ON REFLEXIVE NORMS FOR THE DIRECT PRODUCT OF BANACH SPACES

BY

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Introduction. In a previous paper [7](1), for two Banach spaces E_1, E_2, the Banach spaces $E_1 \otimes E_2$, $E_1' \otimes E_2'$, $E_1'' \otimes E_2''$ [7, p. 205] are constructed. If the norm N [7, Definition 3.1] is defined on $E_1 \otimes E_2$, then the associate norm N' [7, Definition 3.2 and Lemma 3.1] is defined on $E_1' \otimes E_2'$. Similarly N'' denotes the norm on $E_1'' \otimes E_2''$.

Among the unsolved problems (mentioned in [7, §6]), are listed the following two:

A. What are the exact conditions imposed upon a crossnorm [7, Definition 3.3] under which $(E_1 \otimes E_2)' = E_1' \otimes E_2'$ holds?

B. Is the associate with every crossnorm also a crossnorm, or do there exist crossnorms whose associates are not crossnorms?

In the present paper we present a "partial" answer to problem A (which we denote by A*), and a "partial" answer to problem B (which we denote by B*).

A*. A uniformly convex crossnorm N sets up the relation $(E_1 \otimes E_2)' = E_1' \otimes E_2'$ if, and only if, $N'' = N$.

B*. For reflexive Banach spaces (that is, such that $E_1'' = E_1$, $E_2'' = E_2$) the associate with every crossnorm is also a crossnorm.

In this paper we also show that the values of a crossnorm for all expressions of rank not greater than 2 do not necessarily determine the crossnorm.

The following should be mentioned in immediate connection with problem A:

It is evident that for norms for which $(E_1 \otimes E_2)' = E_1' \otimes E_2'$ holds, $N'' = N$. Since in general (for any norm N) all we can state is $(E_1 \otimes E_2)' \supset E_1' \otimes E_2'$ [7, p. 205], we have no basis for assuming that N'' represents the norm in $(E_1 \otimes E_2)''$, or $N'' = N$ for expressions in $\mathfrak{A}(E_1, E_2) \subset \mathfrak{A}(E_1'', E_2'')$ [7, Definition 1.3]. Therefore, $N'' \leqq N$ [7, Lemma 3.2] is the best that can be stated in the general case.

In the present paper we present some results on reflexive norms, that is, such that $N'' = N$.

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(1) Numerals in brackets refer to the bibliography at the end of this paper. Throughout this paper we shall use the results and notation of [7], with a slight modification. Since the printer does not find it easy to handle double and triple bars over expressions by machines, we shall write E' instead of $\bar{E}$, E'' instead of $\bar{\bar{E}}$, E''' instead of $\bar{\bar{\bar{E}}}$. Similarly N' shall take the place of $\bar{N}$, and N'' that of $\bar{\bar{N}}$. Thus $\langle N'' \rangle'$ shall stand for $\overline{\langle \bar{\bar{N}} \rangle}$.

In §1 we prove that norms that are reflexive, minimal, or have an associate property are identical. In §2 we show that for reflexive Banach spaces the associate with every crossnorm is also a crossnorm, that is, there exists a least crossnorm. In §3 we present a method for construction of reflexive crossnorms and prove certain inequalities. An application of the results of this section to Hilbert spaces permits us to construct semi-self-associate crossnorms, that is, for every natural number k a crossnorm S_k is constructed such that $S_k \neq S$ and $S_k = (S_k)' = S$ for all expressions of rank not greater than k, where S denotes the self-associate crossnorm for Hilbert spaces constructed by F. J. Murray and J. v. Neumann [5]. This last result also proves that a crossnorm is not determined by the values which it assumes for all expressions of rank not greater than 2. Finally, in §4 we show that a uniformly convex crossnorm sets up the relation $(E_1 \otimes E_2)' = E_1' \otimes E_2'$ if, and only if, $N'' = N$.

