ON THE OSCILLATION OF DIFFERENTIAL TRANSFORMS. II

CHARACTERISTIC SERIES OF BOUNDARY VALUE PROBLEMS

BY

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INTRODUCTION

1. **Formulation of problem.** G. Pólya and N. Wiener [2](1) have recently made important contributions to the S. Bernstein problem concerning *the relation between the frequency of oscillation of derivatives of high order and the analytic character of the function.* Assuming $f(x)$ of period 2π and denoting the number of sign changes of $f^{(k)}(x)$ in the period by N_k, they show that restrictions in the rate of growth of N_k when $k\to\infty$, imply that the high frequency terms in the Fourier series of $f(x)$ have "small" amplitudes. In particular, *if N_k is bounded, $N_k\leqq N$ for all k,* then the high frequency terms are entirely missing and *$f(x)$ reduces to a trigonometric polynomial of degree at most $N/2$. Conversely, if $f(x)$ is a trigonometric polynomial of degree K, then $N_k=2K$ for all large k.* Their results are less precise when N_k is unbounded. While it is likely that $N_k=O(k)$ is necessary and sufficient for analyticity of $f(x)$, this has not yet been proved, and the best they could do was to show that $N_k=o(k^{1/2})$ implies that $f(x)$ is an entire function.

For these and similar questions G. Szegö has devised a new method of attack, presented in the first paper of this series [4]. This method showed itself capable of giving more precise information when N_k is unbounded. In particular, Szegö could show that $N_k<k(\log k)^{-1}$ implies that $f(x)$ is entire.

The present paper is also closely related to the paper of Pólya and Wiener, but proceeds in a different direction. We aim to preserve the essence of the methods developed by these writers and to apply them to a wider range of problems. There are several features in the investigation of Pólya and Wiener which suggest possible generalizations, in particular, the class of functions considered and the operations applied to them.

Let T be an operator which takes functions $f(x)$ of a certain class F into functions of the same class. Any function $u(x)$ of F such that $Tu(x)=\lambda u(x)$ will be called a *characteristic function of T* corresponding to the *characteristic value* λ and any formal series $\sum f_n u_n(x)$ will be called a *characteristic series of T* if its terms are characteristic functions.

In this terminology we can describe the investigation of Pólya and Wiener

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(1) Numbers in square brackets refer to the references at the end of the paper.

as follows(2). They are concerned with the differential operator D^2 and the characteristic functions of this operator determined by the periodic boundary value problem

$$(D^2 + \mu)y = 0, \qquad y(x + 2\pi) = y(x). \tag{1.1}$$

Any function $f(x) \in C^{(\infty)}(-\infty, \infty)$, satisfying the same condition of periodicity $f(x+2\pi) = f(x)$, can be represented by a characteristic series of the operator,

$$f(x) = (a_0/2) + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.2}$$

to which the operator D^2 can be applied termwise as often as we please. They observe that for $\lambda > 0$, $D^2 - \lambda$ is an *oscillation preserving transformation* in the sense that the transform $(D^2 - \lambda) f(x)$ has at least as many sign changes in the period as $f(x)$ has. This observation is used as follows.

Let m be a positive integer and multiply the nth term of the series (1.2) by the kth power of the factor

$$\left\{\frac{2mn}{m^2 + n^2}\right\}^2. \tag{1.3}$$

A function $F(x, m, k; f)$ results which has at least as many sign changes in the period as $f^{(2k)}(x)$ since

$$(D^2 - m^2)^{2k} F(x, m, k; f) = (2m)^{2k} f^{(2k)}(x).$$

On the other hand, for large values of k the number of sign changes of $F(x, m, k; f)$ can be shown to be at least $2m$ provided the mth term is present in the original expansion (1.2). This is the basis for all their conclusions.

It is obvious from this formulation in what direction we are looking for extensions. Instead of the operator D^2 we shall consider a rather general linear differential operator L. In the present paper we restrict ourselves to second order operators satisfying certain conditions, but first or higher order operators would also be admissible. We define a set of characteristic functions of L by a suitable boundary value problem for L in the basic interval (a, b) and consider the corresponding class of characteristic series, F say, with the restriction that L shall apply termwise to the series as often as we please. It turns out that the operator $L - \lambda$, $\lambda > 0$, is always oscillation preserving in (a, b) with respect to a suitable class of functions which includes F. Even the "root consuming factor" (1.3) has an obvious analogue in terms of characteristic values and the general procedure of Pólya and Wiener can be followed.

(2) Actually Pólya and Wiener work with the operator D and the corresponding characteristic functions $\exp(nix)$. The "root consuming factor" in (1.3) is the square of their factor. The emphasis and terminology have been changed in order to bring out the generalizations.

It should be observed, however, that the method is not constrained to the consideration of characteristic series the terms of which are defined by boundary value problems and consequently orthogonal in the basic interval. The case of almost periodic functions was mentioned by Pólya and Wiener and a non-orthogonal characteristic series figures also in §2.11 of the present paper(3).

2. **Arrangement of material.** Chapter I is devoted to a study of oscillation preserving transformations defined by linear second order differential operators. The basic definitions are found in §1.1 while 1.2 contains a number of lemmas of the classical Sturmian type which are needed for the discussion. In §1.3 the operators L are classified according to their behavior at the end points of the basic interval and to each of the four types considered we introduce function classes $B_\nu^{(k)}$ the elements of which satisfy, together with their L-transforms of order less than k, the corresponding types of boundary conditions. That $L-\lambda$, $\lambda>0$, is oscillation preserving with respect to $B_\nu^{(1)}$ is proved in §§1.4 to 1.7. Various extensions to functions of L are discussed in §1.8 and the corresponding boundary value problems are introduced in 1.9. We call attention to the singular and semi-singular types which appear to be new, though many of the most useful orthogonal systems considered in analysis appear as solutions of such boundary value problems.

Chapter II brings the proof of the analogue of the Pólya-Wiener theorem on finite characteristic series. Here we place the discussion on a rather elaborate postulational basis to make up for our lack of knowledge of the existence and properties of solutions of the singular and semi-singular boundary value problems. We consider systems S consisting of an operator L, a set of characteristic functions $\{u_n(x)\}$ with corresponding characteristic values $\{\mu_n\}$, and a basic interval (a, b). We call the system *admissible* if it satisfies conditions A_1 to A_6 of §2.1. These are conditions which are well known to hold in the case of classical boundary value problems but which, conceivably, may fail in the case of singular ones. We also consider the class F of admissible characteristic series $\sum f_n u_n(x)$ such that $\sum \mu_n^m |f_n| < \infty$ for all m. The convergence theory of such series is discussed in 2.2. The system S is called *conservative* if the set $\{u_n(x), \mu_n\}$ belongs to an appropriate boundary

(3) There are no general results available relating to oscillation problems for non-orthogonal characteristic series. Existing evidence, meager as it is, seems to indicate that the situation is similar to the orthogonal case. In other words, if the frequency of oscillation of $L^k f(x)$ is bounded or has a finite limit inferior, then the frequency of oscillation of the components of $f(x)$ is similarly limited, the main difference being that we may now still have infinitely many components. "Characteristic integrals" can also be studied from this point of view by a suitable modification of the method. A first investigation of this type will be given by J. D. Tamarkin in a later paper in this series. The author wishes to use this opportunity to express his gratitude to his collaborators on the S. Bernstein problem, Professors G. Pólya, A. C. Schaeffer, G. Szegö, and J. D. Tamarkin, with whom he has had many profitable discussions of various points of his work during his stay at Stanford University.

value problem for L in (a, b) and it is shown that the results of Chapter I apply to conservative systems. In particular, $L-\lambda$, $\lambda>0$, and any real polynomial in L with real positive roots are oscillation preserving in (a, b) with respect to the class F. This is proved in §2.3 where we also discuss the relation between F and the classes $B_\nu^{(\infty)}$ introduced in 1.3.

The main theorem is proved in 2.4. *If S is conservative and $f(x)\in F$, then the assumption that the inferior limit of the number of sign changes of $L^k f(x)$ in (a, b) is finite and equals N, implies that $f(x)$ is a linear combination of a finite number of characteristic functions $u_n(x)$, none of which can have more than N* (in an exceptional case possibly $N+1$) *sign changes in (a, b)*. This is the analogue of Theorem I of Pólya and Wiener. In §§2.5 and 2.6 we verify that the *classical boundary value problems lead to conservative systems*. In §§2.7 to 2.11 we give similar verifications for the systems of Legendre, Jacobi, Hermite, Weber-Hermite, and Laguerre, which correspond to singular boundary problems, and that of Bessel which is semi-singular. We call attention, in particular, to the characterization of ordinary polynomials by means of sign change properties given in Theorems 12, 13, and 14. It is analogous to the results of Pólya and Wiener for trigonometric polynomials quoted above.

Extensions to the case in which N_k is unbounded are indicated briefly in §3.1 of the Appendix. The author has extended Theorem III of Pólya and Wiener under fairly general assumptions on the system, but the rather lengthy and complicated analysis is omitted here and the results are stated merely for the singular systems of §§2.7 to 2.10. It turns out that $N_k=o(k^{1/2})$ is again sufficient in order that the corresponding characteristic series shall converge in the finite complex plane and hence represent an entire function. The method of Szegö gives a better result, when it applies, which is to the Legendre and Jacobi cases.

Chapter I. Oscillation preserving transformations

1.1. **Preliminary notions and formulas.** All functions considered in Chapters I and II are real functions of a real variable, defined in a finite or infinite interval $\langle a, b\rangle$ and having certain properties of continuity in $\langle a, b\rangle$. Here $\langle a, b\rangle$ stands for one of the four alternatives (a, b), $(a, b]$, $[a, b)$, and $[a, b]$. The symbols $C^{(k)}\langle a, b\rangle$, with $k=0$, positive integer or ∞, refer to the usual continuity classes. Finally we denote the class of all functions real and holomorphic in $\langle a, b\rangle$ by $A\langle a, b\rangle$.

Let $g(x)\in C^{(0)}(a, b)$. We say that $g(x)$ *has N changes of sign in (a, b), if (a, b) breaks up into exactly $N+1$ subintervals in each of which $g(x)$ keeps a constant sign, the signs being opposite in adjacent intervals.* The subintervals are in general not uniquely determined. The statement that the sign of $g(x)$ in (x_1, x_2) is, for instance, positive is taken in the wide sense, that is, $g(x)\geqq 0$ and actually is greater than 0 in some subinterval of (x_1, x_2). If $g(x)$ is periodic of period $b-a$, this definition should be slightly modified. We map the interval on the circumference of a circle, identifying the end points. Here N intervals of alternating signs determine N sign changes. It is clear that N must be even in the periodic case. If there is no finite N with these properties, we say that $g(x)$ has infinitely many sign changes in (a, b). The number of sign changes of $g(x)$ in (a, b), finite or infinite, is denoted by $V[g(x)]$. The theorem of Rolle implies

LEMMA 1. *If $g(x) \in C^{(1)}(x_1, x_2)$ and $g(x) \to 0$ when $x \to x_1$ and when $x \to x_2$ but $g(x) \neq 0$ in (x_1, x_2), then $g'(x)$ has at least one sign change in (x_1, x_2).*

Let L denote the differential operator defined by

$$L[y] \equiv p_0(x)y + p_1(x)Dy + p_2(x)D^2y, \qquad D = d/dx, \tag{1.1.1}$$

where to start with the coefficients will be subjected to the following two assumptions which will be held fast throughout the paper:

A_1. $p_m(x) \in A(a, b)$, $m = 0, 1, 2$.

A_2. $p_0(x) \leqq 0$, $p_2(x) > 0$ *for* $a < x < b$.

For much of our work in §§1.2 to 1.7 it would be sufficient to assume merely $p_m(x) \in C^{(0)}(a, b)$, but any consideration involving repeated application of the operator requires additional restrictions of $p_m(x)$, so we may just as well assume analyticity from the start(4).

The self-adjoint form of L is L^* where

$$L^*[y] \equiv P(x)L[y] = D[P(x)p_2(x)Dy] + P(x)p_0(x)y, \tag{1.1.2}$$

$$P(x) = \frac{1}{p_2(x)} \exp\left\{ \int^x \frac{p_1(t)}{p_2(t)} dt \right\}. \tag{1.1.3}$$

Here $P(x) > 0$ for $a < x < b$. If $p_1(x) \equiv 0$, we take $P(x) = 1/p_2(x)$.

Any solution of the differential equation

$$(L + \mu)y = 0, \tag{1.1.4}$$

μ constant, will be referred to as a *characteristic function of L corresponding to the characteristic value μ*. The reader should note that the terminology differs from that used in the Introduction according to which $-\mu$ rather than μ would be called the characteristic value. The present convention is preferable when one works with second order linear differential equations.

If $f(x) \in C^{(2)}(a, b)$, then $L[f]$ has a sense and $L[f] \in C^{(0)}(a, b)$. The differential transform $L[f]$ is the first *L-transform* of $f(x)$. The higher L-transforms are defined by recurrence:

$$L^k[f] = L[L^{k-1}[f]], \qquad L^0[f] = f. \tag{1.1.5}$$

If $f(x) \in C^{(2k)}(a, b)$, then $L^k[f]$ exists and belongs to $C^{(0)}(a, b)$. If convenient or desirable we drop the brackets or exhibit the variable. Thus $L^k f$, $L^k f(x)$,

(4) It should be observed in connection with A_2 that the theory goes through with only minor changes if $p_0(x)$ has merely a finite upper bound in (a, b).

$L^k[f(x)]$, $L^k[f]$ all have the same meaning. The reader should observe that the symbol $L^kf(x_0)$, $a \leqq x_0 \leqq b$, denotes the value of $L^k[f]$ for $x = x_0$ and not the result of operating by L^k on the constant $f(x_0)$.

