ON THE OSCILLATION OF DIFFERENTIAL TRANSFORMS. I

BY

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1. Introduction

1.1. In a recent paper G. Pólya and N. Wiener(1) studied the relation between the analytic character of a real periodic function and the number of the sign variations of its derivatives. The purpose of the present paper is to develop another way of attacking this problem different from that used by the authors mentioned. It leads to a new proof of Theorem 1 of the paper of Pólya and Wiener and to refinements of their Theorems 2 and 3 which are in a certain sense best possible results(2).

Let $f(x)$ be a real periodic function with period 2π for which all derivatives $f^{(k)}(x)$ exist. We denote by $2N_k$ the number of the mod 2π distinct values of x for which a sign variation of $f^{(k)}(x)$ takes place. In what follows we give first a new proof of Theorem 1 of Pólya and Wiener. A further, more elaborate, application of our method leads to the following results which correspond to the Theorems 3 and 2, respectively, of the authors mentioned.

Theorem A. *Let $N_k < k/\log k$ provided k is sufficiently large. Then $f(x)$ is an integral function.*

Theorem B. *Let $\rho > 1$ and let $N_k < (k/\rho)^{1/\rho}/2$ provided k is sufficiently large. Then $f(x)$ is an integral function of order not greater than $\rho/(\rho-1)$.*

The following results are more informative.

Theorem A′. *Let for sufficiently large k*

$$N_k < \frac{k}{\log k}\left(1 + \frac{\log\log k - \omega(k)}{\log k}\right) \tag{1.1.1}$$

where $\omega(k) \to +\infty$. Then the conclusion of Theorem A *holds.*

Theorem B′. *Let $\rho > 1$ and let p be a positive number such that $p\rho^{2+1/\rho} > 1$. If for sufficiently large k*

$$N_k < (k/\rho)^{1/\rho}\left(1 - p\,\frac{\log k}{k^{1-1/\rho}}\right), \tag{1.1.2}$$

then the conclusion of Theorem B *holds.*

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(1) G. Pólya and N. Wiener, *On the oscillation of the derivatives of a periodic function*, these Transactions, vol. 52 (1942), pp. 249–256.

(2) See the counterexamples given below, section 7.

The following result contains Theorem A′ (therefore also Theorem A):

THEOREM A″. *Let H be a constant such that*

$$H + \log((1/2)\log 2) > 0, \tag{1.1.3}$$

and let for sufficiently large k

$$N_k < \frac{k}{\log k}\left(1 + \frac{\log\log k - H}{\log k}\right). \tag{1.1.4}$$

Then the function $f(x)$ is analytic in the strip

$$|\Im f(x)| < H + \log((1/2)\log 2). \tag{1.1.5}$$

1.2. In various conversations Professor Hille suggested certain analogues of these problems considering $\vartheta^l f(x)$ instead of $f^{(2l)}(x)$ where ϑ is a given second order differential operator satisfying suitable conditions(3). In the last part of the present paper we illustrate the further applicability of our method by discussing the special operator

$$\vartheta = (1 - x^2)D^2 - 2xD, \qquad D = d/dx, \tag{1.2.1}$$

the "characteristic functions" of which are the classical Legendre functions. Let $f(x)$ be a function having derivatives of all orders in $-1 \leqq x \leqq +1$ and let $N_k = N_{2l}$ denote the number of the sign variations of $\vartheta^l f(x)$ in the interval $-1 < x < +1$. Then we prove(4)

THEOREM C. *If $N_k \leqq N$, k sufficiently large and even, then $f(x)$ must be a polynomial of degree less than or equal to N.*

THEOREM D. *If N_k satisfies the condition of Theorem A″, k even, then $f(x)$ is analytic in an ellipse with foci at -1 and $+1$ the sum of the semi-axes of which is*

$$\exp\{H + \log((1/2)\log 2)\}.$$

These results correspond to Theorem 1 of Pólya and Wiener and to Theorem A″, respectively. The analogue of Theorem B can also be dealt with.

1.3. In what follows we give the proofs of the results formulated above. Section 2 contains a new proof of Theorem 1 of Pólya and Wiener; the underlying idea of this proof is used throughout the present paper. Section 3 contains the proof of Theorem A″, section 4 that of Theorem B′. Sections 5 and 6 are devoted to the proofs of Theorems C and D involving Legendre's operator. Finally in section 7 certain counterexamples are exhibited which

(3) See below, pp. 463–497.

