ON THE OSCILLATION OF DIFFERENTIAL TRANSFORMS. III
OSCILLATIONS OF THE DERIVATIVE OF A FUNCTION

BY
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1. It has been shown by S. Bernstein that if a function which is defined in an open interval is infinitely often differentiable (belongs to C^∞) and each of its derivatives is of constant sign then the function is analytic in the interval(1). As a possible generalization of this result Pólya raised the following question(2): *If in an open interval a function is infinitely often differentiable and no derivative changes sign more than a given number of times is the function analytic in the interval?* The present paper contains an answer to this question and to several related questions. These results are consequences of an inequality (Theorem I of the present paper) which relates the magnitude of the first derivative of a function to the number of times some higher derivative changes sign. All functions considered in this paper will be supposed real for real values of the variable.

In the case in which $f(x)$ is infinitely often differentiable over the interval $(-\infty, \infty)$ let N_k be the maximum number of sign changes of $f^{(k)}(x)$ over any interval of length α, where α is some arbitrary but fixed positive number. Pólya and Wiener(3) made the additional hypothesis that $f(x)$ is periodic and found that if $N_k=O(1)$ then $f(x)$ is a trigonometric polynomial, while if $N_k=o(k^{1/2})$ then $f(x)$ is an entire function. Szegö obtained a new proof of these results and indeed obtained the sharper theorem(4) that if $N_k<k/\log k$ for large k then $f(x)$ is an entire function. Hille(5) has obtained analogous theorems for a very general class of differential operators and functions which satisfy appropriate conditions.

Let us consider for the moment functions which are bounded and infinitely often differentiable over $(-\infty, \infty)$, but are otherwise completely general. It is shown in the present paper that if $N_k=O(1)$ then $f(x)$ is an entire function of exponential type, and if $N_k=O(\log^\gamma k)$, $\gamma<1$, then $f(x)$ is an entire function(6). In the case in which N_k is bounded this is a generalization of

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(1) S. Bernstein, *Leçons sur les propriétes extrémales* · · · , Paris, 1926, pp. 196–197.

(2) At the Stanford University Symposium, August 12, 1941.

(3) G. Pólya and N. Wiener, *On the oscillation of the derivatives of a periodic function*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 249–256.

(4) G. Szegö, *On the oscillation of differential transforms*. I, Ibid. pp. 450–462.

(5) E. Hille, *On the oscillation of differential transforms*. II. *Characteristic series of boundary value problems*, Ibid. pp. 463–497.

(6) A. C. Schaeffer, *Oscillations of the derivatives of a function*, Proc. Nat. Acad. Sci. U. S. A. vol. 28 (1942) pp. 62–63. The results are there stated without proof.

one of the theorems of Pólya and Wiener mentioned above, since an entire function of exponential type which is periodic is a trigonometric polynomial. In the case in which N_k can become infinite as k increases, the results of the present paper overlap those of Pólya and Wiener: we consider a more general class of functions, but, as indicated above, we allow only a much slower rate of growth of N_k.

2. The proofs are based on the following fundamental result.

THEOREM I. *In an interval* $a-L<x<a+L$ *let* $f(x)\in C^n$, $n\geqq 2$, *and let* $|f(x)|\leqq M$. *If*

$$|f'(a)| \geqq (10n)^{2n}M/L \tag{1}$$

then $f^{(n)}(x)$ *changes sign at least* $n-1$ *times in the interval.*

Several inequalities are known for the derivatives of functions which satisfy the conditions of Theorem I and have non-negative nth derivatives(7). We show by an example that there is no constant $A=A(n)$ such that the inequality $|f'(a)|\geqq AM/L$ will imply that $f^{(n)}(x)$ has n (or more) variations in sign. For, with $0<\alpha<1$, let

$$f'(x) = \begin{cases} (1/\alpha)(1 - x^2/\alpha^2)^n, & |x| \leqq \alpha, \\ 0, & |x| > \alpha. \end{cases}$$

Then if we set $f(0)=0$ it follows from simple calculations that, over the entire real axis, $f(x)\in C^n$, $|f(x)|\leqq 1$, and $f^{(n)}(x)$ has $n-1$ changes in sign. But $f'(0)=1/\alpha$, which is unbounded as α approaches zero.

The following lemma will be sufficient for our purpose, although a more precise formulation is known(8).

