

SUFFICIENT CONDITIONS FOR A WEAK RELATIVE MINIMUM IN THE PROBLEM OF BOLZA

BY
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1. Introduction. The past decade has brought forth great advances in the theory of the Bolza problem in the calculus of variations and in the theory of the problems (Lagrange, Mayer, and so on) subsumed under it. Ten years ago the necessary conditions of Weierstrass, Clebsch and Jacobi (or Mayer) were established only for minimizing curves normal on every subarc, while the sufficiency theorems needed even more drastic normality assumptions. Now the sufficiency theorems are established under the assumption that Lagrange multipliers $\lambda_0 \geq 0, \lambda_1(x), \dots, \lambda_m(x)$ exist with which the curve E_{12} satisfies the Euler equation, the transversality condition, and the strengthened Weierstrass, Clebsch and Jacobi conditions. The Euler equation, transversality condition, Weierstrass condition and Clebsch condition are proved necessary with no normality assumptions. Yet normality requirements have not been entirely dispensed with. I have shown⁽¹⁾ that for minimizing curves with order of abnormality 0 or 1 there are multipliers with which all the standard necessary conditions are satisfied. But an example shows that minimizing curves exist, having order of abnormality 2, which do not satisfy all the standard necessary conditions with any multipliers. Thus if the gap is to be closed and the necessary and sufficient conditions brought together for abnormal problems, our only hope is to strengthen the very strong sufficiency theorems of Hestenes, Morse, and Reid.

In a paper to be published in the American Mathematical Monthly, I have considered the problem of minimizing a function $f^0(x) \equiv f^0(x^1, \dots, x^n)$ subject to conditions

$$(1.1) \quad f^\beta(x) = 0 \quad (\beta = 1, \dots, m).$$

Subject to fairly obvious conditions of definition and differentiability, I have shown that the following condition is necessary in order that f^0 have a minimum subject to (1.1) at a point x_0 satisfying (1.1).

(N) *To each set (u^1, \dots, u^n) satisfying the conditions⁽²⁾*

$$(1.2) \quad f_{x^i}^\beta(x_0) u^i = 0 \quad (\beta = 1, \dots, m)$$

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(1) *On the second variation in certain abnormal problems of the calculus of variations*, American Journal of Mathematics, vol. 63 (1941), pp. 516–530.

(2) We use the summation convention, summing on all repeated indices.

there correspond multipliers $l_0 \geq 0, l_1, \dots, l_m$ not all zero such that

$$(1.3) \quad l_\alpha f_{x^i}^{(a)}(x_0) = 0$$

and

$$(1.4) \quad l^\alpha f_{x^i x^j}(x_0) u^i u^j \geq 0.$$

Correspondingly, we show that the following condition is sufficient for $f^0(x)$ to have a proper minimum subject to (1.1) at a point x_0 satisfying (1.1).

(S) To each set of numbers (u^1, \dots, u^n) not all zero satisfying (1.2) there corresponds a set of multipliers $l_0 \geq 0, l_1, \dots, l_m$ such that (1.3) holds and the left member of (1.4) is positive.

Here we have no normality assumptions whatever, and still the gap between conditions (N) and (S) is no greater than that between the necessary condition and the sufficient conditions for a minimum of a function $f(x)$ of a single real variable, without side conditions. The distinctive feature of conditions (N) and (S) is the dependence of the multipliers l_0, \dots, l_m on the solutions u^1, \dots, u^n of equations (1.2).

Let us now consider the Bolza problem of minimizing a functional

$$(1.5) \quad g(x_1, y(x_1), x_2, y(x_2)) + \int_{x_1}^{x_2} f(x, y, y') dx$$

in the class of functions

$$(1.6) \quad y^i = y^i(x) \quad (i = 1, \dots, n; x_1 \leq x \leq x_2)$$

satisfying certain differential equations

$$(1.7) \quad \phi^\alpha(x, y, y') = 0 \quad (\alpha = 1, \dots, m < n; x_1 \leq x \leq x_2)$$

and certain end conditions

$$(1.8) \quad \psi^\mu(x_1, y(x_1), x_2, y(x_2)) = 0 \quad (\mu = 1, \dots, p \leq 2n + 2).$$

Under the usual hypotheses on the functions, we are led by the theorems of the preceding paragraph to the following conjectures.

CONJECTURE (N). If a curve

$$(1.9) \quad E_{12}: \quad y^i = y^i(x), \quad x_1 \leq x \leq x_2,$$

minimizes the functional (1.5) in the class of curves satisfying (1.7) and (1.8) then for each set of functions $\eta^i(x)$ and numbers ξ_1, ξ_2 which satisfy the equations of variation of (1.7) and (1.8) there are multipliers $\lambda^0 \geq 0, \lambda^\alpha(x)$ not all zero such that for the function

$$(1.10) \quad F(x, y, y', \lambda) \equiv \lambda^0 f(x, y, y') + \lambda^\alpha(x) \phi^\alpha(x, y, y')$$

the Euler equations, transversality condition, Weierstrass condition and Clebsch condition are satisfied, and the second variation formed in the usual way from F and η and the end functions is non-negative.

CONJECTURE (S). *In order that a smooth curve (1.9) shall give a strong proper relative minimum to the functional (1.5) in the class of curves satisfying (1.7) and (1.8), it is sufficient that the following condition be satisfied. To each nonidentically zero set $[\eta^i(x), \xi_1, \xi_2]$ satisfying the equations of variation of (1.7) and (1.8) there shall correspond multipliers $\lambda^0 \geq 0, \lambda^\alpha(x)$ with which the Euler equation, transversality condition and strengthened Weierstrass and Clebsch conditions hold, and the second variation is positive.*

These conjectures are now being investigated, the first by Miss Mary Jane Cox and the second by Mr. Franklin G. Myers⁽³⁾. The purpose of the present paper is to establish the analogue of Conjecture (S) for weak relative minima. The proof is made by expansion methods, not as a matter of choice but rather as a matter of necessity. The field theory hardly seems applicable. We cannot even find a conjugate set of accessory extremals; worse, we cannot even set up an accessory problem, because of the dependence of the multipliers on the $\eta^i(x)$.

We shall obtain a sufficiency theorem for the parametric form of the problem, and from this we shall deduce a theorem for the non-parametric problem.

2. Statement of the problem. We shall study the Bolza problem in parametric form. On an open set R_1 of points $(y, r) = (y^0, \dots, y^n, r^0, \dots, r^n)$ in $(2n+2)$ -dimensional space we are given functions

$$f(y, y'), \quad \phi^\beta(y, y') \quad (\beta = 1, \dots, m < n)$$

of class C^2 . We assume that if (y, r) is in R_1 so is (y, kr) for all $k > 0$, and that f and the ϕ^β are positively homogeneous of degree 1 in r . Also, we are given functions

$$\theta(\alpha), \quad T^{is}(\alpha) \quad (i = 0, 1, \dots, n; s = 1, 2)$$

defined and of class C^2 on an open set R_2 in an r -dimensional space of points $(\alpha^1, \dots, \alpha^r)$.

If C is a rectifiable curve, having a representation

$$(2.1) \quad C: \quad y^i = y^i(t) \quad (t_1 \leq t \leq t_2; i = 0, 1, \dots, n)$$

with absolutely continuous functions $y^i(t)$, and α is a point in R_2 , we say that the set (C, α) is admissible, or that C is admissible with parameters α , if for almost all t the point (y, \dot{y}) lies in R_1 and satisfies the equations

⁽³⁾ Added in proof, June 1942: Miss Cox has established Conjecture N; Dr. Myers has shown that the hypotheses of Conjecture S yield a semi-strong minimum, and for certain integrands guarantee a strong minimum.

$$(2.2) \quad \phi^\beta(y(t), \dot{y}(t)) = 0 \quad (\beta = 1, \dots, m),$$

and the end conditions

$$(2.3) \quad y^i(t_s) = T^{is}(\alpha) \quad (i = 0, 1, \dots, n; s = 1, 2)$$

are satisfied.

The problem of Bolza is the problem of minimizing the functional

$$(2.4) \quad J(C, \alpha) = \theta(\alpha) + \int_{t_1}^{t_2} f(y(t), \dot{y}(t)) dt$$

on the class of all admissible sets (C, α) .

Following Carathéodory, we shall denote by $\dot{y}(t)$ the vector $(y^0(t), \dots, y^n(t))$ if this vector is defined and finite, and the vector $(0, \dots, 0)$ otherwise. We shall denote the length of a vector by enclosing the vector between vertical bars; thus

$$\begin{aligned} |y_1 - y_2| &= \left\{ \sum_{i=0}^n (y_1^i - y_2^i)^2 \right\}^{1/2}, \\ |\alpha| &= \left\{ \sum_{h=1}^r (\alpha^h)^2 \right\}^{1/2}. \end{aligned}$$

The concept of weak relative minimum has been given several equivalent formulations. For simplicity of notation, let us suppose that C_0 is a curve of class C^1 , represented by equations

$$(2.5) \quad y^i = y_0^i(t) \quad (t_1 \leq t \leq t_2)$$

in which the functions $y^i(t)$ are of class C^1 and $|y'| \neq 0$. A curve C is in the *first order ϵ -neighborhood* of C_0 if it has a Lipschitzian representation (2.1) such that

$$(2.6) \quad |y(t) - y_0(t)| < \epsilon \quad (t_1 \leq t \leq t_2)$$

and

$$(2.7) \quad \text{l.u.b. } \left| \frac{(y(t) - y_0(t))'}{|y_0'(t)|} \right| < \epsilon.$$

This neighborhood is easily seen to be independent of the particular representation of C_0 .

We say that the functional $J(C, \alpha)$ has a *weak relative minimum* at the admissible set (C_0, α_0) if there is a positive number ϵ such that

$$J(C, \alpha) \geq J(C_0, \alpha_0)$$

for all admissible sets (C, α) having C in the first order ϵ -neighborhood of C_0

and $|\alpha - \alpha_0| < \epsilon$. The minimum is *proper* if equality is excluded from (2.7) except when $(C, \alpha) = (C_0, \alpha_0)$. In this paper we shall set forth conditions which ensure that a set (C_0, α_0) gives a proper weak relative minimum to $J(C, \alpha)$.

3. Statement of the theorem. We use a slight modification of the summation convention. The repetition of an index in a term connotes the summation of the values of that term over all values of the repeated index, except that the indices q and s are exempted; we never sum over values of q or of s . As usual, for each set of numbers $\lambda^0, \dots, \lambda^m$ we define

$$(3.1) \quad F(y, r, \lambda) \equiv \lambda^0 f(y, r) + \lambda^\beta \phi^\beta(y, r).$$

Henceforth we suppose that (C_0, α_0) is an admissible set, the curve C_0 being represented by (2.5) with functions $y_0^i(t)$ of class C^1 which have $|y_0'| > 0$. The curve C_0 satisfies the *Euler equations* with multipliers $\lambda^0, \lambda^1(t), \dots, \lambda^m(t)$ if

$$(3.2) \quad \frac{d}{dt} F_{rt}(y_0(t), y_0'(t), \lambda) = F_{yt}(y_0(t), y_0'(t), \lambda) \quad (i = 0, 1, \dots, n; t_1 \leqq t \leqq t_2).$$

The set (C_0, α_0) satisfies the *transversality condition* with multipliers $\lambda^0, \lambda^1(t), \dots, \lambda^m(t)$ if

$$(3.3) \quad \begin{aligned} \lambda^0 \theta_h(\alpha) + F_{rt}(y_0(t_2), y_0'(t_2), \lambda(t_2)) T_h^{i_2}(\alpha_0) \\ - F_{rt}(y_0(t_1), y_0'(t_1), \lambda(t_1)) T_h^{i_1}(\alpha_0) = 0 \\ (h = 1, \dots, r), \end{aligned}$$

the subscript h denoting partial differentiation with respect to α^h .

For the curve C_0 it is well known that the quadratic form

$$(3.4) \quad F_{rrtj}(y_0(t), y_0'(t), \lambda) v^i v^j$$

vanishes whenever the vector v is linearly dependent on $y_0'(t)$; that is, whenever there is a number k such that

$$v^i = k y_0^{i'}(t) \quad (i = 0, 1, \dots, n).$$

The curve C_0 is said to satisfy the *strengthened Clebsch condition* if for each t in the interval (3.4) and every vector v linearly independent of $y^{i'}(t)$ and satisfying the equations

$$(3.5) \quad \phi_{rt}^\beta(y_0(t), y_0'(t)) v^i = 0 \quad (\beta = 1, \dots, m)$$

the quadratic form (3.4) is positive.