(4) The proof furnishes the conclusion of Theorem C under the condition that $N_k \leqq N$ holds for an infinite number of k values. (The same holds in section 2.)

show that the conditions of Theorems A and B on N_k cannot be replaced by $N_k=O(k)$ and $N_k=O(k^\alpha)$, $\alpha>1/\rho$, respectively.

2. New proof of Theorem 1 of Pólya and Wiener

2.1. Let

$$f(x) = \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x}, \qquad c_{-\nu} = \bar{c}_\nu, \tag{2.1.1}$$

and let $2N_k$ be the number of the mod 2π distinct sign variations of $f^{(k)}(x)$. We assume that k goes to $+\infty$ through a sequence of integers such that N_k has a constant value N. Then we show that $f(x)$ is a trigonometric polynomial of degree less than or equal to N, that is, we prove $c_{N+m}=0$, $m>0$.

Let $x_1, x_2, \cdots, x_{2N}$ denote the mod 2π distinct sign variations of $f^{(k)}(x)$, that is, the values of x for which $f^{(k)}(x)$ changes its sign; $x_\nu = x_\nu(k)$. Let α be real and ([5])

$$\begin{aligned} u(x) &= \sin\frac{x-x_1}{2}\sin\frac{x-x_2}{2}\cdots\sin\frac{x-x_{2N}}{2}(1+\cos^m(x+\alpha)) \\ &= \sum_{\nu=-N-m}^{N+m} u_\nu e^{i\nu x}, \qquad u_{-\nu} = \bar{u}_\nu. \end{aligned} \tag{2.1.2}$$

(In case $N=0$ we write $u(x)=1+\cos^m(x+\alpha)$.) This is a trigonometric polynomial of the fixed degree $N+m$, the sign variations of which are the same as those of $f^{(k)}(x)$. The coefficients $u_\nu = u_\nu(k)$ are bounded as $k\to\infty$; this can easily be showed by multiplying out the expression

$$u(x) = 2^{-2N}\prod_{\nu=1}^{2N}\left(e^{-i(\pi+x_\nu)/2}e^{ix/2}+e^{i(\pi+x_\nu)/2}e^{-ix/2}\right)\left(1+2^{-m}(e^{i\alpha}e^{ix}+e^{-i\alpha}e^{-ix})^m\right). \tag{2.1.3}$$

Also we obtain for the highest coefficient of $u(x)$

$$\begin{aligned} u_{N+m} &= (-1)^N 2^{-2N-m}\exp\left\{-i\sum x_\nu/2 + im\alpha\right\} \\ &= (-1)^N 2^{-2N-m}\exp\left\{ix_0 + im\alpha\right\} \end{aligned} \tag{2.1.4}$$

where the real quantity $x_0 = x_0(k)$ depends on k but it is independent of α.

2.2. Let $c_{N+m}\neq 0$. The sign of

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} f^{(k)}(x)u(x)dx = \sum_{\nu=-N-m}^{N+m}(i\nu)^k c_\nu u_{-\nu} \tag{2.2.1}$$

is *independent* of α, positive say. We determine α in such a way that the last term

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([5]) We could use as well $1+\cos m(x+\alpha)$ or $(1+\cos(x+\alpha))^m$ instead of $1+\cos^m(x+\alpha)$.

$$(2.2.2)\qquad \begin{aligned} &(i(N+m))^k c_{N+m} u_{-N-m} \\ &\qquad = (i(N+m))^k c_{N+m}(-1)^N 2^{-2N-m} \exp\{-ix_0 - im\alpha\}\end{aligned}$$

becomes *real and negative*. Then

$$(2.2.3)\qquad 2(N+m)^k |c_{N+m}| 2^{-2N-m} < \sum_{\nu=-N-m+1}^{N+m-1} |\nu|^k |c_\nu| |u_{-\nu}| = \sum_{\nu=1}^{N+m-1} \nu^k O(1)$$

follows where the bounds $O(1)$ are independent of k. But this involves a contradiction for sufficiently large k.

3. Proof of Theorem A″

3.1. We start with some preliminary remarks.

(a) The constant H must be positive since $\log((1/2)\log 2) < 0$.

