ON SOME TRIGONOMETRIC SUMMABILITY METHODS AND GIBBS' PHENOMENON

BY

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1. **Introduction.** We consider Fourier sine series

$$(1.1)\qquad f(\theta) \sim \sum_1^\infty b_\nu \sin \nu\theta, \qquad \sum_1^n b_\nu \sin \nu\theta = s_n(\theta).$$

Suppose that $f(+0)$ exists and is greater than 0, that is $f(\theta)$ has a simple jump $2f(+0)$ at $\theta=0$. We say that the series (1.1) presents Gibbs' phenomenon, if

$$(1.2)\qquad \limsup s_n(\theta_n) > f(+0), \text{ as } \theta_n \downarrow 0.$$

More generally, if only for some $k \geqq 0$

$$\lim_{\theta \downarrow 0} \frac{2(k+1)}{\theta^{k+1}} \int_0^\theta (\theta - t)^k f(t) dt = j > 0$$

exists, and $\limsup s_n(\theta_n) > j/2$, we say that the series (1.1) presents a generalized Gibbs' phenomenon at $\theta=0$; j is the generalized jump of $f(\theta)$ at $\theta=0$. Our aim is to find general conditions for $f(\theta)$ or its Fourier coefficients, which imply a Gibbs' phenomenon. It is known that the jump j is closely connected with the asymptotic behavior of the sequence $\{nb_n\}$; on the other hand

$$s_n(\theta_n) = \sum_1^n \nu b_\nu \frac{\sin \nu\theta_n}{\nu}$$

is a linear transform of $\{nb_n\}$, with the triangular matrix $a_{n\nu} = \nu^{-1} \sin \nu\theta_n$, $\nu = 1, 2, \cdots, n$. In a previous paper [3](1) we have discussed the relationship of the transform

$$T_n(\theta_n) = \sum_1^n \tau_\nu \frac{\sin \nu\theta_n}{\nu}$$

to the Cesàro means of the sequence $\{\tau_n\}$. As an application we have proved the presence of a Gibbs' phenomenon under general assumptions and we gave new formulae for the determination of the jump $2f(+0)$. More general results are given in the present paper; the knowledge of (3) is not assumed.

We consider trigonometric linear transforms, related to T_n. We discuss in particular the transforms

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(1) Numbers in brackets refer to the literature cited at the end of this paper.

$$B_n = \frac{1}{2}\{T_n(\theta_n) + T_n(\phi_n)\}, \qquad H_n = \frac{1}{n}\sum_{\kappa=1}^{n} T_\kappa(\theta_\kappa),$$
$$S_n = \frac{1}{\theta_n}\int_0^{\theta_n} T_n(t)dt.$$

In §5 we prove a theorem on $(C, 2)$ summability of the sequence $\{nb_n\}$ for Fourier series.

We shall make repeated use of the well known theorem:

THEOREM A. *The convergence of a sequence* $\{\tau_n\}$ *implies the convergence of the transform* $T_n = \sum_1^n a_{n\nu}\tau_\nu$, *if and only if*

$$\lim_{n\to\infty} a_{n\nu} = a_\nu \text{ exists for } \nu = 1, 2, 3, \cdots,$$

$$\sum_{\nu=1}^{n} |a_{n\nu}| = O(1), \quad \text{as} \quad n \to \infty,$$

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} a_{n\nu} = \sigma \text{ exists.}$$

We then have $\lim T_n = \sigma \lim \tau_n + \sum_1^\infty a_\nu(\tau_\nu - \lim \tau_n)$(2).

2. **The average** $(1/2)\{T_n(\theta_n) + T_n(\phi_n)\} = B_n$. On putting $\rho_n = 1$ in Theorem 2 of our paper [3] we get easily the following theorem.

THEOREM 1. *Suppose that*

$$\frac{1}{n}\sum_1^n \tau_\nu \to \tau, \quad \phi_n > 0, \quad \theta_n \geqq 0, \quad n\theta_n = O(1), \quad n\phi_n = O(1), \tag{2.1}$$

and that $\sigma(\alpha) = \int_0^\alpha (t^{-1}\sin t)dt$ *has the same value for all limit points* α *of the sequence* $\{n\theta_n\}$, *and the same value for all limit points* β *of the sequence* $\{n\phi_n\}$; *then*

$$\lim_{n\to\infty} B_n = (\tau/2)(\sigma(\alpha) + \sigma(\beta)). \tag{2.2}$$

For the special case $\phi_n = \pi/n$ see Rogosinski [1, 2].