DEFINITION. *Let F be a subclass of $C^{(2)}(a, b)$ and let λ be fixed real. Then $L-\lambda$ is said to be an oscillation preserving transformation in (a, b) with respect to F if*

$$V[(L - \lambda)f(x)] \geqq V[f(x)] \tag{1.1.6}$$

for every $f(x) \in F$.

It should be observed that there are always functions $f(x) \not\equiv 0$, satisfying (1.1.6). Thus if μ is fixed real and $y(x, \mu)$ is any solution of (1.1.1), then $(L-\lambda)\, y(x, \mu) = -(\lambda+\mu)\, y(x, \mu)$, so that (1.1.6) is trivially satisfied for every $\lambda \neq -\mu$. If $V[f(x)] = \infty$, (1.1.6) is understood to mean merely that the left-hand side is also infinite.

The basic formula in the discussion of the operator $L-\lambda$ is the factorization given by

$$(L - \lambda)f = \frac{p_2 W_2}{W_1} D\left\{\frac{W_1^2}{W_2} D\left[\frac{f}{W_1}\right]\right\}, \tag{1.1.7}$$

for which see L. Schlesinger [3, vol. I, p. 52]. Here, W_1 is a solution of the associated differential equation $(L-\lambda)\, y = 0$, and W_2 is the Wronskian of W_1 and a second linearly independent solution of the same equation. It is permitted to assume that W_2 is real positive in (a, b). The crucial point in the use of formula (1.1.7) lies in the choice of W_1 which we refer to as the *auxiliary solution.*

1.2. **General properties of the auxiliary solution.** We proceed to a discussion of the solutions of the associated differential equation(5)

$$(L - \lambda)y = 0, \qquad \lambda > 0, \tag{1.2.1}$$

in the interval (a, b). Introducing

$$K(x) = P(x)p_2(x), \qquad G(x, \lambda) = P(x)[\lambda - p_0(x)], \tag{1.2.2}$$

we can rewrite the equation in the form

$$D[K(x)Dy] - G(x, \lambda)y = 0. \tag{1.2.3}$$

Under the assumption A_2, $K(x)$ and $G(x, \lambda)$ for $\lambda > 0$ are positive in (a, b).

Two integrated forms of the equation will be useful in the following. First we have obviously

(5) The discussion follows the classical Sturmian pattern, but at least some of the required results do not appear to be available in a convenient form in the literature. The proofs are held down to a minimum.

$$[K(x)y'(x)]_{x_1}^{x_2} = \int_{x_1}^{x_2} G(t, \lambda)y(t)dt. \tag{1.2.4}$$

Secondly, multiplying (1.2.3) by y and integrating we get

$$[K(x)y(x)y'(x)]_{x_1}^{x_2} = \int_{x_1}^{x_2} K(t)[y'(t)]^2 dt + \int_{x_1}^{x_2} G(t, \lambda)[y(t)]^2 dt. \tag{1.2.5}$$

We conclude from (1.2.5) that if $y(x) \not\equiv 0$ is a solution of (1.2.1) in (a, b), then the product $y(x)\ y'(x)$ can vanish at most once in the interval. Hence the real solutions of (1.2.1) are of the following four types in (a, b): (1) monotone solutions of constant sign, (2) solutions of constant sign having a maximum, (3) solutions of constant sign having a minimum, and (4) monotone solutions having a zero. These types are mutually exclusive and exhaust the possibilities. The fourth type is of no interest to us in the following and will be omitted from consideration.

The existence of unbounded solutions is vital in most of our discussion. We introduce the following notation:

$$\begin{gathered} U(x, x_0) = \int_{x_0}^{x} \frac{dt}{K(t)}, \quad Q(x, x_0; \lambda) = \int_{x_0}^{x} G(v, \lambda)dv, \\ R(x, x_0; \lambda) = \int_{x_0}^{x} \frac{Q(t, x_0; \lambda)}{K(t)} dt. \end{gathered} \tag{1.2.6}$$

LEMMA 2. *Let $y(x)$ be the solution of* (1.2.1) *determined by the initial conditions $y(x_0) = 1$, $y'(x_0) = s \geqq 0$, $a < x_0 < b$. A necessary and sufficient condition that $y(x) \to \infty$ when $x \to b$ is that $R(x, x_0; \lambda) \to \infty$ when $x \to b$. If the latter condition is satisfied for a particular choice of x_0 and λ, then it holds for every x_0, $a < x_0 < b$, and every $\lambda > 0$. Moreover, if the condition holds, every solution of* (1.2.1) *such that $y(x)y'(x)$ is ultimately positive for approach to b becomes infinite when $x \to b$. Similarly, if $R(x, x_0; \lambda) \to \infty$ when $x \to a$, then every solution with $y(x)y'(x)$ ultimately negative for approach to a becomes infinite when $x \to a$.*

It is clear from the structure of $R(x, x_0; \lambda)$ that the condition is independent of x_0 and λ. We shall prove the lemma for fixed x_0 and λ and consider only the case $x \to b$. The same method applies at the other end point. The lemma is an immediate consequence of

LEMMA 3. *Under the assumptions of Lemma* 2([6])

$$S(x, x_0; \lambda) < y(x) < \exp \{S(x, x_0; \lambda)\} \tag{1.2.7}$$

([6]) The inequality (1.2.7) does not give very precise information regarding the rate of growth of $y(x)$, but in a certain sense it is the best of its kind. The ratio $y(x)/S(x, x_0; \lambda)$ is bounded in the case of the Legendre operator $L = (1-x^2)D^2 - 2xD$, $a = -1$, $b = 1$, for approach to the singular end points, while $y(x) \exp [-S(x, x_0; \lambda)]$ is bounded away from zero in the case of the Hermite operator $L = D^2 - 2xD$, $a = -\infty$, $b = \infty$. See §§2.7 and 2.9 below.

for $x_0 \leqq x < b$, *where*

$$S(x, x_0; \lambda) = R(x, x_0; \lambda) + K(x_0)y'(x_0)U(x, x_0). \tag{1.2.8}$$

Putting $x_1 = x_0$ and $x_2 = u$ in (1.2.4) and noting that $y(x)$ is increasing and greater than 1 in the interval (x_0, b), we get

$$K(u)y'(u) > K(x_0)y'(x_0) + Q(u, x_0; \lambda).$$

Dividing by $K(u)$ and integrating from x_0 to x, we obtain the first half of (1.2.7). But we have obviously also

$$K(u)y'(u) < K(x_0)y'(x_0) + Q(u, x_0; \lambda)y(u).$$

Dividing by $K(u)y(u)$, dropping $y(u) > 1$ in the first denominator on the right, and then integrating from x_0 to x, we get the second half of the inequality.

This shows that $y(x)$ becomes infinite when $x \to b$ if and only if $S(x, x_0; \lambda)$ has the same property. But both terms on the right in (1.2.8) are positive and a simple calculation shows that $U(x, x_0) \to \infty$ when $x \to b$ implies $R(x, x_0; \lambda) \to \infty$, but not vice versa. This completes the proof of the lemmas.

These inequalities have a relation to the transformation theory of the differential equation which is of some interest for the following. If we introduce a new independent variable in (1.2.1) by putting $u = U(x, x_0)$ and define $y(x) = Y(u)$, then the transformed differential equation is simply

$$\frac{d^2Y}{du^2} - \frac{d^2R}{du^2} Y = 0, \tag{1.2.9}$$

where under our assumptions $d^2R/du^2 > 0$ in the interval (A, B) which is the image of (a, b) under the transformation. If, for instance, $B = \infty$, then it is perfectly trivial that every solution of (1.2.9), which is not positive monotone decreasing in (A, B), becomes infinite with u. This transformation will be useful in the proof of the next lemma which is a comparison theorem of the classical Sturmian type.

LEMMA 4. *Let* $y(x; x_0, s, \lambda)$ *be the solution of* $(L - \lambda)y = 0$, $y(x_0) = 1$, $y'(x_0) = s \geqq 0$. (1) *For fixed* x, x_0, *and* s, $x_0 < x$, $y(x; x_0, s, \lambda)$ *is an increasing function of* λ. (2) *For fixed* λ, *the ratio of* $y(x; x_1, 0, \lambda)$ *to* $y(x; x_2, 0, \lambda)$, $a < x_1 < x_2 < b$, *lies between finite positive bounds depending upon* x_1 *and* x_2 *but not upon* x, $a < x < b$. (3) *For fixed* x_0 *and* λ, *the ratio of* $y(x; x_0, s_1, \lambda)$ *to* $y(x; x_0, s_2, \lambda)$, $0 \leqq s_1 < s_2$, *lies between finite positive bounds depending upon* s_2 *but not upon* x, $x_0 \leqq x < b$.

The first statement follows directly from the formula

$$K(x)\left[y_\mu(x)y_\lambda'(x) - y_\lambda(x)y_\mu'(x)\right] = (\lambda - \mu)\int_{x_0}^{x} P(t)y_\lambda(t)y_\mu(t)dt$$

with obvious notation. The second assertion lies slightly deeper, but follows from the expression for the Wronskian of two solutions $y_1(x)$ and $y_2(x)$ of the equation. Taking $y_1(x) = y(x; x_1, 0, \lambda)$, $y_2(x) = y(x; x_2, 0, \lambda)$, we get

$$K(t)W_2(y_1(t), y_2(t)) = K(x_1)y_2'(x_1) \equiv -C < 0.$$

Dividing through by $[y_2(t)]^2K(t)$ and integrating from x_1 to x where $x_2 < x$, we obtain

$$\frac{y_1(x)}{y_2(x)} = \frac{1}{y_2(x_1)} + C\int_{x_1}^{x} \frac{dt}{K(t)[y_2(t)]^2},$$

so the statement is proved for such values of x if we can show the convergence of the integral when $x \to b$. This is trivial if the integral obtained by suppressing the factor $[y_2(t)]^2$ in the denominator is convergent. Hence we can assume that the function $U(x, x_0)$ of formula (1.2.6) tends to infinity when $x \to b$. Putting $u = U(x, x_1)$ and transforming the differential equation upon the form (1.2.9) we get

$$\int_{x_1}^{x} \frac{dt}{K(t)[y_2(t)]^2} = \int_0^u \frac{dv}{[Y_2(v)]^2}$$

with obvious notation. But $Y_2(v)$ is positive, concave upwards for $v > 0$, and tends to infinity with v. Hence we can find a linear function $\alpha v + \beta$ with $\alpha > 0$ such that $Y_2(v) > \alpha v + \beta$ for $v > v_0$. This proves the convergence of the integral and gives a finite upper bound for the ratio in the interval (x_2, b) where the lower bound is unity. In the interval (a, x_1) we simply interchange $y_1(x)$ and $y_2(x)$ and apply the same method. The interval (x_1, x_2) is trivial. This completes the proof of (2). The same method can be used in proving (3).

1.3. **Boundary conditions.** We shall make no attempt to determine the maximal class with respect to which the operator $L - \lambda$ is oscillation preserving in (a, b). It is likely to be a complicated and none too interesting problem. We shall instead specialize L in various ways and determine certain associated classes of functions by means of appropriate boundary conditions. We shall consider four alternatives which by no means exhaust the field but which at least cover a large number of cases of well established interest.

T_1. Sturm-Liouville type. $p_m(x) \in A[a, b]$, $m = 0, 1, 2$; $p_2(a) \neq 0$, $p_2(b) \neq 0$.

T_2. Periodic type. *Assumptions as under* T_1, *but in addition*, $K(a) = K(b)$, *that is*, $\int_a^b \{p_1(t)/p_2(t)\}\, dt = 0$.

T_3. Singular type. $R(x, x_0; \lambda) \to \infty$ *when* $x \to a$ *and when* $x \to b$ *for some* x_0, $a < x_0 < b$, *and* $\lambda > 0$.

T_4. Semi-singular type. $p_m(x) \in A(a, b]$, $m = 0, 1, 2$; $p_2(b) \neq 0$; *and* $R(x, b; \lambda) \to \infty$ *when* $x \to a$, $\lambda > 0$.

In T_3 and T_4 the functions $R(x, x_0; \lambda)$ and $R(x, b; \lambda)$ are defined by (1.2.6). Lemma 2 shows that the value of x_0 is immaterial and that the condition holds for all $\lambda > 0$ if it holds for a single one. In T_4 the roles of a and b can of course be interchanged.

With each operator L of type T_ν we associate a set of classes $B_\nu^{(k)}\{L; \langle a, b\rangle\}$ of functions $f(x)$ satisfying appropriate boundary conditions. Here k is a positive integer or infinity.

B_1. $B_1^{(k)}\{L; [a, b]\} \subset C^{(2k)}[a, b]$; $L^n f(a) = 0$, $L^n f(b) = 0$, $n = 0, 1, \cdots, k-1$.

B_2. $B_2^{(k)}\{L; [a, b]\} \subset C^{(2k)}[a, b]$; $L^m f(a) = L^m f(b)$, $m = 0, 1, \cdots, k-1, k$; $DL^n f(a) = DL^n f(b)$, $n = 0, 1, \cdots, k-1$.

B_3. $B_3^{(k)}\{L; (a, b)\} \subset C^{(2k)}(a, b)$; $[L^n f(x)]/y(x; x_0, \lambda) \to 0$ *when* $x \to a$ *and when* $x \to b$ *for* $n = 0, 1, \cdots, k-1$, *for some* x_0, $a < x_0 < b$, *and arbitrarily small positive* λ.

B_4. $B_4^{(k)}\{L; (a, b]\} \equiv B_4^{(k)}\{L; C_1, C_2; (a, b]\} \subset C^{(2k)}(a, b]$; $[L^n f(x)]/y(x; b, \lambda) \to 0$ *when* $x \to a$ *for arbitrarily small positive* λ; $C_1 L^n f(b) + C_2 DL^n f(b) = 0$, C_1, C_2 *fixed greater than or equal to* 0, *both conditions holding for* $n = 0, 1, \cdots, k-1$.

