

ON THE OSCILLATION OF THE DERIVATIVES OF A PERIODIC FUNCTION

BY

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1. Let $f(x)$ be a real valued periodic function of period 2π defined for all real values of x and possessing derivatives of all orders. Let N_k denote the number of changes of sign of $f^{(k)}(x)$ in a period. We consider the order of magnitude of N_k as $k \rightarrow \infty$.

(I) If $N_k = O(1)$, $f(x)$ is a trigonometric polynomial.

(II) If $N_k = O(k^\delta)$ where δ is fixed, $0 < \delta < 1/2$, $f(x)$ is an entire function of finite order not exceeding $(1 - \delta)/(1 - 2\delta)$.

(III) If $N_k = o(k^{1/2})$, $f(x)$ is an entire function.

We prove this theorem by consideration of the Fourier series of $f(x)$

$$(1) \quad f(x) = \sum c_n e^{inx},$$

$c_{-n} = \bar{c}_n$ ($n = 0, 1, 2, \dots$). Here, as in what follows, the sign \sum without explicitly stated limits means a summation from $-\infty$ to ∞ . Under the present conditions, the series (1) is absolutely and uniformly convergent for real x , and so are the Fourier series of $f'(x)$, $f''(x)$, \dots , obtained from (1) by term by term differentiation. If we focus our attention on the Fourier series, we may express the general trend of our theorem by saying that *a small amount of oscillation in the higher derivatives implies a rapid decrease in the coefficients*, this decrease being so extreme in case (I) that all coefficients from a certain point onward vanish.

The theorem we have to prove and a few analogous facts⁽¹⁾ point towards a general principle which cannot yet be stated in precise terms but which is not entirely unsuitably expressed by saying that *a small amount of oscillation in the higher derivatives indicates a great amount of simplicity in the analytic nature of the function*.

An analogous theorem may be formulated for almost periodic functions. As in other theorems of this kind, the number of changes of sign in a period is replaced by the density of these changes over the infinite line and a trigonometric polynomial is replaced by an entire function of exponential type. The extension of case (I) of our theorem offers the least difficulty.

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(¹) See S. Bernstein, *Leçons sur les Propriétés Extrêmes*, Paris, 1926, pp. 190–197 and Communications de la Société Mathématique de Kharkow, (4), vol. 2 (1928), pp. 1–11; R. P. Boas and G. Pólya, Proceedings of the National Academy of Sciences, vol. 27 (1941), pp. 323–325.

2. We start with a few preliminary remarks on changes of sign. We consider first a real-valued function $f(x)$ which is defined in an interval $a \leq x \leq b$. We say that this function has N changes of sign in this interval if it is possible to find $N + 1$, and no more, abscissae x_0, x_1, \dots, x_N in the interval such that

$$(2) \quad x_0 < x_1 < \dots < x_{N-1} < x_N,$$

$$(3) \quad f(x_{\nu-1})f(x_\nu) < 0, \quad \nu = 1, 2, 3, \dots, N.$$

If a function has N changes of sign in an interval, its derivative has there at least $N - 1$ changes of sign. This variant of Rolle's theorem is easily proved by considering ξ_ν , such that

$$f(x_\nu) - f(x_{\nu-1}) = (x_\nu - x_{\nu-1})f'(\xi_\nu), \quad x_{\nu-1} < \xi_\nu < x_\nu,$$

and observing that $f(x_\nu)f'(\xi_\nu) > 0$ and that therefore

$$f'(\xi_{\nu-1})f'(\xi_\nu) < 0, \quad \nu = 2, 3, \dots, N.$$

Applying this to the function $e^{ax}f(x)$ (where a is a real constant) and its derivative $[e^{ax}f(x)]' = e^{ax}[af(x) + f'(x)]$, we see that the number of changes of sign of

$$(a + D)f(x)$$

(where D is the symbol of differentiation) is not inferior to $N - 1$, N being the number of changes of sign of $f(x)$.