(b) Let $x = \sigma + it$, σ and t real, and let T denote the unique value such that (2.1.1) converges for $|t| < T$ and diverges for $|t| > T$ (or T is the largest value such that $f(x)$ is analytic in the strip $|t| < T$). We have

$$(3.1.1)\qquad \limsup_{\nu\to+\infty} |c_\nu|^{1/\nu} = e^{-T}.$$

The modifications necessary for $T = 0$ or $T = \infty$ are obvious.

Now another form of the assertion of Theorem A″ is

$$(3.1.2)\qquad T \geqq H + \log((1/2)\log 2).$$

(c) Theorem A′ is obviously a consequence of Theorem A″.

3.2. We assume

$$(3.2.1)\qquad \limsup_{\nu\to+\infty} |c_\nu| e^{\nu\gamma} = +\infty, \qquad \gamma > 0,$$

and show that

$$(3.2.2)\qquad \gamma \geqq H + \log((1/2)\log 2).$$

From (3.2.1) we conclude in a well known manner(6) the existence of a sequence of integers $\{M\}$ such that

$$(3.2.3)\qquad |c_M| e^{M\gamma} > |c_\nu| e^{|\nu|\gamma}, \qquad \pm\nu = 0, 1, 2, \cdots, M-1.$$

Now let ϵ be an arbitrary but fixed positive number. We define a sequence of integers $k = k(M)$ by

$$(3.2.4)\qquad k = k(M) = [M(\log M + H - \epsilon)].$$

(6) Cf. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, 1925, p. 18; p. 173, Problem 107. What is needed here is much less than the lemma used by Pólya and Wiener, loc. cit., p. 252.

Then an easy calculation shows that

$$
\begin{aligned}
N_k &< \frac{k}{\log k}\left(1 + \frac{\log\log k - H}{\log k}\right) \\
&= M\left(1 + \frac{H - \epsilon - \log\log M}{\log M} + O\left(\left(\frac{\log\log M}{\log M}\right)^2\right)\right) \\
&\quad \cdot \left(1 + \frac{\log\log M - H}{\log M} + O\left(\left(\frac{\log\log M}{\log M}\right)^2\right)\right) \\
&= M\left(1 - \frac{\epsilon}{\log M} + O\left(\left(\frac{\log\log M}{\log M}\right)^2\right)\right) < M \qquad (3.2.5)
\end{aligned}
$$

provided M is sufficiently large.

3.3. Let us denote again by $x_1, x_2, \cdots, x_{2N}$, $N = N_k$, the mod 2π distinct sign variations of $f^{(k)}(x)$, $x_\nu = x_\nu(k)$; here $k = k(M)$. We write, α real(7),

$$
(3.3.1) \qquad
\begin{aligned}
u(x; M) &= \sin\frac{x - x_1}{2}\sin\frac{x - x_2}{2}\cdots\sin\frac{x - x_{2N}}{2}(1 + \cos^m(x + \alpha)) \\
&= \sum_{\nu=-M}^{+M} u_\nu e^{i\nu x}, \qquad u_{-\nu} = \bar{u}_\nu, \quad N + m = M.
\end{aligned}
$$

(For $N = 0$ we omit the sine factors.) Since $N < M$, m is positive. The trigonometric polynomial $u(x; M)$ is of degree M and it has the same sign variations as $f^{(k)}(x)$. We prove

LEMMA 1. *Let the trigonometric polynomial* $u(x; M)$ *be defined by* (3.3.1) *and let*

$$
(3.3.2) \qquad U(x; M) = (\cos(x/2))^{2N}(1 + \cos^m x) = \sum_{\nu=-M}^{+M} U_\nu e^{i\nu x} = \sum_{\nu=-M}^{+M} U_\nu \cos \nu x;
$$

then the inequalities

$$
(3.3.3) \qquad |u_\nu| \leqq U_\nu, \qquad \nu = 0, 1, 2, \cdots, M,
$$

hold, with the sign "=" for $\nu = M$.

