ON THE PARTIAL SUMS OF FOURIER SERIES AT POINTS OF DISCONTINUITY

BY

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1. **Introduction.** Consider a Fourier sine series

$$f(\theta) \sim \sum_{1}^{\infty} b_\nu \sin \nu\theta, \qquad b_\nu = (2/\pi) \int_0^\pi f(\theta) \sin \nu\theta d\theta, \qquad 0 < \theta < \pi, \tag{1.1}$$

and write

$$s_n(\theta) = \sum_{1}^{n} b_\nu \sin \nu\theta, \qquad n = 1, 2, 3, \cdots . \tag{1.2}$$

Fejér proved (cf. Zygmund [5, p. 181])(1) that if $f(\theta)$ is of bounded variation, and if $n\theta_n \to \alpha$ as $\theta_n \to 0$, then

$$s_n(\theta_n) \to (2/\pi)f(+0) \int_0^\alpha \frac{\sin t}{t} dt \equiv (2/\pi)f(+0)I(\alpha). \tag{1.3}$$

In particular, choosing α so that $I(\alpha) = \pi/2 = \int_0^\infty t^{-1} \sin t dt$ (thus $\int_\alpha^\infty t^{-1} \sin t dt = 0$), we get $s_n(\theta_n) \to f(+0)$, which is half of the jump of $f(\theta)$ at $\theta = 0$.

On the other hand for $\alpha = \pi$, which gives $I(\alpha)$ its maximal value

$$s_n(\theta_n) \to (2/\pi)f(+0) \int_0^\pi \frac{\sin t}{t} dt = f(+0) \times 1.08949 \cdots .$$

Thus the limit points of the partial sums as $\theta_n \to 0$ cover an interval which extends beyond $f(+0)$, if $f(+0) \neq 0$. This is called Gibbs' phenomenon.

It was also proved by Fejér and Csillag (for references and further results see Szász [4]) that for functions of bounded variation

$$n^{-1} \sum_{1}^{n} \nu b_\nu \to (2/\pi) f(+0), \qquad \text{as } n \to \infty. \tag{1.4}$$

These facts suggest the consideration of

$$s_n(\theta_n) = \sum_{1}^{n} \nu b_\nu \frac{\sin \nu\theta_n}{\nu}, \qquad \theta_n \to 0,$$

as a transform of the sequence $\{nb_n\}$, that is, as a special case of the triangular type transform

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(1) Numbers in brackets refer to the literature at the end of this paper.

$$T_n = \sum_1^n a_{n\nu}\tau_\nu, \tag{1.5}$$

where now $\tau_\nu = \nu b_\nu$, $a_{n\nu} = \nu^{-1} \sin \nu\theta_n$. We shall not restrict ourselves to regularity conditions, and we shall not assume convergence of the sequence $\{\tau_n\}$, but merely Cesàro summability of some order. We then seek simple necessary and sufficient conditions for the convergence of the transform T_n (in general to a different limit). The application to Fourier sine series yields a generalized Gibbs' phenomenon, and also a new device to determine the generalized jump of a function. Our results are in close relationship with some results of Rogosinski [1, 2].

We consider more generally the transform

$$T(\rho_n, \theta_n) = \sum_1^n \tau_\nu \rho_n^\nu \nu^{-1} \sin \nu\theta_n, \qquad \rho_n \to 1,\ \theta_n \to 0, \tag{1.6}$$

which in the case $\tau_\nu = \nu b_\nu$ becomes $\sum_1^n \rho_n^\nu b_\nu \sin \nu\theta_n = s_n(\rho_n, \theta_n)$, where $s_n(\rho, \theta)$ is the nth partial sum of the harmonic series $\sum_1^\infty \rho^\nu b_\nu \sin \nu\theta$.

2. **Permanency with respect to convergent sequences.** It is well known that the convergence of the sequence $\{\tau_n\}$ implies the convergence of the transform T_n, if and only if

$$\lim_{n\to\infty} a_{n\nu} = 0, \qquad \text{for } \nu = 1, 2, 3, \cdots;$$

$$\sum_{\nu=1}^n | a_{n\nu} | = O(1), \qquad \text{as } n \to \infty;$$

$$\lim_{n\to\infty} \sum_1^n a_{n\nu} = \sigma \text{ exists.}$$

We then have $\lim T_n = \sigma \lim \tau_n$. If we restrict ourselves to sequences $\tau_n \to 0$, then the last condition can be omitted. Applied to (1.6) this yields the necessary and sufficient conditions:

$$\sum_1^n \rho_n^\nu \nu^{-1} | \sin \nu\theta_n | = O(1), \qquad \text{as } n \to \infty; \tag{2.1}$$

$$\lim \sum_1^n \rho_n^\nu \nu^{-1} \sin \nu\theta_n = \sigma. \tag{2.2}$$

In particular the last condition is $s_n(\rho_n, \theta_n) \to \sigma$ for the harmonic series $\sum_1^\infty \rho^\nu \nu^{-1} \sin \nu\theta = \arctan \{(\rho \sin \theta)/(1 - \rho \cos \theta)\}$.