Now let $f(x)$ be periodic with the period 2π . We say that the number of changes of sign of $f(x)$ in a period is N , if it is possible to find just $N + 1$, and no more, abscissae x_0, x_1, \dots, x_N such that

$$x_N = x_0 + 2\pi,$$

and (2), (3) hold. Observe that $f(x_N) = f(x_0)$ and that, therefore, N is necessarily even. Hence it follows that the number of changes of sign of $(a + D)f(x)$ in a period is not inferior to that of $f(x)$. We defined N_k in our initial statement; now we see that

$$(4) \quad N_0 \leq N_1 \leq N_2 \leq \dots \leq N_{k-1} \leq N_k \leq \dots$$

Observing that

$$(a + D)(a - D) \sum \frac{c_n e^{inx}}{a^2 + n^2} = \sum c_n e^{inx},$$

we obtain:

LEMMA I. The number of changes of sign of the function (1) in a period is not inferior to that of

$$\sum \frac{c_n e^{inx}}{a^2 + n^2}.$$

3. The series (1) represents a trigonometric polynomial of order m if $c_n = 0$ for $n = m+1, m+2, \dots$. If $f(x)$ is a trigonometric polynomial of order m , it cannot have more than $2m$ roots in a period; this is well known. Observe that for large k , $f^{(k)}(x)$ has actually $2m$ changes of sign, because as $k \rightarrow \infty$, $(im)^{-k} f^{(k)}(x)$ approaches the first or the second of the two expressions

$$(5) \quad -c_{-m}e^{-imx} + c_me^{imx}, \quad c_{-m}e^{-imx} + c_me^{imx},$$

according as k is odd or even. The second of these expressions is of the form $2|c_m| \cos(mx - \gamma)$, with a certain real γ and the first is of the same form except for a factor i .

The case (I) of our theorem characterizes the trigonometric polynomials and can be stated as follows: *A real-valued periodic function $f(x)$ possessing derivatives of all orders is a trigonometric polynomial if and only if the number of changes of sign of $f^{(k)}(x)$ remains bounded for $k \rightarrow \infty$.*

In order to prove this we consider (1). We have to show that some $f^{(k)}(x)$ have an arbitrarily great number of changes of sign if there are $c_n \neq 0$ with arbitrarily large subscripts n . More precisely we shall show this:

If $m > 0$ and $c_m \neq 0$, then all derivatives of (1), from a certain stage onward, have not less than $2m$ changes of sign.

In fact, by repeated application of Lemma I, we ascertain that

$$(6) \quad f^{(k)}(x) = i^k \sum n^k c_n e^{inx}$$

does not have fewer changes of sign than

$$(7) \quad i^k \sum \left(\frac{2mn}{m^2 + n^2} \right)^k c_n e^{inx}.$$

But since it is given that $c_m \neq 0$ and that $\sum c_n$ is absolutely convergent, we have from a certain k onward

$$(8) \quad |c_m| > \left(\sum_{n=1}^{m-1} + \sum_{n=m+1}^{\infty} \right) \left(\frac{2mn}{m^2 + n^2} \right)^k |c_n|.$$

Indeed we have for $n > 0$, $n \neq m$, $0 < 2mn < m^2 + n^2$, and therefore, each term tends to 0 on the right-hand side of (8) for $k \rightarrow \infty$.

But if (8) holds for a certain even k , the sum in (7) has the same sign as the second expression (5) in all those real points x in which this latter reaches $2|c_m|$, the maximum of its absolute value. This maximum is reached with alternating signs, in equidistant points, the distance of two consecutive points being π/m . Therefore (7) has not less than $2m$ changes of sign. We have proved this for even k but the same is true and the proof is nearly the same for odd k . Then, by Lemma I, (6) has not less than $2m$ changes of sign, and case (I) of our theorem is proved.

