

# ON THE OSCILLATION OF DIFFERENTIAL TRANSFORMS. IV JACOBI POLYNOMIALS<sup>(1)</sup>

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1. **Introduction.** In a paper in the *Trans. Amer. Math. Soc.*<sup>(2)</sup>, E. Hille proved the following

THEOREM A. *Let  $\alpha \geq 0, \beta \geq 0, c \geq 0$ . The differential operation*

$$(1.1) \quad \vartheta - c = (1 - x^2)D^2 + [\beta - \alpha - (\alpha + \beta + 2)x]D - c, \quad D = d/dx,$$

*does not diminish the number of the sign changes in the interval  $-1 < x < +1$ .*

More exactly, let  $y = y(x)$  be a real-valued non-constant function of  $x$ ,  $-1 \leq x \leq +1$ , with a continuous second derivative (with one-sided derivatives at the end points  $\pm 1$ ). Then the number of the sign changes of  $Y = (\vartheta - c)y$  in  $-1, +1$  is not less than that of  $y$  in the same interval<sup>(3)</sup>.

First let us observe that under the conditions mentioned  $Y$  cannot vanish identically—this being true even for  $\alpha > -1, \beta > -1$ . More precisely, the solutions of the differential equation  $(\vartheta - c)y = 0$  which are not identically zero cannot have a continuous second derivative in the closed interval  $-1 \leq x \leq +1$ , provided  $c > 0$ ; in the case  $c = 0$  the solution  $y = \text{const.}$  is the only one of the kind mentioned<sup>(4)</sup>. Indeed, let us assume that  $c > 0$ , and let  $u(x)$  and  $v(x)$  be the solutions of the differential equation mentioned regular at  $x = +1$  and  $x = -1$ , respectively, and satisfying the condition  $u(+1) = v(-1) = 1$  [see (2.1)]. Then by means of the table in §2 below we conclude that  $u(x)$  and  $v(x)$  are linearly independent [ $u'(x) \rightarrow \infty, v'(x) = O(1)$  as  $x \rightarrow -1 + 0$  and  $u'(x) = O(1), v'(x) \rightarrow \infty$  as  $x \rightarrow +1 - 0$ ]. Moreover  $\{c_1 u(x) + c_2 v(x)\}' \rightarrow \infty$  either for  $x \rightarrow -1 + 0$  or for  $x \rightarrow +1 - 0$  (or in both cases) unless  $c_1 = c_2 = 0$ .

In the same paper E. Hille proved by means of Theorem A the special case  $c = 0$  of the following

THEOREM B. *Let  $\alpha \geq 0, \beta \geq 0, c \geq 0$  and let  $\vartheta$  have the same meaning as in*

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<sup>(1)</sup> See the previous papers of this series by G. Szegő, E. Hille and A. C. Schaeffer, in the *Trans. Amer. Math. Soc.* vols. 52, 53 (1942–1943). (Cf. below, loc. cit. footnotes 2 and 6.)

<sup>(2)</sup> E. Hille, *On the oscillation of differential transforms. II. Characteristic series of boundary value problems*, *Trans. Amer. Math. Soc.* vol. 52 (1942) pp. 463–497; see §2.8.

<sup>(3)</sup> Regarding the definition of the number of sign changes see, G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, 1925, p. 40.

<sup>(4)</sup> G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939; see p. 61, (4.2.6).

*Theorem A.* We denote by  $f(x)$  a real-valued function possessing derivatives of all orders in  $-1 \leq x \leq +1$ . If the number of the sign changes of the functions  $(\vartheta - c)^k f(x)$ ,  $k = 1, 2, 3, \dots$ , is bounded, say at most  $N$ , then  $f(x)$  is a polynomial of degree at most  $N$ .

The purpose of the present note is to prove

**THEOREM A'.** *Theorem A remains true under the more general condition  $\alpha > -1$ ,  $\beta > -1$ ,  $c \geq 0$ .*

**THEOREM B'.** *Let  $\alpha$  and  $\beta$  be arbitrary real,  $c \geq 0$ . If  $f(x)$  satisfies the conditions of Theorem B,  $f(x)$  must be a polynomial of degree at most  $N + \gamma$ . Here the constant  $\gamma = \gamma(\alpha, \beta, c)$  depends only on  $\alpha, \beta$  and  $c$ .*

Assuming  $\alpha > -1$ ,  $\beta > -1$ , Theorem B' (with  $\gamma = 0$ ) can be derived from Theorem A' in a manner used first by G. Pólya and N. Wiener in case of Fourier series<sup>(5)</sup> and applied later to numerous other instances by E. Hille (loc. cit.). We prefer however a direct proof of Theorem B' based on an idea which was used in the first paper of the present series<sup>(6)</sup>.