Here $L^n f(a)$ and $DL^n f(a)$ are the values of $L^n f(x)$ and $DL^n f(x)$ at $x = a$. In B_3, $y(x; x_0, \lambda) = y(x; x_0, 0, \lambda)$ in the notation of Lemma 4, similarly in B_4 where $x_0 = b$. By virtue of Lemma 4 we should expect that the value of x_0 is immaterial and that small positive values of λ are the decisive ones. It is perfectly obvious that we could consider other classes of functions in connection with these operators. In particular, more general boundary conditions could be allowed at one end point in B_1. If a and b are interchanged in B_4, the sign of C_2 should also be changed. We merely mention these possibilities. Our main object in Chapter I will be to study the four listed types in some detail and to prove Theorem 1 and its various extensions.

THEOREM 1. *If the operator* L *is of type* T_ν *and* $\lambda > 0$, *then for every* $f(x) \in B_\nu^{(1)}\ \{L; \langle a, b\rangle\}$, $\nu = 1, 2, 3, 4$, *we have*(7) $V[(L-\lambda) f(x)] \geqq V[f(x)]$, *that is*, $L - \lambda$ *is oscillation preserving in* (a, b) *with respect to the corresponding class* $B_\nu^{(1)}\{L; \langle a, b\rangle\}$.

1.4. **Discussion of the Sturm-Liouville case.** This case is readily recognized and the proof of Theorem 1 is quite simple. We choose $W_1 = y(x, \lambda)$ in (1.1.7) as the solution of the initial value problem $(L-\lambda)\, y = 0$, $y(b, \lambda) = 1$, $y'(b, \lambda) = 0$. Formula (1.2.5) shows that $y(x, \lambda) > 1$ in (a, b).

The theorem is trivial if $V[f(x)] = \infty$. Suppose then that $V[f(x)] = N < \infty$. We can then find $N+2$ points x_j where $a = x_1 < x_2 < \cdots < x_{N+1} < x_{N+2} = b$, such that $f(x_j) = 0$ and $f(x)$ is not identically zero in anyone of the intervals (x_j, x_{j+1}). Since $y(x, \lambda) \geqq 1$, Lemma 1 shows that $D[f(x)/y(x, \lambda)]$ has at least one sign change in each of the $N+1$ intervals (x_j, x_{j+1}). Multiplication by the positive bounded factor $[y(x, \lambda)]^2/W_2$ does not change this situation and the derivative of the result by Lemma 1 has at least N sign changes in (a, b). Hence $V[(L-\lambda) f(x)] \geqq N$ and the theorem is proved.

If the boundary conditions in B_1 for $k = 1$ be modified so that $f(a) = 0$ is replaced by the condition $C_1 f(a) - C_2 f'(a) = 0$, $C_1 \geqq 0$, $C_2 > 0$, while the condi-

(7) $V[g]$ is to be computed according to the definition for periodic functions when $\nu = 2$ but according to the main definition in the other cases. See §1.1, second paragraph.

tion $f(b)=0$ is left intact, the proof can still be carried through, but the choice of $y(x, \lambda)$ has to be modified accordingly. To each $f(x)$ of the class we determine a corresponding $y(x, \lambda)$ by the condition that it should have the same logarithmic derivative as $f(x)$ at $x=a$. Taking $y(a, \lambda)=1$ as is permissible, we still have $y(x, \lambda)>1$ in (a, b). Then $D[f(x)/y(x, \lambda)]$ will be zero at $x=a$ instead of in the interior of (a, x_1). It consequently still has $N+1$ zeros in (a, b) and does not vanish identically between any consecutive pair of zeros. Thus $(L-\lambda)f(x)$ has N sign changes at least, and the theorem is proved under the more general assumptions. The restriction imposed on the sign of the logarithmic derivative of $f(x)$ at $x=a$ is dictated solely by our concern that the corresponding $y(x, \lambda)$ shall be positive in $[a, b]$. If this condition is known to be satisfied, the restriction can be dropped. It is clear that modifying the boundary conditions at both end points meets with additional difficulties and this problem will not be considered here. It should be observed, however, that the case $f'(a)=0$, $f'(b)=0$, can be handled without difficulty.

1.5. **Discussion of the periodic case.** The name periodic case is to some extent a misnomer, but it is a customary designation for the corresponding type of boundary conditions and the case has close relations to periodicity in the usual sense. Moreover, it includes as a special instance the case in which $K(x)$ and $G(x, \lambda)$ are periodic with period $(b-a)$.

If $f(x)\in B_2^{(1)}\{L; [a, b]\}$, then $f(x)\in C^{(2)}[a, b]$, $f(a)=f(b)$, $f'(a)=f'(b)$, and $Lf(a)=Lf(b)$. We can then find a function $f^*(x)\in C^{(1)}(-\infty, \infty)$ such that $f^*(x+b-a)=f^*(x)$ and $f^*(x)=f(x)$ in $[a, b]$. The second derivative of $f^*(x)$ is continuous everywhere with the possible exception of $x\equiv a \pmod{(b-a)}$ where, however, right- and left-hand derivatives exist. Similarly $Lf(x)$ can be extended periodically as a continuous function and the extension agrees with $Lf^*(x)$.

The definition of $V[g(x)]$ given in §1.1 varied slightly according as $g(x)$ was defined only in $[a, b]$ or could be extended periodically as a continuous function with period $(b-a)$ outside of this interval. In the latter case the definition was such that the number of sign changes in the period would be independent of the choice of the end points. Actually the two definitions are always in agreement except in the case in which $g(a)=g(b)=0$ and $g(x)$ has an odd number, say $2K-1$, sign changes in the interior of the interval. In this case one definition would give $V[g(x)]=2K-1$ and the other $2K$, the zero at $x=a$ being counted as an additional sign change in the definition for periodic functions.

We now agree that if $\nu=2$ the definition for periodic functions shall be used in interpeting the V-symbols in Theorem 1. In other words, the inequality to be proved is actually

$$(1.5.1) \qquad V[(L-\lambda)f^*(x)] \geqq V[f^*(x)].$$

In the subsequent proof $V[g]$ refers to the non-periodic and $V[g^*]$ to the periodic definition. The reader should note that $V[g] \leqq V[g^*] \leqq V[g]+1$ and $V[g^*]$ is always an even number.

For the proof we have to distinguish several subcases. Suppose first that $f(a)=f(b)=0$ but $V[f(x)]=V[f^*(x)]=2K$. The proof given in §1.4 applies without any change and gives $V[(L-\lambda)f(x)] \geqq 2K$ which in turn implies (1.5.1).

Suppose next that $f(a)=f(b)=0$ and $V[f(x)]=2K-1$, $V[f^*(x)]=2K$. We choose the same auxiliary solution $y(x, \lambda)$ as in the preceding case. By Lemma 1, $D[f(x)/y(x, \lambda)]$ has at least $2K$ sign changes in (a, b). It follows that $V[(L-\lambda)f(x)] \geqq 2K-1$. Hence $V[(L-\lambda)f^*(x)] \geqq 2K$ and (1.5.1) follows.

Suppose finally that $f(a)=f(b)>0$ and $V[f(x)]=V[f^*(x)]=2K$. If $f'(a) \geqq 0$ we determine $y(x, \lambda)$ by the initial conditions $y(a, \lambda)=1$, $y'(a, \lambda)=f'(a)/f(a)$. If $f'(a)=f'(b)<0$, we take instead $y(b, \lambda)=1$, $y'(b, \lambda)=f'(b)/f(b)$. In either case $y(x, \lambda) \geqq 1$ in $[a, b]$. Then

$$[y(x, \lambda)]^2 \frac{d}{dx}\left\{\frac{f(x)}{y(x, \lambda)}\right\} = y(x, \lambda)f'(x) - y'(x, \lambda)f(x)$$

has at least $2K-1$ sign changes in (a, b) and, in addition, vanishes at $x=a$ or $x=b$ depending upon the sign of $f'(a)$. It follows that $V[(L-\lambda)f(x)] \geqq 2K-1$ and $V[(L-\lambda)f^*(x)] \geqq 2K$. This completes the proof of Theorem 1 in the periodic case.

The proof is modelled upon that given by Pólya and Wiener for the case $L=D^2$.

1.6. **Discussion of the singular case.** This case is characterized by the presence of singular points of the differential equation at $x=a$ and $x=b$, sufficiently severe to cause the critical function $R(x, x_0; \lambda)$ to become infinite for $\lambda>0$. The class $B_3^{(1)}\{L; (a, b)\}$ consists of all functions $f(x) \in C^{(2)}(a, b)$ such that $f(x)/y(x; x_0, \lambda) \rightarrow 0$ when $x \rightarrow a$ and when $x \rightarrow b$ for arbitrarily small positive values of λ. Here $y(x; x_0, \lambda)$ is determined by $(L-\lambda)y=0$, $y(x_0)=1$, $y'(x_0)=0$, $a<x_0<b$. Let us first assess the influence of x_0 and λ upon the determination of the class $B_3^{(1)}\{L; (a, b)\}$.

Suppose that x_0 and λ are fixed and suppose $f(x) \in C^{(2)}(a, b)$, $f(x)/y(x; x_0, \lambda) \rightarrow 0$, $x \rightarrow a, b$. Denote the class of all such functions for the moment by $F^{(1)}(\lambda, x_0)$. By Lemma 4 the ratio of $y(x; x_1, \lambda)$ to $y(x; x_2, \lambda)$ is bounded away from zero and infinity in (a, b). It follows that $f(x) \in F^{(1)}(\lambda, x_1)$ implies $f(x) \in F^{(1)}(\lambda, x_2)$ and vice versa so that $F^{(1)}(\lambda, x_0)$ is independent of x_0 and can be written simply $F^{(1)}(\lambda)$. Lemma 4 also asserts that $y(x; x_0, \lambda)$ is an increasing function of λ in $x_0 \leqq x<b$. But in our case $s=0$ so that the argument given in Lemma 4, part (1), applies also to the interval (a, x_0). Hence $f(x) \in F^{(1)}(\lambda_1)$ implies $f(x) \in F^{(1)}(\lambda_2)$ for $\lambda_1<\lambda_2$. In other words $F^{(1)}(\lambda_1) \subset F^{(1)}(\lambda_2)$ when $\lambda_1<\lambda_2$. Thus the cross section of all classes $F^{(1)}(\lambda)$ with $\lambda>0$ exists and equals $\lim_{\lambda \rightarrow 0} F^{(1)}(\lambda) \equiv F^{(1)}(+0)$. We can define in the same manner classes $F^{(k)}(\lambda)$ consisting of all functions $f(x)$ of $C^{(2k)}(a, b)$ for which $L^n f(x)/y(x; x_0, \lambda) \rightarrow 0$,

$x \to a, b, n=0, 1, \cdots, k-1$, as well as their cross section for $\lambda > 0$, $F^{(k)}(+0) = \lim_{\lambda \to 0} F^{(k)}(\lambda)$.

Since $y(x; x_0, 0)$ is well defined, so is the class $F^{(k)}(0)$ and it is clear that $F^{(k)}(0) \subset F^{(k)}(+0)$. Ordinarily these sets do not coincide because the sets $F^{(k)}(\lambda)$ are as a rule not continuous in λ. Even if they are continuous for $\lambda > 0$, they may very well lose this property for $\lambda = 0$. Simple examples can be given for both possibilities.

If $p_0(x) \equiv 0$, $y(x; x_0, 0) \equiv 1$ and there is no auxiliary solution (of constant sign) which becomes infinite at both end points for $\lambda = 0$. In this case $F^{(k)}(0)$ reduces simply to the subset of $C^{(2k)}(a, b)$ the elements of which satisfy the boundary conditions $L^n f(x) \to 0$, $x \to a, b$, $n = 0, 1, \cdots, k-1$. It is obvious that this set is a subset of every class $F^{(k)}(\lambda), \lambda > 0$. If $p_0(x) \not\equiv 0$ and $R(x, x_0; 0) \to \infty$, $x \to a, b$, then $F^{(k)}(0)$ certainly contains elements which do not vanish on the boundary together with their L-transforms of order at most $k-1$.

These results allow us to formulate

THEOREM 2. $B_3^{(k)}\{L; (a, b)\} \equiv F^{(k)}(+0) \supset F^{(k)}(0)$.

The proof of Theorem 1 in the singular case can be given in a few lines. We choose $W_1 = y(x; x_0, \lambda)$ and proceed as in the Sturm-Liouville case, the only difference being that the points x_j now figure as zeros of the continuous function $f(x)/y(x; x_0, \lambda)$ rather than of $f(x)$ which of course supplies the sign changes in (a, b). Lemma 1 applies as before and gives $V[(L-\lambda) f(x)] \geq N$(8).

1.7. **Discussion of the semi-singular case.** The discussion follows the same general pattern as in the singular case. The class $B_4^{(1)}\{L; C_1, C_2; (a, b]\}$ is defined as that subclass of $C^{(2)}(a, b]$ the elements of which satisfy at the singular end point $x = a$ the condition $f(x)/y(x; b, \lambda) \to 0$ for arbitrarily small $\lambda > 0$, while at the regular end point $x = b$ we have $C_1 f(b) + C_2 f'(b) = 0$ where $C_1 \geq 0$, $C_2 \geq 0$, $C_1 + C_2 > 0$ are fixed. The auxiliary solution $y(x; b, \lambda)$ satisfies the initial conditions $y(b) = 1$, $y'(b) = 0$.

