that $|s| = \xi$. More specifically, to any such ξ we can make correspond a unique σ -element s_ξ such that $|s_\xi| = \xi$, and if $0 < \xi_1 < \xi_2 \leq |a|$, $s_{\xi_1} < s_{\xi_2}$. Hence further each σ -element corresponds to at most one number ξ .

Proof. Since, for any n , $\sum a_\nu = a$, where $\nu \in \Gamma_n$, if we order the a_ν of Γ_n according to the first differences into a sequence $a_{n,k}$ and if we let $s_{n,k} = \sum_1^n a_{n,i}$ then for each $\xi > 0$ there corresponds a $k = k(\xi, n)$ such that

$$(1) \quad |s_{n,k}| \leq \xi \leq |s_{n,k+1}|,$$

and further

$$(2) \quad s_{n,k_n} < s_{n+1,k_{n+1}} < s_{n+1,k_{n+1}} < s_{n,k_{n+1}}.$$

From (1) and the fact that the elements $a_{n,k}$ are nested, we have

$$(3) \quad \xi - |\lim s_{n,k}| = \lim |a_{n,k}| = 0.$$

Therefore $|\sum s_{n,k}| = \xi$ and since if $\xi_1 < \xi_2$ then for each n $k_{n,\xi_1} < k_{n,\xi_2}$, we also have $\sum s_{n,k(\xi_1)} < \sum s_{n,k(\xi_2)}$, and we can define $s_\xi = \sum s_{n,k_{n,\xi}}$.

13.21. LEMMA. Every element a such that $|a| = \infty$, is the sum of a set of disjoint elements a_n such that $|a_n| = 1$ for each n .

Proof. Every such element a is the sum of a set of elements a'_n such that

$|a_n'| < \infty$, and which we may suppose disjoint and different from θ . Let k_i be a monotonic set of integers such that $|a_{k_i}| \leq i < |a_{k_{i+1}}|$, and let us define a nested sequence of elements a_i^* as follows: If the lower inequality holds, let $a_i^* = a_{k_i}'$; if $i > |a_{k_i}'|$, let $a_i^* = a_{k+a_i'}$, where $a_i'' \leq a_{k_{i+1}}' - a_{k_i}'$, and $|a_i''| = i - |a_{k_i}'|$. In either case, $|a_i^*| = i$, and $a_{k_i} \leq a_i^* < a_{k_{i+1}}$, whence $\lim a_i^* = a$. We can therefore define the required elements a_n as $a_1^*, a_n^* - a_{n-1}^*$.

13.3. DEFINITION. Let ν be any number of the form $k + m/2^n$, where k is a positive integer, $0 \leq m < 2^n$, and $m/2^n$ is expanded dyadically into a sequence of $n(\nu)$ terms. Then a basis (a_ν) of an element $a \neq \theta$ is called regular when $|a| < \infty$ if $0 \leq \nu < 1$ and $|a_\nu| = |a|/2^{n(\nu)}$, and when $|a| = \infty$ if $0 \leq \nu < \infty$ and $|a_\nu| = 1/2^{n(\nu)}$.

13.4. THEOREM. Every $a \neq \theta$ has a regular basis.

Proof. Suppose $|a| < \infty$, and let (a'_i) be any basis satisfying 12.2. Then there exist σ -elements $s_{n,m}$ for $n = 1, 2, \dots$, and $m = 1, 2, \dots, 2^n$, such that $|s_{n,m}| = m|a|/2^n$, and such that $m/2^n > m'/2^{n'}$ implies that $s_{n,m} > s_{n',m'}$. Then if ν is the dyadic expansion of $m/2^n$, $m < 2^n$, then we define a_ν as $s_{n,m+1} - s_{n,m}$, and we make the convention that $a_0 = 0$. If $s'_{n,m} = \sum a_{\nu_i}'$, where ν_i are the dyadic expansions of the numbers $m'/2^{n'}$, $m' \leq m < 2^n$, then $a'_\nu = s'_{n,m+1} - s'_{n,m} = \sum a_{\nu_j}$, where $|s'_{n,m}| \leq \nu_j \leq |s_{n,m+1}|$. Therefore from 12.1, (a_ν) is a basis, which can easily be seen to be regular. If $|a| = \infty$, let (a_k) be a sequence satisfying 13.21, and combine their bases for a basis for a .