Our definition of admissible variations is somewhat more inclusive than the usual definition. We define an *admissible variation set* to be a set $[\eta(t), u] = [\eta^0(t), \dots, \eta^n(t), u^1, \dots, u^k]$ in which the functions $\eta^i(t)$ are ab-

solutely continuous, have derivatives integrable together with their squares, and satisfy the equations of variations of (2.2), that is, the equations

$$(3.6) \quad \Phi^\beta(t, \eta, \dot{\eta}) \equiv \phi_{y^t}^\beta(y_0(t), y_0'(t))\eta^i(t) + \phi_{r^t}^\beta(y_0(t), y_0'(t))\dot{\eta}^i(t) = 0 \\ (\beta = 1, \dots, m; t_1 \leqq t \leqq t_2),$$

for almost all values of t in the interval $[t_1, t_2]$, and the numbers u^h satisfy the equations of variation of (2.3), which are

$$(3.7) \quad \eta^i(t_s) - T_h^{is}(\alpha_0)u^h = 0 \quad (s = 1, 2; i = 0, 1, \dots, n).$$

If $[\eta, u]$ is an admissible variation set, and $\lambda^0, \dots, \lambda^m(t)$ are multipliers, we define the second variation due to $[\eta, u]$ by the equation

$$(3.8) \quad J_2[\eta, u, \lambda] \equiv b_{hk}u^h u^k + \int_{t_1}^{t_2} 2\omega(t, \eta, \dot{\eta})dt,$$

where

$$(3.9) \quad b_{hk} = \theta_{hk}(\alpha_0) + F_{rt}(y_0(t_2), y_0'(t_2), \lambda(t_2))T_h^{i2}(\alpha_0) \\ - F_{rt}(y_0(t_1), y_0'(t_1), \lambda(t_1))T_h^{i1}(\alpha_0)$$

and

$$(3.10) \quad 2\omega(t, \eta, \rho) = F_{yt}y^i\eta^i + 2F_{ytr}\eta^i\rho^i + F_{rt}\rho^i\rho^i,$$

the arguments of the functions in the right member being $(y_0(t), y_0'(t), \lambda(t))$.

An admissible variation set $[\eta, u]$ will be called *essentially null* if

$$(3.11) \quad u^h = 0 \quad (h = 1, \dots, r)$$

and there is a function $\rho(t)$ such that

$$(3.12) \quad \eta^i(t) = \rho(t)y_0^i(t) \quad (t_1 \leqq t \leqq t_2; i = 0, 1, \dots, n).$$

For such sets the second variation is known to have the value 0.

We can now state our principal theorem.

THEOREM I. *Let the following hypotheses be satisfied.*

- (1) *The set $[C_0, \alpha_0]$ is admissible and the curve C_0 is a simple arc of class (4) C^2 , represented by equation (2.5) with functions $y_0^i(t)$ of class C^2 .*
- (2) *For all t in the interval $[t_1, t_2]$ the matrix*

$$\|\phi_{r^t}^\beta(y_0(t), y_0'(t))\|$$

has rank m .

- (3) *To each admissible variation set $[\eta, u]$ which is not essentially null there corresponds a set of continuous multipliers $\lambda^0 \geqq 0, \lambda^1(t), \dots, \lambda^m(t)$ with which*

(4) If we assume C to be of class C^1 , in the presence of hypothesis (3) we can show by the Hilbert differentiability theorem that it must be of class C^2 .

the Euler equations, transversality condition and strengthened Clebsch condition are satisfied, and with which the inequality

$$(3.13) \quad J_2(\eta, u, \lambda) > 0$$

holds.

Then (C_0, α_0) gives $J(C, \alpha)$ a proper weak relative minimum on the class of admissible sets (C, α) .

4. A change of parameter. There is no loss of generality in assuming that (2.3) is the representation of C_0 in terms of arc length, so that $t_1=0$ and t_2 is the length of C_0 and

$$(4.1) \quad |y_0'(t)| = 1 \quad (t_1 \leq t \leq t_2).$$

Our proof will be indirect; we assume the theorem false, and arrive (in §9) at a contradiction.

If the theorem is false, there exists a sequence of admissible sets $[C_q, \alpha_q]$ ($q=1, 2, 3, \dots$) with the following properties. Each set (C_q, α_q) is distinct from (C_0, α_0) . The curves C_q have Lipschitzian representations

$$(4.2) \quad C_q: \quad y^i = y_q^i(t) \quad (t_1 \leq t \leq t_2)$$

such that

$$(4.3) \quad \lim_{q \rightarrow \infty} y_q^i(t) = y_0^i(t)$$

uniformly in $[t_1, t_2]$ and

$$(4.4) \quad \lim_{q \rightarrow \infty} (y_q^i(t) - y_0^i(t)) = 0$$

uniformly in $[t_1, t_2]$. The α_q satisfy

$$(4.5) \quad \lim_{q \rightarrow \infty} \alpha_q^h = \alpha_0^h \quad (h = 1, \dots, r).$$

And finally

$$(4.6) \quad J(C_q, \alpha_q) \leq J(C_0, \alpha_0) \quad (q = 1, 2, \dots).$$

For convenience in our proofs it is highly desirable to choose a particular type of representation of the curves C_q . Specifically, we shall prove the following statement.

(4.7) There is no loss of generality in assuming that the representations (4.2) satisfy the equations

$$(4.8) \quad [y_q^i(t) - y_0^i(t)] y_0^{i'}(t) = A_q t + B_q,$$

where A_q and B_q are constants.

The range of definition of the functions $y_0^i(t)$ may easily be extended to an interval $t_1 - \epsilon < t < t_2 + \epsilon$ in such a way that they remain of class C^2 and the curve $y^i = y_0^i(t)$ remains a simple arc. Consider the equations

$$(4.9) \quad [y^i - y_0^i(t)] y_0^{i'}(t) - At - B = 0.$$

The equations have the initial solutions

$$(4.10) \quad y^i = y_0^i(t); \quad t_1 \leq t \leq t_2; \quad A = B = 0.$$

On a neighborhood of the set (4.10) the left member of (4.9) is of class C^1 ; on the set (4.10) the partial derivative of the left member with respect to t has the value 1. Hence by a known theorem on implicit functions the equation (4.9) has a solution

$$(4.11) \quad t = t(y, A, B)$$

defined and of class C^1 for all (y, A, B) in a set

$$(4.12) \quad |A| < \delta, \quad |B| < \delta, \quad |y - y_0(t)| < \delta \quad (t_1 \leq t \leq t_2; \delta > 0)$$

and assuming values in the interval $(t_1 - \epsilon, t_2 + \epsilon)$.

The constants A_q, B_q of (4.8) are determined by giving t the values t_1, t_2 . By (4.1) and (4.3) the left member of (4.8) approaches zero uniformly, hence A_q and B_q both tend to zero as $q \rightarrow \infty$. We may therefore assume

$$(4.13) \quad |A_q| < \delta, \quad |B_q| < \delta$$

for all q . For all but a finite number of values of q , which we discard, each point $y_q(\tau)$ ($t_1 \leq \tau \leq t_2$) lies within a distance δ of some point of C . Hence the functions

$$(4.14) \quad t_q(\tau) \equiv t(y_q(\tau), A_q, B_q) \quad (t_1 \leq \tau \leq t_2)$$

are defined and satisfy the equation

$$(4.15) \quad [y_q^i(\tau) - y_0^i(t_q(\tau))] y_0^{i'}(t_q(\tau)) = A_q t_q(\tau) + B_q.$$

The functions $t_q(\tau)$ are clearly Lipschitzian. The constants A_q and B_q were so chosen that equation (4.8) is satisfied at t_1 and t_2 , whence

$$(4.16) \quad t_q(t_1) = t_1, \quad t_q(t_2) = t_2.$$

By (4.3), $y_q(\tau)$ tends to $y_0(\tau)$ uniformly in the interval $[t_1, t_2]$, so by the definition (4.14) we have

$$(4.17) \quad \lim_{q \rightarrow \infty} t_q(\tau) = t(y_0(\tau), 0, 0) = \tau$$

uniformly for $t_1 \leq \tau \leq t_2$. Since

$$|y_q(\tau) - y_0(t_q(\tau))| \leq |y_q(\tau) - y_0(\tau)| + |y_0(\tau) - y_0(t_q(\tau))|,$$

relations (4.3) and (4.17) imply

$$(4.18) \quad \lim_{q \rightarrow \infty} |y_q(\tau) - y_0(t_q(\tau))| = 0$$

uniformly on $t_1 \leq \tau \leq t_2$.

Let M_q be the set of values of τ in $t_1 \leq \tau \leq t_2$ for which $y'_q(\tau)$ is defined; this set constitutes almost all of the interval $t_1 \leq \tau \leq t_2$. On this set the function $t_q(\tau)$ also has a derivative, as we see from (4.14). For all τ in M_q the inequality

$$(4.19) \quad |y'_q(\tau) - y'_0(t_q(\tau))| \leq |y'_q(\tau) - y'_0(\tau)| + |y'_0(\tau) - y'_0(t_q(\tau))|$$

holds. So if γ is an arbitrary positive number, for all large q the inequality

$$(4.20) \quad |y'_q(\tau) - y'_0(t_q(\tau))| < \gamma$$

holds on M_q , as follows from (4.19), (4.4), (4.17) and the continuity of y'_0 .

By differentiating both members of (4.15) we find that on M_q the equation

$$(4.21) \quad [A_q + 1 - \{y_q^i(\tau) - y_0^i(t_q(\tau))\} y_0^{i''}(t_q(\tau))] t_q'(\tau) = y_q^{i''}(\tau) y_0^{i''}(t_q(\tau))$$

holds. Let η be an arbitrary positive number less than 1. Since A_q tends to zero and (4.18) holds, the quantity in square brackets in (4.21) lies between $(1 + \eta/2n)^{-1/2}$ and $(1 + \eta/2n)^{1/2}$ for all sufficiently large q . Since $|y'_0| = 1$, by (4.20) with proper choice of⁽⁶⁾ γ the right member of (4.21) lies between the same bounds for all large q . Hence for all sufficiently large q we have

$$(4.22) \quad \left(1 + \frac{\eta}{2n}\right)^{-1} < t_q'(\tau) < 1 + \frac{\eta}{2n}.$$

In particular, if we choose $\eta = 1$ we find that for all but a finite number of values of q (which we discard from further consideration) the value of $t_q'(\tau)$ exceeds $2/3$ on M_q , which is almost all of $t_1 \leq \tau \leq t_2$. Hence $t_q(\tau)$ has a Lipschitzian inverse; we denote it by $\tau_q(t)$. By (4.16) we see that

$$\tau_q(t_1) = t_1, \quad \tau_q(t_2) = t_2.$$

If N_q is the image of M_q under the mapping $t = t_q(\tau)$, then N_q constitutes almost all of $t_1 \leq t \leq t_2$. On it the derivative of $\tau_q(t)$ exists and is the reciprocal of the derivative of $t_q(\tau)$, so by (4.22)

$$(4.23) \quad \left(1 + \frac{\eta}{2n}\right)^{-1} < \tau_q'(t) < 1 + \frac{\eta}{2n}$$

for all t in N_q , q sufficiently large.

We now show that the equations

$$\underline{y^i = y_q^i(\tau_q(t))} \quad (t_1 \leq t \leq t_2)$$

⁽⁶⁾ The choice $\gamma = 1 - (1 + \eta/2n)^{-1/2}$ will serve.

form the desired representation of C_q . By (4.15) the equation (4.8) is satisfied. By (4.18), the new functions satisfy the convergence condition (4.3). Let η be an arbitrary number between 0 and 1. In (4.20) we choose $\gamma = \eta/4n$. For all large q inequalities (4.20) and (4.23) hold on N_q ; and t being arc length on y_0 , we have

$$\begin{aligned} \left| \frac{d}{dt} [y_q(\tau_q(t)) - y_0(t)] \right| &= |y'_q(\tau_q(t))\tau'_q(t) - y'_0(t)| \\ &\leq |y'_q(\tau_q(t)) - y'_0(t)|\tau'_q(t) + |\tau'_q(t) - 1| \\ &< \gamma \left(1 + \frac{\eta}{2n} \right) + \frac{\eta}{2n} \\ &< \frac{\eta}{n}. \end{aligned}$$

By integrating we see that each component

$$y_q^i(\tau_q(t)) - y_0^i(t)$$

satisfies a Lipschitz condition of constant η/n , so the derivative has absolute value at most η/n where it is defined. Thus

$$(4.24) \quad | \{y_q(\tau_q(t)) - y_0(t)\}_i | \leq \eta$$

whenever all the derivatives are defined. Elsewhere inequality (4.24) is trivial, the left member being zero. That is, the left member of (4.24) tends uniformly to zero as $q \rightarrow \infty$, completing the proof.