1. In this section we shall assume that E_1, E_2 are Banach spaces, with no special restrictions.

We introduce the following additional notation:

a. If a norm N is defined on $\mathfrak{A}(E_1, E_2)$, then N'' is defined on $\mathfrak{A}(E_1'', E_2'') \supset \mathfrak{A}(E_1, E_2)$. By $\langle N'' \rangle$ we shall understand N'' considered only on $\mathfrak{A}(E_1, E_2)$. Similarly if N is defined on $\mathfrak{A}(E_1, E_2)$, N''' is defined on $\mathfrak{A}(E_1''', E_2''') \supset \mathfrak{A}(E_1', E_2')$, and $\langle N''' \rangle$ denotes N''' considered only on $\mathfrak{A}(E_1', E_2')$.

b. The set $\mathfrak{A}(E_1, E_2)$ in which there is defined a norm N, we shall denote by $\mathfrak{A}_N(E_1, E_2)$. Thus $\mathfrak{A}_{N'}(E_1', E_2')$ shall denote the set $\mathfrak{A}(E_1', E_2')$ in which there is defined the norm N' associate with N.

c. The symbol $\vdots\, \bar{f} \in \mathfrak{A}^*(E_1, E_2) \,\vdots\, \sup |\overline{F}(\bar{f})| / N(\bar{f})$ shall denote the least upper bound for all numbers $|\overline{F}(\bar{f})| / N(\bar{f})$, obtained when $\bar{f}$ varies over $\mathfrak{A}^*(E_1, E_2)$.

Lemma 1.1. *If N is a norm in $\mathfrak{A}(E_1, E_2)$, then $N' = \langle N'' \rangle' = N'''$ for $\overline{F}$ in $\mathfrak{A}^*(E_1', E_2') \subset \mathfrak{A}^*(E_1''', E_2''')$.*

Proof. Let $\overline{F} \in \mathfrak{A}^*(E_1', E_2')$. Then

$$\begin{aligned} N'(\overline{F}) &= \vdots\, \bar{f} \in \mathfrak{A}^*(E_1, E_2) \,\vdots\, \sup |\overline{F}(\bar{f})| / N(\bar{f}) \\ &\leqq \vdots\, \bar{f} \in \mathfrak{A}^*(E_1, E_2) \,\vdots\, \sup |\overline{F}(\bar{f})| / N''(\bar{f}) = \langle N'' \rangle'(\overline{F}) \\ &\leqq \vdots\, \bar{f} \in \mathfrak{A}^*(E_1'', E_2'') \,\vdots\, \sup |\overline{F}(\bar{f})| / N''(\bar{f}) = N'''(\overline{F}). \end{aligned}$$

On the other hand, $N'''(\overline{F}) \leqq N'(\overline{F})$ [7, Lemma 3.2]. This completes the proof.

Definition 1.1. *A norm N will be termed minimal if, for every norm N^0 for which $N' = (N^0)'$, we have $N^0 \geqq N$.*

Definition 1.2. *A norm N will be termed reflexive if $\langle N'' \rangle = N$.*

Definition 1.3. *A norm N in $\mathfrak{A}(E_1, E_2)$ will be said to have an "associate* *property" if, for a certain norm N^0 in $\mathfrak{A}(E_1', E_2')$, $\mathfrak{A}_N(E_1, E_2) \subset \mathfrak{A}_{(N^0)'}(E_1'', E_2'')$.*

THEOREM 1.1. *For a norm N, the following statements are equivalent:*
(a) *N is minimal,*
(b) *N is reflexive,*
(c) *N has an associate property.*

Proof. We shall prove (a)→(b)→(c)→(a). Let N be minimal. By Lemma 1.1, $\langle N''\rangle' = N'$ for $\overline{F} \in \mathfrak{A}^*(E_1', E_2')$ and any norm N in $\mathfrak{A}(E_1, E_2)$. Thus $\langle N''\rangle$ and N have the same associate. Since N is minimal by hypothesis, this means $\langle N''\rangle \geqq N$. On the other hand, for any norm N, $\langle N''\rangle \leqq N$ [7, Lemma 3.2]. Thus $\langle N''\rangle = N$. We have proved (a)→(b).