Indeed as (2.1.3) shows, the coefficients u_ν of $u(x; M)$ are multilinear functions of $e^{\pm i(\pi + x_\nu)/2}$ and $e^{\pm i\alpha}$ with non-negative coefficients. Thus we do not decrease $|u_\nu|$ by replacing the quantities $e^{\pm i(\pi + x_\nu)/2}$ and $e^{\pm i\alpha}$ by 1, or by replacing the constants x_ν by $-\pi$ and α by 0. This leads precisely to (3.3.2). The assertion regarding $|u_M| = U_M$ is also clear. We have (see (2.1.4))

$$
(3.3.4) \qquad
\begin{aligned}
u_M &= (-1)^N 2^{-2N-m} \exp\{ix_0 + im\alpha\}, \qquad x_0 = -\sum x_\nu/2, \\
U_M &= 2^{-2N-m} = 2^{-N-M}.
\end{aligned}
$$

(7) See Footnote 5.

3.4. Now

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} f^{(k)}(x)u(x;\,M)dx = \sum_{\nu=-M}^{+M} (i\nu)^k c_\nu u_{-\nu} \tag{3.4.1}$$

has a sign *independent* of α. Choosing α in a proper way ($c_M \neq 0$) and using (3.3.3) we obtain

$$\sum_{\nu=-M+1}^{M-1} |\nu|^k |c_\nu| U_\nu \geqq 2M^k |c_M| U_M. \tag{3.4.2}$$

Taking (3.2.3) and the inequality $x \leqq e^{x-1}$ into account we find

$$\begin{aligned} 2U_M &\leqq \sum_{\nu=-M+1}^{M-1} U_\nu \exp\{(|\nu|/M-1)k + (M-|\nu|)\gamma\} \\ &= \sum_{\nu=-M+1}^{M-1} U_\nu Q^{|\nu|-M} < Q^{-M}\sum_{\nu=-M+1}^{M-1} U_\nu(Q^\nu + Q^{-\nu}), \qquad Q = e^{k/M-\gamma}. \end{aligned} \tag{3.4.3}$$

From (3.2.4) we conclude that $Q = Q(M) \to \infty$ or more precisely $Q(M) \cong e^{H-\epsilon-\gamma}M$ as $M \to \infty$. (The symbol $a(M) \cong b(M)$ means that $a(M)[b(M)]^{-1} \to 1$ as $M \to \infty$.) Introducing

$$\xi = \xi(M) = (Q + Q^{-1})/2, \qquad T_{|\nu|}(\xi) = (Q^\nu + Q^{-\nu})/2 \tag{3.4.4}$$

where $T_{|\nu|}(\xi)$ is identical with the Tchebichef polynomial, by virtue of (3.3.2), we can write (3.4.3) in the following form:

$$U_M Q^M \leqq \sum_{\nu=-M+1}^{M-1} U_\nu T_{|\nu|}(\xi) = 2^{-N}(1+\xi)^N(1+\xi^m) - 2U_M T_M(\xi); \tag{3.4.5}$$

hence

$$2U_M Q^M \leqq 2^{-N}(1+\xi)^N(1+\xi^m)$$

or (cf. (3.3.4))

$$2^{1-M}(Q/\xi)^M \leqq (1+\xi^{-1})^N(1+\xi^{-m}) < e^{\xi^{-1}M}(1+\xi^{-1}). \tag{3.4.6}$$

Now let $M \to \infty$. Then

$$2^{-M}(Q/\xi)^M = (1+Q^{-2})^{-M} \to 1 \tag{3.4.7}$$

since $Q^{-2} = O(M^{-2})$. Further $\xi^{-1}M \to 2e^{-H+\epsilon+\gamma}$ so that

$$2 \leqq \exp\{2e^{-H+\epsilon+\gamma}\} \tag{3.4.8}$$

follows. Since ϵ is arbitrarily small, this involves (3.2.2).

4. Proof of Theorem B′

4.1. Let the order λ of $f(x)$ be greater than $\rho/(\rho-1)$. Then using the previous notation (2.1.1),

$$\liminf_{\nu\to+\infty} \frac{\log\log 1/|c_\nu|}{\log \nu} = \frac{\lambda}{\lambda-1} < \rho \tag{4.1.1}$$

holds[8]. Consequently

$$\limsup_{\nu\to+\infty} |c_\nu| e^{\nu^\rho} = \infty. \tag{4.1.2}$$

We obtain now instead of (3.2.3) the inequalities

$$|c_M| e^{M^\rho} > |c_\nu| e^{|\nu|^\rho}, \qquad \pm\nu = 0, 1, 2, \cdots, M-1, \tag{4.1.3}$$

holding for a certain sequence $\{M\}$ of integers.