13.5. If $|a| = |a'| \neq 0$, and if (a_ν) , (a'_ν) are regular bases of a and a' , respectively, and if as previously we denote the corresponding σ -elements $s_\alpha(a) = \sum a_{\nu_i(a)}$ and $s'_\alpha(a') = \sum a'_{\nu_i(a')}$ by s_α and s'_α respectively, then for any set of corresponding pairs it follows from 12.8 and from the fact that $|a_\nu| = |a'_\nu|$ that $|s_\alpha| = |s'_\alpha|$; $|s_1 - s_2| = |s'_1 - s'_2|$, where the differences are symmetric; $|\sum s_i| = |\sum s'_i|$; and $|\prod_i^k s_i| = |\prod_i^k s'_i|$.

14. The isomorphism ϕ . We are now in a position to establish the theorem referred to in §12, namely, that

14.1. THEOREM. If $|a| = |a'|$, then there exists a σ -isomorphism ϕ of $I(a)$ (see 1.1) onto $I(a')$ such that for any $b \leq a$, $|\phi(b)| = |b|$.

Proof. The theorem is trivial if $a = a' = \theta$. Otherwise, let (a_ν) , (a'_ν) , be regular bases of a and a' respectively. Then by 12.8 for any $b \leq a$ there exists a nested sequence of σ -elements $s_k > b$ such that $\lim |s_k - b| = 0$. We define for any $b < a$.

$$(1) \quad \phi(b) \equiv \phi(\prod s_k) \equiv \prod \phi(s'_k).$$

Obviously, ϕ assigns at least one image to every element of $I(a)$ and at least one counterimage to every element of $I(a')$. Further from 13.5 for each k , $|s_k| = |s'_k|$, whence for any image $\phi(b)$

$$(2) \quad |\phi(b)| = \lim |s'_k| = \lim |s_k| = |b|,$$

and the functional $|a|$ is invariant under ϕ . Also, if $b_1 \equiv \prod s_{k,1}$ and $b_2 \equiv \prod s_{k,2}$ are any two elements of $I(a)$, then for each k , $|s_{k,1} - s_{k,2}| = |s'_{k,1} - s'_{k,2}|$ where the differences are symmetric. Passing to the limit with respect to k ,

$$(3) \quad b_1 = b_2 \text{ if and only if } \phi(b_1) = \phi(b_2),$$

that is, the isomorphism $\phi(b)$ is univocal.

Now suppose $b_1 \equiv \prod s_{k,1} \leq b_2 \equiv \prod s_{k,2}$. Then from 12.8, the elements $s_{k,3} = s_{k,1} s_{k,2}$ form a nested sequence of σ -elements such that $b_1 \leq s_{k,3} \leq s_{k,2}$. Again from 13.5 for each k , $s'_{k,3} = s'_{k,1} s'_{k,2}$ so

$$(4) \quad \phi(b_1) = \prod s'_{k,3} \leq \prod s'_{k,2} = \phi(b_2).$$

Applying this same argument with reference to ϕ^{-1} , we see that

$$(5) \quad b_1 \leq b_2 \text{ if and only if } \phi(b_1) \leq \phi(b_2).$$

Let (a_r) be a sequence of elements of $I(a)$, and for each r let $(s_{k,r})$ be a nested sequence of σ -elements such that $a_r = \prod_k s_{k,r}$ and such that $|s_{k,r} - a_r| < 1/2^{k+r}$. Then for each k , $\sum_r s_{k,r}$ is a σ -element and

$$(6) \quad \sum a_r \leq \sum_r s_{k,r} \leq \sum_r s_{k-1,r}$$

$$(7) \quad \left| \sum_r s_{k,r} - \sum a_r \right| = \left| \sum_r \left[s_{k,r} - \left(\sum_r a_r \right) s_{k,r} \right] \right| \leq \sum_r |s_{k,r} - a_r| \leq 1/2^k,$$

and so

$$(8) \quad a_r = \prod_k \left(\sum_r s_{k,r} \right).$$

From (5), for each r , $\phi(a_r) \leq \phi(\sum a_r)$, and so further

$$(9) \quad \sum \phi(a_r) \leq \phi(\sum a_r).$$

But for each k ,

$$(10) \quad \left| \sum_r s'_{k,r} - \sum \phi(a_r) \right| \leq |s'_{k,r} - \phi(a_r)| = \sum (|s'_{k,r}| - |\phi(a_r)|) = \sum (|s_{k,r}| - |a_r|) \leq 1/2^k,$$

and passing to the limit with respect to k gives

$$(11) \quad \sum \phi(a_r) = \phi(\sum a_r).$$

From 12.6 for each finite r the elements $\prod_0^r s_{k,i}$ form a nested sequence of σ -elements such that $\prod_0^\infty a_i \leq \prod_0^r a_i \leq \prod_0^r s_{k,i}$. If for some i $|a_i| < \infty$, and if (r_i) is an increasing sequence of indices such that $|\prod_0^{r_i} a_r| - |\prod_0^\infty a_r| < 1/2^i$,

$$(12) \quad \left| \prod_0^{r_i} s_{k,r} - \prod_0^{\infty} a_{k,r} \right| \leq \left| \prod_0^{r_i} s_{k,r} - \prod_0^{r_i} a_r \right| + \left| \prod_0^{r_i} a_r - \prod_0^{\infty} a_r \right|.$$

