THE CHARACTERISTIC OF A QUADRATIC FORM FOR AN ARBITRARY FIELD

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1. **Introduction**. Ernst Witt [1](1) has shown that for a field K with characteristic not 2 each quadratic form Q can be transformed into a form $G+H$ where

$$G = \sum_{i=1}^{\sigma} (x_i^2 - y_i^2),$$

H is a nonzero form, that is, does not represent zero properly, the rank of Q is the sum of the ranks of G and H, and G has rank 2σ. The number σ is an invariant of Q under nonsingular linear transformations on Q. For the real field the number σ was defined by Loewy [2] as the minimum of the indices(2) of Q and $-Q$, and termed the *characteristic* of Q. Loewy showed that this characteristic could be defined in terms of exponents of elementary divisors of pencils $\{\rho F - Q\}$ formed from Q and real quadratic forms $\{F\}$. The definition of Loewy does not extend to an arbitrary field K, whereas the characteristic of Q for K is arrived at by Witt through the examination of a sequence of quadratic forms $Q, H_1, \cdots, H_\sigma$, where for each s the form Q is G_s+H_s for

$$G_s = \sum_{i=1}^{s} (L_i^2 - M_i^2),$$

the L's and M's being linearly independent linear forms, while the rank of Q is the sum of the ranks of G_s and H_s. In the present paper we shall show that the characteristic of Q can be defined in terms of linearly independent linear forms directly associated with Q. This definition is particularly convenient for treating sums of forms.

With the aid of the viewpoint developed here it is proved (§§3–4) that the characteristic of a quadratic form Q is related to the ranks of the principal minors of matrices associated with Q. By means of this relation we are able to treat the characteristics of sums of forms, and to show in particular (§5) that the characteristic of $Q+\lambda L^2$, L linear, differs at most by 1 from that of Q. This property is likewise possessed by the rank of Q, and, if K is real, also by the index of Q. The above result on the characteristic of $Q+\lambda L^2$ will be used in another paper [3] to show that the characteristic σ of a quadratic

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(1) The numbers in brackets refer to the bibliography at the end of the paper.

(2) The index of Q is the number h of + signs in a canonical form $x_1^2 + \cdots + x_h^2 - x_{h+1}^2 - \cdots - x_r^2$ to which Q is equivalent under a nonsingular linear transformation.

form Q determines the minimum value τ for which Q can be written as

$$\sum_{i=1}^{\tau} L_i M_i$$

where the L's and M's are linear forms. The maximum value which can be attained by the characteristic of Q relative to the rank r of Q is $[r/2]$, where $[r/2]$ designates the largest integer not exceeding $r/2$.

A sum

$$Q = \sum_{i=1}^{r} \lambda_i L_i^2,$$

where the L's are linear, and r is the rank of Q, is a *minimal representation* of Q.

2. **Preliminary definitions and conventions.** Throughout the present paper the usual restriction that the *characteristic* of the field K be different from 2 is made in order that each quadratic form Q may be written as

$$Q = \sum_{i,j=1}^{n} a_{ij} x_i x_j \tag{2.1}$$

where the matrix (a_{ij}) of coefficients is symmetric. We term (a_{ij}) the *matrix* A of Q.

In what follows we shall use the term "equivalent" to mean equivalent under nonsingular linear transformations.

We define the *characteristic* of a quadratic form Q to be the maximum number σ of linearly independent linear forms $L_1, \cdots, L_\sigma$ such that the rank of

$$Q + \lambda_1 L_1^2 + \cdots + \lambda_\sigma L_\sigma^2 \tag{2.2}$$

is identical with the rank of Q for all values of the λ's. In what follows the term "characteristic" will be understood to refer to the invariant just defined, until we have proved (§3) that this invariant is identical with the characteristic of Witt, and for the real field with that of Loewy.

If a form Q with rank r is written as a sum $G+H$ where G has rank 2σ and characteristic σ, and H has rank $r-2\sigma$ and characteristic 0, we term $G+H$ a *characteristic splitting* of Q. If we write Q as

$$\sum_{i=1}^{\sigma} L_i M_i + H, \tag{2.3}$$

where the component in the L's and M's is identical with the form G in a characteristic splitting of Q, we have a *decomposition* of Q.