4. We consider the case (III) of our theorem before case (II).

If the periodic function $f(x)$ is analytic along the whole real axis, it is analytic in a certain horizontal strip bisected by the real axis and the Fourier series (1), which is a Laurent series in

$$z = e^{ix},$$

converges in the interior of the strip. Hence, by examining

$$(9) \quad \limsup_{n \rightarrow \infty} |c_n|^{1/n}$$

we can distinguish the following three cases:

If $f(x)$ is not analytic along the whole real axis, (9) has the value 1.

If $f(x)$ is analytic in a certain horizontal strip of width $2h$ bisected by the real axis, but in no wider horizontal strip, (9) has the value e^{-h} .

If $f(x)$ is an entire function, (9) has the value 0.

In order to prove case (III) of our theorem, we have to show that in the first two cases $N_k = o(k^{1/2})$ is excluded. We prove the following statement.

If there exists a positive number γ such that

$$(10) \quad \limsup_{n \rightarrow \infty} |c_n| e^{n\gamma} = \infty,$$

then there exists a positive number g such that $f^{(k)}(x)$ has, for an infinity of values of k , not less than $(k/g)^{1/2}$ changes of sign.

By the considerations of the foregoing section, $f^{(k)}(x)$ has certainly not less than $2m$ changes of sign if (8) holds. Using (10), we have to find an arbitrarily large m and a corresponding k such that (8) holds. We shall succeed in finding such an m by applying the following known lemma⁽²⁾.

LEMMA II. We consider two infinite sequences $l_1, l_2, \dots, l_n, \dots$ and $s_1, s_2, \dots, s_n, \dots$, and suppose that

$$(11) \quad l_n \geq 0, \quad n = 1, 2, 3, \dots,$$

$$(12) \quad 0 < s_1 < s_2 < s_3 < \dots,$$

$$(13) \quad \lim_{n \rightarrow \infty} l_n = 0,$$

$$(14) \quad \limsup_{n \rightarrow \infty} l_n s_n = \infty.$$

Then there exists an infinity of integers m such that

$$l_m \geq l_{m+\mu}, \quad \mu = 1, 2, 3, \dots,$$

$$l_m s_m \geq l_{m-\mu} s_{m-\mu}, \quad \mu = 1, 2, \dots, m-1.$$

We put

(²) See G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, vol. 1, pp. 18 and 173, Problem 109.

$$|c_n| = l_n, \quad e^{n\gamma} = s_n.$$

This choice satisfies (11), (12), (13), (14); in fact, (13) is satisfied because (1) is convergent, and (14) is satisfied because we have supposed (10). Thus we obtain an infinity of m such that

$$\begin{aligned} |c_{m+\mu}| &\leq |c_m|, & \mu = 1, 2, 3, \dots, \\ |c_{m-\mu}| e^{(m-\mu)\gamma} &\leq |c_m| e^{m\gamma}, & \mu = 1, 2, \dots, m-1. \end{aligned}$$

This we use to estimate the following sum. (Our ultimate aim is to prove (8).)

$$\begin{aligned} (15) \quad & \sum_{\mu=1}^{m-1} \left(\frac{2m(m-\mu)}{m^2+(m-\mu)^2} \right)^k \left| \frac{c_{m-\mu}}{c_m} \right| + \sum_{\mu=1}^{\infty} \left(\frac{2m(m+\mu)}{m^2+(m+\mu)^2} \right)^k \left| \frac{c_{m+\mu}}{c_m} \right| \\ & < \sum_{\mu=1}^{m-1} \left(\frac{2m(m-\mu)}{m^2+(m-\mu)^2} \right)^k e^{\gamma\mu} + \sum_{\mu=1}^{\infty} \left(\frac{2m(m+\mu)}{m^2+(m+\mu)^2} \right)^k \\ & = S_1 + S_2. \end{aligned}$$

We introduced the abbreviations

$$(16) \quad S_1 = \sum_{\mu=1}^{m-1} e^{\gamma\mu} \left(1 + \frac{\mu^2}{2m(m-\mu)} \right)^{-k},$$

$$(17) \quad S_2 = m \sum_{\mu=1}^{\infty} \left(\frac{2(1+\mu/m)}{1+(1+\mu/m)^2} \right)^k \frac{1}{m},$$

and we shall consider S_1 and S_2 in turn.