Let us denote by $G^{(1)}(C_1, C_2; \lambda)$ the class of functions in $C^{(2)}(a, b]$ which satisfy these boundary conditions for a fixed $\lambda \geq 0$. Lemma 4 ensures that the ratio of $y(x; b, 0, \lambda) = y(x; b, \lambda)$ to $y(x; b, -s, \lambda)$ for a fixed positive s is bounded away from zero and infinity in $(a, b]$(9). Hence we have also $f(x)/y(x; b, -s, \lambda) \to 0$, $x \to a$, for any fixed $s > 0$, if $f(x) \in G^{(1)}(C_1, C_2; \lambda)$. As in §1.6 we show that $G^{(1)}(C_1, C_2; \lambda_1) \subset G^{(1)}(C_1, C_2; \lambda_2)$ for $\lambda_1 < \lambda_2$. We find that $G^{(1)}(C_1, C_2; +0) = \lim_{\lambda \to 0} G^{(1)}(C_1, C_2; \lambda)$ is the cross section of all classes $G^{(1)}(C_1, C_2; \lambda)$ for $\lambda > 0$. In a similar manner we define classes $G^{(k)}(C_1, C_2; \lambda)$ and $G^{(k)}(C_1, C_2; +0) = \lim_{\lambda \to 0} G^{(k)}(C_1, C_2; \lambda)$. Here $G^{(k)}(C_1, C_2; \lambda)$ is simply that subclass of $C^{(2k)}(a, b]$ consisting of functions $f(x)$ such that

(8) The same argument applies in case either end point should be regular or the condition $R(x, x_0; \lambda) \to \infty$ should fail to hold, provided $f(x)$ be constrained to vanish at the end point in question. Various intermediary types of operators are covered by this remark.

(9) A change of variable, replacing x by $-x$, reduces the discussion to the case considered in Lemma 4.

$f(x)$, $Lf(x)$, $\cdots$, $L^{k-1}f(x)$ all belong to $G^{(1)}(C_1, C_2; \lambda)$. We have obviously $G^{(k)}(C_1, C_2; 0) \subset G^{(k)}(C_1, C_2; +0)$. We can sum up the result in

THEOREM 3. $B_4^{(k)}\{L; C_1, C_2; (a, b]\} = G^{(k)}(C_1, C_2; +0)$.

The proof of Theorem 1 in the semi-singular case follows the same lines as in the preceding cases. Suppose that $f(x) \in B_4^{(1)}\{L; C_1, C_2; (a, b]\}$. If $C_2 = 0$, that is, if $f(b) = 0$, we choose $y(x; \lambda) = y(x; b, \lambda)$ and proceed as in the singular case. If $C_2 \neq 0$, we set $s = C_1/C_2 = -f'(b)/f(b)$ and take $y(x, \lambda) = y(x; b, -s, \lambda)$. Putting $g(x) = f(x)/y(x, \lambda)$ we see that $g(x) \to 0$ when $x \to a$ and $g'(b) = 0$ since numerator and denominator of the fraction have the same logarithmic derivatives at $x = b$. The proof then proceeds as in the Sturm-Liouville case with generalized boundary conditions.

1.8. **Extensions.** The case $\lambda = 0$ figured briefly in §1.6. It is of some interest to determine function classes for which the operator L itself is oscillation preserving. We arrive at the following result for the proof of which the reader will find the necessary material in the preceding sections.

THEOREM 4. *If the operator L is of type T_ν, there exists a class F_ν with respect to which L is oscillation preserving in (a, b). If $\nu = 1$ or 2 we have $F_\nu \supset B_\nu^{(1)}\{L; [a, b]\}$, while $F_3 \supset F^{(1)}(0)$ and $F_4 \supset G^{(1)}(C_1, C_2; 0)$.*

In the remainder of the paper we shall have to apply a given operator L more than once to the functions under consideration. Here is where the classes $B_\nu^{(k)}\{L; \langle a, b\rangle\}$ with $k > 1$ are required. We note that if $f(x) \in B_\nu^{(k)}\{L; \langle a, b\rangle\}$ and $\lambda > 0$ then $(L - \lambda) f(x) \in B_\nu^{(k-1)}\{L; \langle a, b\rangle\}$. Repeated application of Theorem 1 leads to the following result.

THEOREM 5. *Let $\Pi_k(u)$ be a polynomial in u of degree k, having real coefficients and real positive zeros. If L is of type T_ν, then $\Pi_k(L)$ is an oscillation preserving transformation in (a, b) with respect to the corresponding class $B_\nu^{(k)}\{L; \langle a, b\rangle\}$.*

In particular, we can always allow the class $B_\nu^{(\infty)}\{L; \langle a, b\rangle\}$. It is obvious that $B_\nu^{(k)} \supset B_\nu^{(k+1)} \supset B_\nu^{(\infty)}$ and it can be shown that $B_\nu^{(\infty)}$ is never vacuous[10].

By virtue of Theorem 4 we can also allow the root $u = 0$ with arbitrary multiplicity, in cases T_1 and T_2 without restriction of the class and in cases T_3 and T_4 at least for the corresponding classes $F^{(k)}(0)$ and $G^{(k)}(C_1, C_2; 0)$. We can also extend in a different direction. We can allow operators of the form $E(L)$ where $E(u)$ is a suitably restricted entire function, provided we

[10] For $\nu = 1$ and 2, this follows from Theorems 7, 10, and 11 below. For $\nu = 3$ and 4 the statement is also obvious whenever the corresponding boundary value problems P_3 and P_4 of §1.9 have solutions. In more general cases, the following type of argument leads to functions having the desired properties. Suppose $\nu = 3$, a and b finite and at most poles of the coefficients. Then we can take any function of the form $\exp[-A(x-a)^{-2} - B(x-b)^{-2}]$, $A > 0$, $B > 0$. The modifications necessary in case a or b or both are infinite are obvious. Heavier singularities can be handled by stepping up on the exponential scale. The same type of functions will do for $\nu = 4$.

also restrict $f(x)$ to be analytic. The result, being of no importance for the following, is stated without proof.

THEOREM 6. *Let $E(u)$ be an entire function of order $1/2$ and minimal type*[11]*, having real coefficients and real positive zeros. Let L be of type T_ν. Let $A_\nu\{L; \langle a, b\rangle\}$ be obtained from $B_\nu^{(\infty)}\{L; \langle a, b\rangle\}$ by replacing the requirement $f(x)\in C^{(\infty)}\langle a, b\rangle$ by $f(x)\in A\langle a, b\rangle$. Then $E(L)$ is an oscillation preserving transformation in (a, b) with respect to the class $A_\nu\{L; \langle a, b\rangle\}$.*

What was said above regarding the root $u=0$ applies also, mutatis mutandis, to the case of an entire function.

1.9. **The associated boundary value problems.** With each operator L of type T_ν there is an associated boundary value problem. We refer to the question of determining characteristic functions and characteristic values of the problem

$$(1.9.1) \qquad (L+\mu)u = 0, \qquad u(x) \in B_\nu^{(1)}\{L; \langle a, b\rangle\}.$$

Thanks to the analyticity assumptions for the coefficients of L any solution must also have the property $u(x)\in A_\nu\{L; \langle a, b\rangle\}$. For the sake of clarity, we write out in full the four problems.

P_1. $(L+\mu)u=0$, $u(a)=0$, $u(b)=0$.

P_2. $(L+\mu)u=0$, $u(a)=u(b)$, $u'(a)=u'(b)$.

P_3. $(L+\mu)u=0$, $u(x)/y(x; x_0, \lambda)\to 0$, $x\to a, b$, for every $\lambda>0$.

P_4. $(L+\mu)u=0$, $u(x)/y(x; b, \lambda)\to 0$, $x\to a$, $C_1u(b)+C_2u'(b)=0$, $C_1\geqq 0$, $C_2\geqq 0$.

The problems P_1 and P_2 are classical boundary value problems of the Sturm-Liouville and periodic types, respectively. It is well known that these problems have solutions and the reader will find a short summary of the available information concerning the properties of the solutions, to the extent that is needed for our purposes, in §§2.5 and 2.6 below.

Boundary value problems of types P_3 and P_4 do not seem to have been discussed in the literature though a number of the best known special orthogonal systems used in analysis can be obtained as solutions of such problems. This is not the right place to develop a general theory of problems P_3 and P_4. We restrict ourselves here to pointing out the existence of the problems and will call attention to the special instances as they are encountered in Chapter II.

In the case of problems P_1 and P_2 there is in existence a well developed expansion theory. Thus, for instance, every function $f(x)\in B_1^{(1)}\{L; [a, b]\}$ can be represented by a uniformly convergent series in terms of characteristic functions of P_1. The same is true in the case of P_2. It is natural to expect that

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[11] The statement means that $E(u)\exp(-\epsilon|u|^{1/2})\to 0$ when $|u|\to\infty$ for every $\epsilon>0$. It would be more precise to say that the order is at most $1/2$ and if it equals $1/2$, then the function is of minimal type.

a similar situation holds under fairly general circumstances also in the case of P_3 and P_4. A number of special instances are well known.

CHAPTER II. FINITE CHARACTERISTIC SERIES

2.1. **Admissible systems.** In this chapter we shall start the study of the relationship between the infinitary behavior of the sequence $V[L^k f(x)]$ and the analytical nature of $f(x)$. This will be carried out under rather severe restrictions on L and on $f(x)$. In part the restrictions are dictated by the nature of the problem, but they are also due to our lack of knowledge regarding the boundary problems P_3 and P_4 defined in §1.9. This makes it necessary for us to postulate the existence of a solution of the boundary problems involved with fairly regular properties of characteristic values and functions.

We consider first a system $S=S\{L, u_n(x), \mu_n; (a, b)\}$ consisting of an operator L, a set of characteristic functions $\{u_n(x)\}$ and corresponding characteristic values $\{\mu_n\}$, the interval being (a, b). We say that *S is admissible if it satisfies the assumptions* A_1 *to* A_6 *below and L is of one of the types* T_ν defined in §1.3.

A_1. $p_m(x) \in A(a, b)$, $m=0, 1, 2$.

A_2. $p_0(x) \leqq 0$, $p_2(x) > 0$, *for* $a<x<b$.

A_3. *The functions* $\{[P(x)]^{1/2} u_n(x)\}$ *form a real orthonormal system, complete in* $L_2(a, b)$.

A_4. $0<\mu_n \leqq \mu_{n+1}$. *The series* $\sum_1^\infty \mu_n^{-\alpha}$ *is convergent for some* $\alpha>0$.

A_5. *There exist constants* β *and* γ *and a non-negative function* $U(x) \in C^{(0)}(a, b)$, *such that*

$$| u_n(x) | \leqq \mu_n^\beta U(x), \qquad | u_n'(x) | \leqq \mu_n^\gamma U(x).$$

A_6. *For every fixed interval* (c, d), $a \leqq c < d \leqq b$, $Z_n(c, d)$, *the number of zeros of* $u_n(x)$ *in* (c, d), *tends to infinity with* n. $Z_n(a, b)$ *is finite and a never decreasing function of* n(12).

A number of admissible systems occurring in classical analysis will be exhibited in §§2.5 to 2.11 below.

We also consider a set $F=F\{L, u_n(x), \mu_n; (a, b)\}$ of characteristic series

$$\sum_{n=1}^{\infty} f_n u_n(x). \tag{2.1.1}$$

This set will be called admissible if S is admissible and

C_1. f_n *is a real for all* n,

C_2. $\sum_{n=1}^\infty \mu_n^k |f_n| < \infty$, $k=0, 1, 2, \cdots$.

This condition is obviously equivalent to the convergence of

(12) We have $Z_n(a, b)=V[u_n(x)]$ except possibly in the periodic case when we may have $Z_n(a, b)+1=V[u_n(x)]$.

$$\sum_{n=1}^{\infty} (\mu_n^m f_n)^2$$

for every integral value of m. In other words, the series

$$\sum_{n=1}^{\infty} (-\mu_n)^m f_n u_n(x) \tag{2.1.2}$$

represents a function $f_m(x)$ such that

$$[P(x)]^{1/2} f_m(x) \in L_2(a, b)$$

for $m = 0, 1, 2, \cdots$. It is clear that $f_m(x)$ is obtained from $f(x) = f_0(x)$ by termwise operation with L in the series (2.1.1). *A characteristic series is admissible if its coefficients satisfy* C_1 *and* C_2 *and* S *is admissible. Such a series defines an admissible function.*

2.2. **Convergence in** F. An admissible series converges not merely in weighted mean square but also in the local sense.

LEMMA 5. *If* $f(x) \in F$, *then the series*

$$\sum_{n=1}^{\infty} f_n u_n(x), \qquad \sum_{n=1}^{\infty} f_n u_n'(x), \qquad f_n = \int_a^b P(t) u_n(t) f(t) dt,$$

converge absolutely and uniformly in every fixed interval (a_1, b_1), $a < a_1 < b_1 < b$, *their sums being* $f(x)$ *and* $f'(x)$, *respectively. If the function* $U(x)$ *of* A_5 *can be taken equal to a constant, the convergence is uniform in* $[a, b]$.

The convergence properties follow from assumptions A_5 and C_2. The first series being convergent in (a_1, b_1) both uniformly and in weighted mean square, we conclude that the uniform limit is equivalent to $f(x)$ and can be taken as the definition of $f(x)$ for all x. The sum of the uniformly convergent derived series is then obviously $f'(x)$.

LEMMA 6. *If* $f(x) \in F$, *so does* $L[f(x)]$ *and*

$$L[f(x)] = -\sum_{n=1}^{\infty} \mu_n f_n u_n(x). \tag{2.2.1}$$

For the proof we observe that the second derived series of $f(x)$ also converges absolutely and uniformly in (a_1, b_1) and hence has the sum $f''(x)$. This follows from the identity

$$p_2(x) \sum_{n=j}^{k} f_n u_n''(x) = -p_1(x) \sum_{n=j}^{k} f_n u_n'(x) - \sum_{n=j}^{k} \mu_n f_n u_n(x) - p_0(x) \sum_{n=j}^{k} f_n u_n(x).$$

Hence $L[f(x)]$ exists and

$$L[f(x)] = \sum_{n=1}^{\infty} f_n L[u_n(x)] = -\sum_{n=1}^{\infty} \mu_n f_n u_n(x) \equiv f_1(x)$$

is an element of F.