5. A convergence lemma. For each q we define a non-negative number k_q by the equation

$$(5.1) \quad k_q^2 = |\alpha_q - \alpha_0|^2 + (\max |y_q(t) - y_0(t)|)^2 + \int_{t_1}^{t_2} |\dot{y}_q(t) - \dot{y}_0(t)|^2 dt.$$

These numbers are actually positive; otherwise (C_q, α_q) would be identical with (C_0, α_0) , contrary to hypothesis. Next we define

$$(5.2) \quad u_q^h = (\alpha_q^h - \alpha_0^h)/k_q \quad (h = 1, \dots, r; q = 1, 2, \dots),$$

$$(5.3) \quad \eta_q^i(t) = (y_q^i(t) - y_0^i(t))/k_q \quad (i = 0, 1, \dots, n; q = 1, 2, \dots; t_1 \leq t \leq t_2).$$

From the preceding equations we have at once for each q the equation

$$(5.4) \quad |u_q|^2 + (\max |\eta_q(t)|)^2 + \int_{t_1}^{t_2} |\dot{\eta}_q(t)|^2 dt = 1,$$

so that each summand on the left is at most 1. By the Bolzano-Weierstrass

theorem we can select a subsequence of the u_q which converges to a limit u_0 . There is no loss of generality in supposing that $\{u_q\}$ is already such a sequence, so that

$$(5.5) \quad \lim_{q \rightarrow \infty} u_q^h = u_0^h \quad (h = 1, \dots, r).$$

Let $(\alpha_1, \beta_1), \dots, (\alpha_l, \beta_l)$ be a set of nonoverlapping subintervals of $[t_1, t_2]$, and let E be the set consisting of the sum of these intervals. Then by the inequality of Schwarz

$$(5.6) \quad \begin{aligned} \sum_{i=1}^l |\eta_q(\beta_i) - \eta_q(\alpha_i)| &= \sum_{i=1}^l \left| \int_{\alpha_i}^{\beta_i} \dot{\eta}_q dt \right| \\ &\leq \int_E |\dot{\eta}_q| dt \leq \left[\int_E |\dot{\eta}_q|^2 dt \right]^{1/2} \cdot \left[\int_E 1 dt \right]^{1/2} \\ &\leq 1 \cdot [mE]^{1/2} = \left[\sum_{i=1}^l (\beta_i - \alpha_i) \right]^{1/2}. \end{aligned}$$

In particular, letting $l=1$ we see that the η_q are equi-continuous. Since they have the uniform bound 1 by (5.4), we know by Ascoli's theorem that the sequence $\{\eta_q\}$ contains a subsequence converging uniformly to a limit function $\eta_0(t)$. We may suppose that $\{\eta_q\}$ is already such a subsequence, so that

$$(5.7) \quad \lim_{q \rightarrow \infty} \eta_q^i(t) = \eta_0^i(t) \quad (i = 0, 1, \dots, n)$$

uniformly on the interval $[t_1, t_2]$.

If we let q tend to ∞ in (5.6) we obtain

$$\sum_{i=1}^l |\eta_0(\beta_i) - \eta_0(\alpha_i)| \leq \left[\sum_{i=1}^l (\beta_i - \alpha_i) \right]^{1/2},$$

so that the functions $\eta_0^i(t)$ are absolutely continuous. We wish now to show that their derivatives have integrable squares.

LEMMA 1⁽⁶⁾. *Under the hypotheses on the η_q , the squares of the derivatives of the $\eta_0^i(t)$ are summable, and*

$$(5.8) \quad \int_{t_1}^{t_2} |\dot{\eta}_0|^2 dt \leq \liminf_{q \rightarrow \infty} \int_{t_1}^{t_2} |\dot{\eta}_q|^2 dt \leq 1.$$

(6) This is in fact a corollary of almost any theorem on semi-continuity of integrals in non-parametric form. Lemma 3 is also a consequence of known theorems. But it seems preferable to give the fairly simple proofs of these lemmas rather than refer the reader to some exposition containing complications not essential for our present needs.

For each positive integer k we subdivide the interval $[t_1, t_2]$ into 2^k equal subintervals by points

$$(5.9) \quad t_1 = \tau_1 < \tau_2 < \cdots < \tau_{2k+1} = t_2,$$

and define piecewise linear functions $p_k^i(t)$ which coincide with $\eta^i(t)$ at each τ_l ($l=1, \dots, 2^k+1$) and are linear between. Since the derivatives of these functions are constant on each subinterval we find

$$(5.10) \quad \int_{t_1}^{t_2} |\dot{p}_k|^2 dt = \sum_{l=1}^{2^k} |\eta_0(\tau_{l+1}) - \eta_0(\tau_l)|^2 / (\tau_{l+1} - \tau_l).$$

By Schwarz' inequality,

$$|\eta_q(\tau_{l+1}) - \eta_q(\tau_l)|^2 \leq \left[\int_{\tau_l}^{\tau_{l+1}} |\dot{\eta}_q|^2 dt \right]^2 \leq (\tau_{l+1} - \tau_l) \int_{\tau_l}^{\tau_{l+1}} |\dot{\eta}_q|^2 dt.$$

Hence

$$(5.11) \quad \liminf_{q \rightarrow \infty} \int_{\tau_l}^{\tau_{l+1}} |\dot{\eta}_q|^2 dt \geq |\eta_0(\tau_{l+1}) - \eta_0(\tau_l)|^2 / (\tau_{l+1} - \tau_l).$$

This, with (5.10), yields

$$(5.12) \quad \liminf_{q \rightarrow \infty} \int_{t_1}^{t_2} |\dot{\eta}_q|^2 dt \geq \int_{t_1}^{t_2} |\dot{p}_k|^2 dt.$$

The integrand on the right is non-negative, and except on the set of measure zero on which one or more of the functions η_0, p_1, \dots are non-differentiable its limit as $k \rightarrow \infty$ is $|\dot{\eta}_0(t)|^2$. By Fatou's lemma,

$$(5.13) \quad \liminf_{q \rightarrow \infty} \int_{t_1}^{t_2} |\dot{\eta}_q|^2 dt \geq \liminf_{k \rightarrow \infty} \int_{t_1}^{t_2} |\dot{p}_k|^2 dt \geq \int_{t_1}^{t_2} |\dot{\eta}_0|^2 dt.$$

This establishes the lemma.

6. The equations of variation. In order to show that the $\eta_0(t)$ satisfy the equations of variation (3.6) it is convenient to prove a lemma.

LEMMA 2. *If $g(t)$ is summable together with its square on the interval $[t_1, t_2]$, then*

$$(6.1) \quad \lim_{q \rightarrow \infty} \int_{t_1}^{t_2} g(t)(\eta_q^i - \eta_0^i) dt = \lim_{q \rightarrow \infty} \int_{t_1}^{t_2} g(t)(\dot{\eta}_q^i - \dot{\eta}_0^i) dt = 0 \quad (i = 0, 1, \dots, n).$$

The vanishing of the first limit in (6.1) is easily established, for

$$(6.2) \quad \left| \int_{t_1}^{t_2} g(t)(\eta_q^i - \eta_0^i) dt \right| \leq \max |\eta_q^i - \eta_0^i| \cdot \int_{t_1}^{t_2} |g| dt,$$

and the right member tends to zero by (5.9). Consider then the other limit. Let ϵ be an arbitrary positive number. As is well known, it is possible to find a polynomial $p(t)$ such that

$$(6.3) \quad \int_{t_1}^{t_2} (g(t) - p(t))^2 dt < \epsilon^2/16.$$

By (5.4) and (5.8), we have

$$\left(\int_{t_1}^{t_2} (\dot{\eta}_m^i)^2 dt \right)^{1/2} \leq 1 \quad (m = 0, 1, 2, \dots).$$

So by Minkowski's inequality

$$\left(\int_{t_1}^{t_2} (\dot{\eta}_q^i - \dot{\eta}_0^i)^2 dt \right)^{1/2} \leq 2 \quad (q = 1, 2, \dots).$$

From this and Schwarz' inequality we obtain

$$(6.4) \quad \begin{aligned} & \left| \int_{t_1}^{t_2} g \cdot (\dot{\eta}_q^i - \dot{\eta}_0^i) dt - \int_{t_1}^{t_2} p \cdot (\dot{\eta}_q^i - \dot{\eta}_0^i) dt \right| \\ & \leq \left\{ \int_{t_1}^{t_2} (g - p)^2 dt \right\}^{1/2} \left\{ \int_{t_1}^{t_2} (\dot{\eta}_q^i - \dot{\eta}_0^i)^2 dt \right\}^{1/2} < \epsilon/2. \end{aligned}$$

By integration by parts,

$$(6.5) \quad \int_{t_1}^{t_2} p(t)(\dot{\eta}_q^i - \dot{\eta}_0^i) dt = p(t)(\dot{\eta}_q^i - \dot{\eta}_0^i) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} p'(t)(\eta_q^i - \eta_0^i) dt.$$

The first term on the right tends to zero by (5.7), and the second tends to zero by the part of the lemma already proved. Therefore for all q greater than a certain q_ϵ we have

$$(6.6) \quad \left| \int_{t_1}^{t_2} p(t)(\dot{\eta}_q^i - \dot{\eta}_0^i) dt \right| < \epsilon/2.$$

Now (6.4) and (6.6) imply

$$\left| \int_{t_1}^{t_2} g(t)(\dot{\eta}_q^i - \dot{\eta}_0^i) dt \right| < \epsilon$$

if $q > q_\epsilon$, and our lemma is established.

The curves C_0 and C_q are admissible; so the equations

$$(6.7) \quad \phi^\beta(y_q(t), \dot{y}_q(t)) = \phi^\beta(y_0(t), y'_0(t)) = 0$$

hold for almost all t in the interval $[t_1, t_2]$. Hence, recalling (5.3), the equation

$$(6.8) \quad A_q^{\beta,i}(t)\eta_q^i(t) + B_q^{\beta,i}(t)\dot{\eta}_q^i(t) = 0$$

holds for almost all t , the coefficients being defined by equations

$$(6.9) \quad A_q^{\beta,i}(t) = \int_0^1 \phi_{y^i}^\beta(y_0(t) + \tau[y_q(t) - y_0(t)], y_0'(t) + \tau[\dot{y}_q(t) - y_0'(t)]) d\tau,$$

$$B_q^{\beta,i}(t) = \int_0^1 \phi_{r^i}^\beta(y_0 + \tau[y_q - y_0], y_0' + \tau[\dot{y}_q - y_0']) d\tau.$$

From (4.3) and (4.4) we deduce

$$(6.10) \quad \lim_{q \rightarrow \infty} A_q^{\beta,i}(t) = \phi_{y^i}^\beta(y_0(t), y_0'(t)),$$

$$(6.11) \quad \lim_{q \rightarrow \infty} B_q^{\beta,i}(t) = \phi_{r^i}^\beta(y_0(t), y_0'(t))$$

uniformly on $t_1 \leq t \leq t_2$. For each such t we have, by (6.8),

$$(6.12) \quad \int_{t_1}^t \{A_q^{\beta,i}(t)\eta_q^i(t) + \phi_{r^i}^\beta(y_0, y_0')\dot{\eta}_q^i(t)\} dt$$

$$= \int_{t_1}^{t_2} \{\phi_{r^i}^\beta(y_0, y_0') - B_q^{\beta,i}(t)\} \dot{\eta}_q^i(t) dt.$$

By the Schwarz inequality, with (5.4) and (6.11), the right member of (6.12) approaches 0 as $q \rightarrow \infty$. The limit of the left member is readily found with the help of (6.10) and Lemma 2; we obtain

$$(6.13) \quad \int_{t_1}^t \{\phi_{y^i}^\beta(y_0(t), y_0'(t))\eta_0^i(t) + \phi_{r^i}^\beta\dot{\eta}_0^i(t)\} dt = 0.$$

(Here and henceforth we indicate the arguments only in the first function in a bracketed expression whenever the remaining terms have the same arguments.) By differentiating both members of (6.13) we find that $\eta_0(t)$ satisfies equations (3.6) for almost all t in the interval $t_1 \leq t \leq t_2$.

Since the sets (C_0, α_0) and (C_q, α_q) are admissible, the equations

$$(6.14) \quad \begin{aligned} y_0^i(t_s) &= T^{is}(\alpha_0) \\ y_q^i(t_s) &= T^{is}(\alpha_q) \end{aligned} \quad (i = 0, 1, \dots, n; s = 1, 2)$$

are satisfied. With (5.2) and (5.3), this implies

$$(6.15) \quad k_q \eta_q^i(t_s) = T^{is}(\alpha_0 + k_q u_q) - T^{is}(\alpha_0).$$

So by the theorem of mean value there is an $\bar{\alpha}_q$ on the line segment joining α_0 and α_q such that

$$(6.16) \quad \eta_q^i(t_s) = T_h^{is}(\bar{\alpha}_q) u_q^h.$$

By passage to the limit we find

$$(6.17) \quad \eta_0^i(t_s) = T_h^{is}(\alpha_0) u_0^h \quad (i = 0, 1, \dots, n; s = 1, 2).$$

Thus it has been shown that $[\eta_0, u_0]$ is an admissible set, as defined in §3.