Suppose N is reflexive, that is, $\langle N''\rangle = N$. Then obviously N has an associate property, because

$$\mathfrak{A}_N(E_1, E_2) = \mathfrak{A}_{\langle N''\rangle}(E_1, E_2) \subset \mathfrak{A}_{N''}(E_1'', E_2'').$$

Therefore (b)→(c).

Suppose finally that N defined in $\mathfrak{A}(E_1, E_2)$ has an associate property, that is, for a certain norm N^0 defined in $\mathfrak{A}(E_1', E_2')$, $\mathfrak{A}_N(E_1, E_2) \subset \mathfrak{A}_{(N^0)'}(E_1'', E_2'')$. Let the norm N^{00} in $\mathfrak{A}(E_1, E_2)$ be such that $(N^{00})' = N'$. Then, for $\overline{F} \in \mathfrak{A}^*(E_1', E_2')$, we have

$$\begin{aligned}(N^{00})'(\overline{F}) = N'(\overline{F}) &= \vdots\, \bar{f} \in \mathfrak{A}^*(E_1, E_2) \,\vdots\, \sup \,|\overline{F}(\bar{f})|\,/N(\bar{f}) \\ &\leqq \vdots\, \bar{f} \in \mathfrak{A}^*(E_1'', E_2'') \,\vdots\, \sup \,|\overline{F}(\bar{f})|\,/(N^0)'(\bar{f}) \\ &= (N^0)''(\overline{F}) \leqq N^0(\overline{F}).\end{aligned}$$

Thus, $(N^{00})' \leqq N^0$ throughout $\mathfrak{A}(E_1', E_2')$, and therefore $(N^{00})'' \geqq (N^0)'$ throughout $\mathfrak{A}(E_1'', E_2'')$. Consequently, $N^{00} \geqq (N^{00})'' \geqq (N^0)' = N$, for $\bar{f}$ in $\mathfrak{A}^*(E_1, E_2) \subset \mathfrak{A}^*(E_1'', E_2'')$. Thus N is minimal. Therefore (c)→(a). This completes the proof.

COROLLARY 1. *If N and N^0 denote two reflexive norms, such that $N \leqq N^0$ and $N \neq N^0$, then $N' \geqq (N^0)'$ and $N' \neq (N^0)'$.*

Proof. Obviously $N' \geqq (N^0)'$. If it were $N' = (N^0)'$, then $N'' = (N^0)''$ and $\langle N''\rangle = \langle (N^0)''\rangle$. By hypothesis N and N^0 are reflexive. Thus Definition 1.2 gives $N = N^0$. This contradicts our assumption.

COROLLARY 2. *If N and N^0 denote two norms in $\mathfrak{A}(E_1, E_2)$, of which N is reflexive, and $N \leqq N^0$ throughout $\mathfrak{A}(E_1, E_2)$, then $N \leqq \langle (N^0)''\rangle$.*

Proof. By assumption, $N \leqq N^0$. Therefore $N' \geqq (N^0)'$ and $N'' \leqq (N^0)''$ [7, Lemma 3.3]. Thus $\langle N''\rangle \leqq \langle (N^0)''\rangle$. N is reflexive by assumption. Therefore $N = \langle N''\rangle$, and $N \leqq \langle (N^0)''\rangle$. This completes the proof.

Remark. Property (c) of Theorem 1.1 suggests the existence of an infinite number of different reflexive crossnorms. We shall prove later that this is the case, and present a method for construction of reflexive crossnorms(2).

2. Throughout the rest of this paper we shall assume $E_1=E_1''$, $E_2=E_2''$. In this case $\mathfrak{A}(E_1, E_2)=\mathfrak{A}(E_1'', E_2'')$, and for any norm N in $\mathfrak{A}(E_1, E_2)$, $\langle N''\rangle=N''$, $\langle N'''\rangle=N'''$. It should be noticed, however, that many results of the following sections are also valid for general Banach spaces.

Let $N_L(E_1, E_2)$, $N_G(E_1, E_2)$ denote the least crossnorm whose associate is also a crossnorm and the greatest crossnorm in $\mathfrak{A}(E_1, E_2)$, respectively; in the case where there is no fear of misunderstanding, we shall write simply N_L, N_G [7, Definitions 4.1, 4.2, Lemmas 4.1, 4.2 and Theorem 4.1].