It follows that all functions $L^m[f(x)] = f_m(x)$ exist and belong to F whenever $f(x)$ does. Thus we can apply the operation L as often as we please termwise to an admissible characteristic series and the result will stay in F. Such a series can also be differentiated termwise arbitrarily often, but it is not a priori obvious that the result is always in F, though this appears to be true in simple cases.

The class F could evidently be characterized by descriptive properties. Its elements are real in (a, b) and $[P(x)]^{1/2}f(x) \in L_2(a, b)$. F is invariant under the operation L. It is a linear vector space with real multipliers and contains the basis $\{u_n(x)\}$. However, for our purposes it is simpler and more natural to start from the characteristic series.

2.3. **Conservative systems.** We need a couple of additional assumptions linking the classes $A_\nu\{L; \langle a, b\rangle\}$ of Theorem 6 with the systems S and F. They read as follows.

D_ν. *If L is of type T_ν, then $u_n(x) \in A_\nu\{L; \langle a, b\rangle\}$ for all n.*

E_3. *If $\nu = 3$ there exists a finite positive $C(x_0, \lambda)$ such that $U(x) \leqq C(x_0, \lambda) y(x; x_0, \lambda)$ for $a < x < b$, $\lambda > 0$.*

E_4. *If $\nu = 4$ there exists a finite positive $C(\lambda)$ such that $U(x) \leqq C(\lambda)\, y(x; b, \lambda)$ for $a < x < b$, $\lambda > 0$.*

It is worth while stating explicitly what D_ν amounts to in the various cases. Since $u_n(x)$ is a characteristic function of L, the denumerable infinity of boundary conditions entering into the definition of $A_\nu\{L; \langle a, b\rangle\}$ reduces to a single pair. We get:

D_1. $u_n(a) = 0$, $u_n(b) = 0$.

D_2. $u_n(a) = u_n(b)$, $u_n'(a) = u_n'(b)$.

D_3. $u_n(x)/y(x; x_0, \lambda) \to 0$, $x \to a, b$, *for all* $\lambda > 0$.

D_4. $u_n(x)/y(x; b, \lambda) \to 0$, $x \to a$, *for all* $\lambda > 0$, *and* $C_1 u_n(b) + C_2 u_n'(b) = 0$.

In other words, D_ν asserts the existence of a solution of the corresponding boundary value problem P_ν of §1.9 and that this solution is given by $\{u_n(x), \mu_n\}$. If $\nu = 1$ or 2, the function $U(x)$ of A_5 can be taken equal to a constant. This explains the absence of any conditions E_1 and E_2. In the two remaining cases we need an inequality between $U(x)$ and the auxiliary solution which is supplied by E_3 and E_4.

Definition. *An admissible system S is called conservative if it satisfies the conditions D_ν and E_ν corresponding to its type T_ν.*

Thus a conservative system satisfies conditions A_1 to A_6, one of the conditions T_ν, $\nu = 1, 2, 3$ or 4, and the corresponding conditions D_ν and E_ν.

THEOREM 7. *If* $S=S\{L,\ u_n(x),\ \mu_n;\ (a,\ b)\}$ *of type* T_ν *is conservative and* $F=F\{L,\ u_n(x),\ \mu_n;\ (a,\ b)\}$ *is the corresponding admissible set of functions, then* $F\subset B_\nu^{(\infty)}\{L;\ \langle a,\ b\rangle\}$. *If* $\nu=1$ *or* 2, $F=B_\nu^{(\infty)}$.

Suppose first that $\nu=1$ and $f(x)\in F$. We can then find a constant U_1 such that $|u_n(x)|\leqq U_1$, $a\leqq x\leqq b$, for all n(13). Formula (1.2.4) with $\lambda=-\mu_n$ shows that $|u_n'(x)|\leqq\mu_n U_2$ for a suitably chosen constant U_2. Thus we can take $U(x)=U=\max(U_1,\ U_2)$, $\beta=0$, $\gamma=1$ in A_5. By Lemma 5 the characteristic series of $f(x)$ converges uniformly in $[a,\ b]$. Since every partial sum of the series vanishes for $x=a$ and $x=b$ by D_1, we have $f(a)=f(b)=0$. Further the first derived series converges uniformly in $[a,\ b]$ so that $f'(x)$ is also continuous in $[a,\ b]$. By Lemma 6, $L^kf(x)\in F$ for all k. This means that $f(x)$ has derivatives of all orders continuous in $[a,\ b]$ and $L^kf(a)=L^kf(b)=0$ for all k. Hence $f(x)\in B_1^{(\infty)}\{L;\ [a,\ b]\}$.

Suppose, conversely, that $f(x)\in B_1^{(\infty)}\{L;\ [a,\ b]\}$. This means that $f(x)\in C^{(\infty)}[a,\ b]$ and $L^kf(a)=L^kf(b)=0$ for all k. Since $L^kf(x)\in C^{(\infty)}[a,\ b]$ and vanishes at the end points, we have

$$L^kf(x) = \sum_{n=1}^{\infty} f_{n,k}u_n(x),$$

uniformly convergent in $[a,\ b]$. But here we can use the classical identity of Lagrange (for the notation, see formulas (1.1.2) and (1.2.2)):

$$gL^*[h] - hL^*[g] = D\{K(gh' - hg')\}.$$

If g and h belong to $B_1^{(\infty)}\{L;\ [a,\ b]\}$, integration from a to b gives

$$\int_a^b g(t)L^*[h(t)]dt = \int_a^b h(t)L^*[g(t)]dt$$

or

$$\int_a^b g(t)P(t)L[h(t)]dt = \int_a^b h(t)P(t)L[g(t)]dt$$

and by iteration

$$\int_a^b g(t)P(t)L^k[h(t)]dt = \int_a^b h(t)P(t)L^k[g(t)]dt$$

for every integer $k\geqq 0$. Putting in particular $g(x)=u_n(x)$, $h(x)=f(x)$ we get $f_{n,k}=(-\mu_n)^kf_n$. Since $f_{n,k}$ is real and $\sum f_{n,k}^2$ converges for every k, we see that the coefficients f_n satisfy conditions C_1 and C_2. Hence $f(x)\in F$ and the theorem is proved for $\nu=1$.

The same type of argument applies if $\nu=2$, where of course periodicity plays the same role as vanishing on the boundary did when $\nu=1$.

(13) This follows from property (3) of §2.5.

Suppose now that $\nu=3$ and that $f(x)\in F$. In order to prove that $f(x)\in B_3^{(\infty)}\{L;\ (a,\ b)\}$ it is enough to show that $f(x)/y(x;\ x_0,\ \lambda)\to 0,\ x\to a,\ b,$ for every $\lambda>0$; the L-transforms of $f(x)$ will then automatically satisfy the same conditions. But using A_5 and E_3 we have

$$\left|\frac{f(x)}{y(x;\ x_0,\ \lambda)}-\sum_1^N f_n\frac{u_n(x)}{y(x;\ x_0,\ \lambda)}\right|=\left|\sum_{N+1}^{\infty} f_n\frac{u_n(x)}{y(x;\ x_0,\ \lambda)}\right|$$
$$\leqq C(x_0,\ \lambda)\sum_{N+1}^{\infty}|f_n|\,\mu_n^{\beta}.$$

By C_2 we can choose N so large that the last member is less than any preassigned ϵ, and by D_3 the finite series in the first member tends to zero when $x\to a$. This completes the proof.

Suppose finally $\nu=4$ and $f(x)\in F$. Since $y(x;\ b,\ \lambda)$ is bounded in $[a+\delta,\ b]$, $\delta>0$, for fixed λ, assumption E_4 shows that $U(x)$ is bounded in $[a+\delta,\ b]$. Hence by Lemma 5 the series for $f(x)$ and $f'(x)$ are uniformly convergent in $[a+\delta,\ b]$. The partial sums satisfy the boundary condition $C_1S_n(b)+C_2S_n'(b)=0$ for all n. Hence we have also $C_1f(b)+C_2f'(b)=0$ and the same boundary condition is satisfied by $L^kf(x)$ for all k. The proof that $f(x)/y(x;\ b,\ \lambda)\to 0$ when $x\to a$ for every $\lambda>0$ goes through as when $\nu=3$.

We cannot assert that $F=B_\nu^{(\infty)}$ when $\nu=3$ or 4. The following example disproves such a conjecture. We take for L the Hermite-Weber operator D^2-x^2, $a=-\infty$, $b=\infty$; $\{u_n(x)\}$ is the set obtained by orthogonalizing and normalizing the Hermite polynomials and $\mu_n=2n+1$. It is shown in §2.9 that this system is conservative and of type T_3. If $f(x)=1$ then $L^kf(x)$ is an even polynomial of degree $2k$. Referring to formula (2.9.3) which gives the asymptotic behavior of $y(x;\ 0,\ \lambda)$ for large x, we see that $f(x)=1$ belongs to $B_3^{(\infty)}\{L;\ (-\infty,\ \infty)\}$, but it does not satisfy the boundary conditions (2.9.7) for $k=0$; so it cannot belong to F.

Similarly $S\{D^2-x^2,\ u_{2n}(x),\ 4n+1;\ (0,\ \infty)\}$ is a conservative system of type T_4, the regular boundary condition being $u'(0)=0$. Again $f(x)=1$ belongs to $B_4^{(\infty)}\{D^2-x^2;\ 0,\ 1;\ [0,\ \infty)\}$ but not to the corresponding class F for which the singular boundary condition is still given by (2.9.7).

Combining Theorems 5 and 7 we get

THEOREM 8. *Let $S=S\{L,\ u_n(x),\ \mu_n;\ (a,\ b)\}$ be a conservative system and let F be the corresponding set of admissible functions. Let $\Pi(u)$ be a polynomial in u with real coefficients and real positive zeros. Then $\Pi(L)$ is an oscillation preserving transformation in $(a,\ b)$ with respect to F.*

2.4. **The main theorem.** We shall now prove

THEOREM 9. *Let $S=S\{L,\ u_n(x),\ \mu_n;\ (a,\ b)\}$ be a conservative system and let $F=F\{L,\ u_n(x),\ \mu_n;\ (a,\ b)\}$ be the corresponding set of admissible functions. Let $f(x)\in F$ and suppose that*

$$\liminf_{k\to\infty} V[L^k f(x)] = N < \infty. \tag{2.4.1}$$

Then there exists an integer $M = M(N)$ such that $f_n = 0$ for $n > M$, that is

$$f(x) = \sum_{n=1}^{M} f_n u_n(x). \tag{2.4.2}$$

If all characteristic values are simple,

$$V[u_M(x)] \leqq N, \tag{2.4.3}$$

otherwise it is at most $N+1$. Conversely, if $f(x)$ is given by (2.4.2), *then*

$$V[L^k f(x)] \geqq V[u_M(x)] \tag{2.4.4}$$

for all large k.

For the proof we employ the device of Pólya and Wiener [2] in suitable modification. To the given function $f(x) \in F$ with Fourier coefficients f_n we form the auxiliary function

$$\Phi(x, k, m; f) = \sum_{n=1}^{\infty} \left\{ \frac{4\mu_m \mu_n}{(\mu_m + \mu_n)^2} \right\}^k f_n u_n(x) \tag{2.4.5}$$

where $k \geqq 0$, $m \geqq 1$ are arbitrary integers. We have $\Phi(x, 0, m; f) \equiv f(x)$. The multipliers are positive numbers less than or equal to 1 and equal to 1 only when $\mu_n = \mu_m$. Since every characteristic value is at most double, this means for at most two values of n. It follows that the coefficients of Φ also satisfy conditions C_1 and C_2 so that $\Phi \in F$. We can consequently apply the operator $(L - \mu_m)$ termwise as often as we please to the series (2.4.5). We find in particular that

$$(L - \mu_m)^{2k} \Phi(x, k, m; f) = (-4\mu_m)^k L^k f(x). \tag{2.4.6}$$

But $\mu_m > 0$ and by Theorem 8 $(L - \mu_m)^{2k}$ is an oscillation preserving transformation with respect to the class F in (a, b). Hence

$$N_k = V[L^k f(x)] \geqq V[\Phi(x, k, m; f)] \tag{2.4.7}$$

for every k and m.

So far m was arbitrary. We suppose now that $f_m \neq 0$. In order to take care of the slightly more complicated case in which there are double characteristic values, let us suppose $\mu_{m-1} = \mu_m$ and that also $f_{m-1} \neq 0$. We then write $\Phi = S_1 + S_2 + S_3$. Here S_1 is the finite sum from $n = 1$ to $n = m-2$, $S_2 = f_{m-1}u_{m-1}(x) + f_m u_m(x)$, while S_3 is the remainder. The trivial modification necessary if μ_m is simple is obvious. We shall estimate S_1 and S_3. The idea of the proof is to show that for sufficiently large values of k, $|S_1 + S_3|$ is dominated by $|S_2|$ at the maxima of the latter, provided that we restrict ourselves to a fixed interior interval (a_1, b_1), and that consequently the oscillatory properties of Φ in this interval are essentially the same as those of S_2. The latter, however, are regulated by assumption A_6 which ensures that the number of sign changes of S_2 in (a_1, b_1) tends to infinity with m. This will lead to a contradiction for suitable values of m.

We consider now an arbitrary interior interval (a_1, b_1), $a<a_1<b_1<b$. Let $B=\max U(x)$ for $a_1 \leqq x \leqq b_1$. Let

$$\delta = \max \frac{4\mu_m\mu_n}{(\mu_m + \mu_n)^2}$$

for $n \neq m-1$ and m ($n\neq m$, if μ_m is simple). We have $\delta<1$. By assumption A_5

$$|S_1 + S_3| \leqq \delta^k \sum{}' |f_n u_n(x)| < \delta^k \sum_1^\infty |f_n| \mu_n^\beta \cdot U(x) \leqq \delta^k B \sum_1^\infty |f_n| \mu_n^\beta \equiv \delta^k T$$

for $a_1 \leqq x \leqq b_1$. Here the prime after the summation sign indicates that $n \neq m-1$ and m.