We first assume

$$0 < \liminf \rho_n^n \leqq \limsup \rho_n^n < \infty; \tag{2.3}$$

in this case for some $c_1>0$, $c_2>0$

$$c_1\sum_1^n \nu^{-1}|\sin\nu\theta| < \sum_1^n \rho_n^\nu \nu^{-1}|\sin\nu\theta| < c_2\sum_1^n \nu^{-1}|\sin\nu\theta|,$$

thus (2.1) reduces to

$$\sum_1^n \nu^{-1}|\sin\nu\theta_n| = O(1), \qquad \text{as } n\to\infty. \tag{2.4}$$

Now for any $\theta>0$

$$\sum_1^n \nu^{-1}|\sin\nu\theta| < n\theta, \tag{2.5}$$

hence $n\theta_n=O(1)$ implies (2.4). To prove the converse let $\theta_n<1<\theta_n(n-1)$, and put $[\theta_n^{-1}]=\kappa_n=\kappa$, so that $\kappa\leqq\theta_n^{-1}<\kappa+1\leqq n$. Now $\sum_1^n\nu^{-1}|\sin\nu\theta_n| >(1/2)\sum_1^n\nu^{-1}(1-\cos 2\nu\theta_n)$, and

$$\left|\sum_1^n \nu^{-1}\cos 2\nu\theta_n\right| < \sum_1^\kappa \nu^{-1} + \left|\sum_{\kappa+1}^n \nu^{-1}\cos 2\nu\theta_n\right|$$

$$< 1+\log\kappa + (1/(\kappa+1))\max_{\kappa<\lambda\leqq n}\left|\sum_{\kappa+1}^\lambda \cos 2\nu\theta_n\right|$$

$$< 1+\log\theta_n^{-1} + \theta_n/\sin\theta_n < 3-\log\theta_n.$$

Thus

$$2\sum_1^n \nu^{-1}|\sin\nu\theta_n| > \log n+\log\theta_n - 3 = -3+\log(n\theta_n);$$

hence (2.4) implies $n\theta_n=O(1)$. For null sequences only this is required.

To satisfy (2.2) consider the case that 0 is a limit point of the sequence $\{n\theta_n\}$; for a subsequence of indices n: $n\theta_n\to 0$, and for that subsequence, using (2.5)

$$\sum_1^n \rho_n^\nu\nu^{-1}\sin\nu\theta_n = O\left(\sum_1^n \nu^{-1}|\sin\nu\theta_n|\right) = O(n\theta_n) = o(1).$$

Hence σ, if it exists, is 0 and then every convergent sequence is transformed into a null sequence. Next assume lim inf $n\theta_n>0$. We choose a subsequence of integers $n=n'$ for which ρ_n^n and $n\theta_n$ have limits $n'\theta_{n'}\to\beta>0$, $\rho_{n'}^{n'}\to e^\gamma$ say; by (2.3) γ is finite. Furthermore from $\log\rho/(\rho-1)\to 1$ as $\rho\to 1$, $n'(\rho_{n'}-1)\to\gamma$.

Suppose first $\gamma=0$, that is $\rho_{n'}^{n'}\to 1$, and $n'(\rho_{n'}-1)\to 0$. Now, as n runs through the sequence $\{n'\}$

$$\left|\sum_1^n (\rho_n^\nu - 1)\nu^{-1}\sin\nu\theta_n\right| < |\rho_n - 1|\,O(n)\sum_1^n \nu^{-1}|\sin\nu\theta_n| = o(1)O(n\theta_n) = o(1),$$

hence

$$\lim \sum_1^n \rho_n^{\nu} \nu^{-1} \sin \nu\theta_n = \lim \sum_1^n \nu^{-1} \sin \nu\theta_n$$

for $n = n' \to \infty$, if either side exists. But

$$\sum_1^n \nu^{-1} \sin \nu\theta = \int_0^\theta \left(\sum_1^n \cos \nu t \right) dt = -(1/2)\theta + \int_0^\theta \frac{\sin (n + 1/2)t}{2 \sin (t/2)} dt$$
$$= -(1/2)\theta + \int_0^{(n+1/2)\theta} \frac{\sin u du}{(2n + 1) \sin (u/(2n + 1))},$$

hence

$$\sum_1^n \nu^{-1} \sin \nu\theta_n = -(1/2)\theta_n + \int_0^\beta \frac{\sin u}{u} \cdot \frac{u du}{(2n + 1) \sin (u/(2n + 1))} + o(1)$$
$$\to \int_0^\beta \frac{\sin u}{u} du$$

as $n \to \infty$ through the sequence $\{n'\}$. The consideration of the case $\gamma \neq 0$ remains; we write

$$\sum_1^n \rho^\nu \nu^{-1} \sin \nu\theta$$
$$= \int_0^\theta \left(\sum_1^n \rho^\nu \cos \nu t \right) dt$$
$$= \int_0^\theta \frac{\cos t - \rho^2 + \rho^{n+2} \cos nt - \rho^{n+1} \cos (n+1)t}{1 - 2\rho \cos t + \rho^2} dt$$
$$= \int_0^\theta \frac{1 - \rho^2 - (1 - \cos t) + \rho^{n+1}[\cos nt - \cos (n+1)t] - (1-\rho)\rho^{n+1} \cos nt}{(1-\rho)^2 + 2\rho(1 - \cos t)} dt,$$

thus

$$\begin{aligned} \sum_1^n \rho^\nu \nu^{-1} \sin \nu\theta = (1 - \rho^2) &\int_0^\theta \frac{dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)} \\ &- 2 \int_0^\theta \frac{\sin^2 (t/2) dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)} \\ &- (1 - \rho)\rho^{n+1} \int_0^\theta \frac{\cos nt dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)} \\ &+ 2\rho^{n+1} \int_0^\theta \frac{\sin (t/2) \sin (n + 1/2)t dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)}. \end{aligned} \tag{2.6}$$