7. A semi-continuity proof. In the course of our proof we shall have need of a generalization of Lemma 1. Let us suppose that $a_{ij}(t)$, $b_{ij}(t)$, $c_{ij}(t)$ ($i, j = 0, 1, \dots, n$) are functions defined and continuous on the interval $t_1 \leq t \leq t_2$. We define

$$(7.1) \quad I(\eta) = \int_{t_1}^{t_2} [a_{ij}(t) \eta^i(t) \eta^j(t) + 2b_{ij}\eta^i \dot{\eta}^j + c_{ij}\dot{\eta}^i \dot{\eta}^j] dt.$$

Concerning such integrals we prove a sequence of three lemmas, the last of which is the one needed later.

LEMMA 3. *If for each t in the interval $[t_1, t_2]$ the quadratic form $c_{ij}v^i v^j$ is positive definite, then*

$$(7.2) \quad \liminf_{q \rightarrow \infty} I(\eta_q) \geq I(\eta_0).$$

It is evident that there is no loss of generality in assuming

$$(7.3) \quad a_{ij}(t) = a_{ji}(t), \quad c_{ij}(t) = c_{ji}(t) \quad (t_1 \leq t \leq t_2; i, j = 0, 1, \dots, n).$$

We readily compute

$$(7.4) \quad \begin{aligned} a_{ij}\eta_q^i\eta_q^j + 2b_{ij}\eta_q^i\dot{\eta}_q^j + c_{ij}\dot{\eta}_q^i\dot{\eta}_q^j &= a_{ij}\eta_0^i\eta_0^j + 2b_{ij}\eta_0^i\dot{\eta}_0^j + c_{ij}\dot{\eta}_0^i\dot{\eta}_0^j \\ &\quad + 2(a_{ij}\eta_0^i + b_{ij}\dot{\eta}_0^j)(\eta_q^i - \eta_0^i) \\ &\quad + 2(b_{ij}\eta_0^i + c_{ij}\dot{\eta}_0^j)(\dot{\eta}_q^i - \dot{\eta}_0^i) \\ &\quad + a_{ij}(\eta_q^i - \eta_0^i)(\eta_q^j - \eta_0^j) \\ &\quad + 2b_{ij}(\eta_q^i - \eta_0^i)(\dot{\eta}_q^j - \dot{\eta}_0^j) \\ &\quad + c_{ij}(\dot{\eta}_q^i - \dot{\eta}_0^i)(\dot{\eta}_q^j - \dot{\eta}_0^j). \end{aligned}$$

The last term is non-negative, hence

$$(7.5) \quad \begin{aligned} I(\eta_q) &\geq I(\eta_0) + 2 \int_{t_1}^{t_2} (a_{ij}\eta_0^i + b_{ij}\dot{\eta}_0^j)(\eta_q^i - \eta_0^i) dt \\ &\quad + 2 \int_{t_1}^{t_2} (b_{ij}\eta_0^i + c_{ij}\dot{\eta}_0^j)(\dot{\eta}_q^i - \dot{\eta}_0^i) dt \\ &\quad + \int_{t_1}^{t_2} a_{ij}(\eta_q^i - \eta_0^i)(\eta_q^j - \eta_0^j) dt \\ &\quad + 2 \int_{t_1}^{t_2} b_{ij}(\eta_q^i - \eta_0^i)(\dot{\eta}_q^j - \dot{\eta}_0^j) dt. \end{aligned}$$

The second and third terms in the right member approach zero by Lemma 2. The fourth tends to zero by (5.7). The last term can be shown to approach zero by using Schwarz' inequality and recalling (5.7) and (5.4). This establishes the lemma.

LEMMA 4. *If for each t in the interval $[t_1, t_2]$ the quadratic form*

$$c_{ij}v^i v^j$$

is positive for all non-null vectors v which satisfy the equations

$$(7.6) \quad \phi_{r,i}^\beta(y_0(t), y'_0(t))v^i = 0 \quad (\beta = 1, \dots, m),$$

then inequality (7.2) is satisfied.

Let E be the set of points (t, v) in $(n+2)$ -dimensional space satisfying the conditions $t_1 \leq t \leq t_2$, $|v| = 1$. For each positive integer p we define U_p to be the set of all points (t, v) at which

$$(7.7) \quad c_{ij}(t)v^i v^j + p \sum_{\beta=1}^m (\phi_{r,i}^\beta(y_0(t), y'_0(t))v^i)^2 > 0.$$

Every point of E which satisfies (7.6) is in U_p for all p ; every point of E at which (7.6) is false is in U_p if p is large enough. By the Borel theorem a finite number of the U_p , say those with subscripts p_1, \dots, p_k , cover E . Let N be the greatest of these subscripts; then

$$(7.8) \quad c_{ij}(t)v^i v^j + N \sum_{\beta=1}^m (\phi_{r,i}^\beta v^i)^2 > 0$$

for $t_1 \leq t \leq t_2$ and $|v| = 1$. By homogeneity, (7.8) continues to hold if $t_1 \leq t \leq t_2$ and $|v| \neq 0$.

Now define

$$(7.9) \quad I_N(\eta) = \int_{t_1}^{t_2} \left\{ a_{ij}\dot{\eta}^i \dot{\eta}^j + 2b_{ij}\dot{\eta}^i \dot{\eta}^j + c_{ij}\dot{\eta}^i \dot{\eta}^j + N \sum_{\beta=1}^m (\phi_{y,i}^\beta + \phi_{r,i}^\beta)^2 \right\} dt.$$

The terms quadratic in $\dot{\eta}$ constitute the left member of (7.8), so I_N satisfies the hypotheses of Lemma 3, and

$$(7.10) \quad \liminf I_N(\eta_q) \geq I_N(\eta_0).$$

Since η_0 satisfies the equations (3.6), the terms added to $I(\eta_0)$ in (7.9) vanish, so that

$$(7.11) \quad I(\eta_0) = I_N(\eta_0).$$

This is not true of η_q . But from (6.8), with (7.9) and (7.11), we find

$$\begin{aligned}
 I_N(\eta_q) - I(\eta_q) &= \int_{t_1}^{t_2} N \left\{ \phi_y^\beta \phi_y^\beta \eta_q^i \dot{\eta}_q^j + 2\phi_y^\beta \phi_r^\beta \eta_q^i \dot{\eta}_q^j + \phi_r^\beta \phi_r^\beta \eta_q^i \dot{\eta}_q^j \right\} dt \\
 (7.12) \quad &= N \int_{t_1}^{t_2} \left\{ [\phi_y^\beta \phi_y^\beta - A_q^{\beta,i} A_q^{\beta,j}] \eta_q^i \dot{\eta}_q^j \right. \\
 &\quad \left. + 2[\phi_y^\beta \phi_r^\beta - A_q^{\beta,i} B_q^{\beta,j}] \eta_q^i \dot{\eta}_q^j + [\phi_r^\beta \phi_r^\beta - B_q^{\beta,i} B_q^{\beta,j}] \eta_q^i \dot{\eta}_q^j \right\} dt.
 \end{aligned}$$

The quantities in square brackets tend uniformly to zero, by (6.10) and (6.11), and the integrals of the absolute values of their coefficients are bounded; so

$$(7.13) \quad \lim_{q \rightarrow \infty} [I_N(\eta_q) - I(\eta_q)] = 0.$$

From (7.10), (7.11) and (7.13) we obtain the conclusion (7.2) of our lemma.

LEMMA 5. *If for each t in the interval $[t_1, t_2]$ the quadratic form $c_{ij}v^i v^j$ is non-negative whenever the vector v is linearly independent of $y'_0(t)$ and satisfies equations (7.6), then inequality (7.2) is satisfied.*

Let ϵ be an arbitrary positive number, and let

$$(7.14) \quad I_\epsilon(\eta) = \int_{t_1}^{t_2} [a_{ij}\eta^i \dot{\eta}^j + 2b_{ij}\eta^i \dot{\eta}^j + c_{ij}\eta^i \dot{\eta}^j + \epsilon \eta^i \dot{\eta}^i] dt.$$

The quadratic form $c_{ij}v^i v^j$ is here replaced by

$$c_{ij}v^i v^j + \epsilon v^i v^i.$$

This is positive for all non-null vectors v which satisfy equations (7.6). For the second term is positive, while the first is non-negative, by hypothesis if v is linearly independent of y'_0 and by continuity if v is a multiple of y'_0 . So I_ϵ satisfies the hypotheses of Lemma 4, and

$$(7.15) \quad \lim I_\epsilon(\eta_q) \geq I_\epsilon(\eta_0).$$

From (5.4) we deduce

$$(7.16) \quad 0 \leq \int_{t_1}^{t_2} \dot{\eta}_q^i \dot{\eta}_q^i dt \leq 1.$$

Hence

$$(7.17) \quad I(\eta_0) \leq I_\epsilon(\eta_0) \leq \liminf_{q \rightarrow \infty} I_\epsilon(\eta_q) \leq \liminf_{q \rightarrow \infty} I(\eta_q) + \epsilon.$$

Since ϵ is an arbitrary positive number, this implies inequality (7.2), and the proof of the lemma is complete.

8. First case. We now distinguish two possible cases.

Case I. Either $u_0 \neq (0, \dots, 0)$ or else the functions $\eta_0^i(t)$ do not all vanish identically.

Case II. The numbers u_0^h are all 0 and the functions $\eta_0^i(t)$ all vanish identically.

In this section we shall discuss the first case.

Under the hypotheses of Case I, the set $[\eta, u]$ is not essentially null. This is obvious from the definition if some u^h is not zero. If all the u^h have the value 0, we have by (4.8) and (5.3)

$$(8.1) \quad \eta_q^i(t) y_0^{i'}(t) = (A_q/k_q)t + (B_q/k_q).$$

The left member converges uniformly as $0 \rightarrow \infty$, so its limit $\eta_0^i y_0^{i'}$ must be linear in t . But since $u_0 = 0$ equations (6.17) show that this linear function vanishes at t_1 and at t_2 , so it is identically zero:

$$(8.2) \quad \eta_0^i(t) y_0^{i'}(t) \equiv 0 \quad (t_1 \leq t \leq t_2).$$

Now $\eta_0^i(t)$ cannot have the form $\rho(t)y_0^{i'}(t)$ ($i=0, \dots, n$); for then (8.2) would give $\rho \equiv 0$, contrary to the hypothesis that the η_0^i do not all vanish identically.

By the hypotheses of Theorem I, there are multipliers $\lambda^0 \geq 0, \lambda^1(t), \dots, \lambda^n(t)$ with which the set (C_0, α_0) satisfies the Euler equations, transversality condition and strengthened Clebsch condition and with which $J_2(\eta_0, u_0, \lambda)$ is positive. This last condition will not be used until the last paragraph of this section.

Since λ^0 is non-negative, inequality (4.6) implies

$$(8.3) \quad \lambda^0 J(C_q, \alpha_q) - \lambda^0 J(C_0, \alpha_0) \leq 0.$$

Recalling that along an admissible curve we have $F = \lambda^0 f$, this can be written in the form

$$(8.4) \quad \lambda^0 \theta(\alpha_q) - \lambda^0 \theta(\alpha_0) + \int_{t_1}^{t_2} \{F(y_q, \dot{y}_q, \lambda) - F(y_0, \dot{y}_0, \lambda)\} dt \leq 0.$$

By Taylor's theorem with integral form of remainder, this implies (with (5.2) and (5.3))

$$(8.5) \quad \begin{aligned} & \lambda^0 \theta_h(\alpha_0) k_q u_q^h + k_q^2 \lambda^0 \int_0^1 (1-\tau) \theta_{hk}(\alpha_0 + \tau k_q u_q) u_q^h \dot{u}_q^h d\tau \\ & + k_q \int_{t_1}^{t_2} \{F_{y^i}(y(t), \dot{y}(t), \lambda(t)) \eta_q^i + F_{r^i} \dot{\eta}_q^i\} dt \\ & + k_q^2 \int_{t_1}^{t_2} \int_0^1 (1-\tau) \{F_{y^i y^j}(y + \tau k_q \eta_q, \dot{y} + \tau k_q \dot{\eta}_q, \lambda) \eta_q^i \eta_q^j \\ & + 2F_{y^i r^j} \eta_q^i \dot{\eta}_q^j + F_{r^i r^j} \dot{\eta}_q^i \dot{\eta}_q^j\} d\tau dt \leq 0. \end{aligned}$$

To the third term in (8.5) we apply the usual integration by parts. Since

by hypothesis the Euler equations are satisfied, this term has the value

$$(8.6) \quad k_q F_{rt\eta_q} \Big|_{t_1}^{t_2}.$$

But

$$(8.7) \quad \begin{aligned} k_q \eta_q^i(t_s) &= y_q^i(t_s) - y_0^i(t_s) \\ &= T^{is}(\alpha_q) - T^{is}(\alpha_0) \\ &= k_q T_h^{is}(\alpha_0) u_q^h + k_q^2 \int_0^1 (1-\tau) T_{hk}^{is}(\alpha_0 + \tau k_q u_q) u_q^h u_q^k d\tau. \end{aligned}$$