Let us write $Z_m(a_1, b_1) = j_m$. Assumption A_6 asserts that $j_m \to \infty$ with m. Since μ_m is a double characteristic value, $S_2(x) = f_{m-1}u_{m-1}(x) + f_m u_m(x)$ is a real solution of the differential equation $(L+\mu_m)\, y=0$. Consequently it has at least j_m-1 and at most j_m+1 zeros in (a_1, b_1) by the classical oscillation theorems. Let the actual number be i_m. We can suppose without essential restriction of the generality that the zeros of $S_2(x)$ are interior to the interval (a_1, b_1) and that $S_2(a_1)>0$. Then $\operatorname{sgn} S_2(b_1) = (-1)^{i_m}$. Let the zeros occur at the points x_α, $a_1<x_1<x_2< \cdots <x_{i_m}<b_1$. Let ξ_α be the uniquely determined point between x_α and $x_{\alpha+1}$ where $S_2'(x)=0$. If $S_2'(x)=0$ at a point in (a_1, x_1), we denote this point by ξ_0; otherwise we set $\xi_0=a_1$. Similarly ξ_{i_m} is either the point where $S_2'(x)=0$ in (x_{i_m}, b_1) or b_1 itself. We note that the points ξ_0 and ξ_{i_m} are uniquely determined. Now let

$$\sigma = \min |S_2(\xi_\alpha)|, \qquad \alpha = 0, 1, \cdots, i_m.$$

We then choose k so large that

$$\delta^k T < \sigma,$$

which is obviously possible. But this means that for such values of k and m

$$\operatorname{sgn} \Phi(\xi_\alpha, k, m; f) = (-1)^\alpha, \qquad \alpha = 0, 1, \cdots, i_m.$$

Hence $\Phi(x, k, m; f)$ has at least i_m sign changes in (a_1, b_1) and a fortiori in (a, b). Since $(L-\mu_m)^{2k}$ is an oscillation preserving transformation, formulas (2.4.6) and (2.4.7) show that $L^k f(x)$ also has at least i_m sign changes in (a, b) for all sufficiently large k. Hence $i_m \leqq N_k$ for all large k. But (2.4.1) asserts that $N_k=N$ for infinitely many values of k. This implies $i_m \leqq N$.

This is a contradiction, however, for large m since i_m tends to infinity with m. Since $i_m \geqq j_m - 1$, this gives a contradiction for $j_m > N+1$. We are thus led to the conclusion that the characteristic series of $f(x)$ cannot contain any term $u_m(x)$ having more than $N+1$ zeros in (a_1, b_1). But here (a_1, b_1) is a perfectly arbitrary interior interval. It follows that the series of $f(x)$ contains no term $u_m(x)$ with $Z_m(a, b) > N+1$. If there are no multiple characteristic values, we can replace $N+1$ by N since then $j_m = i_m$. In the case of double characteristic values, however, it would seem possible for the finite sum to end with two terms corresponding to the same characteristic value, either term having $N+1$ zeros while their sum has only N zeros. Whether or not this exceptional case can ever arise must be left an open question.

This argument proves formula (2.4.3) except in the periodic case. Here $N = 2K$ is an even integer. Further $Z_n(a, b) \leqq V[u_n(x)] \leqq Z_n(a, b)+1$ and $V[u_n(x)]$ is even. If there are no double characteristic values then the inequality $Z_M(a, b) \leqq N = 2K$ implies $V[u_M(x)] \leqq 2K$ and formula (2.4.3) is proved. If however, $\mu_M = \mu_{M-1}$, then the previous proof shows that $S_2(x)$ cannot have more than $2K$ sign changes in (a, b) and hence $V[S_2(x)] \leqq 2K$. Now $u_{M-1}(x)$, $u_M(x)$, and $S_2(x)$ are solutions of the same differential equation $(L+\mu)\, u = 0$ for $\mu = \mu_M = \mu_{M-1}$. Hence the three quantities $V[u_{M-1}(x)]$, $V[u_M(x)]$ and $V[S_2(x)]$ can differ by at most one unit and, being even integers, they must consequently be equal. This shows that $V[u_M(x)] \leqq 2K$ and formula (2.4.3) is proved.

Suppose, conversely, that $f(x)$ is a finite sum of characteristic functions given by (2.4.2). We choose $m = M$ and form $\Phi(x, k, M; f)$. For (a_1, b_1) we take an interval containing all zeros of $u_M(x)$ or $S_2(x)$ as the case may be. Proceeding as above, we see that $\Phi(x, k, M; f)$ has at least as many sign changes in (a_1, b_1) as the last term or group of terms has for large values of k. Combining with (2.4.7) we see that (2.4.4) holds. The argument is evidently also valid in the periodic case. It is often possible to exclude the sign of inequality both in (2.4.3) and in (2.4.4). This completes the proof of Theorem 9.

Pólya and Wiener proved that if $f(x)$ is periodic and $V[D^{2k}f(x)]$ is bounded with respect to k, then the Fourier series of $f(x)$ cannot contain any high frequency terms. Theorem 9 shows that this result has analogues for general orthogonal series defined by boundary value problems relating to linear second order differential equations, the operator D^2 being replaced by L.

In §§2.5 to 2.11 below we shall give special instances of the theory.

2.5. **Sturm-Liouville operators.** We have the following simple results([14]):

THEOREM 10. *Let $p_m(x) \in A\,[a, b]$, $p_0(x) \leqq 0$, $p_2(x) > 0$, $a \leqq x \leqq b$. Let $\{u_n(x)\}$ and $\{\mu_m\}$, $n = 0, 1, 2, \cdots$, be the sets of characteristic functions and corresponding characteristic values of the boundary value problem*

$$(L + \mu)u = 0, \qquad u(a) = 0, \qquad u(b) = 0.$$

([14]) We state the assumptions in Theorems 10 and 11 explicitly since they are so simple. Formulations in terms of the previous postulates are given below.

Then $S\{L, u_n(x), \mu_n; (a, b)\}$ *is a conservative system. If* $f(x) \in B_1^{(\infty)}\{L; [a, b]\}$, *that is, if* $f(x) \in C^{(\infty)}[a, b]$ *and* $L^n f(a) = 0$, $L^n f(b) = 0$ *for* $n = 0, 1, 2, \cdots$, *and*

$$\liminf_{k \to \infty} V[L^k f(x)] = N,$$

then

$$f(x) = \sum_{n=0}^{N} f_n u_n(x).$$

We are assuming the validity of A_1, A_2, T_1, and D_1 and have first to show that they imply A_3 to A_6. Now this is the classical Sturm-Liouville system except for the restrictive assumptions of analytical coefficients which are unnecessary in the boundary value problem but desirable for our special needs. Referring to the literature for proofs (see for instance E. L. Ince [1, §§10.61, 10.7, and 11.4]), we list the following properties of the solutions. We put

$$z = (1/\omega) \int_a^x [p_2(t)]^{-1/2} dt, \qquad \omega = (1/\pi) \int_a^b [p_2(t)]^{-1/2} dt,$$

$$\rho_n^2 = \omega^2 \mu_n.$$

Then:

(1) The characteristic values are real, positive, and simple.

(2) $\rho_n = n + 1 + O(1/n)$.

(3) $u_n(x) = A_n [P^2(x)\, p_2(x)]^{-1/4} \{\sin(\rho_n z) + O(1/n)\}$, where A_n is a normalizing factor, independent of x and bounded with respect to n[15].

(4) $V[u_n(x)] = n$.

(5) $\{[P(x)]^{1/2} u_n(x)\}$ is complete in $L_2(a, b)$.

These properties show that conditions A_3 to A_6 are amply satisfied. Thus S is a conservative system and Theorem 9 holds for the corresponding set F of admissible series. It was shown in Theorem 7, however, that $F = B_1^{(\infty)}\{L; [a, b]\}$. Finally, $M(N) = N$ by virtue of property (4). This completes the proof of Theorem 10.

The same result is valid for more general boundary conditions, for example, $u(a) = 0$, $C_1 u(b) + C_2 u'(b) = 0$, $C_1 \geqq 0$, $C_2 > 0$ and, for $u'(a) = 0$, $u'(b) = 0$.

2.6. **Periodic operators.** Here we also have simple results.

THEOREM 11. *Let* $p_m(x) \in A[a, b]$, $m = 0, 1, 2$; $p_2(a) \neq 0$, $p_2(b) \neq 0$, $K(a) = K(b)$. *Let* $\{u_n(x)\}$ *and* $\{\mu_n\}$ *be the sets of characteristic functions and corresponding characteristic values of the boundary value problem*

$$(L + \mu)u = 0, \qquad u(a) = u(b), \qquad u'(a) = u'(b).$$

Then $S\{L, u_n(x), \mu_n; (a, b)\}$ *is a conservative system. If* $f(x) \in B_2^{(\infty)}\{L; [a, b]\}$, *that is, if* $f(x) \in C^{(\infty)}[a, b]$ *and* $L^n f(a) = L^n f(b)$, $DL^n f(a) = DL^n f(b)$, $n = 0, 1,$

(15) $A_n \to \{2/(\pi\omega)\}^{1/2}$ when $n \to \infty$.

2, $\cdots$, *and if*

$$\liminf_{k\to\infty} V[L^k f(x)] = 2K,$$

then

$$f(x) = \sum_{n=0}^{2K} f_n u_n(x).$$

Here we assume A_1, A_2, T_2, and D_2, and want to conclude that A_3 to A_6 hold. In the present section $V[g]$ is to be determined according to the definition for periodic functions. The available information concerning the solutions of the boundary value problem is quite precise in the case of equations with periodic coefficients and only slightly less so in the general case. We refer to E. L. Ince [1, §§10.8, 10.81, and 11.4], where the reader will find further references to the literature. In the notation of the preceding section we obtain:

(1) The characteristic values are real, non-negative but need not be simple.

(2) $\rho_n = [(n+1)/2] + O(1)$.

(3) $|u_n(x)| \leqq U, a \leqq x \leqq b, n = 0, 1, 2, \cdots$.

(4) $V[u_0(x)] = 0$, $V[u_{2m-1}(x)] = V[u_{2m}(x)] = 2m$.

(5) $\{[P(x)]^{1/2} u_n(x)\}$ is complete in $L_2(a, b)$.

If $K(x)$ and $G(x, \lambda)$ are even periodic functions of period $(b-a)$, the remainder term in (2) can be replaced by $O(1/n)$ and (3) can be replaced by formulas of type (3), in §2.5 with sine replaced by cosine when n is even([16]). The properties as listed are, however, more than sufficient to prove that A_3 to A_6 are satisfied so that S is a conservative system. Since $F = B_2^{(\infty)}\{L; [a, b]\}$ by Theorem 7, and $M(N) = 2K$ by (4), Theorem 11 is proved.

The simplest of all operators satisfying the conditions of Theorem 11 is $L = D^2$. In this case the theorem reduces to Theorem I of Pólya and Wiener. A less trivial instance is given by the operator of Mathieu

$$L = D^2 - (A + B \cos 2x)$$

to which corresponds expansions in terms of the *functions of the elliptic cylinder.*

The remaining sections of the chapter will be devoted to special instances of singular and semi-singular operators.

2.7. **The Legendre operator.** We consider the case

$$L[y] = (1 - x^2)D^2y - 2xDy, \qquad a = -1, \qquad b = +1. \tag{2.7.1}$$

The end points of the interval are singular and we find that $R(x, 0; \lambda) = -(\lambda/2) \log (1 - x^2) \to \infty$ when $x \to \pm 1$. It follows that the problem is of type T_3. The corresponding singular boundary value problem

([16]) It is not difficult to show that similar formulas hold also in the general case considered in §2.6. We have merely to replace $\rho_n z$ by $\rho_n z + \phi_n$ where ϕ_n is a suitable phase angle, determinable with an error which is $O(1/n)$. Property (3) is an immediate consequence of such formulas.

(2.7.2) $$D[(1-x^2)Du]+\mu u=0,\quad u(x)/\log(1-x^2)\to 0,\quad x\to\pm 1$$

has as its characteristic functions the Legendre polynomials $P_n(x)$ with corresponding characteristic values $n(n+1)$, $n=0, 1, 2, \cdots$ [17]. We take $u_n(x)=(n+(1/2))^{1/2}P_n(x)$. We shall prove that the system $S\{L, u_n(x), \mu_n; (-1, 1)\}$ is conservative. We know to start with that A_1, A_2, T_3, and D_3 are satisfied. It is well known that A_3 holds and so does obviously A_4, except for that fact that the least μ_n is zero. This is immaterial, however[18]. We have $|P_n(x)|\leqq 1$, $|P_n'(x)|\leqq n(n+1)/2$, so that A_5 and E_3 are satisfied. Further the n zeros of $P_n(x)$ are all located in $(-1, 1)$ and the maximal distance between consecutive zeros is $O(1/n)$ so that A_6 is valid. Thus the system is actually conservative and Theorem 9 holds for the corresponding class F.