Now

$$\int_0^{\theta_n} \frac{\sin^2 (t/2)dt}{(1-\rho_n)^2 + 4\rho_n \sin^2 (t/2)} < \frac{\theta_n}{4\rho_n} \to 0 \qquad \text{as } n \to \infty.$$

Next

$$(\rho_n^2 - 1)\int_0^{\theta_n} \frac{dt}{(1-\rho_n)^2 + 4\rho_n \sin^2 (t/2)}$$

$$= n(\rho_n - 1)(\rho_n + 1)\int_0^{n\theta_n} \frac{du}{n^2[(\rho_n - 1)^2 + 4\rho_n \sin^2 (u/2n)]}$$

$$\to 2\gamma \int_0^{\beta} \frac{du}{\gamma^2 + u^2} = 2 \text{ arc tan } (\beta/\gamma).$$

Similarly

$$(\rho_n - 1)\rho_n^{n+1}\int_0^{\theta_n} \frac{\cos ntdt}{(1-\rho_n)^2 + 4\rho_n \sin^2 (t/2)}$$

$$= n(\rho_n - 1)\rho_n^{n+1}\int_0^{n\theta_n} \frac{\cos udu}{n^2[(\rho_n - 1)^2 + 4\rho_n \sin^2 (u/2n)]} \to \gamma e^{\gamma}\int_0^{\beta} \frac{\cos udu}{\gamma^2 + u^2},$$

and

$$\rho_n^{n+1}\int_0^{\theta_n} \frac{\sin (t/2) \sin (n + 1/2)tdt}{(1-\rho_n)^2 + 4\rho_n \sin^2 (t/2)}$$

$$= \rho_n^{n+1}\int_0^{(n+1/2)\theta_n} \frac{(2n+1) \sin \{u/(2n+1)\} \cdot \sin udu}{n(2n+1)[(\rho_n - 1)^2 + 4\rho_n \sin^2 \{u/(2n+1)\}]}$$

$$\to (1/2)e^{\gamma}\int_0^{\beta} \frac{u \sin udu}{\gamma^2 + u^2}.$$

Summarizing

$$\sum_1^n \rho_n^{\nu} \nu^{-1} \sin \nu\theta_n \to \int_0^{\beta} \frac{\gamma e^{\gamma} \cos u + e^{\gamma} u \sin u - 2\gamma}{\gamma^2 + u^2} du$$

$$= \int_0^{\beta/\gamma} \frac{e^{\gamma}(t \sin \gamma t + \cos \gamma t) - 2}{1 + t^2} dt.$$

The case (2.3) is now completely discussed. We next assume

$$\limsup_{n\to\infty} \rho_n^n = \infty,$$

so that for a subsequence $n = n' : \rho_{n'}^{n'} \to \infty$. We first prove that (2.1) implies $n'\theta_{n'} \to 0$. Otherwise for a subsequence n'' of $n' : n''\theta_{n''} \to \beta > 0$. For these indices

$$\sum_1^n \rho_n^\nu \nu^{-1} |\sin \nu\theta_n| > \sum_{\nu \leqq \alpha/\theta_n} \rho_n^\nu \nu^{-1} |\sin \nu\theta_n|,$$

where α is so chosen that $0<\alpha<\beta$ and $\alpha \leqq \pi/2$. Now

$$\sum_{\nu \leqq \alpha/\theta_n} \rho_n^\nu \nu^{-1} |\sin \nu\theta_n| > (2/\pi)\theta_n \sum \rho_n^\nu = \frac{2\rho_n\theta_n}{\pi} \frac{\rho_n^{[\alpha\theta_n^{-1}]} - 1}{\rho_n - 1},$$

hence (2.1) implies

$$\theta_n \rho_n^{\alpha n/2\beta} = O(\rho_n - 1),$$

which by virtue of $\log \rho_n/(\rho_n - 1) \to 1$ yields $n\theta_n = o(1)$. Furthermore

$$(2/\pi)\theta_n \sum \rho_n^\nu < \sum_1^n \rho_n^\nu \nu^{-1} |\sin \nu\theta_n| < \theta_n \sum_1^n \rho_n^\nu,$$

thus, if for a subsequence of indices $\rho_n^n \to \infty$, then for these indices (2.1) is equivalent to

$$\theta_n \rho_n^n = O(\rho_n - 1).$$

If this condition is satisfied, then in view of $n\theta_n \to 0$

$$0 < \theta_n \sum_1^n \rho_n^\nu - \sum_1^n \rho_n^\nu \nu^{-1} \sin \nu\theta_n = \theta_n \sum_1^n \rho_n^\nu \left(1 - \frac{\sin \nu\theta_n}{\nu\theta_n}\right)$$
$$< \theta_n \left(1 - \frac{\sin n\theta_n}{n\theta_n}\right) \sum_1^n \rho_n^\nu \to 0,$$

hence (2.2) holds if and only if $\lim \theta_n \rho_n^n/(\rho_n - 1)$ exists, which is then the value of σ.