LEMMA I (S. Bernstein). *If in a closed interval of length* $2L$, $f(x)\in C^n$ *and* $|f(x)|\leqq 1$ *then there is at least one point in the interior of the interval for which*

$$|f^{(n)}(x)| \leqq (2/L)^n n!.$$

One proof, which is undoubtedly known to many, is as follows. Let the interval be $[1, 1]$ and let

$$g(x) = f(x) - \sum_{\nu=1}^{n+1} f(x_\nu)T_{n+1}(x)/((x - x_\nu)T'_{n+1}(x_\nu))$$

where $T_{n+1}(x) = \cos((n+1)\arccos x) = 2^n(x - x_1)\cdots(x - x_{n+1})$,

(7) E. Landau, *Über einen Satz von Herrn Esclangon*, Math. Ann. vol. 102 (1930) pp. 177–188. R. P. Boas, *Functions with positive derivatives*, Duke Math. J. vol. 8 (1941) pp. 163–172. R. P. Boas and G. Pólya, *Influence of the signs of the derivatives of a function on its analytic character*, Duke Math. J. vol. 9 (1942) pp. 406–424.

(8) S. Bernstein, loc. cit. p. 10. J. Shohat, *A simple proof of a formula of Tchebycheff*, Tôhoku Math. J. vol. 36 (1932–1933) pp. 230–235. R. P. Boas and G. Pólya, loc. cit. pp. 413–414. With more detail the proof in the text can be made to yield more information.

$x_\nu = \cos((\nu - 1/2)\pi/(n+1))$, is the $(n+1)$st Tchebycheff polynomial. Then $g(x)$ vanishes at $n+1$ or more points, so repeated use of Rolle's Theorem shows that its nth derivative has at least one zero, $g^{(n)}(\alpha) = 0$. Since $T_{n+1}(x)/(x - x_\nu)$ is a polynomial of degree n with leading coefficient 2^n, we obtain

$$0 = f^{(n)}(\alpha) - 2^n n! \sum_{\nu=1}^{n+1} f(x_\nu)/T'_{n+1}(x_\nu).$$

The summation is bounded by 1 since $|T'_{n+1}(x_\nu)| = (n+1)(1-x_\nu^2)^{-1/2} \geqq n+1$, and the lemma follows.

Proof of Theorem I. It is sufficient to consider the case in which $M=1$ and the interval is $(-1, 1)$. We suppose that $f'(0) \leqq -(10n)^{2n}$ and prove that $f^{(n)}(x)$ must change sign at least $n-1$ times in $(-1, 1)$. According to Lemma I, each closed interval of length $1/(5n)$ lying in $(-1, 1)$ must contain at least one point in its interior where

$$|f^{(k)}(x)| \leqq (20n)^k k! < (20n)^n n^n; \qquad k = 0, 1, 2, \cdots, n. \tag{2}$$

The point x in this inequality depends on k and on the interval.

Let

$$\phi(x) = f(x)(20n)^{-n} n^{-n}.$$

Then

$$\phi'(0) \leqq -10^{2n}20^{-n} = -5^n \tag{3}$$

and from inequality (2) it follows that each closed interval of length $1/(5n)$ contains at least one point in its interior where

$$|\phi^{(k)}(x)| < 1; \qquad k = 0, 1, 2, \cdots, n. \tag{4}$$

It is to be shown that for $k = 1, 2, \cdots, n$ there are points $x_0, x_1, \cdots, x_{k+1}$, depending on k, such that

$$\begin{gathered} -k/(5n) < x_0 < x_1 < \cdots < x_{k+1} < k/(5n), \\ (-1)^\nu \phi^{(k)}(x_\nu) > 3, \qquad \nu = 1, 2, \cdots, k, \\ |\phi^{(k)}(x_0)| < 1, \qquad |\phi^{(k)}(x_{k+1})| < 1. \end{gathered} \tag{5}$$

In the case $k=n$ these relations will imply that $f^{(n)}(x)$ changes sign at least $n-1$ times, so the theorem will follow. In the case $k=1$ these relations are a consequence of inequalities (3) and (4) where we have $x_1 = 0$. We now suppose that (5) is true for some k, $1 \leqq k \leqq n-1$, and proceed by induction.