LEMMA 2.1. *$N_L(E_1, E_2)=N_L(E_1'', E_2'')$ and $N_G(E_1, E_2)=N_G(E_1'', E_2'')$.*

Proof. This is a consequence of the definition of the least and greatest crossnorms [7, Definitions 4.1, 4.2].

LEMMA 2.2 *For any crossnorm N in $\mathfrak{A}(E_1, E_2)$ for which $N\geqq N_L(E_1, E_2)$, we have $N'\geqq N_L(E_1', E_2')$, $N''\geqq N_L(E_1'', E_2'')=N_L(E_1, E_2)$, $\cdots$.*

Proof. Let N denote a crossnorm whose associate is also a crossnorm, that is, $N\geqq N_L$ [7, Theorem 4.1]. Since $N''\leqq N$ [7, Lemma 3.2], $N''(f\otimes\varphi)\leqq N(f\otimes\varphi)=\|f\|\,\|\varphi\|$ for $f\in E_1$, $\varphi\in E_2$. But N'' is the associate of the crossnorm N'. Therefore $N''(f\otimes\varphi)\geqq\|f\|\,\|\varphi\|$ [7, Lemma 4.3]. Thus N'' is a crossnorm, or the associate with the crossnorm N' in $\mathfrak{A}(E_1', E_2')$ is a crossnorm. This gives $N'\geqq N_L(E_1', E_2')$ [7, Theorem 4.1]. From $N\geqq N_L(E_1, E_2)$ we conclude $N'\geqq N_L(E_1', E_2')$. Similarly from $N'\geqq N_L(E_1', E_2')$ we conclude $N''\geqq N_L(E_1'', E_2'')=N_L(E_1, E_2)$, by Lemma 2.1. This completes the proof.

LEMMA 2.3. *N_L is reflexive.*

Proof. Obviously $N_L\geqq N_L$. Therefore $(N_L)''\geqq N_L$, by virtue of Lemma 2.2. But $(N_L)''\leqq N_L$ [7, Lemma 3.2]. Therefore $(N_L)''=N_L$.

LEMMA 2.4. *The associate with the greatest crossnorm is the least crossnorm, that is, $(N_G(E_1, E_2))'=N_L(E_1', E_2')$.*

Proof. Since $N_G(E_1, E_2)\geqq N_L(E_1, E_2)$, $(N_G(E_1, E_2))'\geqq N_L(E_1', E_2')$ as follows from Lemma 2.2. But $(N_L(E_1', E_2'))'$ is a crossnorm [7, Theorem 4.1]. Thus $(N_L(E_1', E_2'))'\leqq N_G(E_1'', E_2'')=N_G(E_1, E_2)$ [7, Theorem 4.2], or $N_L(E_1', E_2')=(N_L(E_1', E_2'))''\geqq(N_G(E_1, E_2))'$, as follows from Lemma 2.3. Therefore $(N_G(E_1, E_2))'=N_L(E_1', E_2')$. This completes the proof.

COROLLARY. *N_L and N_G'' are associate with each other.*

Proof. The associate with N_G'' is $N_G'''=N_G'=N_L$, by Lemmas 2.4 and 1.1.

(2) The question of existence of non-reflexive crossnorms is not settled in this paper. It should be noticed that for every non-reflexive norm, $E_1'\otimes E_2'$ would be a proper subset of $(E_1\otimes E_2)'$. In particular it is not settled whether the greatest crossnorm is reflexive. It can be shown, however, that when E_1, E_2 are Hilbert spaces, then the greatest crossnorm is reflexive.

The associate with N_L is $N_L' = N_G''$, by Lemma 2.4.

LEMMA 2.5. *If, for a certain crossnorm* N, $N' \geqq N_L$, *then* $N \geqq N_L$.

Proof. Lemma 2.2 gives $N'' \geqq N_L$, therefore $N \geqq N'' \geqq N_L$. This completes the proof.

THEOREM 2.1. *Every crossnorm* $N \geqq N_L$.