**2. Proof of Theorem A'.** First we assume  $c > 0$ . Let  $u(x)$  be the uniquely determined solution of  $(\vartheta - c)y = 0$  which is regular at  $x = +1$  and for which  $u(+1) = 1$  holds; we have as well known

$$(2.1) \quad u(x) = F(k, k'; l; (1-x)/2) \\ = \sum_{n=0}^{\infty} \frac{k(k+1) \cdots (k+n-1) k'(k'+1) \cdots (k'+n-1)}{l(l+1) \cdots (l+n-1) \cdot 1 \cdot 2 \cdots n} ((1-x)/2)^n$$

where  $k$  and  $k'$  are the roots of the quadratic equation  $k(-k + \alpha + \beta + 1) = c$  and  $l = \alpha + 1$ . Since  $(k + \nu)(k' + \nu) = \nu(\nu + \alpha + \beta + 1) + c > 0$ ,  $\nu = 0, 1, 2, \dots$ , we have  $u(x) > 0$  and  $u'(x) < 0$  in  $-1 < x \leq +1$ . Incidentally,  $k$  and  $k'$  are different from  $0, -1, -2, \dots$ ;  $l > 0$ .

Let us investigate the behavior of  $u(x)$  and  $u'(x)$  as  $x \rightarrow -1 + 0$ . Since

$$(2.2) \quad \frac{k(k+1) \cdots (k+n-1) k'(k'+1) \cdots (k'+n-1)}{l(l+1) \cdots (l+n-1) \cdot 1 \cdot 2 \cdots n} \cong \frac{\Gamma(l)}{\Gamma(k)\Gamma(k')} n^{k+k'-l-1} \\ = \frac{\Gamma(l)}{\Gamma(k)\Gamma(k')} n^{\beta-1}, \quad n \rightarrow \infty,$$

Cesàro's theorem<sup>(7)</sup> can be applied to  $u(x)$  provided  $\beta \geq 0$  and to  $u'(x)$  provided  $\beta > -1$ . We obtain the following table:

<sup>(5)</sup> G. Pólya and N. Wiener, *On the oscillation of the derivatives of a periodic function*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 249-256.

<sup>(6)</sup> G. Szegö, *On the oscillation of differential transforms. I*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 450-462.

<sup>(7)</sup> See, for instance, G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, 1925, p. 14, Problem 85.

(2.3)

	$u(x) \sim$	$-u'(x) \sim$
$\beta > 0$	$(1+x)^{-\beta}$	$(1+x)^{-\beta-1}$
$\beta = 0$	$-\log(1+x)$	$(1+x)^{-1}$
$-1 < \beta < 0$	1	$(1+x)^{-\beta-1}$

The symbol  $f(x) \sim g(x)$  means that  $f(x)/g(x)$  approaches a positive limit as  $x \rightarrow -1+0$ .

We also note the identity

(2.4) 
$$\begin{cases} Y = (\vartheta - c)y = (1-x)^{-\alpha}(1+x)^{-\beta}\{u(x)\}^{-1}u'(x), \\ t(x) = H(x)(y'u - yu'), \quad H(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}. \end{cases}$$

Now let  $y$  have  $N$  sign changes in  $-1 < x < +1$ ,  $N > 0$ ; then  $N$  abscissae  $\alpha_\nu$  exist,  $\alpha_0 = 1 > \alpha_1 > \alpha_2 > \dots > \alpha_N > \alpha_{N+1} = -1$ , such that  $y$  is alternately less than or equal to 0 and greater than or equal to 0 in the intervals  $\alpha_{\nu+1}, \alpha_\nu$ , without being identically zero in these intervals. We may assume that in an arbitrary small left-hand neighborhood of  $\alpha_\nu$  there are abscissae for which  $y \neq 0$ ,  $1 \leq \nu \leq N$ . (By this condition the  $\alpha_\nu$  are uniquely determined.) Obviously  $y(\alpha_\nu) = 0$ ,  $1 \leq \nu \leq N$ . Then by Rolle's theorem we conclude the existence of at least  $N-1$  zeros for  $u^2(y/u)' = y'u - yu'$  hence also for  $t(x)$  between  $\alpha_1$  and  $\alpha_N$  separating the abscissae  $\alpha_\nu$ ; in addition  $\lim t(x) = 0$  as  $x \rightarrow -1-0$ .