The previous proof needs only unessential modifications.

4.2. We write

$$k = k(M) = \left[\rho M^\rho\left(1 + q\frac{\log M}{M^{\rho-1}}\right)\right] \tag{4.2.1}$$

where q is a fixed constant satisfying the conditions

$$1/\rho < q < p\rho^{1+1/\rho}. \tag{4.2.2}$$

An easy calculation shows that for large k

$$\begin{aligned} N_k &< (k/\rho)^{1/\rho}\left(1 - p\frac{\log k}{k^{1-1/\rho}}\right) \\ &= M\left(1 + q\rho^{-1}\frac{\log M}{M^{\rho-1}} + o\left(\frac{\log M}{M^{\rho-1}}\right)\right) \\ &\quad \cdot\left(1 - p\rho^{1/\rho}\frac{\log M}{M^{\rho-1}} + o\left(\frac{\log M}{M^{\rho-1}}\right)\right) \\ &= M\left(1 - (p\rho^{1/\rho} - q\rho^{-1})\frac{\log M}{M^{\rho-1}} + o\left(\frac{\log M}{M^{\rho-1}}\right)\right) < M. \end{aligned} \tag{4.2.3}$$

Using the same notation and the same argument as in §3.4 we obtain instead of (3.4.3)

$$2U_M \leqq \sum_{\nu=-M+1}^{M-1} U_\nu \exp\left\{(|\nu|/M - 1)k + M^\rho - |\nu|^\rho\right\}. \tag{4.2.4}$$

Since $M^\rho - |\nu|^\rho \leqq (M - |\nu|)\rho M^{\rho-1}$ we find as before

$$2U_M \leqq R^{-M}\sum_{\nu=-M+1}^{M-1} U_\nu(R^\nu + R^{-\nu}), \qquad R = e^{k/M - \rho M^{\rho-1}}. \tag{4.2.5}$$

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(8) See Pólya and Wiener, loc. cit., p. 254.

On account of (4.2.1) we have $R=R(M)\cong M^{\rho q}\to\infty$ as $M\to\infty$.

Now let

$$(4.2.6)\qquad \eta = \eta(M) = (R + R^{-1})/2 \to \infty \qquad \text{as } M \to \infty;$$

then we obtain (cf. (3.4.6))

$$(4.2.7)\qquad 2^{1-M}(R/\eta)^M \leqq e^{\eta^{-1}M}(1 + \eta^{-1}).$$

But $\rho q>1$ so that $(1+R^{-2})^{-M}\to 1$. Moreover

$$(4.2.8)\qquad \eta^{-1}M \cong 2M^{1-\rho q} \to 0.$$

This furnishes the contradictory inequality $2\leqq 1$.

5. Proof of Theorem C

5.1. The proofs of Theorems C and D are based on arguments similar to those followed in the previous part. Instead of trigonometric series, expansions in Legendre series are used.

Let

$$(5.1.1)\qquad f(x) = \sum_{\nu=0}^{\infty} c_\nu P_\nu(x),$$

c_ν real, be the Legendre expansion of $f(x)$ where $P_\nu(x)$ is the Legendre polynomial in the customary notation. By using the notation (1.2.1)

$$(5.1.2)\qquad \vartheta^l f(x) = \sum_{\nu=0}^{\infty} (-\lambda_\nu)^l c_\nu P_\nu(x), \qquad \lambda_\nu = \nu(\nu+1),$$

follows. Let $N_k=N_{2l}$ denote the number of the sign variations of $\vartheta^l f(x)$ in $-1<x<+1$ and let $N_k=N_{2l}=N$ be fixed as $l\to\infty$ through a proper sequence of integers. We show then that $c_{N+m}=0$, $m>0$.

Let $x_1, x_2, \cdots, x_N$ be the sign variations of $\vartheta^l f(x)$ in $-1<x<+1$. We form[9]

$$(5.1.3)\qquad v(x) = (x-x_1)(x-x_2)\cdots(x-x_N)(1+\delta x^m) = \sum_{\nu=0}^{N+m} v_\nu P_\nu(x)$$

where δ is either $+1$ or -1. This is a polynomial of degree $N+m$ with the same sign variations as $\vartheta^l f(x)$. The coefficients $v_\nu=v_\nu(l)$ are bounded as $l\to\infty$. Furthermore $v_{N+m}=\delta h_{N+m}$ if h_ν denotes the highest coefficient in the Legendre expansion of x^ν.