(1) Split the sum S_1 in two parts, μ being less than or equal to $m/2$ in the first part and greater than $m/2$ in the second. Using the fact that

$$(1+x)^{-1} < e^{-x/2}, \quad 0 < x < 1,$$

we obtain

$$\begin{aligned} S_1 &< \sum_1^{m/2} e^{\gamma\mu - k\mu^2/4m^2} + \sum_{m/2}^{m-1} e^{\gamma\mu} (4/5)^k \\ &< \sum_1^{m/2} e^{-[(k/4m^2) - \gamma]\mu} + m e^{\gamma m} (4/5)^k \\ &< e^{-(\sigma-\gamma)/(1-e^{-(\sigma-\gamma)})} + m e^{\gamma m} (4/5)^{4\sigma m^2}. \end{aligned}$$

We put

$$(18) \quad k = 4gm^2$$

where g is a positive integer, $g > \gamma$. We choose a fixed g such that for sufficiently great m

$$(19) \quad S_1 < 1/2.$$

(2) The function $2x(1+x^2)^{-1}$ decreases for $x > 1$. Therefore by (17)

$$S_2 < m \int_1^\infty \left(\frac{2x}{1+x^2} \right)^k dx \sim \frac{m}{2} \left(\frac{2\pi}{k} \right)^{1/2} = \frac{1}{2} \left(\frac{\pi}{2g} \right)^{1/2}.$$

We used a well known asymptotic evaluation of definite integrals⁽³⁾ and (18). If $g \geq 2$, which we assume, we obtain for sufficiently great m

$$(20) \quad S_2 < 1/2.$$

But (15), (19), (20) show that (8) is true so that $f^{(k)}(x)$ has not fewer changes of sign than

$$2m = (k/g)^{1/2}.$$

5. We now proceed to the proof of case (II).

LEMMA III. *The Fourier series (1) represents an entire function of the finite order λ , $\lambda > 1$, if and only if*

$$(21) \quad \liminf_{n \rightarrow \infty} \frac{\log \log (1/|c_n|)}{\log n} = \frac{\lambda}{\lambda - 1}.$$

The proof consists of two parts. Both parts follow familiar lines; so we do not give all the details.

(1) Assume that $f(x)$ is entire and of order λ . Then for a fixed positive ϵ and for sufficiently great $|x|$,

$$(22) \quad |f(x)| < e^{|x|^{\lambda+\epsilon}}.$$

If we evaluate c_n and shift the line of integration (periodicity and Cauchy's formula), we obtain as a result that if r is any positive number,

$$(23) \quad \begin{aligned} c_n &= \frac{1}{2\pi} \int_{-ir-\pi}^{-ir+\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} e^{-nr} \int_{-\pi}^{\pi} f(-ir+u) e^{-inu} du, \\ |c_n| &\leq e^{(r+\pi)\lambda+\epsilon-nr}. \end{aligned}$$

Here we use (22). We choose r , for given n , so that this right-hand side of (23) shall be a minimum. It follows by straight-forward calculation that (21) holds with " \geq " instead of " $=$."

(2) Assume that

$$\liminf_{n \rightarrow \infty} \frac{\log \log (1/|c_n|)}{\log n} = \kappa > 0.$$

(*) See, for example, G. Pólya and G. Szegő, loc. cit., vol. 1, pp. 78 and 244, Problem 201.