Finally assume that $\liminf \rho_n^n = 0$; thus for a subsequence of indices $\rho_n^n \to 0$ ($\gamma = -\infty$). If $n\theta_n = O(1)$, then $\sum_1^n \rho_n^\nu \nu^{-1} |\sin \nu\theta_n| = O(\sum_1^n \nu^{-1} |\sin \nu\theta_n|) = O(1)$, which is (2.1). If on the other hand for a subsequence $n\theta_n \to \infty$, then

$$\sum_1^n \rho_n^\nu \nu^{-1} |\sin \nu\theta_n| > \sum_{\theta_n^{-1} < \nu}^{\leqq n} \rho_n^\nu \nu^{-1} \frac{1 - \cos 2\nu\theta_n}{2};$$

but $\rho_n^\nu \nu^{-1} \downarrow$ as $\nu \uparrow$, hence for $\theta_n < 1$

$$\left| \sum_{\theta_n^{-1} < \nu}^{\leqq n} \rho_n^\nu \nu^{-1} \cos 2\nu\theta_n \right| < \rho_n^{1+[\theta_n^{-1}]} \frac{1}{1 + [\theta_n^{-1}]} \max_{\lambda \leqq n} \left| \sum_{\theta_n^{-1} < \nu}^{\leqq \lambda} \cos 2\nu\theta_n \right|$$
$$< \frac{\theta_n}{\sin \theta_n} < \pi/2.$$

Furthermore

$$\sum_{\theta_n^{-1}<\nu}^{\leq n} \rho_n^\nu \nu^{-1} = \sum_{\theta_n^{-1}<\nu}^{\leq n} \nu^{-1} \exp(\nu \log \rho_n) > \sum \int_\nu^{\nu+1} u^{-1} \exp(-u \log \rho_n^{-1}) du$$

$$= \int_{1+[\theta_n^{-1}]}^{n+1} u^{-1} \exp(-u \log \rho_n^{-1}) du = \int_{(1+[\theta_n^{-1}])\log \rho_n^{-1}}^{(n+1)\log \rho_n^{-1}} t^{-1}e^{-t}dt;$$

thus in this case (2.1) implies $\theta_n = O(\log 1/\rho_n)$, or $\theta_n = O(1-\rho_n)$. If this condition is satisfied, then

$$\sum_1^n \rho_n^\nu \nu^{-1} |\sin \nu\theta_n| = O\left(\theta_n \sum_1^n \rho_n^\nu\right) = O(\theta_n/(1-\rho_n)) = O(1),$$

hence (2.1) holds. To satisfy (2.2) now, we note that

$$\sum_{n+1}^\infty \rho_n^\nu \nu^{-1} \sin \nu\theta_n = O\left(\theta_n \frac{\rho_n^n}{1-\rho_n}\right) = O(\rho_n^n) = o(1),$$

hence (2.2) holds if and only if

$$\sum_1^\infty \rho_n^\nu \nu^{-1} \sin \nu\theta_n = \arctan \frac{\rho_n \sin \theta_n}{1-\rho_n \cos \theta_n}$$

has a limit, and σ is then this limit. But

$$\frac{\rho_n \sin \theta_n}{1-\rho_n \cos\theta_n} = \frac{\rho_n \sin\theta_n}{1-\rho_n+\rho_n(1-\cos\theta_n)} \sim \frac{\theta_n}{1-\rho_n} \cdot \frac{1}{1+O(1-\rho_n)} \sim \frac{\theta_n}{1-\rho_n},$$

hence σ exists, if and only if $\lim \theta_n/(1-\rho_n) = \delta < +\infty$. We then have $\sigma = \lim \arctan \{\theta_n/(1-\rho_n)\} = \arctan \delta$. To summarize our results put

(a)
$$\sigma(\beta, 0) = \int_0^\beta \frac{\sin u}{u} du,$$
$$\sigma(\beta, \gamma) = \int_0^\beta \frac{\gamma e^\gamma \cos u + e^\gamma \sin u - 2\gamma}{\gamma^2 + u^2} du, \text{ for finite } \gamma \neq 0,$$

(b)
$$\sigma(0, \infty) = \lim \frac{\theta_n \rho_n^n}{\rho_n - 1},$$

(c)
$$\sigma(\delta, -\infty) = \lim \arctan \frac{\theta_n}{1-\rho_n} = \arctan \delta < \pi/2.$$

We then have

THEOREM 1. *Necessary and sufficient conditions that for every convergent sequence $nb_n \to \tau$ the transform $\sum_1^n \rho_n^\nu b_\nu \sin \nu\theta_n$ has a limit, are that one of the following three cases holds:*

(a′) $n(\rho_n-1)\to\gamma$ *finite*, $n\theta_n\to\beta<\infty$,
(b′) $n(\rho_n-1)\to+\infty$, $\lim \theta_n\rho_n^n(\rho_n-1)^{-1}$ *exists*,
(c′) $n(\rho_n-1)\to-\infty$, $\lim \theta_n(1-\rho_n)^{-1}=\delta$ *exists*, $0\leqq\delta<\infty$.

The limit of the transform is then $\tau\sigma$, where σ is defined above for the respective cases. Different subsequences may belong to different cases (β, γ) if only the corresponding σ attain the same value, and with the restriction $n\theta_n=O(1)$ in case (a′).