Proof. Since N is a crossnorm, $N \leqq N_G$ [7, Lemma 4.2]. Therefore $N' \geqq N_G' = N_L$, as follows from Lemma 2.4. Thus Lemma 2.5 gives $N \geqq N_L$. This completes the proof.

THEOREM 2.2. *The associate with every crossnorm is also a crossnorm.*

Proof. This is a consequence of Theorem 2.1 and [7, Theorem 4.1].

THEOREM 2.3. *There exists a least crossnorm.*

Proof. It is obviously N_L, as follows from Theorem 2.1 and [7, Definition 4.2 and Lemma 4.4].

COROLLARY. *For every crossnorm* N *in* $\mathfrak{A}(E_1, E_2)$,

$$\left\| \sum_{i=1}^{n} F(f_i)\varphi_i \right\| \leqq \|F\| N\left(\sum_{i=1}^{n} f_i \otimes \varphi_i \right) \qquad \text{for } \overline{F} \in E_1.$$

Proof. This is a consequence of Theorem 2.1 and [7, Theorem 4.1.1].

The question of existence of a norm not less than N_L which is not a crossnorm, and whose associate is a crossnorm, remains open. It is clear that if such a norm exists, it must be non-reflexive.

3. DEFINITION 3.1. *Let* E_1, E_2 *denote two Banach spaces, and* N *a crossnorm in* $\mathfrak{A}(E_1', E_2')$. *We define a sequence of functions* $\{N_k\}$ *for expressions in* $\mathfrak{A}(E_1, E_2) = \mathfrak{A}(E_1'', E_2'')$ *in the following way*:

$$N_k\left(\sum_{i=1}^{n} f_i \otimes \varphi_i \right) = \sup \left| \left(\sum_{j=1}^{k} F_j \otimes \phi_j \right) \left(\sum_{i=1}^{n} f_i \otimes \varphi_i \right) \right| \Big/ N\left(\sum_{j=1}^{k} F_j \otimes \phi_j \right)$$

where sup, *that is, the least upper bound, is taken for all sequences of* k *terms* $F_1, \cdots, F_k$ *in* E_1' *and* $\phi_1, \cdots, \phi_k$ *in* E_2'.

THEOREM 3.1. *For every natural* k, N_k *is a crossnorm in* $\mathfrak{A}(E_1, E_2)$.

Proof. The proof is similar to that of [7, Theorem 7.2].

From the definition, $N_L = N_1 \leqq N_2 \leqq N_3 \leqq \cdots$ and $\lim_{k\to\infty} N_k = N'$.

THEOREM 3.2. *For every natural* k, *the crossnorm* N_k *is reflexive.*

Proof. Suppose that for a certain expression $\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0 \in \mathfrak{A}(E_1, E_2)$, $N_k''(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0) < N_k(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0) - \epsilon$, where ϵ is a certain positive number.

Then for a given expression $\sum_{j=1}^{m} F_j \otimes \phi_j$ in $\mathfrak{A}(E_1', E_2')$, there exists an expression $\sum_{i=1}^{n} f_i \otimes \varphi_i$, such that

$$\frac{\left|\left(\sum_{j=1}^{m} F_j \otimes \phi_j\right)\left(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0\right)\right| N_k\left(\sum_{i=1}^{n} f_i \otimes \varphi_i\right)}{\left|\left(\sum_{j=1}^{m} F_j \otimes \phi_j\right)\left(\sum_{i=1}^{n} f_i \otimes \varphi_i\right)\right|} < N_k\left(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0\right) - \epsilon.$$