But  $t(x)$  must have also a zero in  $-1 < x < \alpha_N$ . Assume the contrary, for instance  $t(x) < 0$  or  $(y/u)' < 0$  in  $-1 < x < \alpha_N$ . Then  $y/u$  is decreasing in this interval and since  $y(\alpha_N) = 0$  we must have  $y > 0$  in  $-1 < x < \alpha_N$  and  $y > hu$  in  $-1 < x \leq \alpha_N - \epsilon$  [ $0 < \epsilon < \alpha_N + 1$ ,  $h = h(\epsilon) > 0$ ].

In case  $\beta \geq 0$  we conclude that  $y \rightarrow +\infty$  as  $x \rightarrow -1+0$  [see table (2.3)] which is a contradiction.

In case  $-1 < \beta < 0$  we obtain  $y > h'$  ( $h' > 0$ ) for  $-1 < x \leq \alpha_N - \epsilon$ . But in this case  $-u/u' \sim (1+x)^{\beta+1} \rightarrow 0$  as  $x \rightarrow -1+0$  so that

(2.5) 
$$\frac{t(x)}{-(1+x)^{\beta+1}yu'} = \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}(y'u - yu')}{-(1+x)^{\beta+1}yu'} \rightarrow 2^{\alpha+1},$$

hence  $t(x) > 0$  when  $x$  is sufficiently near  $-1$ . This is again a contradiction.

Recapitulating, we have found certain zeros  $\beta_0, \beta_1, \dots, \beta_N$  of  $t(x)$  satisfying the inequalities  $\beta_0 = 1 > \beta_1 > \dots > \beta_{N-1} > \beta_N > -1$  and  $\alpha_{\nu+1} < \beta_\nu < \alpha_\nu$ ,  $1 \leq \nu \leq N$ . Repeated application of Rolle's theorem furnishes at least  $N$  sign changes of  $Y$ . Note that  $t(x)$  cannot be identically 0 in  $\beta_{\nu+1}, \beta_\nu$  since this would imply  $y/u \equiv \text{const.}$ , hence  $y \equiv 0$  on account of  $y(\alpha_{\nu+1}) = 0$ . But  $y \neq 0$  at suitable points to the left from  $\alpha_{\nu+1}$ .

The remaining case  $c = 0$  can easily be settled. The identity (2.4) holds then with  $u(x) = 1$ , that is,  $t(x) = H(x)y'$ . In this case  $t(x)$  has at least  $N-1$  zeros in the interior of  $-1, +1$  and in addition the zeros  $x = \pm 1$ .



After these preliminaries we proceed to the proof of Theorem B'. First let us exclude the case  $\alpha + \beta = -l - 1$ ,  $l$  positive integer. We expand  $f^{(m)}(x)$  in a series of Jacobi polynomials  $P_n^{(\alpha+m, \beta+m)}(x)$ :

$$(3.6) \quad f^{(m)}(x) = \sum_{n=0}^{\infty} f_n P_n^{(\alpha+m, \beta+m)}(x).$$

Term-by-term integration and use of (3.2) furnishes

$$(3.7) \quad f(x) = \phi(x) + \sum_{n=0}^{\infty} 2^m \{ (n + \alpha + \beta + 2m)(n + \alpha + \beta + 2m - 1) \cdots (n + \alpha + \beta + m + 1) \}^{-1} f_n P_{n+m}^{(\alpha, \beta)}(x)$$

where  $\phi(x)$  is a polynomial of degree  $m - 1$  [for  $m = 0$  we have  $\phi(x) = 0$ ]. Since in this case  $P_n^{(\alpha, \beta)}(x)$  is of the precise degree  $n$  we can write

$$(3.8) \quad f(x) = \sum_{n=0}^{\infty} \phi_n P_n^{(\alpha, \beta)}(x).$$

Obviously

$$(3.9) \quad (\vartheta - c)^k f(x) = \sum_{n=0}^{\infty} (-1)^k [c + n(n + \alpha + \beta + 1)]^k \phi_n P_n^{(\alpha, \beta)}(x).$$