Now

$$(5.1.4)\qquad \int_{-1}^{+1}\{\vartheta^l f(x)\}v(x)dx = \sum_{\nu=0}^{N+m}(\nu+1/2)^{-1}(-\lambda_\nu)^l c_\nu v_\nu.$$

(9) We could use $1+\delta P_m(x)$ instead of $1+\delta x^m$. See Footnote 5.

This expression has the same sign whether $\delta=+1$ or $\delta=-1$. We obtain by a suitable choice of δ

$$(5.1.5)\qquad \sum_{\nu=0}^{N+m-1}(\nu+1/2)^{-1}\lambda_\nu^l|c_\nu||v_\nu| \geqq (N+m+1/2)^{-1}\lambda_{N+m}^l|c_{N+m}|h_{N+m}.$$

Division through by λ_{N+m}^l leads to a contradiction as $l\to\infty$ unless $c_{N+m}=0$.

6. Proof of Theorem D

6.1. For the proof of Theorem D we follow again the previous argument. Under the assumption (3.2.1) we obtain a sequence $\{M\}$ of integers such that (3.2.3) holds. The definition of $k=2l$ is in this case slightly different from (3.2.4), namely

$$(6.1.1)\qquad k = 2l = 2[(M/2)(\log M + H - \epsilon)].$$

Then k is even and $N=N_k<M$. Now we define m by $N+m=M$ and $v(x)=v(x;\,M)$ by (5.1.3). We prove

Lemma 2. *Let the rational polynomial $v(x)=v(x;\,M)$ be defined by* (5.1.3) *and let*

$$(6.1.2)\qquad V(x;\,M) = (x+1)^N(1+x^m) = \sum_{\nu=0}^{M} V_\nu P_\nu(x);$$

then the inequalities

$$(6.1.3)\qquad |v_\nu| \leqq V_\nu, \qquad \nu = 0, 1, 2, \cdots, M,$$

hold with the sign "=" for $\nu=M$.

It is well known[10] that $P_r(x)P_s(x)$ expanded in terms of Legendre polynomials has non-negative coefficients. Multiplying out

$$(6.1.4)\qquad v(x;\,M) = \prod_{\nu=1}^{N}(P_1(x)-x_\nu)\{1+\delta(P_1(x))^m\}$$

we see that the coefficients v_ν of $v(x;\,M)$ are multilinear functions of $-x_\nu$ and δ with non-negative coefficients. Obviously we do not decrease $|v_\nu|$ by replacing $-x_\nu$ and δ by 1 which leads precisely to (6.1.2).

6.2. Starting from (5.1.4) we obtain (5.1.5) and

$$(6.2.1)\qquad (M+1/2)^{-1}h_M \leqq \sum_{\nu=0}^{M-1}(\nu+1/2)^{-1}(\lambda_\nu/\lambda_M)^{k/2}e^{(M-\nu)\gamma}V_\nu.$$

([10]) See J. C. Adams, *On the expression of the product of any two Legendre's coefficients by means of a series of Legendre's coefficients*, Proceedings of the Royal Society, vol. 27 (1878), pp. 63–71; *Collected Scientific Papers*, vol. 1, pp. 487–496. See E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edition, 1935, p. 331, Problem 11.

Now let

$$g_0 = 1; \qquad g_\nu = \frac{1\cdot 3 \cdots (2\nu - 1)}{2\cdot 4 \cdots 2\nu}, \qquad \nu = 1, 2, 3, \cdots. \tag{6.2.2}$$

Then $(\nu+1/2)g_\nu\lambda_\nu^{-1}$ is decreasing as ν increases since

$$\frac{\nu + 1/2}{\nu - 1/2}\,\frac{g_\nu}{g_{\nu-1}}\left(\frac{\lambda_\nu}{\lambda_{\nu-1}}\right)^{-1} = \frac{(\nu + 1/2)(\nu - 1)}{(\nu + 1)\nu} < 1. \tag{6.2.3}$$