3. **Permanency with respect to (C, κ) summability.** Given the sequence $\{\tau_n\}$, write

$$\overset{0}{\tau}_n = \tau_n, \qquad \overset{\kappa}{\tau}_n = \sum_{\nu=1}^{n} \overset{\kappa-1}{\tau}_n, \qquad n, \kappa = 1, 2, 3, \cdots ;$$

also

$$A_n^{\kappa} = C_{n+\kappa,n} = \frac{(\kappa+1)\cdots(\kappa+n)}{n!} \sim \frac{n^{\kappa}}{\kappa!}, \qquad \text{as } n\to\infty. \tag{3.1}$$

The sequence $\{\tau_n\}$ is summable (C, κ) to the value τ, if $\overset{\kappa}{\tau}_n/A_n^{\kappa}\to\tau$ as $n\to\infty$; $(C, 0)$ is evidently convergence.

We write

$$\Delta^0\tau_n = \tau_n, \qquad \Delta^1\tau_n = \Delta\tau_n = \tau_n - \tau_{n+1}, \qquad \Delta^{\kappa}\tau_n = \Delta(\Delta^{\kappa-1}\tau_n);$$

then by induction

$$\Delta^{\kappa}\tau_n = \sum_{\nu=0}^{\kappa} (-1)^{\nu} C_{\kappa,\nu}\tau_{n+\nu}, \qquad \kappa = 0, 1, 2, \cdots . \tag{3.2}$$

Abel's transformation yields for finite sums

$$\sum_1^n \alpha_\nu\tau_\nu = \sum_1^n \overset{1}{\tau}_\nu\Delta\alpha_\nu = \sum_1^n \overset{2}{\tau}_\nu\overset{2}{\Delta}\alpha_\nu = \cdots ,$$

where $\alpha_{n+1}=0$, $\alpha_{n+2}=0$, $\cdots$. Applying this to (1.5) we get

$$T_n = \sum_{\nu=1}^{n} \overset{\kappa}{\tau}_\nu \overset{\kappa}{\Delta} a_{n\nu},$$

where $a_{n\nu}=0$ for $\nu>n$. Thus the transform converges for every (C, κ) summable sequence if in addition to the conditions of §2

$$\sum_{\nu=1}^{n} A_\nu^{\kappa}\left| \overset{\kappa}{\Delta} a_{n\nu}\right| = O(1) \qquad \text{as } n\to\infty.$$

In particular for the transform (1.6) we have the conditions (2.1), (2.2) and

$$\sum_{\nu=1}^{n-\kappa} A_\nu^{\kappa}\left| \overset{\kappa}{\Delta}\rho_n^{\nu}\nu^{-1}\sin\nu\theta_n\right| + \sum_{n-\kappa+1}^{n} A_\nu^{\kappa}\left|\delta_\nu\right| = O(1) \qquad \text{as } n\to\infty, \tag{3.3}$$

where, from (3.2)

$$\delta_\lambda = \sum_{\nu=0}^{n-\lambda} (-1)^\nu C_{\kappa,\nu} \rho_n^{\lambda+\nu} \frac{\sin (\lambda+\nu)\theta_n}{\lambda+\nu}, \qquad n-\kappa < \lambda \leqq n.$$

We first consider $(C, 1)$ summability $(\kappa=1)$. Now (3.3) becomes

$$\sum_1^{n-1} (\nu+1) \left| \Delta \rho_n^\nu \nu^{-1} \sin \nu\theta_n \right| + (n+1)\rho_n^n n^{-1} \left| \sin n\theta_n \right| = O(1),$$

or

$$(3.4) \quad \sum_1^{n-1} \nu \left| \rho_n^\nu \nu^{-1} \sin \nu\theta_n - \rho_n^{\nu+1}(\nu+1)^{-1} \sin (\nu+1)\theta_n \right| + \rho_n^n \left| \sin n\theta_n \right| = O(1).$$

We consider in succession the different cases of Theorem 1.

(a′) For a sequence of indices $n\theta_n \to \beta < \infty$, $n(\rho_n - 1) \to \gamma$ finite, that is, $\rho_n^n \to e^\gamma > 0$. Thus $\rho_n^n \sin n\theta_n$ is $O(1)$, and

$$\sum_1^{n-1} \nu\rho_n^\nu \left| \nu^{-1} \sin \nu\theta_n - \rho_n(\nu+1)^{-1} \sin (\nu+1)\theta_n \right|$$
$$\leqq \sum_1^{n-1} \nu\rho_n^\nu \left| \nu^{-1} \sin \nu\theta_n - (\nu+1)^{-1} \sin (\nu+1)\theta_n \right|$$
$$+ \left| 1-\rho_n \right| \sum_1^{n-1} \rho_n^\nu \left| \sin (\nu+1)\theta_n \right|.$$

Now

$$\left| 1-\rho_n \right| \sum_1^{n-1} \rho_n^\nu \left| \sin (\nu+1)\theta_n \right| < \left| 1-\rho_n \right| \sum_1^{n} \rho_n^\nu < \rho_n \left| 1-\rho_n^n \right| = O(1),$$

and

$$\nu \left| \nu^{-1} \sin \nu\theta_n - (\nu+1)^{-1} \sin (\nu+1)\theta_n \right|$$
$$= \left| (\nu+1)^{-1} \sin (\nu+1)\theta_n - 2 \sin (1/2)\theta_n \cos ((2\nu+1)/2)\theta_n \right| < 2\theta_n;$$

hence

$$\sum_1^{n-1} \nu\rho_n^\nu \left| \nu^{-1} \sin \nu\theta_n - (\nu+1)^{-1} \sin (\nu+1)\theta_n \right| < 2\theta_n \sum_1^n \rho_n^\nu = O(n\theta_n) = O(1).$$

Hence in this case no additional condition results.