We substitute this in (8.6), and substitute expression (8.6) for the third term in (8.5). Since by hypothesis the transversality condition is satisfied, inequality (8.5) yields, after division by $k_q^2/2$,

$$(8.8) \quad \begin{aligned} D_q &\equiv 2 \int_0^1 (1-\tau) \{ \lambda^0 \theta_{hk}(\alpha_0 + \tau k_q u_q) \\ &\quad + F_{rt}(y_0(t_2), y'_0(t_2), \lambda(t_2)) T_{hk}^{i2}(\alpha_0 + \tau k_q u_q) \\ &\quad - F_{rt}(y_0(t_1), y'_0(t_1), \lambda(t_1)) T_{hk}^{i1}(\alpha_0 + \tau k_q u_q) \} u_q^h u_q^k d\tau \\ &\quad + 2 \int_{t_1}^{t_2} \int_0^1 (1-\tau) \{ F_{yt,yj}(y_0 + \tau k_q \eta_q, y'_0 + \tau k_q \dot{\eta}_q, \lambda) \eta_q^i \dot{\eta}_q^i \\ &\quad + 2F_{yt,j}\eta_q^i \dot{\eta}_q^i + F_{rt,rj}\eta_q^i \dot{\eta}_q^i \} d\tau dt \leq 0. \end{aligned}$$

Therefore it is clear that

$$(8.9) \quad \limsup_{q \rightarrow \infty} D_q \leq 0.$$

Since by (5.3)

$$y_0^i(t) + \tau k_q \eta_0^i(t) = y_0^i(t) + \tau [y_q^i(t) - y_0^i(t)],$$

with a like equation for the derivatives, we see by (4.3) and (4.4) that

$$(8.10) \quad \lim_{q \rightarrow \infty} F_{rt,rj}(y_0 + \tau k_q \eta_q, y'_0 + \tau k_q \dot{\eta}_q, \lambda) = F_{rt,rj}(y_0, y'_0, \lambda)$$

uniformly for $0 \leq \tau \leq 1$ and $t_1 \leq t \leq t_2$. Relations similar to (8.10) hold for the other coefficients in (8.8). Hence if ϵ is an arbitrary positive number, for all sufficiently large values of q the replacement of k_q by 0 in (8.8) alters each coefficient by less than ϵ . But after this replacement the variable τ has disappeared from (8.8) except in the factors $(1-\tau)$, whose integral from 0 to 1 has the value $1/2$. The result is that the left member of (8.8) takes the form $J_2(\eta_q, u_q, \lambda)$ (cf. (3.8)). Since each coefficient was changed by less than ϵ , we thus find with the help of Schwarz' inequality

$$(8.11) \quad \begin{aligned} |D_q - J_2(\eta_q, u_q, \lambda)| &\leq \epsilon \left(\sum_{h=1}^r u_q^h \right)^2 + \epsilon \int_{t_1}^{t_2} \left(\sum_{i=0}^n \{ |\eta_q^i| + |\dot{\eta}_q^i| \} \right)^2 dt \\ &\leq \epsilon \left\{ r |u_q|^2 + 2(n+1) \int_{t_1}^{t_2} (|\eta_q|^2 + |\dot{\eta}_q|^2) dt \right\}. \end{aligned}$$

The coefficient of ϵ is bounded, since (5.4) is satisfied, and ϵ is arbitrary. Hence

$$(8.12) \quad \lim_{q \rightarrow \infty} [D_q - J_2(\eta_q, u_q, \lambda)] = 0.$$

Relations (8.9) and (8.12) imply

$$(8.13) \quad \limsup_{q \rightarrow \infty} J_2(\eta_q, u_q, \lambda) \leq 0.$$

By (5.5) we have

$$(8.14) \quad \lim_{q \rightarrow \infty} b_{hk} u_q^h u_q^k = b_{hk} u_0^h u_0^k.$$

The integral in (3.8) satisfies the hypotheses of Lemma 3, since the strengthened Clebsch condition holds. Hence from Lemma 3 and equation (8.14) we obtain

$$(8.15) \quad \liminf_{q \rightarrow \infty} J_2(\eta_q, u_q, \lambda) \geq J_2(\eta_0, u_0, \lambda).$$

But now we have reached our desired contradiction. For by the choice of the λ^0 and $\lambda^\alpha(t)$ the right member of (8.15) is positive, so inequalities (8.13) and (8.15) are incompatible.

9. Second case. We still have to dispose of Case II, in which the u_0 and $\eta_0(t)$ are all zero. The hypotheses of Theorem I do not mention such variation sets, so we must prove a lemma.

LEMMA 6. *Under the hypotheses of Theorem I, there exist admissible variation sets which are not essentially null.*

Bliss⁽⁷⁾ has shown that there exist functions $\phi^\gamma(t, r)$ of class C^2 ($\gamma = m + 1, \dots, n+1$) such that the determinant

$$(9.1) \quad \begin{vmatrix} \phi_{rt}^\beta(y_0(t), y'_0(t)) \\ \phi_{rt}^\gamma(t, y'_0(t)) \end{vmatrix}$$

does not vanish on the interval $t_1 \leq t \leq t_2$. In the interval we choose $n+3$ points τ_i such that

$$t_1 \leq \tau_1 < \tau_2 < \dots < \tau_{n+3} \leq t_2.$$

(7) G. A. Bliss, *The problem of Mayer with variable end points*, these Transactions, vol. 19 (1918), pp. 305–314.

For $l=1, \dots, n+2$ we choose functions $\xi_i^k(t)$ ($k=1, \dots, n+1$) with the following properties. If $k=1, \dots, m$, then $\xi_i^k(t)$ is identically zero. Except on $[\tau_l, \tau_{l+1}]$ the other $\xi_i^k(t)$ also vanish identically. On $[\tau_l, \tau_{l+1}]$ the $\xi_i^k(t)$ are constants, and the vector

$$(9.2) \quad (\xi_1^1(t), \dots, \xi_1^{n+1}(t))$$

is linearly independent of

$$(9.3) \quad (\phi_{r,i}^1 y_0^{i'}(\tau_l), \dots, \phi_{r,i}^{n+1} y_0^{i'}(\tau_l)).$$

This last condition can be satisfied since by hypothesis m is less than n , so that we have the free choice of at least the last two components of the vector (9.2).

By known theorems on differential equations, for $l=1, \dots, n+2$ the equations

$$(9.4) \quad \phi_{y,i}^\beta(y_0, y_0') H_i^i + \phi_{r,i}^\beta(y_0, y_0') H_i^{i'} = 0, \quad \phi_{r,i}^\gamma(t, y_0') H_i^{i'} = \xi_i^\gamma(t) \quad (\beta = 1, \dots, m; \gamma = m+1, \dots, n+1)$$

have unique solutions H_i^i vanishing at t_1 . The $n+1$ homogeneous equations

$$(9.5) \quad c_i H_i^i(t_2) = 0 \quad (i = 0, 1, \dots, n)$$

in the $n+2$ unknowns c_i have a non-trivial solution. We define

$$(9.6) \quad \bar{\eta}^i(t) = c_i H_i^i(t) \quad (t_1 \leq t \leq t_2).$$

Then $\bar{\eta}(t)$ satisfies the equations (3.6), because of (9.4). Since $\bar{\eta}$ vanishes at t_1 and at t_2 , it satisfies (3.7) with $\bar{u}=0$. Hence $(\bar{\eta}, \bar{u})$ is an admissible variation set. If it were essentially null, there would be a function $\rho(t)$ such that

$$(9.7) \quad \bar{\eta}^i(t) = \rho(t) y_0^{i'}(t) \quad (t_1 \leq t \leq t_2).$$

This $\rho(t)$ is easily seen to be continuous and to have corners only at the points τ_l . If λ is the least integer such that $c_\lambda \neq 0$, then $\bar{\eta}^i(t)$ is identically zero on $[t_1, \tau_\lambda]$. By (9.7) and (9.6)

$$H_\lambda^{i'}(\tau_\lambda+) = \bar{\eta}^{i'}(\tau_\lambda+)/c_\lambda = \rho'(\tau_\lambda+) y_0^{i'}(\tau_\lambda)/c_\lambda.$$

Since $H_\lambda^i(t_\lambda)$ vanishes, by (9.4) we see that for $l=\lambda$ the vector (9.2) is a multiple of (9.3), contrary to its choice. Lemma 6 is therefore established.

Now by Lemma 6 and hypothesis (3) of Theorem I there are multipliers $\lambda^0 \geq 0, \lambda^1(t), \dots, \lambda^m(t)$ with which the Euler equations, transversality condition and strengthened Clebsch condition hold. From the last mentioned condition we see that the form

$$(9.8) \quad F_{rt,ri}(y_0(t), y'_0(t), \lambda) v^i v^i$$

is positive on the set of unit vectors v which are orthogonal to y'_0 and satisfy equations (3.5). This set of vectors is bounded and closed, so on it the form (9.8) has a positive lower bound, which we denote by 2ϵ . It follows that on the set of vectors v just described the inequality

$$(9.9) \quad F_{rt,ri}v^i v^i - \epsilon v^i v^i > 0$$

holds, since the coefficient of ϵ has the value 1. By homogeneity, (9.9) continues to hold all non-null vectors v which are orthogonal to y'_0 and satisfy equations (3.5).

Let v be any vector which satisfies (3.5) and is linearly independent of y'_0 . It can be resolved into components

$$(9.10) \quad v^i = v_0^i + \delta y_0^{i'}(t),$$

where v_0 is orthogonal to y'_0 . Since v is linearly independent of y'_0 , the component v_0 is not null. The homogeneity of F and ϕ^β entails the well known consequence

$$(9.11) \quad y_0^{i'}(t) F_{rt,ri}(y_0, y'_0, \lambda) = 0, \quad y_0^{i'} \phi_{rt}^\beta = 0.$$

Now y'_0 and v both satisfy the linear equations (3.5), hence by (9.10) v_0 also satisfies those equations. Therefore (9.9) holds with v_0 in place of v , and with the help of (9.11) we deduce

$$(9.12) \quad F_{rt,ri}(y_0, y'_0, \lambda) v^i v^i - \epsilon [v^i v^i - (y_0^{i'} v^i)^2] = F_{rt,ri} v_0^i v_0^i - \epsilon v_0^i v_0^i > 0.$$

Inequality (9.12) shows that the integral

$$(9.13) \quad I_\epsilon(\eta) = \int_{t_1}^{t_2} \{ 2\omega(t, \eta, \dot{\eta}) - \epsilon [\eta^i \dot{\eta}^i - (y_0^{i'} \dot{\eta}^i)^2] \} dt$$

satisfies the hypotheses of Lemma 3, so that

$$(9.14) \quad \liminf I_\epsilon(\eta_q) \geq I_\epsilon(\eta_0) = 0.$$

Since u_q approaches $u_0 = 0$ as $q \rightarrow \infty$, this and (3.8) together imply

$$(9.15) \quad \liminf \left\{ J_2(\eta_q, u_q, \lambda) - \epsilon \int_{t_1}^{t_2} [\dot{\eta}_q^i \dot{\eta}_q^i - (y_0^{i'} \dot{\eta}_q^i)^2] dt \right\} \geq 0.$$

By (4.8) and (5.3) the function

$$y_0^{i'}(t) \eta_q^i(t)$$

is linear in t , and it converges uniformly to zero, since η_0 is zero. Except on a set of measure zero we have

$$y_0^{\frac{i}{i}, \frac{i}{i}} \eta_q = (y_0^{\frac{i}{i}, \frac{i}{i}})' - y_0^{\frac{i}{i}, \frac{i}{i}} \eta_q,$$

and both terms on the right tend uniformly to zero. So the last term in the square bracket in (9.15) can be omitted without affecting the limit. In (5.4) the first and second terms tend to zero, so the third term tends to 1 as $q \rightarrow \infty$. Thus (9.15) implies

$$(9.16) \quad \liminf J_2(\eta_q, u_q, \lambda) - \epsilon \geq 0.$$

On the other hand, the considerations leading to inequality (8.13) are applicable to Case II as well to Case I, so inequality (8.13) must hold. This contradicts (9.16). Hence in each of the two possible cases we have arrived at a contradiction, and Theorem I is established.