Relations (5) imply that $0 < x_{\nu+1} - x_\nu < 2k/(5n) < 2/5$. The mean value theorem shows that there is a point y_1 such that

$$\phi^{(k)}(x_1) - \phi^{(k)}(x_0) = (x_1 - x_0)\phi^{(k+1)}(y_1), \qquad x_0 < y_1 < x_1.$$

The left-hand side is less than $-3+1 = -2$ so

$$\phi^{(k+1)}(y_1) < -2/(2/5) < -3.$$

Again using the mean value theorem we have (if $k>1$)

$$6 < \phi^{(k)}(x_2) - \phi^{(k)}(x_1) = (x_2 - x_1)\phi^{(k+1)}(y_2), \qquad x_1 < y_2 < x_2,$$

from which we conclude that $\phi^{(k+1)}(y_2) > 3$, and so on. Thus there are points

$$-k/(5n) < y_1 < y_2 < \cdots < y_{k+1} < k/(5n)$$

such that

$$(-1)^\nu \phi^{(k+1)}(y_\nu) > 3; \qquad \nu = 1, 2, \cdots, k+1.$$

Then each of the intervals $[y_1 - 1/(5n), y_1]$ and $[y_{k+1}, y_{k+1}+1/(5n)]$ contains at least one interior point where $|\phi^{(k+1)}| < 1$. We call these points y_0 and y_{k+2} respectively and (5) is true with k replaced by $k+1$, which completes the induction.

3. THEOREM II. *If $f(x) \in C^\infty$ in an open interval and there exists an integer p such that no derivative of $f(x)$ changes sign more than p times then the function is analytic in the interval.*

Proof. It is sufficient to show that the function is analytic in the interior of every closed interval in the given interval. (In this way we avoid considering the possibility of the function becoming infinite near the end points.) Also, by a simple transformation the problem is reduced to the case in which the interval is $(-1, 1)$ and $|f(x)| \leqq 1$, $-1 < x < 1$.

We show by induction that

$$|f^{(k)}(x)| \leqq a^k k!(1 - |x|)^{-k}, \qquad -1 < x < 1, \tag{6}$$

for $k = 0, 1, 2, \cdots$ where $a = 3\{10(p+2)\}^{2(p+2)}$. This inequality is true for $k=0$. If it is true for some $k \geqq 0$ and if $|x_0| < 1$, then in the interval $|x - x_0| < (1 - |x_0|)/(k+1)$ we have

$$(1 - |x|)^{-k} < (1 - |x_0|)^{-k}(1 - 1/(k+1))^{-k} < 3(1 - |x_0|)^{-k},$$

so, by (6),

$$|f^{(k)}(x)| < 3a^k k!(1 - |x_0|)^{-k}.$$

Theorem I with $L = (1 - |x_0|)/(k+1)$ and $n = p+2$ implies that

$$|f^{(k+1)}(x_0)| < \{10(p+2)\}^{2(p+2)} 3a^k k!(1 - |x_0|)^{-k}(k+1)/(1 - |x_0|)$$

or

$$|f^{(k+1)}(x_0)| < a^{k+1}(k+1)!(1 - |x_0|)^{-k-1}.$$

Thus (6) is true for all k; and Taylor's expansion of $f(x)$ about any point in the interior of $(-1, 1)$ converges to the function in an entire neighborhood of the point.

THEOREM III. *Let $f(x)\in C^\infty$ $(-\infty, \infty)$ and let the number of variations in sign of $f^{(k)}(x)$ in every interval of length α be less than β. If*

$$\overline{\lim_{x\to\pm\infty}}\ |x|^{-1}\log|f(x)| < \infty \tag{7}$$

then $f(x)$ is an entire function of exponential type. Here α and β are arbitrary but fixed positive numbers.

Proof. From (7) we conclude that there are constants A and B such that $|f(x)| < Ae^{B|x|}$ for all real x. There is an integer p such that no derivative of $f(x)$ changes sign more than p times in any interval of length 2.

We suppose that for some k

$$|f^{(k)}(x)| < A\lambda^k e^{B|x|}, \qquad -\infty < x < \infty, \tag{8}$$

where

$$\lambda = e^B\{10(p+2)\}^{2(p+2)}, \tag{9}$$

and use induction. Now in an interval of length 2 with center at x_0, inequality (8) shows that

$$|f^{(k)}(x)| < Ae^B\lambda^k e^{B|x_0|}.$$

Then Theorem I shows immediately that (8) is true for the next higher integer. Since (8) is true for $k=0$ it is true for all k.