(b′) $n(\rho_n-1) \to +\infty$, $\theta_n\rho_n^n(\rho_n-1)^{-1} \to \sigma$. Hence $n\theta_n \to 0$, and now $\rho_n^n \sin n\theta_n = O(1)$ is equivalent to $n\theta_n\rho_n^n = O(1)$. Thus $\theta_n\rho_n^n(\rho_n-1)^{-1} = n\theta_n\rho_n^n n^{-1}(\rho_n-1)^{-1} \to 0$, that is, $\sigma = 0$. Now

$$\left| 1-\rho_n \right| \sum_1^{n-1} \rho_n^\nu \left| \sin (\nu+1)\theta_n \right| = O\left[(\rho_n - 1)\theta_n \sum_1^n \nu\rho_n^\nu \right]$$
$$= O\left[\theta_n\rho_n^n(\rho_n-1)^{-1}\right] = o(1);$$

furthermore

$$\theta_n \sum_1^n \rho_n^\nu = O\left[\theta_n \rho_n^n (\rho_n - 1)^{-1}\right] = o(1);$$

hence (3.4) holds. Finally:

(c′) If $\lim \theta_n/(1-\rho_n) < \infty$ exists, and $n(\rho_n - 1) \to -\infty$, that is, $\rho_n^n \to 0$, then $\rho < \rho_n^n \sin n\theta_n \to 0$, $\left|1-\rho_n\right| \sum_1^{n-1} \rho_n^\nu \left| \sin (\nu+1)\theta_n \right| < (1-\rho_n)\sum_0^\infty \rho_n^\nu = 1$, and

$$\theta_n \sum_1^n \rho_n^\nu < \frac{\theta_n}{1 - \rho_n} = O(1).$$

No additional condition appears in this case. Summarizing, we have

THEOREM 2. *Necessary and sufficient conditions that when* $\lim n^{-1}\sum_1^n \nu b_\nu = \tau$ *exists the transform* $\sum_1^n \rho_n^\nu b_\nu \sin \nu\theta_n$ *has a limit*: $\tau\sigma$, *are either of the alternatives*:

(a″) $n(\rho_n - 1) \to \gamma$, *finite*, $n\theta_n \to \beta < \infty$,

(b″) $n(\rho_n - 1) \to +\infty$, $n\theta_n \rho_n^n = O(1)$,

(c″) $n(\rho_n - 1) \to -\infty$, $\lim \theta_n(1-\rho_n)^{-1}$ *exists.*

The value of σ is in the cases (a″) and (c″) given by (a) and (c). In case (b″) $\sigma = 0$. Different subsequences may belong to different cases if only σ has the same value, with the restriction $n\theta_n = O(1)$ in case (a″).

We now consider (C, κ) summability for $\kappa > 1$. First of all, to satisfy (3.3) we must have

$$n^\kappa \sum_{\nu=0}^m (-1)^\nu C_{\kappa,\nu} \rho_n^{n-m+\nu} \frac{\sin (n - m + \nu)\theta_n}{n - m + \nu} = O(1), \quad m = 0, 1, \cdots, \kappa - 1. \tag{3.5}$$

Or

$$n^{\kappa-1} \rho_n^n \sin n\theta_n = O(1),$$

$$n^\kappa \rho_n^n \left\{ \frac{\sin (n-1)\theta_n}{n-1} - \kappa\rho_n \frac{\sin n\theta_n}{n} \right\} = O(1),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$n^\kappa \rho_n^n \left\{ \frac{\sin (n - \kappa + 1)\theta_n}{n - \kappa + 1} - \cdots + (-1)^{\kappa-1} \kappa \rho_n^{\kappa-1} \frac{\sin n\theta_n}{n} \right\} = O(1).$$

This is equivalent to

$$\begin{aligned} n^{\kappa-1} \rho_n^n \sin n\theta_n &= O(1), \\ n^{\kappa-1} \rho_n^n \sin (n-1)\theta_n &= O(1), \\ &\cdots\cdots\cdots\cdots, \\ n^{\kappa-1} \rho_n^n \sin (n - \kappa + 1)\theta_n &= O(1). \end{aligned} \tag{3.6}$$