This means

$$\frac{\left|\left(\sum_{j=1}^{m} F_j \otimes \phi_j\right)\left(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0\right)\right| \left|\left(\sum_{j=1}^{k} \hat{F}_j \otimes \hat{\phi}_j\right)\left(\sum_{i=1}^{n} f_i \otimes \varphi_i\right)\right|}{\left|\left(\sum_{j=1}^{m} F_j \otimes \phi_j\right)\left(\sum_{i=1}^{n} f_i \otimes \varphi_i\right)\right| N\left(\sum_{j=1}^{k} \hat{F}_j \otimes \hat{\phi}_j\right)} < N_k\left(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0\right) - \epsilon$$

for every sequence of k terms $\hat{F}_1, \cdots, \hat{F}_k$ in E_1'; $\hat{\phi}_1, \cdots, \hat{\phi}_k$ in E_2'. Taking originally for $\sum_{j=1}^{m} F_j \otimes \phi_j$ an expression of the form $\sum_{j=1}^{k} \hat{F}_j \otimes \hat{\phi}_j$ we get $|(\sum_{j=1}^{k} \hat{F}_j \otimes \hat{\phi}_j)(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0)| / N(\sum_{j=1}^{k} \hat{F}_j \otimes \hat{\phi}_j) < N_k(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0) - \epsilon$ for every sequence of k terms $\hat{F}_1, \cdots, \hat{F}_k$ in E_1', $\hat{\phi}_1, \cdots, \hat{\phi}_k$ in E_2'. Therefore $N_k(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0) \leqq N_k(\sum_{i=1}^{s} f_i^0 \otimes \varphi_i^0) - \epsilon$, as follows from Definition 3.1. The last inequality cannot hold. This completes the proof.

THEOREM 3.3. *If N is a reflexive crossnorm, then $(N_k)' = N$ for all expressions of rank not greater than k.*

Proof. Obviously $(N_k)' \leqq N$ for all expressions of rank not greater than k as a consequence of Definition 3.1 for N_k, and that of an associate with a given norm. On the other hand $N_k \leqq N'$ everywhere in $\mathfrak{A}(E_1, E_2)$. Therefore $(N_k)' \geqq N'' = N$. This completes the proof.

THEOREM 3.4. *If N is a reflexive crossnorm, then N_k is the least crossnorm whose associate equals N for all expressions of rank not greater than k.*

Proof. Theorem 3.3 gives $(N_k)' = N$ for all expressions of rank not greater than k. On the other hand it is easy to see that if a crossnorm N^0 satisfies the inequality $N^0(\sum_{i=1}^{n} f_i \otimes \varphi_i) < N_k(\sum_{i=1}^{n} f_i \otimes \varphi_i)$ for a certain expression $\sum_{i=1}^{n} f_i \otimes \varphi_i$, then there exists an expression $\sum_{j=1}^{k} F_j \otimes \phi_j$, such that $(N^0)'(\sum_{j=1}^{k} F_j \otimes \phi_j) > N(\sum_{j=1}^{k} F_j \otimes \phi_j)$. Thus $(N^0)' = N$, for all expressions of rank not greater than k, implies $N^0 \geqq N_k$. This completes the proof.

It is not difficult to see that the last theorem is a generalization of [7, Theorem 4.1].

In the case E_1 and E_2 are Hilbert spaces, we may assume $\mathfrak{A}(E_1, E_2) = \mathfrak{A}(E_1', E_2')$. In this case, N and N' are defined in $\mathfrak{A}(E_1, E_2)$. For the case of Hilbert spaces, F. J. Murray and J. v. Neumann define in $\mathfrak{A}(E_1, E_2)$ a self-associate crossnorm, which we shall denote by S [5].

THEOREM 3.5. *For every natural p, $S_p = (S_p)' = S$ for all expressions of rank not greater than p.*

Proof. Since $S = S'$, $S = S''$. Thus S is reflexive. Theorem 3.3 gives $(S_p)' = S$ for all expressions of rank not greater than p. Now suppose that for a certain expression $\sum_{i=1}^n f_i \otimes \varphi_i$ and a certain crossnorm $N \leqq S$, $N'(\sum_{i=1}^n f_i \otimes \varphi_i) = S(\sum_{i=1}^n f_i \otimes \varphi_i)$. Since $N \leqq S$, $N' \geqq S' = S$. But $S^2 \leqq NN'$ [7, p. 213]. Therefore $N(\sum_{i=1}^n f_i \otimes \varphi_i) = S(\sum_{i=1}^n f_i \otimes \varphi_i)$. An application of the last remark to the fact that $S_p \leqq S$ and $(S_p)' = S$, for all expressions of rank not greater than p, gives $S_p = S$ for all expressions of rank not greater than p. Thus $S_p = (S_p)' = S$ for all expressions of rank not greater than p. This completes the proof.