Taylor's expansion of $f(x)$ about the origin then shows that it is an entire function of exponential type λ,

$$|f(z)| \leqq A\sum_0^\infty \lambda^k|z|^k/k! = Ae^{\lambda|z|},$$

which proves the theorem.

4. The constant λ determines the rate of growth of $f(z)$, and in relation (9) we have an explicit expression for λ in terms of the frequency of the variation in sign of $f^{(k)}(x)$. This expression is not the "best possible," but, at least in the case in which $f(z)$ is bounded on the real axis, a "best possible" inequality for the rate of growth of $f(z)$ can be obtained by use of function theory.

THEOREM IV. *Let $f(x)\in C^\infty$ $(-\infty, \infty)$ and let it be bounded. If no derivative of $f(x)$ changes sign more than p times in any interval of length π then $f(x)$ is an entire function of exponential type p,*

$$|f(z)| \leqq Me^{p|y|}.$$

We prove the theorem with p an integer, it will then follow for non-integer p. The proof will depend on two lemmas.

LEMMA II (Pólya-Szegö)(9). *Let $f(z)$ be an entire function which satisfies*

$$f(z) = O(e^{\alpha|z|}), \qquad |f(x)| \leqq M,$$

then

$$|f(z)| \leqq Me^{\alpha|y|}.$$

LEMMA III (S. Bernstein)(10). *Under the conditions of the previous lemma*

$$|f'(x)| \leqq M\alpha, \qquad -\infty < x < \infty,$$

and the equality holds only if

$$f(z) \equiv \sin(\alpha z + \beta).$$

Note that if $f(z)$ satisfies the conditions of Lemma II then its derivative is also an entire function of exponential type α. Hence if to $f(z)$ we apply Lemma III and then Lemma II we obtain $|f'(z)| \leqq M\alpha e^{\alpha|y|}$ which is apparently a stronger statement than Lemma III.

Proof of Theorem IV. A function which satisfies the conditions of Theorem IV is, according to Theorem III, an entire function of exponential type. Let it be of precise type α, that is let

$$\alpha = \overline{\lim_{r\to\infty}}\, r^{-1} \max_{\theta} \log |f(re^{i\theta})|.$$

Then $f(z) = O(e^{(\alpha+\epsilon)|z|})$ if $\epsilon > 0$, and so by Lemma II $|f(z)| \leqq Me^{(\alpha+\epsilon)|y|}$, where M is the upper bound of $f(x)$. Then letting ϵ approach zero, we find that $|f(z)| \leqq Me^{\alpha|y|}$. Let

$$M_k = \sup_{-\infty<x<\infty} |f^{(k)}(x)|$$

and

$$r = \overline{\lim_{k\to\infty}}\, M_{k+1}/M_k.$$

We first show that

$$r = \alpha. \tag{10}$$

Now $f^{(k)}(z)$ is an entire function of exponential type since $f(z)$ is, and $f^{(k)}(z)$ is bounded by M_k on the real axis. Then, from Lemma III we find that $M_{k+1} \leqq \alpha M_k$ and so $r \leqq \alpha$. On the other hand, if $r < \alpha$ there is a constant ρ

(9) G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, vol. 2, pp. 35–36 and 218–219 (Problems 201, 202). R. Duffin and A. C. Schaeffer, *Some properties of functions of exponential type*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 236–240, where a more precise formulation is given (for functions which are real on the real axis).

(10) S. Bernstein, loc. cit. pp. 97–102. G. Pólya and G. Szegö, loc. cit. vol. 2 pp. 35 and 218–219 (Problem 201). R. Duffin and A. C. Schaeffer, loc. cit. p. 239. In the discussion of the case of equality it is essential that the functions be real on the real axis.

such that $M_{k+1}/M_k<\rho<\alpha$, $k\geqq k_0$. This clearly implies that $M_k<A\rho^k$, $k=0, 1, 2, \cdots$, for some constant A. Then using Taylor's expansion of $f(z)$ about the origin, we have

$$|f(z)| \leqq A\sum_0^\infty \rho^k |z|^k/k! = Ae^{\rho|z|},$$

which implies that $f(z)$ is of exponential type $\rho<\alpha$. This proves (10).