Now let  $k$  belong to a certain infinite sequence such that the corresponding functions  $(\vartheta - c)^k f(x)$  have a fixed number,  $M$  say, sign changes;  $M \leq N^{(8)}$ . We denote the abscissae at which these sign changes take place by  $x_1, x_2, \dots, x_M; x_\nu = x_\nu(k)$ . Then if  $\delta = +1$  or  $-1$  is properly chosen,

$$(3.10) \quad \delta \int_{-1}^{+1} (1 - x)^{\alpha+m} (1 + x)^{\beta+m} \{ (\vartheta - c)^k f(x) \} (x - x_1) \cdots (x - x_M) (1 \pm x^\rho) dx > 0.$$

Here  $\rho$  is an arbitrary non-negative integer and  $\delta$  does not depend on  $\rho$ . Substituting for  $(\vartheta - c)^k f(x)$  its expansion (3.9) the arising integrals will all vanish provided  $n > 2m + M + \rho$ . However for  $n = n' = 2m + M + \rho$  we obtain

$$\pm \delta (-1)^k [c + n'(n' + \alpha + \beta + 1)]^k \cdot \phi_{n'} \int_{-1}^{+1} (1 - x)^{\alpha+m} (1 + x)^{\beta+m} P_{n'}^{(\alpha, \beta)}(x) x^{M+\rho} dx,$$

and the last integral is different from 0 because of (3.5). Hence if  $\phi_{n'} \neq 0$  we find for  $k \rightarrow \infty$

$$[c + n'(n' + \alpha + \beta + 1)]^k = O(1) \max_{0 \leq \nu \leq n'-1} |c + \nu(\nu + \alpha + \beta + 1)|^k$$

(8) From here on we use the argument of the paper cited in footnote 6.

which is impossible provided

$$|c + n'(n' + \alpha + \beta + 1)| > \max_{0 \leq \nu \leq n'-1} |c + \nu(\nu + \alpha + \beta + 1)|.$$

This is the case if  $n' \geq n_0 = n_0(\alpha, \beta, c)$ .

The previous argument furnishes  $\phi_n = 0$  for  $n \geq 2m + M$ ,  $n \geq n_0$ , which is equivalent to the assertion of Theorem B'.

In case  $\alpha + \beta = -l - 1$ ,  $l$  positive integer, this proof needs a slight modification. We integrate then only the terms  $n \geq m + 1$  in (3.6) and conclude (3.7) with the modification that the summation is now extended over the range  $n \geq m + 1$  and  $\phi(x)$  is a polynomial of degree  $2m$ . [The expression in the braces of (3.7) is then positive since  $2m + \alpha + \beta + 2 > 0$ .] As a further addition to the previous argument we have to show that

$$(\vartheta - c)^k \phi(x) = O(1) |c + n'(n' + \alpha + \beta + 1)|^k, \quad k \rightarrow \infty,$$

uniformly for  $-1 \leq x \leq +1$  provided  $n'$  is sufficiently large,  $n' \geq n_1 = n_1(\alpha, \beta, c)$ . But  $(\vartheta - c)^k \phi(x)$  is a polynomial of degree  $2m$  and the last assertion follows if we can show that the coefficients of this polynomial have moduli at most  $RS^k$ ; here  $R > 0$  depends on  $f(x)$ ,  $\alpha$ ,  $\beta$ ,  $c$  and  $S > 0$  depends only on  $\alpha$ ,  $\beta$ ,  $c$ . Now

$$(3.11) \quad \begin{aligned} (\vartheta - c)x^h &= h(h - 1)(1 - x^2)x^{h-2} \\ &+ h[\beta - \alpha - (\alpha + \beta + 2)x]x^{h-1} - cx^h; \end{aligned}$$

hence with arbitrary constants  $\lambda_h$

$$(3.12) \quad (\vartheta - c) \sum_{h=0}^{2m} \lambda_h x^h \ll S \cdot \max |\lambda_h| \cdot \sum_{h=0}^{2m} x^h$$

where

$$(3.13) \quad S = 2 \cdot 2m(2m - 1) + 2m|\beta - \alpha| + 2m|\alpha + \beta + 2| + |c|.$$

This furnishes the statement by taking for  $R$  the maximum modulus of the coefficients of  $\phi(x)$  and choosing  $S$  according to (3.13).

Theorems B and B' remain of course true if the condition regarding  $(\vartheta - c)^k f(x)$  is satisfied only for an infinite number of values of  $k$ .

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