THEOREM 3.6. *For every natural p, $S_p \neq S_{p+1}$.*

Proof. That $S_2 \neq S_3$(3) is shown in the following manner: Consider an expression $f_1 \otimes \varphi_1 \cdot + \cdot f_2 \otimes \varphi_2 \cdot + \cdot f_3 \otimes \varphi_3$, where f_1, f_2, f_3, and $\varphi_1, \varphi_2, \varphi_3$ form orthonormal sets. A calculation shows that

$$S_2(f_1 \otimes \varphi_1 \cdot + \cdot f_2 \otimes \varphi_2 \cdot + \cdot f_3 \otimes \varphi_3) = \sup \frac{|(h_1 \otimes \chi_1 \cdot + \cdot h_2 \otimes \chi_2)(f_1 \otimes \varphi_1 \cdot + \cdot f_2 \otimes \varphi_2 \cdot + \cdot f_3 \otimes \varphi_3)}{S(h_1 \otimes \chi_1 \cdot + \cdot h_2 \otimes \chi_2)}$$

where sup, that is, the least upper bound, is taken for all pairs h_1, h_2, in $\mathfrak{M}_1$ (the closed linear manifold determined by f_1, f_2, f_3) and χ_1, χ_2, in $\mathfrak{M}_2$ (the closed linear manifold determined by $\varphi_1, \varphi_2, \varphi_3$). From this it is easily concluded that $S_2(f_1 \otimes \varphi_1 \cdot + \cdot f_2 \otimes \varphi_2 \cdot + \cdot f_3 \otimes \varphi_3) \leqq 2^{1/2}$ and

$$S_3(f_1 \otimes \varphi_1 \cdot + \cdot f_2 \otimes \varphi_2 \cdot + \cdot f_3 \otimes \varphi_3) = S(f_1 \otimes \varphi_1 \cdot + \cdot f_2 \otimes \varphi_2 \cdot + \cdot f_3 \otimes \varphi_3) = 3^{1/2}.$$

A similar reasoning can be applied to prove $S_p \neq S_{p+1}$ for any natural p.

COROLLARY. $S_p \neq S$ for $p = 1, 2, 3, \cdots$.

Proof. Obviously $S_p \leqq S_{p+1} \leqq S$. But $S_p \neq S_{p+1}$. Therefore $S_p \neq S$. This completes the proof(4).

(3) This result and proof is due to F. J. Murray.

(4) For a reflexive (or not) norm N, N_k is reflexive. There exist, however, reflexive norms, for instance S, such that for no reflexive norm N is $S = N_k$. Proof. For a reflexive N, Definition 3.1 and Theorem 3.3 give $((N_k)')_k = N_k$. If it were $S = N_k$ for a reflexive N, then $(N_k)' = S' = S$ and $S_k = S$ as a consequence of the preceding relation. We have shown above that this is not the case. This completes the proof.

Remark. From Theorems 3.5 and 3.6 it is evident that we have constructed three different reflexive crossnorms, which are equal for all expressions of rank not greater than p; namely S_p, $(S_p)'$, S.

S_p and $(S_p)'$ are reflexive, hence associate with each other.

Thus for every natural p we have constructed reflexive "semi-self-associate" crossnorms S_p and $(S_p)'$, that is not self-associate, but equal to their associates for all expressions of rank not greater than p.

Incidentally, this result also proves that the values of a crossnorm for all expressions of rank not greater than p (where p denotes any natural number) do not necessarily determine the crossnorm.

4. In the introduction of this paper, it was pointed out that if a norm sets up the relation $(E_1 \otimes E_2)' = E_1' \otimes E_2'$, then $N'' = N$ for expressions in $\mathfrak{A}(E_1, E_2) \subset \mathfrak{A}(E_1'', E_2'')$, or N is reflexive. In the present section we consider a "converse" problem.