We have now to determine what functions are represented in $[-1, 1]$ by expansion of the form

$$\sum_{n=0}^{\infty} a_nP_n(x), \qquad \sum_{1}^{\infty} n^m|a_n| < \infty$$

for all m, a_n being real. It is obvious that the series as well as all derived series converge uniformly in $[-1, 1]$ so that $f(x)\in C^{(\infty)}[-1, 1]$. Conversely, if $f(x)\in C^{(\infty)}[-1, 1]$ so do all its L-transforms. From this we conclude readily that $f(x)\in F$. Hence we have shown that

(2.7.3) $$F\{D(1-x^2)D, (n+(1/2))^{1/2}P_n(x), n(n+1); (-1, 1)\}\equiv C^{(\infty)}[-1, 1].$$

This fact gives the following formulation of Theorem 9 for the Legendre operator:

THEOREM 12. *If $L=(1-x^2)D^2-2xD$, $f(x)\in C^{(\infty)}[-1, 1]$, and* $\liminf_{k\to\infty} V[L^kf(x)]=N$, *then $f(x)$ is a polynomial of degree N. Conversely, every real polynomial of exact degree N has the property $V[L^kf(x)]=N$ for all large k.*

In order to prove the converse, we merely express the given polynomial in terms of Legendre polynomials. The expression will involve the term $P_N(x)$ with a coefficient different from zero. By (2.4.4) $V[L^kf(x)]\geqq N$ for all large k. Since $L^kf(x)$ is also a polynomial of degree N, we must have $V[L^kf(x)]=N$ for all large k.

Theorem 12 has also been proved by Szegö [4, Theorem C] by a different method.

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(17) The proof of this statement goes as follows. The only solution $u(x)$ satisfying the boundary condition at $x=1$ is a multiple of $F(a+1, -a, 1, (1-x)/2)$ where $a(a+1)=\mu$. This solution becomes logarithmically infinite at $x=-1$ unless a is an integer when it reduces to $P_n(x)$.

(18) The fact that $\mu_0=0$ means merely that the case $N=0$ is not covered by Theorem 9. But if $N=0<1$ we can still conclude that $f(x)=a+bx$ and since $L^k(a+bx)=(-2)^kbx$, we must have $b=0$. Hence Theorem 12 is also valid for $N=0$. A similar argument takes care of the other cases encountered below in which the least characteristic value is zero.

2.8. **Jacobi operators.** Analogous results can be proved for the Jacobi operator

$$L = (1 - x^2)D^2 + [\beta - \alpha - (\alpha + \beta + 2)x]D \tag{2.8.1}$$

for the interval $(-1, 1)$. Here the end points are again singular. A simple calculation shows that $R(x, 0; \lambda) \to \infty$ when $x \to 1$ if and only if $\alpha \geqq 0$, and when $x \to -1$ if and only if $\beta \geqq 0$. Thus the problem is of type T_3 if and only if $\alpha \geqq 0$, $\beta \geqq 0$. We shall suppose $\alpha > 0$, $\beta > 0$, since the limiting case $\alpha = 0$, $\beta = 0$, reduces to Legendre's operator(19). The corresponding singular boundary value problem can then be formulated as follows:

$$(L + \mu)u = 0, \quad u(x)(1 - x)^\alpha(1 + x)^\beta \to 0, \quad x \to \pm 1, \tag{2.8.2}$$

since the calculation shows that $(1-x)^\alpha(1+x)^\beta\, y(x;\, 0, \lambda)$ is bounded away from zero and infinity in $(-1, 1)$. The solutions are given by the Jacobi polynomials $u_n(x) = A_n P_n^{(\alpha,\beta)}(x)$, $\mu_n = n(n+\alpha+\beta+1)$, where A_n is a normalizing factor. The reader will find in the treatise by Szegö [5, §§3.1, 7.32, and 8.9], the necessary information regarding Jacobi polynomials required to show that $S\{L, u_n(x), \mu_n; (-1, 1)\}$ is a conservative system. We show as in §2.7 that the corresponding set F of admissible functions is identical with $C^{(\infty)}[-1, 1]$.

It follows that Theorem 12 remains valid if we replace the Legendre operator by the general Jacobi operator (2.8.1) provided $\alpha \geqq 0$, $\beta \geqq 0$. Professor Szegö has kindly informed me that the theorem actually remains true for $\alpha > -1$, $\beta > -1$ and that this can be proved both by his method used in [4] and by a suitable modification of mine. We note, in particular, the case $\alpha = \beta = -1/2$ which leads to the polynomials of Tchebycheff. By his method Szegö is also able to prove that if α and β are arbitrary real numbers, then the assumptions $f(x) \in C^{(\infty)}[-1, 1]$ and $\liminf_{k\to\infty} V[L^k f(x)] = N$ imply that $f(x)$ is a polynomial of degree at most $N + M(\alpha, \beta)$ where $M(\alpha, \beta)$ is an integer depending only upon α and β. Detailed proofs will be presented in a later note in this series.

2.9. **The Hermite and Hermite-Weber operators.** We consider next the two operators

$$L_1 = D^2 - 2xD, \qquad L_2 = D^2 - x^2, \tag{2.9.1}$$

which we refer to as the Hermite and Hermite-Weber operators respectively. The interval (a, b) will be $(-\infty, \infty)$. Since

$$e^{x^2/2}L_2(e^{-x^2/2}y) = (L_1 - 1)y, \tag{2.9.2}$$

the two operators can be treated simultaneously. The Hermite-Weber case

(19) The cases $\alpha = 0$, $\beta > 0$ and $\alpha > 0$, $\beta = 0$ can also be handled by the same method. One of the powers occurring in (2.8.2) should then be replaced by a logarithm.

is easier to handle directly, but the final result is more striking if expressed in terms of the Hermite operator.

We concentrate the attention on L_2. The point at infinity is singular and $R(x, 0; \lambda) \to \infty$ when $|x| \to \infty$ for $\lambda > 0$. The problem, therefore, is of type T_3. The function $y(x, 0; \lambda)$ is a constant multiple of $D_n(2^{1/2}x) + D_n(-2^{1/2}x)$ where D_n is the *parabolic cylinder function* of Whittaker and $n = -(1+\lambda)/2$. For large values of $|x|$ we have consequently (cf. E. T. Whittaker and G. N. Watson [6, §16.52])

$$(2.9.3) \qquad y(x, 0; \lambda) = B(\lambda) \mid x \mid^{(\lambda-1)/2} \exp\left[x^2/2 \right] \{1 + o(1)\}.$$

The singular boundary value problem

$$(2.9.4) \quad (D^2 + \mu - x^2)u = 0, \; u(x) \mid x \mid^{(1-\lambda)/2} \exp\left[- x^2/2 \right] \to 0, \; \mid x \mid \to \infty,$$

has for solutions the Weber-Hermite functions([20])

$$u_n(x) = [\pi^{1/2} 2^n n!]^{-1/2} (-1)^n e^{x^2/2} D^n(e^{-x^2}), \qquad \mu_n = 2n + 1.$$

It is well known that this system is complete in $L_2(-\infty, \infty)$. Condition A_5 is fulfilled since([21])

$$(2.9.5) \qquad \mid u_n(x) \mid \leqq B_1, \qquad \mid u_n'(x) \mid \leqq B_2 n^{1/2}.$$

All zeros of $u_n(x)$, zeros of the nth Hermite polynomials, are real and located in the interval $(-\mu_n^{1/2}, \mu_n^{1/2})$. They are densest towards the center of the interval, but the minimum distance between consecutive zeros is of the same order as the average distance. It follows that the conditions A_3 to A_6 and E_3 are satisfied. Hence S is a conservative system and Theorem 9 applies to the corresponding set F.

The determination of the class F is much more laborious than in the Legendre case. We know that

$$f(x) = \sum_{n=0}^{\infty} f_n u_n(x), \qquad \sum_{n=1}^{\infty} n^m \mid f_n \mid < \infty$$

for all m. The series is uniformly convergent in the infinite interval by virtue of (2.9.5) and the terms tend to zero as $|x| \to \infty$. It follows that $f(x) \in C_0^{(0)}[-\infty, \infty]$ where the subscript 0 indicates that $f(x) \to 0$ when

([20]) For the proof it is enough to observe that the only solution which satisfies the boundary condition when $x \to \infty$ is a multiple of $D_\nu(2^{1/2}x)$, $\nu = (\mu - 1)/2$, and that this solution, as is seen from its asymptotic representation, does not satisfy the boundary condition for $x \to -\infty$ unless μ is a positive odd integer.

([21]) Better estimates are available. For the first inequality see Szegö [5, Theorem 8.91.3]. The second inequality follows from the first combined with formula (2.9.6) below. For the properties of Hermite polynomials used in this discussion see also §§5.5, 5.7, 6.31, and 6.32 of Szegö's treatise.

$|x| \to \infty$. It is obvious that all L-transforms also belong to $C_0^{(0)}[-\infty, \infty]$. But much more can be asserted.

To this end we note that $f(x) \in F$ implies $x f(x)$ and $f'(x) \in F$. This is a consequence of the relations

$$\left.\begin{matrix} xu_n(x) \\ u_n'(x) \end{matrix}\right\} = -2^{-1/2}\{n^{1/2}u_{n-1}(x) \pm (n+1)^{1/2}u_{n+1}(x)\}, \tag{2.9.6}$$

which in their turn follow from the recurrence formulas for Hermite polynomials plus the relation $H_n'(x) = 2n\, H_{n-1}(x)$. Hence

$$\left.\begin{matrix} xf(x) \\ f'(x) \end{matrix}\right\} = -2^{-1/2}\sum_{n=0}^{\infty}[(n+1)^{1/2}f_{n+1} \pm n^{1/2}f_{n-1}]u_n(x),$$

and these series are clearly elements of F. By induction we show that $x^m f^{(k)}(x) \in F$ for every k and m. But this implies $x^m f^{(k)}(x) \in C_0^{(0)}[-\infty, \infty]$ for all k and m.

Conversely, suppose that $x^m f^{(k)}(x) \in C_0^{(0)}[-\infty, \infty]$ for all k and m. This implies that $x^m L^k f(x)$ has the same property and consequently $L^k f(x) \in L_2(-\infty, \infty)$ for every k. But if $g(x)$ is a function satisfying the same conditions as $f(x)$, an application of Lagrange's identity combined with the boundary conditions gives

$$\int_{-\infty}^{\infty} g(x)L^k f(x)dx = \int_{-\infty}^{\infty} f(x)L^k g(x)dx,$$

and in particular

$$\int_{-\infty}^{\infty} u_n(x)L^k f(x)dx = \int_{-\infty}^{\infty} f(x)L^k u_n(x)dx = (-1)^k(2n+1)^k f_n.$$

We conclude that the coefficients f_n must satisfy conditions C_1 and C_2 and that $f(x) \in F$. Consequently we have proved:

The class $F\{D^2 - x^2, u_n(x), 2n+1; (-\infty, \infty)\}$ is that subset of $C^{(\infty)}[-\infty, \infty]$ the elements of which satisfy the boundary conditions

$$\lim_{|x|\to\infty} x^m f^{(k)}(x) = 0, \qquad k, m = 0, 1, 2, \cdots. \tag{2.9.7}$$

From this we get without difficulty[22]:

The class $F\{D^2 - 2x\, D, A_n H_n(x), 2n; (-\infty, \infty)\}$ is that subset of $C^{(\infty)}(-\infty, \infty)$, the elements of which satisfy the boundary conditions

$$\lim_{|x|\to\infty} x^m \exp[-x^2/2] f^{(k)}(x) = 0, \quad k, m = 0, 1, 2, \cdots. \tag{2.9.8}$$

We can consequently formulate Theorem 9 as follows for the case of the Hermite operator.

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(22) A_n is a normalization factor.

THEOREM 13. *Let* $L=D^2-2x\,D$. *Let* $f(x)\in C^{(\infty)}(-\infty,\ \infty)$ *and satisfy the boundary conditions* (2.9.8). *If* $\liminf_{k\to\infty} V[L^kf(x)]=N$, *then* $f(x)$ *is a polynomial of degree* N. *Conversely, every real polynomial of exact degree* N *has the property* $V[L^kf(x)]=N$ *for all large values of* k.

2.10. **The Laguerre operator.** Our last example of a singular operator is that of Laguerre

$$L = xD^2 + (1-x)D, \tag{2.10.1}$$

the interval being $(0,\ \infty)$. The equation

$$(L-\lambda)y \equiv xy'' + (1-x)y' - \lambda y = 0$$

has singular points at 0 and ∞. A simple computation shows that $R(x,\ 1;\ \lambda)\to\infty$ at both points, so the problem is of type T_3. The origin is a regular singular point with indicial equation $\rho^2=0$. Hence there is a solution which becomes infinite as $\log(1/x)$ while the other solution is regular at $x=0$. The point at infinity is irregular-singular. Assuming x and λ positive, we have one solution tending to zero as $x^{-\lambda}$ and another tending to infinity as $x^{\lambda-1}e^x$ when $x\to\infty$. It follows that

$$y(x,\ 1;\ \lambda) = \begin{cases} A(\lambda)\log(1/x)\{1+o(1)\}, & x\to 0, \\ B(\lambda)x^{\lambda-1}e^x\{1+o(1)\}, & x\to\infty. \end{cases}$$

The singular boundary value problem

$$(L+\mu)u = 0, \qquad \lim_{x\to 0}\frac{u(x)}{\log(1/x)} = 0, \qquad \lim_{x\to\infty} u(x)x^{1-\lambda}e^{-x} = 0 \tag{2.10.2}$$

(for every $\lambda>0$) determines the Laguerre polynomials $L_n(x)$ corresponding to the characteristic values $\mu_n=n,\ n=0,\ 1,\ 2,\ \cdots$ [23].