If the rank of the form (2.2) is identical with the rank r of Q for all values of the λ's, the forms $L_1, \cdots, L_\sigma$ clearly depend only on the variables which occur in Q. A stronger statement can be made. We let

$$\mu_1 M_1^2 + \cdots + \mu_r M_r^2$$

be a minimal representation of Q. Since $M_1, \cdots, M_r$ may be taken as the variables in terms of which Q is expressed, it follows that the L's are linear forms in the M's, whence the characteristic of Q cannot exceed r.

3. **Characteristic splittings.** In the present section we shall relate the characteristic of a quadratic form Q to the results of Witt.

We recall that a matrix D of order c and rank d has *nullity* $c-d$.

LEMMA 3.1. *The characteristic of a quadratic form Q of rank r is the maximum σ for which Q is equivalent to a form F with rth order matrix C where complementary principal minors C_{11}, C_{22} of C have order and nullity σ, respectively.*

In considering matrices $\{C\}$ with complementary principal minors C_{11} and C_{22}, it will be no restriction if we take C_{11} to be a leading minor so that

$$C = \begin{Vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{Vmatrix}. \tag{3.1}$$

We write Q as in (2.1) with $n=r$. We suppose that Q has characteristic σ. We may assume without restriction that the rank of (2.2) with $L_i \equiv x_i$ for each i is identically equal to r for all values of the λ's. We assume that $\sigma \geqq 1$. We let M denote the minor of the matrix A of Q obtained from A by deleting the first σ rows and σ columns of A. Expanding the determinant of the form (2.2) we find that the rank of the form (2.2) is r for all values of the λ's if and only if each principal minor determinant of A containing M vanishes, except the determinant $|A|$ itself. The rank of M is a number b, where $b<r-\sigma$. We may suppose that M is in the shape

$$\begin{Vmatrix} D & 0 \\ 0 & 0 \end{Vmatrix} \tag{3.2}$$

where D is a nonsingular minor of order b. We suppose, for the moment, that $b \geqq r-2\sigma$, and consider the minor determinants of A of the type

$$\begin{vmatrix} * & * & E' \\ * & D & 0 \\ E & 0 & 0 \end{vmatrix}, \tag{3.3}$$

where E has order $r-\sigma-b$, and E' denotes the transpose of E. If $b>r-2\sigma$, the minors of type (3.3) are distinct from $|A|$. Since the vanishing (3.3) implies the singularity of E, the last $r-\sigma-b$ columns of A are linearly dependent, a contradiction. Thus $b \leqq r-2\sigma$. If $b<r-2\sigma$, the matrix A is singular. Thus $b=r-2\sigma$.

Conversely, if Q has the matrix C where C_{22} has rank $r-2\sigma$ and order $r-\sigma$, we adjoin elements from q rows and q columns of C to C_{22} to obtain a minor of C of order m, where $m=r-\sigma+q$, with rank at most r', where $r'=r-2\sigma+2q$. If $\sigma>0$, the relation $q<\sigma$ implies that $r'<m$, whence each square minor of C containing C_{22}, except C, is singular. The rank of (2.2) with $L_i\equiv x_i$ for each i is now r for all choices of the λ's, whence the characteristic of Q is at least σ.

THEOREM 3.1. *A quadratic form Q with rank r has characteristic σ if and only if Q has the characteristic splitting $G+H$, where G has rank 2σ and characteristic σ, while H has rank $r-2\sigma$ and characteristic* 0.

By a result of Witt, quoted in the introduction, the form Q is equivalent to a sum

$$\sum_{i=1}^{\rho} u_i v_i + H, \tag{3.4}$$

where H is a nonzero form with rank $r-2\rho$. The number ρ (by the theory of Witt) is uniquely determined by Q. Further, if Q is equivalent to a form (3.4) where H is a zero form with rank $r-2\rho$, the form Q is equivalent to a sum (3.4) with ρ replaced by a larger number ρ', where H now is a nonzero form with rank $r-2\rho'$.

We let σ denote the characteristic of Q. Since the rank of

$$Q+\lambda_1 u_1^2+\cdots+\lambda_\rho u_\rho^2$$

is r for all values of the λ's we have $\sigma\geqq\rho$.