In case (a'') the first condition becomes $\sin n\theta_n = O(n^{1-\kappa})$, as $n \to \infty$; in particular $\sin n\theta_n \to 0$, thus in view of (a'') $n\theta_n \to \lambda\pi$, λ a positive integer or zero. On putting $n\theta_n = \lambda\pi + \epsilon_n$, we get $\sin \epsilon_n = O(n^{1-\kappa})$, or $n\theta_n - \lambda\pi = O(n^{1-\kappa})$. From the second condition now $\cos n\theta_n \sin \theta_n = O(n^{1-\kappa})$, as $n \to \infty$, or $\lambda\pi + \epsilon_n = O(n^{2-\kappa})$; hence for $\kappa = 2$, (3.6) reduces to

$$n\theta_n = \lambda\pi + O(n^{-1}). \tag{3.7}$$

For $\kappa > 2$ we must have

$$n\theta_n - \lambda\pi = \epsilon_n = O(n^{1-\kappa}) \quad \text{and} \quad \lambda\pi + \epsilon_n = O(n^{2-\kappa}),$$

hence $\lambda = 0$, and

$$n\theta_n = O(n^{1-\kappa}). \tag{3.7'}$$

It then follows that

$$n^{\kappa-1} \sin (n - \nu)\theta_n = O(1) \qquad \text{for } \nu = 0, 1, \cdots, \kappa - 1.$$

Furthermore, for the rest of (3.3)

$$\sum_1^{n-\kappa} A_\nu^\kappa \left| \Delta^\kappa \rho_n^\nu \nu^{-1} \sin \nu\theta_n \right| = O\left(\sum_1^{n-\kappa} \nu^\kappa \left| \Delta^\kappa \rho_n^\nu \nu^{-1} \sin \nu\theta_n \right| \right).$$

Now

$$\Delta^\kappa \rho^\nu \nu^{-1} \sin \nu\theta = \Delta^\kappa \rho^\nu \int_0^\theta \cos \nu t dt = R \int_0^\theta \Delta^\kappa z^\nu dt, \qquad z = \rho e^{it},$$

and, using (3.2)

$$\Delta^\kappa \rho^\nu \nu^{-1} \sin \nu\theta = R \int_0^\theta \sum_{\lambda=0}^{\kappa} (-1)^\lambda C_{\kappa,\lambda} z^{\nu+\lambda} dt = R \int_0^\theta z^\nu (1 - z)^\kappa dt,$$

hence

$$\begin{aligned} \left| \Delta^\kappa \rho^\nu \nu^{-1} \sin \nu\theta \right| &< \rho^\nu \int_0^\theta \left| 1 - \rho e^{it} \right|^\kappa dt < \rho^\nu \int_0^\theta \left\{ (1 - \rho)^2 + \rho t^2 \right\}^{\kappa/2} dt \\ &< \theta \rho^\nu \left\{ (1 - \rho)^2 + \rho\theta^2 \right\}^{\kappa/2}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_1^{n-\kappa} \nu^\kappa \left| \Delta^\kappa \rho_n^\nu \nu^{-1} \sin \nu\theta_n \right| &< \left(\sum_1^n \nu^\kappa \rho_n^\nu \right) \left\{ (1 - \rho_n)^2 + \rho_n \theta_n^2 \right\}^{\kappa/2} \theta_n \\ &< \left\{ n^2 (1 - \rho_n)^2 + \rho_n n^2 \theta_n^2 \right\}^{\kappa/2} \theta_n \sum_1^n \rho_n^\nu \\ &= O\left(\theta_n \sum_1^n \rho_n^\nu \right), \end{aligned} \tag{3.8}$$

and, from $\rho_n^n = O(1)$,

$$\theta_n \sum_{1}^{n} \rho_n^{\nu} = O(n\theta_n) = O(1).$$

Hence in case (a$''$) the additional condition is (3.7) for $\kappa=2$, and (3.7$'$) for $\kappa>2$.

In case (b$''$): $n(\rho_n-1)\to+\infty$, $n\theta_n\rho_n^n=O(1)$, as $n\to\infty$; hence $\rho_n^n\to+\infty$, and $n\theta_n\to 0$. Now (3.6) becomes

$$n^{\kappa}\rho_n^{n}\theta_n = O(1). \tag{3.9}$$

For large n evidently $\rho_n>1$, and

$$\theta_n \sum_{1}^{n} \rho_n^{\nu} < n\theta_n\rho_n^{n} = O(1)$$

(from (3.9)). In view of (3.8) now (3.3) holds. Thus in this case the additional condition is (3.9) (for $\kappa\geqq 2$).

Finally, in case (c$''$): $n(\rho_n-1)\to-\infty$ (that is $\rho_n^n\to 0$), and $\lim \theta_n/(1-\rho_n)=\delta<\infty$ exists. Now $\rho_n^n<1/(n(1-\rho_n))$, hence $n\theta_n\rho_n^n=O(1)$; thus for $\kappa=2$ condition (3.6) reduces to $n\rho_n^n \sin n\theta_n=O(1)$. While for $\kappa>2$ (3.6) reduces to $n^{\kappa-1}\rho_n^n \sin n\theta_n=O(1)$ and $n^{\kappa-1}\theta_n\rho_n^n=O(1)$. Furthermore, as $\rho_n<1$, $\theta_n\sum_0^n\rho_n^\nu<\theta_n/(1-\rho_n)=O(1)$, hence, in view of (3.8) now (3.3) is satisfied.