The system $\{e^{-x/2}L_n(x)\}$ is complete in $L_2(0,\ \infty)$. It was proved by Szegö that

$$e^{-x/2}\,|\,L_n(x)\,| < 1, \qquad x > 0.$$

Since

$$L_n'(x) = -\sum_{\nu=0}^{n-1} L_\nu(x), \tag{2.10.3}$$

[23] To prove that no other solutions exist is fairly complicated. We shall merely outline an argument which the interested reader will be able to complete. There exist two formal but asymptotic solutions of the form $x^\mu\mathfrak{P}_1(1/x)$ and $e^xx^{-1-\mu}\mathfrak{P}_2(1/x)$, of which only the first one satisfies the boundary condition at infinity. The series are easily computed. If $\mu=n$ the first series terminates and reduces to a multiple of $L_n(x)$. For other values of μ it may be summed by Borel's method which leads to the result $u(x)=x^{\mu+1}\int_0^\infty F(-\mu,\ -\mu,\ 1,\ -t)e^{-xt}dt$. The behavior of the integral for small positive x is determined by that of the hypergeometric function for large t. If μ is not zero or a positive integer, $F(-\mu,\ -\mu,\ 1,\ -t)=A(\mu)t^\mu\log t[1+o(1)]$, $A(\mu)\neq 0$, for large t, and $u(x)$ becomes logarithmically infinite when $x\to 0$. Thus the Laguerre polynomials are the only solutions of the boundary value problem.—For the properties of $L_n(x)$ used in this section, see Szegö [5, §§5.1, 5.7, 6.31, and 7.21].

we get

$$e^{-x/2}\left| L_n'(x) \right| < n, \qquad x > 0.$$

These inequalities show that A_5 and E_3 are satisfied if we take $U(x)=e^{x/2}$. For later use we note the recurrence formula

$$(n+1)L_{n+1}(x) + (x - 2n - 1)L_n(x) + nL_{n-1}(x) = 0. \tag{2.10.4}$$

The zeros of $L_n(x)$ are all real positive and the mth zero equals $C_{m,n}(m+1)^2(n+1)^{-1}$ where $1/4 \leqq C_{m,n} \leqq 4$. It follows that A_6 also holds and S is consequently a conservative system.

It remains to determine the class F. If $f(x)\in F$ then

$$f(x) = \sum_{n=0}^{\infty} f_n L_n(x), \qquad \sum_{n=1}^{\infty} n^m \left| f_n \right| < \infty \tag{2.10.5}$$

for all m. Multiplying on both sides in (2.10.5) by $e^{-x/2}$ we obtain a series which is uniformly convergent in $[0, \infty]$ and the terms of which tend to zero when $x\to\infty$. It follows that $e^{-x/2}f(x)$ is continuous in $[0, \infty]$ and tends to zero when $x\to\infty$. We show next that $x f(x)$ and $f'(x)$ must belong to F whenever $f(x)$ does. Multiplying both sides of (2.10.5) by x, reducing with the aid of (2.10.4) and rearranging, we obtain the series

$$xf(x) = \sum_{n=0}^{\infty} \left[(2n+1)f_n - nf_{n-1} - (n+1)f_{n+1}\right]L_n(x)$$

which clearly belongs to F. Similarly we obtain the series

$$f'(x) = - \sum_{n=0}^{\infty}\left(\sum_{\nu=n+1}^{\infty} f_\nu \right) L_n(x)$$

from (2.10.5) with the aid of (2.10.3). This is also an element of F. It follows that any function of the form $x^m f^{(k)}(x)\in F$ and

$$\lim_{x\to\infty} x^m e^{-x/2} f^{(k)}(x) = 0, \qquad k, m = 0, 1, 2, \cdots. \tag{2.10.6}$$

At the origin we find of course that $f^{(k)}(x)$ tends to a finite limit for every k.

Conversely, if $f(x)\in C^{(\infty)}[0, \infty)$ and satisfies the boundary conditions (2.10.6), then $e^{-x/2}L^k f(x) \in L_2(0, \infty)$ for every k. Using Lagrange's identity we verify that

$$\int_0^\infty e^{-x/2} L_n(x) L^k f(x)\,dx = (-n)^k f_n$$

and from this we conclude that $f(x)\in F$.

The class $F\{xD^2+(1-x)D,\ L_n(x),\ n;\ (0, \infty)\}$ *equals the subset of* $C^{(\infty)}[0, \infty)$ *the elements of which satisfy conditions* (2.10.6).

THEOREM 14. *Let* $L = x D^2 + (1-x) D$. *Let* $f(x)\in C^{(\infty)}[0, \infty)$ *and satisfy the* *boundary conditions* (2.10.6). *If* $\liminf_{k\to\infty} V[L^k f(x)] = N$, *then* $f(x)$ *is a polynomial of degree* N. *Conversely, if* $f(x)$ *is a real polynomial of exact degree* N, *then* $V[L^k f(x)] = N$ *for all large values of* k(24).

Theorems 12, 13, and 14 give three distinct unique characterizations of real ordinary polynomials in terms of their behavior with respect to certain second order differential operators. This is analogous to the unique characterization of trigonometric polynomials by means of the operator D^2 given by Pólya and Wiener.

2.11. **Bessel operators.** Our last examples will deal with semi-singular operator problems related to the theory of Bessel functions. In this theory we find essentially three different types of expansions, conventionally referred to as the Bessel-Fourier, the Neumann, and the Schlömilch series. Only the first type falls directly under our theory, but the third type is also accessible to the methods of Pólya and Wiener.

We start with the operator

$$L = D^2 + D/x - m^2/x^2 \tag{2.11.1}$$

where $m \geqq 0$ is fixed. We take the interval (0, 1) of which one end point is singular and the other regular. It is easily seen that $R(x, 1; \lambda) \to \infty$ when $x \to 0$ so the problem is of type T_4. We have $y(x; 1, \lambda) \sim B(\lambda) \log(1/x)$ or $B(\lambda)x^{-m}$ at $x=0$ according as $m=0$ or is greater than 0. The corresponding boundary value problem for $m>0$ is

$$(L + \mu)u = 0, \qquad \lim_{x\to 0} x^m u(x) = 0, \qquad C_1 u(1) + C_2 u'(1) = 0, \tag{2.11.2}$$

where $C_1 \geqq 0$, $C_2 \geqq 0$, $C_1 + C_2 > 0$. If $m=0$ the factor x^m should be replaced by $[\log(1/x)]^{-1}$. The problem has as its solution the set $J_m(\mu_n x)$, where μ_n runs through the positive roots of the equation

$$C_1 \mu J_m(\mu) + C_2 J_m(\mu) = 0. \tag{2.11.3}$$

If $m=0$, $C_2=0$, we have to add $\mu_0 = 0$ with $u_0(x) = 1$(25).

Using any standard text on Bessel functions, the reader will have no difficulties in proving that the corresponding system S is conservative. We note in particular that $Z_n(0, 1) = V[J_m(\mu_n x)] = n$ so that $M(N) = N$ in Theorem 9. We shall not state the corresponding form of the theorem, but we shall determine the class F.

If $f(x) \in F \subset B_4^{(\infty)}\{L; C_1, C_2; (0, 1]\}$ then we must have

$$C_1 L^n f(1) + C_2 D L^n f(1) = 0, \qquad n = 0, 1, 2, \cdots, \tag{2.11.4}$$

(24) Similar results hold for the general Laguerre operator $L = xD^2 + (1+\alpha-x)D$, $\alpha > -1$.

(25) The only solution which satisfies the boundary condition at $x=0$ is a multiple of $J_m(\mu x)$. The values of μ are determined by the second condition. The root $\mu=0$ figures if and only if $C_2/C_1 = -m$. Owing to our sign restrictions, this case occurs only if $m=0$, $C_2=0$.

with our usual notation. At the singular end point we can write $f(x)=x^m g(x)$. A simple computation shows that if $g(x)$ is defined in $[-1, +1]$ by the convention $g(-x)=g(x)$ then $g(x)\in C^{(\infty)}[-1, +1]$.

Suppose, conversely, that $f(x) = x^m g(x)$ where $g(-x) = g(x)$, $g(x) \in C^{(\infty)}[-1, +1]$, and (2.11.4) is satisfied, The computation shows that $Lf(x)$ satisfies the same conditions. We can then apply the identity of Lagrange and find that

$$\int_0^1 xu_n(x)L^k f(x)dx = \int_0^1 xf(x)L^k u_n(x)dx = (-\mu_n)^k f_n,$$

since all intermediary integrated expressions vanish at both end points of (0, 1). It follows that the coefficients f_n satisfy the conditions C_1 and C_2 of §2.1 and $f(x)\in F$.

Thus, *the class* $F\{L, A_nJ_m(\mu_n x), \mu_n; (0, 1)\}$ *consists of all functions of the form* $f(x)=x^m g(x)$, *satisfying* (2.11.4), *such that* $g(-x)=g(x)$ *and* $g(x) \in C^{(\infty)}[-1, +1]$.

The Neumann series

$$\sum_{n=0}^{\infty} a_n J_n(x) \tag{2.11.5}$$

does not give rise to any interesting oscillation problems for the simple reason that in any fixed interval $(0, b)$ the function $J_n(x)$ is ultimately non-oscillatory since the least positive zero of $J_n(x)$ exceeds n.

We get more interesting results for the Schlömilch series

$$(f_0/2) + \sum_{n=1}^{\infty} f_n J_0(nx) \tag{2.11.6}$$

the terms of which are characteristic functions of the operator (2.11.1) with $m=0$ corresponding to the characteristic values n^2. The corresponding system S is not admissible in the technical sense of §2.1, since the functions $\{J_0(nx)\}$ do not form an orthogonal system. But the methods employed in the present paper nevertheless apply and lead to a result which we state without proof([26]).

THEOREM 15. *Let F be the class of functions defined by the formula*

$$f(x) = (2/\pi)\int_0^{\pi/2} g(x \sin t)dt, \qquad g(u) = (f_0/2) + \sum_{n=1}^{\infty} f_n \cos nu, \tag{2.11.7}$$

where $g(u)$ *is any real even function of period* 2π *belonging to* $C^{(\infty)}(-\infty, \infty)$. *Let* $N_k(R)$ *be the number of sign changes of* $\{D^2+(1/x)D\}^k f(x)$ *in the interval*

([26]) See E. T. Whittaker and G. N. Watson [6, §17.82], for the relation between the series (2.11.6) and (2.11.7).

(0, R). *If*

$$\liminf_{k\to\infty} \limsup_{R\to\infty} N_k(R)/R = C < \infty,$$

then $g(u)$ is a trigonometric polynomial of degree at most $C\pi$.

Appendix

3.1. **Characteristic series representing entire functions.** Pólya and Wiener [2, Theorem III], proved for periodic functions $f(x)$ that the assumption $V[D^{2k}f(x)] = o(k^{1/2})$ implies that $f(x)$ is an entire function. This result also admits of far reaching generalizations but cannot be true for arbitrary conservative systems. It is obviously necessary that the functions $u_n(x)$ themselves are entire. It is also necessary to have some definite information concerning the convergency properties of the series $\sum f_n u_n(z)$ for complex values. In this direction it is enough to know that a condition of the form

$$\limsup_{n\to\infty} |f_n| \exp(\tau\mu_n^{1/2}) = \infty$$

for some finite τ, prevents the convergence of the series in the whole finite plane, while on the other hand the finiteness of the limit superior for every τ implies that the series does converge in the whole plane. The matter is complicated by the fact that an entire function may have a characteristic series which is not convergent outside of the real interval (a, b) or even anywhere([27]). In addition it is desirable to have more precise information concerning the characteristic values and the degree of regularity of the oscillations of the characteristic functions in fixed interior intervals. It is not worth while stating here in precise form the assumptions under which we have succeeded in extending Theorem III of Pólya and Wiener. It is enough to mention that the results apply to the operators of Legendre, Jacobi, Hermite and Laguerre. We state without proof:

THEOREM 16. *The condition $V[L^k f(x)] = o(k^{1/2})$ is sufficient in order that $f(x) = \sum_{n=0}^{\infty} f_n u_n(x)$ shall define an entire function, the series being convergent in the finite complex plane, provided $S = S\{L, u_n(x), \mu_n; (a, b)\}$ is one of the five systems considered in* §§2.7 *to* 2.10.

For the case of the Legendre operator, this theorem has also been proved by Szegö ([4], special case of his Theorem D) and with a much less restrictive condition on the rate of growth of N_k. His method would also apply to the Jacobi case, at least for $\alpha > -1$, $\beta > -1$, with a similar improvement of the rate of growth condition. His method, however, does not apply to the Hermite, Hermite-Weber, and Laguerre operators.

([27]) This happens, for instance, in the case of expansions in terms of Hermite and Laguerre polynomials, but not in the Jacobi and Legendre cases.

3.2. **Upper limits for the frequency of oscillation.** It has been conjectured (by Pólya, at least for the operator D) that $o(k)$ is the correct order in theorems of the type of our Theorem 16 and that this order cannot be raised to $O(k)$. The latter part of the conjecture has been proved by Pólya and Szegö [4, §7]. It is very easy to verify that $O(k)$ is not admissible in the case of two rather wide classes of second order operators.

Suppose first that (a, b) is a finite or semi-infinite interval and that the coefficients $p_m(x)$ of L are polynomials. Take $f(x)=1/(x-c)$, where c is real and outside of $[a, b]$. A simple computation shows that the L^k-transform of $f(x)$ is a rational function whose denominator is $(x-c)^{2k+1}$ while the numerator is a polynomial of degree at most Ak, where A is a constant depending only upon the degree of the polynomials $p_m(x)$. It is clear that for this function $V[L^kf(x)]\leqq Ak$ and $f(x)$ is not entire. If $(a, b)=(-\infty, \infty)$, we take $f(x)=1/(x^2+c^2)$ instead.

If $(a, b)=(-\pi, \pi)$ and the coefficients $p_m(x)$ are trigonometric polynomials, we have similar results. We take $f(x)=1/(2-\sin x)$ instead. Here $L^kf(x)$ is the quotient of two trigonometric polynomials, the degree of the numerator being at most Ak. Hence $V[L^kf(x)]\leqq 2Ak$ and $f(x)$ is not entire.

Finally it should be observed that all the available evidence so far supports the conjecture that $V[L^kf(x)]=O(k)$ is a necessary and sufficient condition in order that an admissible characteristic series shall define an analytic function.

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