By Lemma 3.1 the form Q is equivalent to a form Q' with the matrix C, given in (3.1), where the order of C_{11} and the nullity of C_{22} equal σ. In view of the nullity of C_{22} we may assume that C_{22} is written as (3.2) where the order of D is equal to $r-2\sigma$. We write Q' as (2.1) with $n=r$, whence $(a_{ij})=C$. The form Q' is now the sum $G'+H'$, where

$$G'=\sum_{i=1}^{\sigma} x_i L_i, \tag{3.5}$$

the L's are linear forms and H' is the form with matrix D. Since Q' has rank r, the variables $x_1,\cdots,x_\sigma$ in G', as well as $x_{\sigma+1},\cdots,x_{r-\sigma}$ in H', and $L_1,\cdots,L_\sigma$ comprise a set of linearly independent forms, whence these may be taken as the variables in terms of which Q' is expressed. It follows from the Witt theory that $\rho\geqq\sigma$, whence $\rho=\sigma$.

It is readily seen that the component in the x's and L's in (3.5) has characteristic σ, whereas H' has characteristic 0.

From Theorem 3.1 there are a number of immediate consequences valid for an arbitrary field K. The characteristic of a quadratic form Q of rank r does not exceed $[r/2]$. The *characteristic* of a quadratic form Q of rank r attains the *maximum value* $r/2$ if and only if Q is equivalent to the canonical form

$$\sum_{i=1}^{r/2} x_i y_i.$$

A quadratic form Q is a *nonzero form if and only if* the *characteristic* of Q is 0.

For the complex field the characteristic of a quadratic form Q is $[r/2]$. For the real field the characteristic is clearly the minimum of the indices of Q and $-Q$. The index and characteristic of Q may thus be distinct. In fact the concept of index of Q is identical with that of characteristic and the type $(\pm)$ of definiteness of a nonzero component H in a canonical splitting of Q.

From a result of Dickson [4] one can readily show that the quadratic forms

$$Q = \sum_{i=1}^{n} \alpha_i x_i^2, \qquad E = \sum_{i=1}^{n} e_i y_i^2,$$

with $\alpha_1 = e_1$ and rank n, are equivalent if and only if the subforms

$$\sum_{i=2}^{n} \alpha_i x_i^2, \qquad \sum_{i=2}^{n} e_i y_i^2$$

are equivalent. Witt [1] proved this same result by different methods, and showed that quadratic forms with canonical splittings $G+H$ and $G'+H'$, where H and H' are nonzero forms, are equivalent if and only if they have the same characteristic, and H is equivalent to H'. Thus H is uniquely determined up to equivalence. The study of the equivalence of quadratic forms thus reduces to that of nonzero forms, so extensively treated in the literature(3).

4. **Characteristics and principal minors.** To treat characteristics of sums of forms we shall need some properties of principal minors developed here.

LEMMA 4.1. *If the matrix of order r of a quadratic form Q of rank r has a principal minor of nullity σ, the characteristic of Q is at least σ.*

Without restriction on the generality of the method we may suppose that the matrix of order r of $Q(x_1, \cdots, x_r)$ is given by

$$(4.1) \qquad A = \begin{Vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & 0 \\ D_{31} & 0 & 0 \end{Vmatrix},$$

where the bottom right zero represents a minor of order σ, and D_{22} is nonsingular. We let t designate the number of rows of D_{13}. By Lemma 3.1 we may restrict ourselves to the case where $t > \sigma$. By a nonsingular linear transformation affecting only the variables $x_1, \cdots, x_t$, the form Q can be brought into a form Q' with matrix B, where the principal minor of B obtained by

(3) See, for example, [5].

striking out the first t rows and columns of B is identical with this minor for A, and D_{13} is replaced by a nonsingular σth order minor followed by rows of zeros. The matrix B is of the type C given in (3.1), where C_{11}, C_{22} have order and nullity σ, respectively, whence by Lemma 3.1 the characteristic of Q is at least σ.