10. Statement of problem in non-parametric form. From Theorem I we can deduce its analogue for problems in non-parametric form. We use the formulation due to Morse and Myers^(8,9). The functions

$$\begin{aligned} f(x, z, p) &= f(x, z^1, \dots, z^n, p^1, \dots, p^n) \\ \phi^\beta(x, z, p) &= \phi^\beta(x, z^1, \dots, z^n, p^1, \dots, p^n) \quad (\beta = 1, \dots, m < n) \end{aligned}$$

will be supposed to be defined and of class C^2 on an open point set S_1 in (x, z, p) -space. The functions $T^{is}(\alpha)$ ($i = 0, \dots, n$; $s = 1, 2$) are defined and of class C^2 on an open set R_2 in $(\alpha^1, \dots, \alpha^n)$ -space. An admissible set $[z, \alpha]$ is a set of functions $z^i(x)$ absolutely continuous on an interval $[x_1, x_2]$ such that for almost all x in $[x_1, x_2]$ the point $(x, z(x), \dot{z}(x))$ is in S_1 and satisfies the equations

$$(10.1) \quad \phi^\beta(x, z(x), \dot{z}(x)) = 0 \quad (\beta = 1, \dots, m),$$

together with a set of parameters α in R_2 with which the end conditions

$$(10.2) \quad x_s = T^{0s}(\alpha), \quad z^c(x_s) = T^{cs}(\alpha) \quad (c = 1, \dots, n; s = 1, 2)$$

are satisfied.

The problem of Bolza in non-parametric form is the problem of minimizing the functional

$$(10.3) \quad J[z, \alpha] \equiv \theta(\alpha) + \int_{x_1}^{x_2} f(x, z, z') dx$$

in the class of admissible sets $[z, \alpha]$.

Let $[z_0, \alpha_0]$ be an admissible set in which the functions

$$(10.4) \quad z^c = z_0^c(x) \quad (x_{1,0} \leq x \leq x_{2,0}; c = 1, 2, \dots, n)$$

⁽⁸⁾ M. Morse and S. B. Myers, *The problems of Lagrange and Mayer with variable end points*, Proceedings of the American Academy of Arts and Sciences, vol. 66 (1931), pp. 235–253.

⁽⁹⁾ M. R. Hestenes, *Sufficient conditions for the problem of Bolza in the calculus of variations*, these Transactions, vol. 36 (1934), pp. 793–818.

are of class C^1 . The set $[z_0, \alpha_0]$ satisfies the Euler equations with multipliers $\lambda^0, \lambda^1(x), \dots, \lambda^m(x)$ if the functions

$$(10.5) \quad F(x, z, p, \lambda) \equiv \lambda^0 f(x, z, p) + \lambda^s \phi^s(x, z, p)$$

satisfy the equations

$$(10.6) \quad \frac{d}{dx} F_{pc}(x, z_0, z'_0, \lambda) = F_{zc}(x, z_0, z'_0, \lambda) \quad (c = 1, \dots, n).$$

It satisfies the transversality conditions with multipliers $\lambda^0, \lambda^1(x), \dots, \lambda^m(x)$ if

$$(10.7) \quad \lambda^0 \theta_h(\alpha_0) + [(F - z_0^{cs} F_{pc}) T_h^{0s} + F_{pc} T_h^{cs}]_1^2 = 0 \quad (h = 1, \dots, r).$$

As usual, c has the range $1, \dots, n$. The square-bracketed symbol in (10.7) is to be understood as follows. The functions F , and so on are first evaluated at $(x_{s,0}, z_0(x_{s,0}), z'_0(x_{s,0}), \lambda(x_{s,0}))$, $s=1, 2$. Then the value of the sum inside the square bracket is evaluated for $s=1$ and for $s=2$, and the former value subtracted from the latter.

The set $[z_0, \alpha_0]$ satisfies the *strengthened Clebsch condition* with multipliers $\lambda^0, \lambda^1(x), \dots, \lambda^m(x)$ if for each x in $[x_{1,0}, x_{2,0}]$ the inequality

$$(10.8) \quad F_{pcpd}(x, z_0(x), z'_0(x), \lambda(x)) v^{cpd} > 0$$

(summed over $c, d = 1, \dots, n$) holds for all sets of numbers $(v^1, \dots, v^n) \neq (0, \dots, 0)$ satisfying the equations

$$(10.9) \quad \phi_{pc}^\beta(x, z_0(x), z'_0(x)) v^c = 0 \quad (\beta = 1, \dots, m).$$

An *admissible variation set* $[\xi(x), u]$ is a set consisting of n functions, absolutely continuous and having derivatives whose squares are summable over $[x_{1,0}, x_{2,0}]$ and satisfying the equations

$$(10.10) \quad \phi_{zc}^\beta(x, z_0, z'_0) \xi^c(x) + \phi_{pc}^\beta(x, z_0, z'_0) \xi^c(x) = 0 \quad (\beta = 1, \dots, m)$$

for almost all x , and a set of numbers (u^1, \dots, u^r) with which the equations

$$(10.11) \quad \xi^c(x_{s,0}) = [T_h^{cs}(\alpha_0) - z_0^{cs}(x_{s,0}) T_h^{0s}] u^h \quad (s = 1, 2; h = 1, \dots, r).$$

hold.

If $[\xi, u]$ is an admissible variation set, we define the second variation due to $[\xi, u]$ by the equation

$$(10.12) \quad J_2[\xi, u, \lambda] = b_{hk} u^h u^k + \int_{x_{1,0}}^{x_{2,0}} 2\omega(x, \xi, \dot{\xi}) dx,$$

where

$$(10.13) \quad b_{hk} = \lambda^0 \theta_{hk}(\alpha_0) + [(F_z - z_0^c F_{zc}) T_h^{0s} T_k^{0s} + (F - z_0^c F_{pc}) T_h^{0s} \\ + F_{zc}(T_h^{0s} T_k^{cs} + T_k^{0s} T_h^{cs}) + F_{pc} T_{hk}^{cs}]_1$$

and

$$(10.14) \quad 2\omega(x, \zeta, \pi) = F_{zc} z_d(x, z_0, z'_0, \lambda) \zeta^c \zeta^d + 2F_{zc} p^d \zeta^c \pi^d + F_{pc} p^d \pi^c \pi^d.$$

The concept of a weak relative minimum will be carried over unchanged from the parametric problem. Let the functions (10.4) be of class C^1 , and consider another set of absolutely continuous functions

$$(10.15) \quad z^c = z^c(x), \quad x_1 \leq x \leq x_2.$$

We can use these functions to define a curve C in $(n+1)$ -space by means of the equation

$$(10.16) \quad x = t, \quad z^c = z^c(t) \quad (x_1 \leq t \leq x_2, c = 1, \dots, n),$$

and likewise for the set (10.4). For these curves the concept of first order ϵ -neighborhood has already been defined in §2, and so has the concept of weak relative minimum.

For problems in non-parametric form we shall establish the following analogue of Theorem I.

THEOREM II. *Let the following hypotheses be satisfied.*

(1) *The set $[z_0, \alpha_0]$ defined by (10.4) is admissible, and the functions $z_0^c(x)$*

($c = 1, \dots, n$) are of class C^2 .

(2) *For each x in the interval $[x_{1,0}, x_{2,0}]$ the matrix*

$$(10.17) \quad \left\| \phi_{pc}^\beta(x, z_0(x), z'_0(x)) \right\| \quad (\beta = 1, \dots, m; c = 1, \dots, n)$$

has rank m .

(3) *To each nonidentically vanishing admissible variation set $[\zeta, u]$ there corresponds a set of absolutely continuous multipliers $\lambda^0 \geq 0, \lambda^1(x), \dots, \lambda^m(x)$ with which the set $[z_0, \alpha_0]$ satisfies the Euler equations, transversality condition and strengthened Clebsch condition, and with which the inequality*

$$(10.18) \quad J_2[\zeta, u, \lambda] > 0$$

is satisfied.

Then the set $[z_0, \alpha_0]$ gives $J[z, \alpha]$ a proper weak relative minimum on the class of admissible sets $[z, \alpha]$.

In the next two sections we shall show that this theorem is in fact a consequence of Theorem I.

11. Transformation into parametric form. We prove Theorem II by replacing the non-parametric problem of §10 by an equivalent parametric problem. The symbols y^0, y^1, \dots, y^n will be used as alternative names for the

x, z^1, \dots, z^n axes in $(n+1)$ -space. In the $(2n+2)$ -dimensional space of points $(y, r) = (y^0, \dots, y^n, r^0, \dots, r^n)$ we define R_1 to be the set of points (y, r) having $r^0 > 0$ and such that $(y^0, y^1, \dots, y^n, r^1/r^0, \dots, r^n/r^0)$ is in S_1 . On R_1 we define functions $g(y, r), \psi^\beta(y, r)$ by the equations

$$(11.1) \quad \begin{aligned} g(y, r) &= r^0 f(y^0, y^1, \dots, y^n, r^1/r^0, \dots, r^n/r^0), \\ \psi^\beta(y, r) &= r^0 \phi^\beta(y^0, y^1, \dots, y^n, r^1/r^0, \dots, r^n/r^0). \end{aligned}$$

These have the continuity and homogeneity properties specified in §2.

From (11.1) we deduce

$$(11.2) \quad g(y^0, \dots, y^n, 1, r^1, \dots, r^n) = f(y^0, y^n, r^1, \dots, r^n),$$

from which by differentiation we obtain similar identities for all existing partial derivatives not involving differentiation with respect to r^0 ; for instance,

$$(11.3) \quad g_{y^0}(y^0, \dots, y^n, 1, r^1, \dots, r^n) = f_x(y^0, \dots, y^n, r^1, \dots, r^n).$$

Moreover, identities analogous to (11.2) and its corollaries are also valid for each of the functions ϕ^β .

We now have the data needed for setting up the parametric problem of minimizing the functional

$$(11.4) \quad I(C, \alpha) \equiv \theta(\alpha) + \int_{t_1}^{t_2} g(y(t), \dot{y}(t)) dt$$

in the class of sets (C, α) for which the curves $C: y = y(t)$ have (y, \dot{y}) in R_1 for almost all t , satisfy the differential equations

$$(11.5) \quad \psi^\beta(y, \dot{y}) = 0 \quad (\beta = 1, \dots, m)$$

for almost all t , and satisfy the end conditions

$$(11.6) \quad y^i(t_s) = T^{i_s}(\alpha) \quad (i = 0, 1, \dots, n; s = 1, 2).$$

From (11.1) it is evident that if the set $[z, \alpha]$ with functions $z(t)$ defined by (10.15) is admissible, and C is defined by (10.16), then (C, α) is admissible for the parametric problem, and

$$(11.7) \quad I(C, \alpha) = J[z, \alpha].$$

(The converse also is true; if (C, α) is admissible for the parametric problem, C can be represented in the form (10.16), and the functions $z^c(x)$ thus obtained are admissible with parameters α for the non-parametric problem. But we do not need this.) Theorem II will therefore be established if we can show that its hypotheses imply that the corresponding parametric problem satisfies the hypotheses of Theorem I.

12. Verification of the hypotheses of Theorem I. Hypotheses (1) and (2) of Theorem I follow at once from the corresponding hypotheses of Theorem II.

Let $[\eta(t), u]$ be an admissible variation set for the parametric problem. We define

$$(12.1) \quad \xi^i(x) = \eta^i(x) - y_0^{i'}(x)\eta^0(x) \quad (i = 0, \dots, n; x_{1,0} \leq x \leq x_{2,0}).$$

(The notations x and t for the independent variable are interchangeable, because of (10.16); and for the same reason $z^c(x)$ and $y^c(x)$ are identical, $c=1, 2, \dots, n$.)

If we recall that

$$(12.2) \quad y_0^{0'}(t) \equiv 1$$

a well known consequence of the homogeneity of the ψ^β implies (with (3.6) and the analogues of (11.1))

$$\begin{aligned} & \phi_{zc}^\beta(x, z_0, z_0')\xi^c + \phi_{pc}^\beta(x, z_0, z_0')\xi^c \\ &= \psi_{y^i}^\beta(y_0, y_0')(\eta^i - y_0^{i'}\eta^0) + \psi_{r^i}^\beta(\eta^i - y_0^{i'}\eta^0 - y_0^{i''}\eta^0) \\ (12.3) \quad &= -\eta^0[\psi_{y^i}^\beta y_0^{i'} + \psi_{r^i}^\beta y_0^{i''}] \\ &= -\eta^0[\psi^\beta(y_0, y_0')]' \\ &= 0. \end{aligned}$$

So equations (10.10) hold. Equations (10.11) follow at once from (12.1) and (3.7). Therefore, by hypothesis (3) of Theorem II there exist multipliers $\lambda^0 \geq 0, \lambda^1(x), \dots, \lambda^n(x)$ with which the Euler equations, transversality condition and strengthened Clebsch condition hold for the non-parametric problem, and also the inequality (10.18).