Let $k_1<k_2<\cdots$ be a sequence of positive integers such that

$$M_{k_\nu+1}/M_{k_\nu} > \alpha(1 - 1/\nu).$$

Then at some point $f^{(k_\nu+1)}(x)$ will be near its upper bound. Let

$$|f^{(k_\nu+1)}(x_\nu)| > (1 - 1/\nu)M_{k_\nu+1},$$

and let

$$\phi_\nu(z) = f^{(k_\nu)}(x_\nu + z)/M_{k_\nu}.$$

Then $\phi_\nu(z)$ is an entire function of exponential type α and it is bounded by 1 on the real axis, so by Lemma II

$$|\phi_\nu(z)| \leqq e^{\alpha|y|}. \tag{11}$$

Also, according to the construction,

$$|\phi_\nu'(0)| = |f^{(k_\nu+1)}(x_\nu)|/M_{k_\nu} > (1 - 1/\nu)M_{k_\nu+1}/M_{k_\nu} > \alpha(1 - 1/\nu)^2.$$

Inequality (11) together with Vitali's convergence theorem implies that there is a subsequence of the functions $\phi_\nu(z)$ which converges to an entire function $\phi(z)$ with the properties that

$$|\phi(z)| \leqq e^{\alpha|y|}$$

and

$$|\phi'(0)| \geqq \alpha.$$

The case of equality in Lemma III then shows that

$$\phi(z) \equiv \pm \sin \alpha z.$$

If $\alpha>p$ then $\phi(x)$ would change sign at least $p+1$ times in an interval $(-\epsilon, \pi-\epsilon)$ where ϵ is some sufficiently small positive number, and then $f^{(k_\nu)}(x)$ would change sign at least $p+1$ times in an interval of length π if k_ν is large. This is impossible, so $\alpha\leqq p$ and the theorem follows.

5. In the following let N_k be such that $f^{(k)}(x)$ does not change sign more than N_k times in any interval of length α, where α is an arbitrary but fixed positive number.

THEOREM V. *Let* $f(x)\in C^\infty(-\infty, \infty)$ *and let* $N_k=O(\log^\gamma k)$, $\gamma<1$. *If*

$$\overline{\lim_{x\to\pm\infty}} \ |x|^{-1} \log |f(x)| < \infty$$

then $f(x)$ is an entire function of order not exceeding one.

It is sufficient to consider the case in which $\alpha=2$. There are constants M_0 and B such that

$$|f(x)| < M_0e^{B|x|}, \qquad -\infty < x < \infty.$$

If x_0 is any real number then $f(x)$ is bounded by $M_0e^Be^{B|x_0|}$ in the interval $|x-x_0|<1$. Then in virtue of the conditions of this theorem there is an integer k such that $N_k<k-2$, and so Theorem I gives a dominant for $f'(x)$ of the form $|f'(x)|<M_1e^{B|x|}$, $-\infty<x<\infty$. Repetition of this argument shows that for $k=0, 1, 2, \cdots$

$$(12) \qquad |f^{(k)}(x)| < M_ke^{B|x|}, \qquad -\infty < x < \infty,$$

and the object is to prove that the M_k can be chosen to increase at a sufficiently slow rate so that $f(x)$ is an entire function of order no larger than one.

If $\gamma<\mu<1$ then for large k,

$$N_k < \log^\mu k.$$

Also, if $\mu<\lambda<1$ and we define n by

$$(13) \qquad n = [\log^\lambda k]$$

then for large k,

$$N_{k+n} < \log^\mu (k + \log^\lambda k) < n - 2.$$

If k is large (12) shows that $f^{(k)}(x)$ is bounded by $M_ke^Be^{B|x_0|}$ in the interval $|x-x_0|<1$, so, applying Theorem I to the function $f^{(k)}(x)$ with n defined by (13), we have

$$|f^{(k+1)}(x_0)| < (10n)^{2n}M_ke^Be^{B|x_0|}.$$

Thus

$$M_{k+1} \leqq e^B(10n)^{2n}M_k,$$

and after some simplification this becomes

$$M_{k+1} \leqq k^\epsilon M_k, \qquad \epsilon > 0,$$

for large k. Finally,

$$M_k = O((k!)^\epsilon)$$

for every $\epsilon>0$, and this shows that $f(x)$ is an entire function of order no greater than one.

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