In Lemma 3.1 we showed how the characteristic σ of a quadratic form Q is the maximum value σ for which certain minors C_{11}, C_{22} possess given properties. We shall show how a further examination of these minors reveals whether or not the maximum value σ is attained for them. In the following theorem the characteristic of C_{22} is understood to be the characteristic of the quadratic form associated with C_{22}.

THEOREM 4.1. *Suppose that the order and nullity of complementary principal minors* C_{11}, C_{22} *of the matrix of order* r *of a quadratic form* Q *of rank* r *equal* σ, *respectively. The characteristic of* Q *is* σ *if and only if the characteristic of* C_{22} *is* 0.

We write Q as in (2.1) with $n=r$. Since C_{22} has nullity σ there is a nonsingular matrix M such that $MC_{22}M'$ is the minor (3.2) where D has order $r-2\sigma$. It will thus be no restriction on the generality of the method to suppose that the matrix A of Q has the shape (4.1) with $D_{22}=D$, and the order of D_{11} equal to σ. We can thus write Q as a sum $G'+H'$, where G' is given by (3.5), H' is a form with the matrix D, and the rank of Q is the sum of the ranks of G' and H'. The characteristic of G' is clearly σ.

If Q has characteristic σ, the sum $G'+H'$ is a characteristic splitting of Q, whence H' has characteristic 0. It follows that C_{22} has characteristic 0.

If conversely, the characteristic of C_{22} is 0, the sum $G'+H'$ is again a canonical splitting, whence the characteristic of Q is σ.

We consider the canonical splitting $G+H$, where

$$G = \sum_{i=1}^{\sigma} x_i x_{r-\sigma+i},$$

and H is a form in $x_{\sigma+1}, \cdots, x_{r-\sigma}$. Since the matrix of $G+H$ is of the type (3.1) with the order of C_{11} and the nullity of C_{22} equal to σ, and the characteristic of C_{22} equal to 0, the characteristic of a quadratic form Q is σ *if and only if* Q is equivalent to a form with the matrix (3.1) where complementary principal minors C_{11} and C_{22} have the properties just mentioned.

5. **The characteristic of a sum of forms.** Each quadratic form Q is equivalent to a quadratic form with a diagonal matrix. This is the same as the property that each quadratic form Q has a minimal representation. It follows that the study of the effect of the addition of a quadratic form F to a quadratic form Q reduces to the study of the addition of a term λL^2, L linear, to Q.

THEOREM 5.1. *Under addition of a term* λL^2, L *linear, to a quadratic form* Q *the characteristic* σ *of* Q *changes at most by* 1.

We suppose that the characteristic of $Q+\lambda L^2$ is at least $\sigma+2$. We let q designate the rank of $Q+\lambda L^2$. By Lemma 3.1 the pair (Q, L) can be transformed nonsingularly into a pair (Q', M), where the matrix C of $Q'+\lambda M^2$ is of order q, and is of the shape (3.1), while C_{11} and C_{22} have order and nullity $\sigma+2$, respectively.

The rank of a quadratic form Q changes at most by 1 under addition of a term λL^2, L linear, to Q. We shall assume, to begin with, that $q=r$, where r is the rank of Q. The addition of $-\lambda M^2$ to $Q'+\lambda M^2$ changes the nullity of C_{22} by at most 1. Thus C_{22} goes into a minor D_{22} with nullity at least $\sigma+1$. By Lemma 4.1 the characteristic of Q' is at least $\sigma+1$, a contradiction. It follows that the characteristic t of $Q+\lambda L^2$ does not exceed $\sigma+1$. Thus

$$\sigma + 1 \geqq t \geqq \sigma - 1.$$

We now consider the case where $q=r+1$, and designate the variables in $Q'+\lambda M^2$ by $y_1, \cdots, y_{r+1}$ where the rank of

$$Q' + \lambda M^2 + \lambda_1 y_1^2 + \cdots + \lambda_{\sigma+2} y_{\sigma+2}^2$$