Hence

$$(\nu + 1/2)g_\nu\lambda_\nu^{-1} > (M + 1/2)g_M\lambda_M^{-1}, \qquad \nu = 0, 1, \cdots, M - 1, \tag{6.2.4}$$

so that from (6.2.1)

$$g_M h_M \leqq \sum_{\nu=0}^{M-1} g_\nu(\lambda_\nu/\lambda_M)^{k/2-1}e^{(M-\nu)\gamma}V_\nu. \tag{6.2.5}$$

But

$$\lambda_\nu^{1/2} - \nu = (\nu(\nu + 1))^{1/2} - \nu = (1 + (1 + \nu^{-1})^{1/2})^{-1} \tag{6.2.6}$$

is positive and increasing with ν so that

$$\lambda_\nu/\lambda_M \leqq \exp\left\{2(\lambda_\nu^{1/2} - \lambda_M^{1/2})\lambda_M^{-1/2}\right\} \leqq \exp\left\{2(\nu - M)\lambda_M^{-1/2}\right\}. \tag{6.2.7}$$

Hence

$$g_M h_M \leqq \sum_{\nu=0}^{M-1} g_\nu V_\nu S^{\nu-M}, \qquad S = e^{(k-2)\lambda_M^{-1/2}-\gamma}. \tag{6.2.8}$$

From (6.1.1) we conclude that $S = S(M) \cong e^{H-\epsilon-\gamma}\, M$ as $M \to \infty$.

Writing

$$\zeta = (S + S^{-1})/2 \to \infty, \qquad M \to \infty, \tag{6.2.9}$$

we obtain by using a classical representation of Legendre polynomials([11])

$$P_\nu(\zeta) \geqq g_\nu S^\nu, \qquad \nu = 0, 1, 2, \cdots, \tag{6.2.10}$$

so that from (6.2.8) and (6.1.2)

$$g_M h_M \leqq S^{-M}\sum_{\nu=0}^{M-1} V_\nu P_\nu(\zeta) = S^{-M}(1 + \zeta)^N(1 + \zeta^m) - S^{-M}h_M P_M(\zeta). \tag{6.2.11}$$

The representation mentioned furnishes also $g_M h_M = 2^{-M}$ so that using (6.2.10) we again conclude

([11]) See, for instance, G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, p. 92, equation (4.9.4).

$$2^{1-M} \leqq S^{-M}(1+\zeta)^N(1+\zeta^m)$$

or

(6.2.12) $$2^{1-M}(S/\zeta)^M \leqq (1+\zeta^{-1})^N(1+\zeta^{-m}) \leqq e^{\zeta^{-1}N}(1+\zeta^{-1}).$$

Now let $M\to\infty$. Then (cf. section 3.4)

$$2^{-M}(S/\zeta)^M = (1+S^{-2})^{-M} \to 1,$$
$$\zeta^{-1}M \to 2e^{-H+\epsilon+\gamma}$$

so that (3.4.8) follows. Consequently (3.1.1), (3.1.2) hold.

If the coefficients c_ν of the Legendre expansion (5.1.1) satisfy (3.1.1), then $f(x)$ must be analytic in an ellipse with foci at -1 and $+1$ the sum of the semi-axes of which is e^T([12]).

This establishes the proof of Theorem D.

7. Counterexamples

7.1. In this section we show that the conditions $N_k<k/\log k$ and $N_k<(k/\rho)^{1/\rho}/2$ of Theorems A and B cannot be replaced by $N_k=O(k)$ and $N_k=O(k^\alpha)$, respectively, where $\alpha>1/\rho$. I owe the necessary counterexamples to a suggestion of Professor Pólya.

7.2. The first assertion can be proved by considering the non-integral periodic function

(7.2.1) $$f(x) = (1-2h\cos x+h^2)^{-1}, \qquad 0<h<1.$$

We see by mathematical induction that

(7.2.2) $$f^{(k)}(x) = t_k(x)(1-2h\cos x+h^2)^{-k-1}$$

where $t_k(x)$ is a trigonometric polynomial of degree k. Consequently in this case $N_k\leqq 2k$.

7.3. Let $p>1$. The integral function

(7.3.1) $$f(x) = \sum_{n=1}^{\infty} e^{-n^p}\cos nx$$

is([13]) of order $p/(p-1)$ and as we shall prove $N_k=O(k^{1/p})$. This furnishes, indeed, the desired counterexample by assuming $\alpha<1$ and choosing p according to the conditions $1/\rho<1/p<\alpha$; then $\rho/(\rho-1)<p/(p-1)$.