We next assume $(C, 2) \lim \tau_n = \tau$. Let $\tau_n' = \sum_1^n \tau_\nu$, $\tau_n^2 = \sum_1^n \tau_\nu^1$, then our assumption is: $2\tau_n^2/n^2 \to \tau$. We write $\Delta^0\tau_n = \tau_n$, $\Delta\tau_n = \tau_n - \tau_{n+1}$, $\Delta^2\tau_n = \Delta(\Delta\tau_n)$; then

$$\begin{aligned} B_n = \sum_1^{n-2} \tau_\nu^2 \Delta^2 \frac{\sin \nu\theta_n + \sin \nu\phi_n}{\nu} &+ \tau_{n-1}^2 \left\{ \frac{\sin (n-1)\theta_n + \sin (n-1)\phi_n}{n-1} \right. \\ &\left. - 2\frac{\sin n\theta_n + \sin n\phi_n}{n} \right\} + \tau_n^2 \frac{\sin n\theta_n + \sin n\phi_n}{n}. \end{aligned} \tag{2.3}$$

(2) The statement at the beginning of §2 in [3] should be corrected accordingly; it does not affect the rest of the paper.

It follows from Theorem A that, if in addition to the assumptions of Theorem 1,

$$\sum_1^{n-2} \nu^2 \left| \Delta^2 \frac{\sin \nu\theta_n + \sin \nu\phi_n}{\nu} \right| + n^2 \left| \Delta \frac{\sin (n-1)\theta_n + \sin (n-1)\phi_n}{n-1} \right|$$
$$+ n \left| \sin n\theta_n + \sin n\phi_n \right| = O(1),$$

then again (2.2) holds.

We now assume $n(\phi_n - \theta_n) = (2\mu - 1)\pi + \delta_n$, $\delta_n = O(1/n)$, μ an integer $\geqq 1$. Then

$$\sin n\theta_n + \sin n\phi_n = 2 \sin \frac{n}{2}(\theta_n + \phi_n) \cos \frac{n}{2}(\theta_n - \phi_n) = O\left(\frac{1}{n}\right);$$

also

$$\cos \frac{n-1}{2}(\theta_n - \phi_n) = \cos \frac{n}{2}(\theta_n - \phi_n) \cos \frac{\theta_n - \phi_n}{2}$$
$$+ \sin \frac{n}{2}(\theta_n - \phi_n) \sin \frac{\theta_n - \phi_n}{2} = O\left(\frac{1}{n}\right).$$

Furthermore, for $\theta > 0$,

$$\Delta^2 \left(\frac{\sin \nu\theta}{\nu}\right) = \Delta^2 \int_0^\theta \cos \nu t dt = R \int_0^\theta \Delta^2 z^\nu dt, \qquad z = c^{it}.$$

Hence

$$\Delta^2 \frac{\sin \nu\theta}{\nu} = R \int_0^\theta z^\nu (1 - z)^2 dt,$$

and

$$\left| \Delta^2 \frac{\sin \nu\theta}{\nu} \right| < \int_0^\theta \left| 1 - z \right|^2 dt = \int_0^\theta \left| e^{-it/2} - c^{it/2} \right|^2 dt$$
(2.4)
$$= 4 \int_0^\theta \sin^2 \frac{t}{2} dt < \int_0^\theta t^2 dt = \frac{\theta^3}{3} \cdot$$

Thus in view of (2.1)

$$\sum_1^{n-2} \nu^2 \left| \Delta^2 \frac{\sin \nu\theta_n + \sin \nu\phi_n}{\nu} \right| < (\theta_n^3 + \phi_n^3) \sum_1^n \nu^2 = O(1).$$

We have thus proved the following theorem.

THEOREM 2. *Suppose that* $(C, 2)$ $\lim \tau_n = \tau$, $\phi_n > 0$, $\theta_n \geqq 0$, $n\theta_n = O(1)$, $n\phi_n = n\theta_n + (2\mu - 1)\pi + O(1/n)$, μ *an integer greater than or equal to* 1, *and that* $\sigma(\alpha) = \int_0^\alpha t^{-1} \sin t dt$ *has the same value for all limit points* α *of the sequence* $\{n\theta_n\}$. *Then* $\lim B_n = (\tau/2)\{\sigma(\alpha) + \sigma(\alpha + (2\mu - 1)\pi)\}$.