is $r+1$ for all values of $\lambda_1, \cdots, \lambda_{\sigma+2}$. By the development used in the proof of Lemma 3.1 the form $Q'+\lambda M^2$ has a matrix C of order $r+1$ as given in (3.1) where C_{11} and C_{22} are minors of order and nullity $\sigma+2$, respectively. If M is linearly independent of $y_1, \cdots, y_{\sigma+2}$, we may suppose that $M\equiv y_{\sigma+3}$. Removal of the row and column of C corresponding to $y_{\sigma+3}$ yields the matrix of Q' of order r. The minor obtained from C_{22} by removal of this row and column has nullity at least $\sigma+1$. By Lemma 4.1 the characteristic of Q' is at least $\sigma+1$, a contradiction. If, on the other hand, the form M is linearly dependent on $y_1, \cdots, y_{\sigma+2}$, the minor C_{22} of C is a minor of a matrix C^* of order $r+1$ of Q'. Since C_{22} has nullity $\sigma+2$, there is a nonsingular matrix N such that $N'C^*N=A$, where A is given in (4.1) the minors D_{13} and D_{31} being of order $\sigma+2$, while D_{22} is nonsingular. Since A is singular, D_{13} is singular. It follows that there is a nonsingular matrix M such that $M'AM$ is identical in shape with A except that the last column of D_{13} is replaced by a column of zeros, and a corresponding remark holds for the last row of D_{31}. We drop the last row and column of $M'AM$ to obtain a matrix B whose lower principal minor of order $r-\sigma-3$ has nullity $\sigma+1$. By Lemma 4.1 and the invariance of the characteristic under nonsingular linear transformations we have again arrived at a contradiction. Thus in any event, when $q=r+1$, the characteristic of $Q+\lambda L^2$ does not exceed $\sigma+1$.

Since the index of Q is σ there are linearly independent linear forms $L_1, \cdots, L_\sigma$ such that the rank of (2.2) is r for all values of the λ's. If $q=r+1$, the form L is linearly independent of the variables in Q, whence the rank of

$$Q + \lambda L^2 + \lambda_1 L_1^2 + \cdots + \lambda_\sigma L_\sigma^2$$

is $r+1$ for all values of $\lambda_1, \cdots, \lambda_\sigma$. Thus if the rank of $Q+\lambda L^2$ exceeds the rank of Q the characteristic of $Q+\lambda L^2$ is at least as great as that of Q.

The case where $q=r-1$ reverts to the preceding.

COROLLARY 5.1. *If ρ and R denote the characteristic and rank of the quadratic forms Q and F, respectively, the characteristic of $Q+F$ satisfies the inequalities:*

$$\rho - R \leqq \sigma \leqq \rho + R.$$

We shall show that Theorem 5.1 is valid if we replace the characteristic of Q by the index of Q for the real field. In the following theorem all coefficients are understood to be in the real field.

THEOREM 5.2 (Analogue of Theorem 5.1). *Under addition of a term λL^2, L linear, to a real quadratic form Q, the index of Q changes at most by* 1.

It is readily seen that Theorem 5.2 is true when the ranks of Q and $Q+\lambda L^2$ are distinct. We suppose, therefore, that these ranks are identical. We have

$$Q + \lambda L^2 \equiv M_1^2 + \cdots + M_q^2 - M_{q+1}^2 - \cdots - M_r^2,$$

where r is the rank of Q, while q is the index of $Q+\lambda L^2$, and the M's are linear forms. We suppose that $q>h+1$, where h is the index of Q. We may suppose that Q is written as

$$\sum_{i=1}^{h} x_i^2 - \sum_{i=h+1}^{r} x_i^2.$$

We set $x_1= \cdots =x_h=0$. Since $M_1, \cdots, M_q$ are linearly independent to begin with, at most h of these vanish. We may assume, therefore, that M_{h+1} and M_{h+2} are linearly independent of $x_1, \cdots, x_h$. We also set $L=M_{q+1}= \cdots =M_r=0$. We have

$$-\sum_{i=h+1}^{r} x_i^2 \equiv \sum_{i=h+1}^{q} M_i^2. \tag{5.1}$$

We have imposed at most $r-q+h+1$ independent conditions on the variables in Q. It follows that the left and right members in (5.1) do not vanish identically, whence we have a contradiction.

The theorems above and well known rank theory now imply that the rank, characteristic, and, if the field K is real, also the index of a quadratic form Q, have the common property that they change at most by 1 under addition of a term λL^2, L linear, to Q.

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