Let us define

$$(12.4) \quad G(y, r, \lambda) = \lambda^0 g(y, r) + \lambda^\beta \psi^\beta(y, r).$$

The Euler equations for the non-parametric problem constitute the last n of the $n+1$ Euler equations for the parametric problem. But from the homogeneity of G it can be shown that the equation

$$(12.5) \quad y^{i'}(t) \left\{ \frac{d}{dt} G_{r^i}(y, r', \lambda) - G_{y^i}(y, r', \lambda) \right\} = 0$$

holds for any admissible curve $y^i = y^i(t)$ of class C^2 . Since (12.2) holds and the last n of the factors in braces vanish for $y_0(t)$, the first also vanishes, and all $n+1$ Euler equations are satisfied.

From the homogeneity relation

$$(12.6) \quad G(y, r, \lambda) = r^i G_{r^i}(y, r, \lambda),$$

with (12.2) and (11.1), we find that

$$(12.7) \quad G_{r^0}(y_0, y_0', \lambda) = F(x, z_0, z_0', \lambda) - z_0^c F_{pc}(x, z_0, z_0', \lambda).$$

This shows that the transversality conditions (10.7) imply the transversality conditions (3.3) for the parametric problem.

Let (v^0, \dots, v^n) be a vector linearly independent of $y'_0(t)$ and satisfying the equations (3.5) with ψ^β in place of ϕ^β . Define

$$(12.8) \quad w^c = v^c - v^0 y_0^c(t) \quad (c = 1, \dots, n).$$

By the analogue of (12.6) and (12.2),

$$\begin{aligned} \phi_{pc}^\beta(x, z_0, z'_0) w^c &= \psi_r^\beta(v^i - v^0 y_0^i) \\ &= \psi_r^\beta v^i - v^0 \psi^\beta(y_0, y'_0) \\ &= 0. \end{aligned}$$

Also, the w^c are not all zero, since (v^0, \dots, v^n) is not a multiple of $y'_0(t)$. Since by hypothesis the strengthened Clebsch condition holds,

$$(12.9) \quad F_{pcpd}(x, z_0, z'_0, \lambda) w^c w^d > 0.$$

A consequence of the homogeneity of G is

$$(12.10) \quad G_{rti}(y, r, \lambda) r^i = 0.$$

From this and (12.8) we have

$$(12.11) \quad G_{rti}(y_0, y'_0, \lambda) v^i v^j = F_{pcpd}(x, z_0, z'_0, \lambda) w^c w^d,$$

which is positive by (12.9). Therefore the strengthened Clebsch condition holds for the parametric problem.

Let us define

$$(12.12) \quad 2\omega^*(t, \eta, \rho) \equiv G_{rti}(y_0, y'_0, \lambda) \eta^i \eta^i + 2G_{rtm} \eta^i \rho^i + G_{rtsp} \rho^i \rho^i.$$

This is a symmetric quadratic form in (η, ρ) , whence

$$(12.13) \quad 2\omega^*(t, \eta, \rho) = \eta^i \omega_{\eta i}^*(t, \eta, \rho) + \rho^i \omega_{\rho i}^*(t, \eta, \rho)$$

and

$$(12.14) \quad \bar{\eta}^i \omega_{\eta i}^*(t, \eta, \rho) + \bar{\rho}^i \omega_{\rho i}^*(t, \eta, \rho) = \eta^i \omega_{\eta i}^*(t, \bar{\eta}, \bar{\rho}) + \rho^i \omega_{\rho i}^*(t, \bar{\eta}, \bar{\rho}).$$

(These identities can also be established easily by direct computation.) It is well known that for every set of functions $\eta^i(t)$ of class C^2 the equations

$$(12.15) \quad y_0^{i'}(t) \left\{ \frac{d}{dt} \omega_{\rho i}^*(t, \eta, \eta') - \omega_{\eta i}^*(t, \eta, \eta') \right\} = 0$$

are satisfied identically. We now prove a lemma.

LEMMA 7. *If the functions $\gamma(t)$, $\eta^0(t)$, \dots , $\eta^n(t)$ are absolutely continuous and the squares of their derivatives are summable, then*

$$(12.16) \quad \int_{t_1}^{t_2} \{ (\gamma y_0^{i'}) \cdot \omega_{\rho^i}^*(t, \eta, \dot{\eta}) + (\gamma y_0^{i'}) \omega_{\eta^i}^*(t, \eta, \eta') \} dt \\ = \gamma(t) \eta^i(t) G_{y^i}(y_0(t), y_0'(t), \lambda(t)) \Big|_{t_1}^{t_2}.$$

Let us first suppose that the $\eta^i(t)$ are of class C^2 . By an integration by parts, with use of (12.15), the left member of (12.16) is reduced to

$$(12.17) \quad \gamma y_0^{i'} \omega_{\rho^i}^* \Big|_{t_1}^{t_2}.$$

The analogue of (12.6) holds for G_{y^i} , and with the help of this and (12.10) the expression (12.17) transforms into the right member of (12.16). Hence the lemma holds if the $\eta^i(t)$ are of class C^2 .

By hypothesis, the derivatives of the η^i are of class $L^{(2)}$, so by a known property of the Lebesgue integral there is a sequence of polynomials

$$\hat{p}_q^{i'}(t) \quad (i = 0, 1, \dots, u; q = 1, 2, \dots)$$

such that

$$(12.18) \quad \lim_{q \rightarrow \infty} \int_{t_1}^{t_2} |\dot{\eta} - \hat{p}_q'|^2 dt = 0.$$

By the inequality of Schwarz, the functions

$$\hat{p}_q^i(t) \equiv \eta^i(t_1) + \int_{t_1}^t \hat{p}_q^{i'}(t) dt$$

converge uniformly to $\eta^i(t)$. The coefficients of the form ω^* are continuous, so

$$(12.19) \quad \lim_{q \rightarrow \infty} \int_{t_1}^{t_2} \sum_{i=0}^n \{ [\omega_{\eta^i}^*(t, \hat{p}_q, \hat{p}_q') - \omega_{\eta^i}^*(t, \eta, \dot{\eta})]^2 \\ + [\omega_{\rho^i}^*(t, \hat{p}_q, \hat{p}_q') - \omega_{\rho^i}^*(t, \eta, \dot{\eta})]^2 \} dt = 0.$$

For each p_q the analogue of (12.16) holds, so by Schwarz' inequality

$$(12.20) \quad \int_{t_1}^{t_2} \{ (\gamma y_0^{i'}) \cdot \omega_{\rho^i}^*(t, \eta, \dot{\eta}) + (\gamma y_0^{i'}) \omega_{\eta^i}^*(t, \eta, \eta') \} dt \\ = \lim_{q \rightarrow \infty} \int_{t_1}^{t_2} \{ (\gamma y_0^{i'}) \cdot \omega_{\rho^i}^*(t, p_q, p_q') + (\gamma y_0^{i'}) \omega_{\eta^i}^*(t, p_q, p_q') \} dt \\ = \lim_{q \rightarrow \infty} \gamma p_q^i G_{y^i}(y_0, y_0', \lambda) \Big|_{t_1}^{t_2} \\ = \gamma \eta^i G_{y^i}(y_0, y_0', \lambda) \Big|_{t_1}^{t_2}.$$

This establishes the lemma.

By definition (3.8),

$$(12.21) \quad I_2(\eta, u, \lambda) = \lambda^0 b_{hk}^* u^h u^k + \int_{t_1}^{t_2} 2\omega^*(t, \eta, \dot{\eta}) dt,$$

where

$$(12.22) \quad b_{hk}^* = \lambda^0 \theta_{hk} + [G_{rt}(t_s) T_{hk}^{is}(\alpha_0)]_1^2.$$

From (12.13), (12.21) and (12.1) we deduce

$$(12.23) \quad \begin{aligned} I_2(\eta, u, \lambda) &= \lambda^0 b_{hk}^* u^h u^k + \int_{t_1}^{t_2} \{ \xi^i \omega_{\eta i}^*(t, \xi, \dot{\xi}) + \xi^i \omega_{\rho i}^*(t, \xi, \dot{\xi}) \} dt \\ &\quad + \int_{t_1}^{t_2} \{ \xi^i \omega_{\eta i}^*(t, \eta^0 y'_0, (\eta^0 y'_0)') + \xi^i \omega_{\rho i}^*(t, \eta^0 y'_0, (\eta^0 y'_0)') \} dt \\ &\quad + \int_{t_1}^{t_2} \{ \eta^0 y_0^i \omega_{\eta i}^*(t, \xi, \dot{\xi}) + (\eta^0 y_0^i)' \omega_{\rho i}^*(t, \xi, \dot{\xi}) \} dt \\ &\quad + \int_{t_1}^{t_2} \{ \eta^0 y_0^i \omega_{\eta i}^*(t, \eta^0 y'_0, (\eta^0 y'_0)') \\ &\quad \quad \quad + (\eta^0 y_0^i)' \omega_{\rho i}^*(t, \eta^0 y'_0, (\eta^0 y'_0)') \} dt. \end{aligned}$$

The integrand in the first integral in the right member is $2\omega^*(t, \xi, \dot{\xi})$, which by (12.12) is the same as $2\omega(t, \xi, \dot{\xi})$, since ξ^0 vanishes identically. By Lemma 7, the third integral has the value

$$(12.24) \quad \eta^0 \xi^i G_{\nu i}(y_0, y'_0, \lambda) \Big|_{t_1}^{t_2}.$$

Since $\xi^0 = 0$, by (12.4) and (11.1) this is equal to

$$(12.25) \quad \eta^0 \xi^c F_{zc}(x, z_0, z'_0, \lambda) \Big|_{x_{1,0}}^{x_{2,0}}.$$

By the same argument, the fourth integral has the value

$$(12.26) \quad \eta^0 \eta^0 y_0^i G_{\nu i}(y_0, y'_0, \lambda) \Big|_{x_{1,0}}^{x_{2,0}}.$$

By (12.14), the second integral has the same value (12.25) as the third. We substitute these evaluations in (12.23), and for the end values of η^0 and ξ^c we substitute the values given by (3.7) and (10.11). On collecting terms and recalling (12.7), (10.12) and (10.13), we find

$$(12.27) \quad I_2(\eta, u, \lambda) = J_2[\xi, u, \lambda],$$

which is positive by hypothesis.

We have now verified all the hypotheses of Theorem I for the parametric formulation of our problem, so by that theorem the set (C_0, α_0) gives a proper weak relative minimum to $J(C, \alpha)$ on the class of admissible sets (C, α) . This immediately implies that $[z_0, \alpha_0]$ gives $J[z, \alpha]$ a proper weak relative minimum on the class of admissible sets $[z, \alpha]$, and Theorem II is established.

13. **A corollary.** If it were not for our unusually inclusive definition of admissible variation set, Theorem II would at once include Hestenes' sufficiency theorem for weak relative minima⁽¹⁰⁾. For Hestenes assumes the hypotheses of Theorem II, with the additional requirement that the multipliers can be chosen independently of the sets $[\xi, u]$. However, it requires some proof to show that in this case the assumption that the second variation is positive for all variation sets admissible in the sense of §3 is necessarily satisfied if the second variation is positive whenever $[\xi, u]$ is an admissible set and the $\xi^c(x)$ are of class D^1 . We establish this for the normal case; as Bliss has shown⁽¹¹⁾, the theorem of Hestenes can be deduced from the sufficiency theorem for the normal problem.

For each real number a , let us define

$$(13.1) \quad Q_a[\xi, u, \lambda] = \lambda^0 b_{hk} u^h u^k + a u^h u^h + \int_{x_{1,0}}^{x_{2,0}} \{2\omega(x, \xi, \dot{\xi}) + a \xi^c \dot{\xi}^c\} dx.$$

Clearly

$$(13.2) \quad Q_0[\xi, u, \lambda] = J_2[\xi, u, \lambda].$$

As in the discussion of (7.7), we can show that if a is sufficiently large the quadratic form

$$(13.3) \quad \lambda^0 b_{hk} u^h u^k + a u^h u^h$$

is positive definite and

$$(13.4) \quad 2\omega(x, \xi, \dot{\xi}) + a \xi^c \dot{\xi}^c$$

is positive for all nonidentically zero sets (ξ, v) satisfying (10.9). We choose such an a ; then there is a positive ϵ such that

$$(13.5) \quad \lambda^0 b_{hk} u^h u^k + a u^h u^h \geq \epsilon u^h u^h$$

and

$$(13.6) \quad 2\omega(x, \xi, \dot{\xi}) + a \xi^c \dot{\xi}^c \geq \epsilon [\xi^c \dot{\xi}^c + \dot{\xi}^c \xi^c]$$

whenever ξ satisfies (10.9).

Let K_1 be the collection of admissible sets $[\xi, u]$ such that

⁽¹⁰⁾ Hestenes, loc. cit., p. 816.