Lemma 4.1. *If N is a reflexive crossnorm in $\mathfrak{A}(E_1, E_2)$, then $E_1 \otimes E_2 \subset E_1'' \otimes E_2''$.*

Proof. The linear set $\mathfrak{A}^*(E_1'', E_2'')$ in which there is defined the norm N'' is an extension of the linear set $\mathfrak{A}^*(E_1, E_2)$ in which there is defined the norm N. Thus the closure of $\mathfrak{A}^*(E_1'', E_2'')$ is an extension of the closure of $\mathfrak{A}^*(E_1, E_2)$. This completes the proof.

Lemma 4.2. *If N is a reflexive crossnorm in $\mathfrak{A}(E_1, E_2)$, then $E_1' \otimes E_2'$ forms a fundamental subset of $(E_1 \otimes E_2)'$, that is if, for all $\bar{F}$ in $E_1' \otimes E_2'$ and a certain $\bar{f}_0$ in $E_1 \otimes E_2$, $\bar{F}(\bar{f}_0) = 0$, then $\bar{f}_0 = 0$.*

Proof. For $\bar{F}_0 \in E_1' \otimes E_2'$, we have $N'(\bar{F}_0) = \sup |\bar{F}_0(\bar{f})|/N(\bar{f})$ where sup is taken over the set of all $\bar{f}$'s in $E_1 \otimes E_2$ [7, Lemma 3.4]. Similarly, for $\bar{F}_0^+ \in E_1'' \otimes E_2''$, $N''(\bar{F}_0^+) = \sup |\bar{F}_0^+(\bar{F})|/N'(\bar{F})$ where sup is taken over the set of all $\bar{F}$'s in $E_1' \otimes E_2'$. N is reflexive by assumption. Therefore for $\bar{f}_0 \in E_1 \otimes E_2$, Lemma 4.1 gives $N''(\bar{f}_0) = \sup |\bar{F}(\bar{f}_0)|/N'(\bar{F})$, where sup is taken over the set of all $\bar{F}$'s in $E_1' \otimes E_2'$. The second part of our assumption gives $N''(\bar{f}_0) = 0$. But $N'' = N$. Therefore $N(\bar{f}_0) = 0$. Thus $\bar{f}_0 = 0$. This completes the proof.

Lemma 4.3. *If a reflexive crossnorm sets up the relation $(E_1 \otimes E_2)'' = E_1 \otimes E_2$, then $(E_1 \otimes E_2)' = E_1' \otimes E_2'$.*

Proof. Suppose that $(E_1 \otimes E_2)'' = E_1 \otimes E_2$, and $\bar{F}^*$ is an element of $(E_1 \otimes E_2)'$ which does not belong to $E_1' \otimes E_2'$. From the construction of $E_1' \otimes E_2'$ follows that the set is closed in $(E_1 \otimes E_2)'$. Hence, there exists a linear functional $\mathcal{J}$ on $(E_1 \otimes E_2)'$, such that $\mathcal{J}(\bar{F}^*) = 1$ and $\mathcal{J}(\bar{F}) = 0$, for $\bar{F}$ in $E_1' \otimes E_2'$ [1, p. 57]. Since $(E_1 \otimes E_2)'' = E_1 \otimes E_2$, there exists an $\bar{f}_0$ in $E_1 \otimes E_2$, corresponding to $\mathcal{J}$, such that $\bar{F}^*(\bar{f}_0) = 1$ and $\bar{F}(\bar{f}_0) = 0$ for $\bar{F}$ in $E_1' \otimes E_2'$. The last condition implies $\bar{f}_0=0$ by Lemma 4.2. This contradicts $\bar{F}^*(\bar{f}_0)=1$. This completes the proof.

THEOREM 4.1. *If a reflexive crossnorm N defined on* $\mathfrak{A}(E_1, E_2)$ *is uniformly convex* [2], *then* $(E_1\otimes E_2)'=E_1'\otimes E_2'$.

Proof. By continuity it follows that N is uniformly convex in $E_1\otimes E_2$, hence [3, 4, 6] $(E_1\otimes E_2)''=E_1\otimes E_2$. An application of Lemma 4.3 completes the proof.

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