Let k be even. We apply Jensen's theorem to the function

(7.3.2) $$f^{(k)}(x) = (-1)^{k/2}\sum_{n=1}^{\infty} n^k e^{-n^p}\cos nx$$

in the circle $|x|\leqq 2\pi$. Since

([12]) See Szegö, loc. cit., p. 238, Theorem 9.1.1.

([13]) See Footnote 8.

$$|f^{(k)}(x)| \leqq \sum_{n=1}^{\infty} n^k e^{-n^p+2\pi n} \tag{7.3.3}$$

and

$$|f^{(k)}(0)| = \sum_{n=1}^{\infty} n^k e^{-n^p} \tag{7.3.4}$$

we find for the number $N(\pi)$ of the roots of $f^{(k)}(x)$ in the circle $|x| \leqq \pi$

$$2^{N(\pi)} \sum_{n=1}^{\infty} n^k e^{-n^p} \leqq \sum_{n=1}^{\infty} n^k e^{-n^p+2\pi n}. \tag{7.3.5}$$

Obviously $N_k \leqq N(\pi)$.

In order to find a suitable bound for $N(\pi)$ let us consider the function $\lambda(\nu) = \nu^k e^{-\nu^p}$ of the continuous variable ν, $\nu \geqq 1$. It is increasing for $\nu < \nu_0$ and decreasing for $\nu > \nu_0$ where

$$\nu_0 = \nu_0(k) = (k/p)^{1/p}. \tag{7.3.6}$$

The maximum of $\lambda(\nu)$ is $\exp\{(k/p)\log(k/p) - k/p\}$.

The function $\lambda^*(\nu) = \nu^k e^{-\nu^p/2}$ assumes its maximum for $\nu_0^1 = \nu_0^1(k) = (2k/p)^{1/p}$.

Now let ω be fixed, $\omega > (2/p)^{1/p}$, $\log \omega - \omega^p/2 < -(\log p + 1)/p$. Then for $k \to \infty$

$$\mathrm{I} = \sum_{n \leqq \omega k^{1/p}} n^k e^{-n^p+2\pi n} \leqq e^{2\pi\omega k^{1/p}} |f^{(k)}(0)|. \tag{7.3.7}$$

Further $\lambda^*(\nu)$ is decreasing for $\nu > \omega k^{1/p} > \nu_0'(k)$ so that

$$\begin{aligned} \mathrm{II} &= \sum_{n > \omega k^{1/p}} n^k e^{-n^p+2\pi n} \leqq \lambda^*(\omega k^{1/p}) \sum_{n=1}^{\infty} e^{-(n^p/2)+2\pi n} \\ &= O(1) \exp\{k \log \omega + (k/p) \log k - \omega^p k/2\}. \end{aligned} \tag{7.3.8}$$

By use of the mean value theorem we find

$$\log \lambda(\nu_0) \leqq \log \lambda([\nu_0] + 1) + Ck^{1-1/p} \tag{7.3.9}$$

where $C > 0$ is independent of k. But for large k

$$\begin{aligned} k \log \omega + (k/p) \log k - \omega^p k/2 &< (k/p) \log (k/p) - k/p - Ck^{1-1/p} \\ &< \log \lambda([\nu_0] + 1), \end{aligned} \tag{7.3.10}$$

hence

$$\begin{aligned} \mathrm{II} &= O(1)|f^{(k)}(0)|, \\ \mathrm{I} + \mathrm{II} &= O(1) e^{2\pi\omega k^{1/p}} |f^{(k)}(0)|. \end{aligned} \tag{7.3.11}$$

Consequently

$$2^{N(\pi)} = O(1)e^{2\pi\omega k^{1/p}} \tag{7.3.12}$$

from which $N(\pi)=O(k^{1/p})$ follows.

7.4. In case of an odd k we have $f^{(k)}(0)=0$. Then in Jensen's theorem $f^{(k)}(0)$ has to be replaced by

$$f^{(k+1)}(0) = (-1)^{(k+1)/2}\sum_{n=1}^{\infty} n^{k+1}e^{-n^p}. \tag{7.4.1}$$

The previous argument holds good except that k has to be replaced by $k+1$.

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