We summarize our results in

THEOREM 3. *In order that* $\lim \sum_1^n \rho_n^\nu b_\nu \sin \nu\theta_n=\tau\sigma$ *exists, whenever* (C,κ) $\lim nb_n=\tau$ *for some* $\kappa\geqq 2$, *necessary and sufficient conditions are the alternatives*:

(a$'''$) $n(\rho_n-1)\to\gamma$, *finite, and for* $\kappa=2$: $n\theta_n=\lambda\pi+O(n^{-1})$, λ *an integer, for* $\kappa>2$: $\theta_n=O(n^{-\kappa})$;

(b$'''$) $n(\rho_n-1)\to+\infty$, $n^\kappa\rho_n^n\theta_n=O(1)$;

(c$'''$) $n(\rho_n-1)\to-\infty$, $\lim \theta_n/(1-\rho_n)=\delta<\infty$ *exists, and for* $\kappa=2$: $n\rho_n^n \sin n\theta_n=O(1)$, *for* $\kappa>2$: $n^{\kappa-1}\rho_n^n(\theta_n+|\sin n\theta_n|)=O(1)$.

The value of σ is given in case (a$'''$) by (a), where for $\kappa=2$: $\beta=\lambda\pi$, for $\kappa>2$: $\beta=0$, $\sigma=0$; in case (b$'''$): $\sigma=0$; in case (c$'''$): $\sigma=\arctan\delta$.

4. **Application to Fourier series.** First consider a function of bounded variation and its Fourier sine series (1.1). It follows from the introduction that $\lim nb_n$, if it exists, is $(2/\pi)f(+0)$. Under the assumptions of Theorem 1 on ρ_n and θ_n, $s_n(\rho_n,\theta_n)\to\tau\sigma=(2\sigma/\pi)f(+0)$. In particular whenever $\sigma>\pi/2$, then we have an analogue of Gibbs' phenomenon. It is known that for functions of bounded variation

$$(1/n)\sum_{1}^{n} \nu b_\nu \to (2/\pi)f(+0);$$

more generally if (cf. Szász [3, Lemma 6])

(4.1) $$2f_1(\theta) = (2/\theta)\int_0^\theta f(t)dt \to j, \qquad \text{as } \theta \downarrow 0;$$

and

(4.2) $$\lim_{\delta\downarrow 0} \liminf_{n\to\infty} \min_{0<\kappa<\delta_n} \sum_n^{n+\kappa} b_\nu \geqq 0,$$

then

$$(1/n)\sum_1^n \nu b_\nu \to j/\pi.$$

Hence, applying Theorem 2 we have

$$s_n(\rho_n, \theta_n) \to (j/\pi)\sigma(\beta, \gamma), \qquad \text{as } n(\rho_n - 1) \to \gamma \text{ and } n\theta_n \to \beta;$$

j is the generalized jump of $f(\theta)$ at $\theta=0$. For $\gamma=0$ this yields a generalization of formula (1.4). Note that

$$f_1(\theta) = \theta^{-1}\int_0^\theta f(t)dt = \sum_1^\infty b_\nu \frac{1-\cos\nu\theta}{\nu\theta} = (\theta/2)\sum_1^\infty \nu b_\nu \left(\frac{\sin(\nu\theta/2)}{\nu\theta/2}\right)^2;$$

$(2\theta/\pi)\{s_0/2+\sum_1^\infty((\sin\nu\theta)/\nu\theta)^2 s_\nu\}$ is called the Riemannian mean of the second kind corresponding to the sequence $\{s_n\}$. It is a regular transform, as is seen from the identity

$$\frac{2\theta}{\pi}\left\{1/2 + \sum_1^\infty\left(\frac{\sin\nu\theta}{\nu\theta}\right)^2\right\} = 1.$$

If we assume only that $(C, 2)\lim nb_n = j/\pi$ exists, then Theorem 3 yields again a Gibbs' phenomenon in the case (a''') and $\lambda>0$.

In this connection we introduce two lemmas.

LEMMA 1. *If*

(4.3) $$(1-r)\sum_1^\infty \tau_n r^n \to \tau \qquad \text{as } r\uparrow 1,$$

and

(4.4) $$\tau_n' = \sum_1^n \tau_\nu > -pn,$$

for some $p>0$, and all $n>0$, then

(4.5) $$(C, 2)\lim \tau_n = \tau.$$

We have from (4.3)

$$(1-r)^2\sum_1^\infty \tau_n' r^n \to \tau \qquad \text{as } r \uparrow 1,$$

hence

$$(1-r)^2\sum_1^\infty (\tau_n' + pn)r^n \to \tau + p \qquad \text{as } r \uparrow 1;$$

in view of (4.4) a theorem of Hardy and Littlewood yields

$$\sum_1^n (\tau_\nu' + p\nu) \sim (1/2)(\tau + p)n^2,$$

or

$$\sum_1^n \tau_\nu' \sim (1/2)\tau n^2, \qquad \text{as } n \to \infty,$$

which is (4.5).

LEMMA 2. *If* (4.1) *holds, then* $(1-r)\sum_1^\infty nb_n r^n \to j/\pi$. [3, Lemma 5].

Combining these two lemmas it is seen that (4.1) and the assumption

$$\sum_1^n \nu b_\nu > -pn \qquad \text{for some } p > 0 \text{ and all } n > 0, \tag{4.6}$$

imply $(C, 2)$ lim $nb_n = j/\pi$. With reference to Theorem 3 the assumptions (4.1) and (4.6) again yield a Gibbs' phenomenon.

In closing we remark that the existence of $f(+0)$ implies itself $(C, 2)$ lim $nb_n = (1/2)f(+0)$. A more general result will be given elsewhere.

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