Therefore we have, for a given positive ϵ and all sufficiently great n

$$|c_n e^{inx}| < e^{-n^{\kappa-\epsilon} + |x|n}.$$

We choose n , for a given x , so that the right-hand side is a maximum. This maximum gives the right order of magnitude because the terms of (1) whose index surpasses a certain multiple of the index of the maximum term, yield a negligible contribution. We find that the order λ of $f(x)$ satisfies the inequality

$$\lambda \leq \frac{\kappa}{\kappa - 1}.$$

This gives (21) with " \leq " instead of " $=$."

6. Now we are prepared to prove case (II) of our theorem. We have to show that *if the entire function $f(x)$ is of order λ , and $\epsilon > 0$ then, for an infinity of k ,*

$$N_k > k^{(\lambda-1)/(2\lambda-1)-\epsilon}.$$

Put $\lambda/(\lambda-1) + \eta = \gamma$, η being positive and small. By Lemma III, the fact we have to show can be stated as follows:

If there exists a positive number γ , $\gamma > 1$, such that

$$\limsup_{n \rightarrow \infty} |c_n| e^{n^\gamma} = \infty,$$

then there exists a positive g such that $f^{(k)}(x)$ has, for an infinity of k , more than $(k/g)^{1/(\gamma+1)}$ changes of sign.

We apply Lemma II, whose conditions are satisfied by

$$l_n = |c_n|, \quad s_n = e^{n^\gamma}.$$

We obtain the result that for an infinity of m ,

$$\begin{aligned} |c_m| &\geq |c_{m+\mu}|, & \mu &= 1, 2, \dots, \\ |c_m| e^{m^\gamma} &\geq |c_{m-\mu}| e^{(m-\mu)^\gamma}, & \mu &= 1, 2, \dots, m-1. \end{aligned}$$

Hence, using the fact that $\gamma > 1$, we obtain

$$\begin{aligned} &\sum_{\mu=1}^{m-1} \left(\frac{2m(m-\mu)}{m^2 + (m-\mu)^2} \right)^k \left| \frac{c_{m-\mu}}{c_m} \right| + \sum_{\mu=1}^{\infty} \left(\frac{2m(m+\mu)}{m^2 + (m+\mu)^2} \right)^k \left| \frac{c_{m+\mu}}{c_m} \right| \\ (24) \quad &< \sum_{\mu=1}^{m-1} \left(\frac{2m(m-\mu)}{m^2 + (m-\mu)^2} \right)^k e^{\gamma m^{\gamma-1} \mu} + \sum_{\mu=1}^{\infty} \left(\frac{2m(m+\mu)}{m^2 + (m+\mu)^2} \right)^k \\ &= S'_1 + S_2. \end{aligned}$$

S_2 has the same meaning as before (see (17)), and

$$\begin{aligned}
 S_1' &= \sum_{\mu=1}^{m-1} e^{\gamma m^{\gamma-1} \mu} \left(1 + \frac{\mu^2}{2m(m-\mu)} \right)^{-k} \\
 (25) \quad &< \sum_{\mu=1}^{m/2} e^{\gamma m^{\gamma-1} \mu - k \mu^2 / 4m^2} + m e^{\gamma m^{\gamma} (4/5)^k} \\
 &= \sum_{\mu=1}^{m/2} e^{-m^{\gamma-1} (g \mu^2 - \gamma \mu)} + m e^{\gamma m^{\gamma} (4/5)^{4g} m^{\gamma+1}}.
 \end{aligned}$$

We put

$$k = 4gm^{\gamma+1},$$

and we choose g so that

$$\gamma + 1 \leq g < \gamma + 2,$$

and so that k is an integer. This choice assures that

$$S_1' \rightarrow 0, \quad S_2 \rightarrow 0$$

for $m \rightarrow \infty$ (see (25) and the considerations preceding (20)). Therefore, by (24), (8) is true and $f^{(k)}(x)$ has not fewer changes of sign than

$$2m > 2[k/4(\gamma + 2)]^{1/(\gamma+1)}.$$

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