⁽¹¹⁾ G. A. Bliss, *Normality and abnormality in the calculus of variations*, these Transactions, vol. 43 (1938), pp. 365-376.

$$(13.7) \quad u^h u^h + \int_{x_{1,0}}^{x_{2,0}} \xi^c \xi^c dx = 1.$$

For any $[\xi, u]$ in K_1 and any number b we have

$$(13.8) \quad Q_a[\xi, u, \lambda] - Q_b[\xi, u, \lambda] = a - b.$$

Since the forms (13.3) and (13.4) are non-negative, Q_a has a non-negative lower bound m on the class K_1 . Let $[\xi_q, u_q]$ be a sequence of sets in K_1 for which Q_a tends to its lower bound m on K_1 . By (13.5) and (13.6),

$$(13.9) \quad Q_a[\xi, u, \lambda] \geq \epsilon \left\{ u^h u^h + \int_{x_{1,0}}^{x_{2,0}} (|\xi|^2 + |\dot{\xi}|^2) dx \right\},$$

so the value of the expression in braces is bounded on the sequence $[\xi_q, u_q]$. The boundedness of the integral of $|\dot{\xi}_q|^2$ implies the equi-continuity of the ξ_q , as in (5.6). The boundedness of $|u_q|$ implies the boundedness of $|\xi_q(x_{1,0})|$, and this with the equi-continuity of the ξ_q implies their uniform boundedness. By Ascoli's theorem, we can select a subsequence converging uniformly to a limit $\xi_0(x)$; we suppose $[\xi_0, u_0]$ such a sequence. We may also suppose that the u_q converge to a limit u_0 . As in §5, the ξ_0^c are absolutely continuous, and the squares of their derivatives are summable, and by Lemma 2

$$(13.10) \quad Q_a[\xi_0, u_0, \lambda] \leq \liminf_{q \rightarrow \infty} Q_a[\xi_q, u_q, \lambda] = m.$$

But $[\xi_0, u_0]$ also belongs to K_1 , so inequality is impossible in (13.10). That is, $[\xi_0, u_0]$ minimizes Q_a on the class K_1 .

By (13.8), $[\xi_0, u_0]$ also minimizes Q_{a-m} on K_1 , and

$$Q_{a-m}[\xi_0, u_0, \lambda] = 0.$$

Since Q_{a-m} is homogeneous of degree 2 in $[\xi, \dot{\xi}, u]$ we see that Q_{a-m} is non-negative for all admissible variation sets, and on this class $[\xi_0, u_0]$ minimizes Q_{a-m} .

Now we need only a slight extension of Bliss'(¹²) proof of the multiplier rule to show that there are absolutely continuous multipliers $\mu^0, \mu^1(x), \dots, \mu^m(x)$ such that for the function

$$2\Omega(x, \xi, \pi, \mu) = 2\mu^0 \omega(x, \eta, \pi) + \mu^\beta [\phi_z^\beta(x, z_0, z'_0) \xi^c + \phi_p^\beta \pi^c]$$

the du Bois-Reymond relation is satisfied:

$$\Omega_{\pi^c}(x, \xi_0, \dot{\xi}_0) = a_c + \int_{x_{0,1}}^x \Omega_{\xi^c}(x, \xi_0, \dot{\xi}_0) dx,$$

(¹²) G. A. Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 673-744.

where a_c is a constant ($c = 1, \dots, n$). But if the minimizing curve is normal, so is $[\xi_0, u_0]$. So we may suppose $\mu_0 = 1$. It follows that

$$\Omega_{\pi c \pi d}(x, \xi_0, \dot{\xi}_0) v^c v^d = F_{p c p d}(x, z_0, z'_0) v^c v^d,$$

and this last is positive for all nonzero sets (v^1, \dots, v^n) satisfying (10.9). From this we deduce without trouble that $\dot{\xi}_0$ must be continuous, so that ξ_0 is of class C^1 .

Now by (13.2) and (13.8) we have

$$J_2[\xi_0, u_0, \lambda] = Q_{a-m}[\xi_0, u_0, \lambda] - a + m = m - a.$$

But since ξ_0 is of class C^1 , by hypothesis the left member is positive. Hence $m - a$ is positive, and the minimum of J_2 on the class K_1 is positive. It follows by homogeneity that J_2 is positive for all nonidentically vanishing admissible variation sets $[\eta, u]$, as was to be proved.

Since the proofs in this section are rather long, the following remark may not be amiss. Most of the theory of the calculus of variations, originally developed for functions of class D^1 , can be extended without difficulty to absolutely continuous functions. It seems reasonable to suppose that in the majority of specific problems in which the inequality $J_2[\xi, u, \lambda] > 0$ can be established for functions ξ of class D^1 , it will be possible to use essentially the same proof to establish the positiveness of J_2 for all variation sets admissible in the sense of §3.

14. An example. We now exhibit an example of a problem to which Theorem II applies, but which is not covered by any previously published theorem. A simpler example could be given, but this one will have the virtue of being non-trivial. It is convenient to use subscripts instead of superscripts for enumeration of variables, and so on, since we need to use exponents.

Interior to the interval $[0, 1]$ we choose three disjoint closed intervals $\delta_1, \delta_2, \delta_3$ all of the same length ϵ (necessarily less than $1/3$). It is easy to find functions $\gamma_1(x), \gamma_2(x)$ of class C^2 on $[0, 1]$ such that γ_1 has the value 1 on δ_1 and the value 0 on δ_2 and δ_3 , while γ_2 has the value 1 on δ_2 and the value 0 on δ_1 and δ_3 . We define

$$(14.1) \quad f(x, z_1, \dots, z_5, p_1, \dots, p_5) \equiv p_1^2 + p_2^2 + p_3^2 - 4z_1 p_1 - 4z_2 p_2.$$

The problem is to minimize

$$(14.2) \quad J[z, \alpha] \equiv \int_0^1 f(x, z, z') dx$$

in the class of sets $[z_1, \dots, z_5, \alpha_1, \alpha_2, \alpha_3]$ with absolutely continuous functions $z_i(x)$ ($i = 1, \dots, 5$) which satisfy the end conditions

$$(14.3) \quad \begin{aligned} x_1 &= z_i(x_1) = 0 & (i = 1, \dots, 5), \\ x_2 &= 1, & z_4(x_2) = z_5(x_2) = 0, \\ z_1(x_2) &= \alpha_1, & z_2(x_2) = \alpha_2, & z_3(x_2) = \alpha_3, \end{aligned}$$

and the differential equations

$$(14.4) \quad \begin{aligned} \phi_1(x, z, z') &\equiv z'_4 + z_1z'_1 - z_2z'_2 + \gamma_1(x)z'_3 = 0, \\ \phi_2(x, z, z') &\equiv z'_5 + z_1z'_2 + z_2z'_1 + \gamma_2(x)z'_3 = 0. \end{aligned}$$

By use of Theorem II we shall show that the set

$$(14.5) \quad z_i(x) \equiv 0, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0 \quad (0 \leq x \leq 1)$$

gives $J[z, \alpha]$ a weak relative minimum on the class of admissible sets $[z, \alpha]$.

As usual, we define $F = \lambda_0 f + \lambda_1(x)\phi_1 + \lambda_2(x)\phi_2$. The Euler equations simplify to the form

$$\lambda'_1 = \lambda'_2 = 0,$$

and therefore are satisfied for all sets of constant multipliers. Henceforth we assume λ_1 and λ_2 constant. The transversality condition is identically satisfied. The quadratic form (10.8) is

$$(14.6) \quad 2\lambda_0(v_1^2 + v_2^2 + v_3^2),$$

while equations (10.9) become

$$v_4 = v_5 = 0.$$

Hence the strengthened Clebsch condition holds if and only if λ_0 is positive.

Equations (10.10) take the form

$$\xi_4 = 0, \quad \xi_5 = 0,$$

and are therefore satisfied by any set (ξ_1, \dots, ξ_5) with ξ_4 and ξ_5 constant. Equations (10.11) become

$$(14.7) \quad \begin{aligned} \xi_i(0) &= 0 & (i = 1, \dots, 5), \\ \xi_1(1) = u_1, \quad \xi_2(1) = u_2, \quad \xi_3(1) = u_3, \quad \xi_4(1) = \xi_5(1) &= 0. \end{aligned}$$

Thus an admissible variation set is a set $[\xi, u]$ in which ξ_4 and ξ_5 vanish identically and the other three ξ_i vanish at $x=0$ and are absolutely continuous and have derivatives summable with their squares, and in which the u_h satisfy (14.7).

The coefficients b_{hk} are all zero, so if we observe that certain of the terms in 2ω are perfect differentials and make use of (14.7) we obtain

$$(14.8) \quad J_2 [\xi, u] = 2\lambda_0 \int_0^1 (\xi_1^2 + \xi_2^2 + \xi_3^2) dx - 4\lambda_0(u_1^2 + u_2^2) + \lambda_1(u_1^2 - u_2^2) + 2\lambda_2 u_1 u_2.$$

If for every nonidentically null admissible variation set we can choose constants $\lambda_0 > 0$, λ_1 and λ_2 for which this is positive, all the hypotheses of Theorem II will be verified.

If the derivatives ζ_i vanish almost everywhere, the ζ_i are constants. In this case, by (14.7) the ζ_i vanish identically and the u_i also vanish, so $[\zeta, u]$ is identically null. Therefore if $[\zeta, u]$ is not identically null, the integral in (14.8) is positive. If $u_1 = u_2 = 0$, the choice $\lambda_0 = 1, \lambda_1 = \lambda_2 = 0$ serves. Otherwise we could choose $\lambda_0 = u_1^2 + u_2^2, \lambda_1 = 4(u_1^2 - u_2^2), \lambda_2 = 8u_1u_2$. Then

$$J_2[\zeta, u] = 2(u_1^2 + u_2^2) \int_{t_1}^{t_2} (\dot{\zeta}_1^2 + \dot{\zeta}_2^2 + \dot{\zeta}_3^2) dx > 0.$$

It is easy to show⁽¹³⁾ that it is not possible to choose any one set of multipliers with which the second variation is positive for every nonidentically zero admissible variation set $[\zeta, u]$.

To show that the problem is not a trivial one, in which the extremal (14.5) is isolated, and also that the problem does not impose any hidden end conditions, let us choose any three numbers $\alpha_1, \alpha_2, \alpha_3$. Let z_1, z_2 be any Lipschitzian functions vanishing at $x = 0$ and assuming the respective values α_1, α_2 at $x = 1$. We determine three numbers, a_1, a_2, a_3 by the conditions

$$(14.9) \quad \begin{aligned} \epsilon a_1^3 + (1/2)(\alpha_1^2 - \alpha_2^2) &= 0, \\ \epsilon a_2^3 + \alpha_1 \alpha_2 &= 0, \\ \epsilon(a_1 + a_2 + a_3) - \alpha_3 &= 0. \end{aligned}$$

(The number ϵ was defined in the second paragraph of this section.) Let \dot{z}_3 be the function which has the value a_i on the interval δ_i ($i = 1, 2, 3$) and is zero elsewhere, and let

$$z_3(x) = \int_0^1 \dot{z}_3(x) dx.$$

By the last of equations (14.9) we have $z_3(1) = \alpha_3$. The functions z_4, z_5 are determined by (14.4), with the initial values 0. If we integrate from 0 to 1 in (14.4), by (14.9) we find $z_4(1) = z_5(1) = 0$. Hence the functions $z_i(x)$ satisfy the conditions (14.3) and (14.4). Furthermore, it is clear that they can be made to lie in an arbitrarily small first order neighborhood of (14.5) by restricting $|\dot{z}_1|$ and $|\dot{z}_2|$ to be uniformly small and restricting α_1, α_2 and α_3 to lie near zero.

15. Extension to rectifiable curves. The proofs of our sufficiency theorems did not depend in any essential way on the continuity of the derivatives $y'_0(t)$ or $z'_0(x)$. Theorem I, for example, can be generalized by letting C_0 be a rectifiable curve. This of course requires an investigation of the concept of first order neighborhood in the space of rectifiable curves. Such an investiga-

(13) E. J. McShane, *On the second variation in certain normal problems of the calculus of variations*, American Journal of Mathematics, vol. 63 (1941), §5.

tion has already been made⁽¹⁴⁾. Also the formulation of the strengthened Clebsch condition must be altered. The condition as stated in §3 is equivalent, when the functions $y_0(t)$ are of class C^1 , to the following.

There is a positive number ϵ such that the inequality

$$(15.1) \quad F_{r^i r^j}(y_0(t), y'_0(t), \lambda)v^i v^j \geq \epsilon |v|^2$$

holds for all t and all vectors v orthogonal to $y'_0(t)$.

It is this latter form which seems appropriate for extension to the case of rectifiable curves. We say that a rectifiable curve $C_0: y^i = y_0^i(t)$, $t_1 \leq t \leq t_2$, satisfies the strengthened Clebsch condition if (15.1) holds for almost all t such that $y'_0(t)$ is defined and is different from $(0, \dots, 0)$ and for all v orthogonal to $y'_0(t)$.

(14) E. J. McShane, *Curve-space topologies associated with variational problems*, Annali della R. Scuola Normale Superiore di Pisa, (2), vol. 9 (1940), pp. 45–60.

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