But since $s_{k,r} \leq \prod_0^i a_r + \sum_0^i (s_{k,r} - a_r)$, (12) yields

$$(13) \quad \left| \prod_0^{r_i} s_{k,r} - \prod_0^{\infty} a_r \right| < 1/2^{k+i},$$

whence

$$(14) \quad \prod_0^{\infty} a_r = \prod_{k=0}^{\infty} \left(\prod_{r=0}^{r_k} s_{k,r} \right).$$

From (5), for each r , $\phi(\prod_r a_r) \leq \phi(a_r)$, and so

$$(15) \quad \phi\left(\prod a_r\right) \leq \prod \phi(a_r).$$

On the other hand,

$$(16) \quad \sum \phi(a_r) = \prod_{r=0}^{\infty} \prod_{k=0}^{\infty} s'_{k,r} = \prod_{k=0}^{\infty} \prod_{r=0}^{\infty} s'_{k,r} \leq \prod_{k=0}^{\infty} \prod_{r=0}^{r_k} s'_{k,r} = \phi\left(\prod a_r\right),$$

which with (15) gives

$$(17) \quad \prod \phi(a_r) = \phi\left(\prod a_r\right),$$

if for some i , $|a_i| < \infty$. If for all r , $|a_r| = \infty$, let $(a_{1,j})$ be a sequence of elements such that $\sum a_{1,j} = a_1$ and $|a_{1,j}| < \infty$ for all j 's. Then from (11) and (17),

$$(18) \quad \phi\left(\prod a_r\right) = \phi\left[\sum_j \left(\prod_r a_{1,j} a_r\right)\right] = \sum_j \left[\prod_r \phi(a_{1,j}) \phi(a_r)\right] = \prod \phi(a_r),$$

and this holds for any sequence (a_r) .

It is now verified by statements (2), (3), (11), and (18) that ϕ satisfies the conditions required in Theorem 14.1, the proof of which is now completed. We have as an immediate corollary to 14.1 that

14.11. The class Φ of all σ -isomorphisms ϕ between pairs of principal ideals of M is non-void and satisfies (β) of §1.

15. Application to Part II. Examples.

15.1. If now m is a field of sets upon which a measure function $|a|$ is defined, and if n is the ideal of sets for which $|a| = 0$, then we define a functional $|a|$ on the residue class ring $M = m/n$ as the common value of $|a|$ for all the sets $a \in a$. If $|a|$ is such that (ζ) holds, then by Theorem 15.1 we have a measuring system (m, n, Φ) (§6). If as before we denote the measure in the sense of Part II of any $a \in M$ by a , then $a = b$ if and only if $|a| = |b|$, $a = 0$ if and only if $a = \theta$, that is, if and only if $|a| = 0$; and $a = \sum a_n$ if and only if $|a| = \sum |a_n|$. Further from 13.2, a is bounded in the sense of Part II if, and from 15.21 only if, $|a| < \infty$. From the fact that $a < b$ if and only if $|a| < |b|$,

it is clear that the system (m, n, Φ) satisfies (δ) (§9) and (ϵ) (§10). Therefore the correspondence $f(\mathbf{a}) \equiv f(a) = |\mathbf{a}|$ is an isomorphism of the type described in Theorem 10.1 of the set \mathfrak{M} of measure values onto a closed interval. It follows from 10.1 that the ratios $|a|/|b|$ are uniquely determined by the system (m, n, Φ) as the numbers $\mu(a, b)$ of the lemma to 10.1, and so that $|\mathbf{a}|$ is determined up to a multiplicative constant.

15.2. EXAMPLES. Let s be any locally compact metric spaces and let $|\mathbf{a}|$ be a positive functional on the open sets with compact closures such that $|\mathbf{a}| = 0$ only on the empty set, and such that with any set of concentric spheres \mathbf{a}_ϵ of radius ϵ , $\lim |\mathbf{a}_\epsilon| = 0$. Then the measure generated in the usual fashion from $|\mathbf{a}|$ satisfies conditions (ζ) and so on can be obtained as in Part II. The Haar measure on a locally compact group is also covered by Part II, since the decomposition theorem of Banach and Tarski can be extended⁽³⁾ to this case; if the fundamental space is not a group, but metric and locally compact⁽⁴⁾, we need make only a trivial examination of the case in which points have positive measure.

15.3. The theorem of Banach and Tarski mentioned in the Introduction can be generalized under conditions of 15.1 in the following sense: Let (s_ν) be a regular basis of s (13.3). An isomorphism ϕ of Φ is called a *congruence* if ϕ is defined between two base elements, say s_ν and $s_{\nu'}$, where obviously $n(\nu) = n(\nu')$, and satisfies the following condition:

Let ν, ν' , be the dyadic expansions of $m/2^n, m'/2^n$, respectively. The elements $s_\gamma < s_\nu$, $[s_{\gamma'} < s_{\nu'}]$ are those such that $m/2^n \leq \gamma < m + 1/2^n$, $[m'/2^n \leq \gamma' < m' + 1/2^n]$. We define $\phi(s_\gamma)$ for all such γ as $s_{\gamma'}$, where $\gamma' = m'/2^n + (\gamma - m/2^n)$. Obviously, ϕ is completely determined for any $s' < s$ by the above relation. Then under conditions (ζ) and with this definition for congruence, a necessary and sufficient condition that two elements a and a' be of equal measure is that there exist sequences $(a_n), (a'_n)$ of disjoint elements such that $a = \sum a_n$, $a' = \sum a'_n$ and a_n and a'_n are congruent for each n .

The sufficiency is obvious. The necessity will be proved for the case for which $|a| = |a'| < \infty$ and both a and a' are different from s , from which the general case follows at once. The required proof then reduces to that of the following statements: a. If $|a| = |a'| < \infty$, then each element a or a' is the limit of a nested sequence of σ -elements of finite measure, of which corresponding elements have equal measure. b. Each of two σ -elements of equal measure is the sum of a family of disjoint base elements, corresponding elements having equal measure and so being congruent. For if a and b hold, and if we let the class of all sums of congruences, which class can be seen to satisfy (β) , be Φ , then we can apply Theorem 4.11 (which states that the respective limits of

⁽³⁾ See Footnote 1.

⁽⁴⁾ As in the "Note on Haar's measure," by S. Banach, in Saks' *Theory of the Integral*, Appendix II.

monotonic sequences of bounded equivalent elements are equivalent) and so get the desired theorem.

To prove a, we consider nested sequences of σ -elements $s_k > a$, and $s'_k > a'$, such that for some $\epsilon > 0$, $|s| > \epsilon \geq |s_k - a| \geq 2|s_{k+1} - a|$, and $|s'_k - a'| < |s_k - a|$. It follows that $(|s_k| - |s'_k|) < (|s_{k-1}| - |s'_{k-1}|)$. But for each p , the elements (s_ν) for which $n(\nu) = p$ can be ordered into a sequence $(s_{n,p})$. Then for each k and p , there is an integer $i = i_{k,p}$ for which $|\sum_{j=0}^{i-1} s_{j,p} + s'_k| \leq |s_k| - |s'_k| < |\sum_{j=0}^i s_{j,p} + s'_k|$, and the proof proceeds as in 13.2. Relation b is obvious in the case of two finite sums of base elements, for each can be expressed as the sum of elements s_ν such that $n(\nu) = n^*$, where n^* is the maximum value of $n(\nu)$ occurring in the given summands. Otherwise we write both summands as infinite sums. But two infinite sums can, as a result of 13.21 and the regularity of the basis (s_ν) , be expressed as sums of infinite disjoint sets of finite sums, of which corresponding finite sums have equal measure. Hence our assertion is proved.

In the case of euclidean r -space (ξ_1, \dots, ξ_r) we may consider our class Φ as sums of rigid motions and the regular basis as the family of cubes $[m/2^n \leq \xi_i < (m+1)/2^n]$, $i = 1, \dots, r$. Our result then reduces to the theorem of Banach and Tarski from the fact that every open set is the union of a set of disjoint half-open cubes, and that every measurable set is G_δ , modulo the null sets.

It can be deduced from the results of Part III that under the conditions of 15.1 that $M = m/n$ and the totality Φ of measure-preserving transformations on M are respectively isomorphic to the field M' of measurable sets modulo the null sets in euclidean space, and to the totality Φ' of measure-preserving transformations on M' .

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