3. **The average** $(1/n) \sum_{\kappa=1}^n T_\kappa(\theta_\kappa) = H_n$. We have

$$(3.1)\qquad nH_n = \sum_{\kappa=1}^{n}\sum_{\nu=1}^{\kappa} \nu^{-1}\tau_\nu \sin \nu\theta_\kappa = \sum_{\nu=1}^{n} \nu^{-1}\tau_\nu \sum_{\kappa=\nu}^{n} \sin \nu\theta_\kappa = \sum_{\nu=1}^{n} C_{n\nu}\tau_\nu,$$

where

$$(3.2)\qquad C_{n\nu} = \nu^{-1}\sum_{\kappa=\nu}^{n} \sin \nu\theta_\kappa, \qquad \nu = 1, 2, \cdots, n.$$

Thus $nH_n = \sum_1^{n-1}(C_{n\nu} - C_{n,\nu+1})\tau_\nu' + C_{nn}\tau_n'$. We first prove the following theorem.

THEOREM 3. *If* $(C, 1) \lim \tau_n = \tau$, *and*

$$(3.3)\qquad \frac{1}{n}\sum_{1}^{n} |\nu\theta_\nu - \alpha| \to 0,$$

then

$$(3.4)\qquad H_n \to \tau \int_0^\alpha \frac{\sin t}{t}\, dt.$$

(3.3) is strong summability $(C, 1)$ of the sequence $\{n\theta_n\}$ to α; it implies that $\theta_n \to 0$, hence $\lim_{n\to\infty} n^{-1}C_{n\nu} = 0$ for each ν. Furthermore

$$\begin{aligned}
&\sum_{1}^{n-1} \nu |C_{n\nu} - C_{n,\nu+1}| + n |C_{nn}| \\
&\qquad = \sum_{1}^{n-1} \left| \sum_{\kappa=\nu}^{n} \sin \nu\theta_\kappa - \frac{\nu}{\nu+1}\sum_{\nu+1}^{n} \sin (\nu+1)\theta_\kappa \right| + |\sin n\theta_n| \\
&\qquad < \sum_{1}^{n-1} |\sin \nu\theta_\nu| + \sum_{1}^{n-1} \left| \sum_{\kappa=\nu+1}^{n} (\sin \nu\theta_\kappa - \sin (\nu+1)\theta_\kappa) \right| \\
&\qquad\quad + \sum_{1}^{n-1} \frac{1}{\nu+1} \left| \sum_{\nu+1}^{n} \sin (\nu+1)\theta_\kappa \right| + 1 \\
&\qquad < \sum_{1}^{n} \nu\theta_\nu + 2\sum_{1}^{n-1}\sum_{\nu+1}^{n} \left| \sin \frac{\theta_k}{2} \cos \frac{2\nu+1}{2}\theta_k \right| + \sum_{1}^{n-1}\sum_{\nu+1}^{n} \theta_k + 1 \\
&\qquad < 3\sum_{1}^{n} \nu\theta_\nu + 1 = O(n), \quad \text{as} \quad n \to \infty,
\end{aligned}$$

hence $H_n = O(1)$. Thus, in order that H_n has a limit whenever $n^{-1}\tau_n'$ tends to a limit τ, the additional condition is that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\nu=1}^{n} C_{n\nu} = \lim \frac{1}{n}\sum_{\nu=1}^{n} \nu^{-1}\sum_{\kappa=\nu}^{n} \sin \nu\theta_\kappa = \sigma$$

exists. We then have $H_n \to \sigma\tau$ (cf. Theorem A). Now

$$\frac{1}{n}\sum_{1}^{n} C_{n\nu} = \frac{1}{n}\sum_{\kappa=1}^{n} S_\kappa(\theta_k),$$

and

$$S_\kappa(\theta_\kappa) = \sum_{\lambda=1}^{\kappa} \lambda^{-1} \sin \lambda\theta_\kappa = \int_0^{\theta_\kappa}\left(\sum_{1}^{\kappa} \cos \lambda t\right) dt$$
$$= -\frac{\theta_\kappa}{2} + \int_0^{\theta_\kappa} \frac{\sin (\kappa + 1/2)t}{2 \sin (t/2)} dt.$$

Furthermore

$$\int_0^{\theta} \frac{\sin (\kappa + 1/2)t}{2 \sin (t/2)} dt = \int_0^{\theta} \frac{\sin (\kappa + 1/2)t}{t} dt$$
$$+ \int_0^{\theta} \sin (\kappa + 1/2)t\left\{\frac{1}{2 \sin (t/2)} - \frac{1}{t}\right\} dt,$$

and, from the mean value theorem,

$$(3.5) \qquad 0 < \frac{1}{2 \sin (t/2)} - \frac{1}{t} = \frac{t - 2 \sin (t/2)}{2t \sin (t/2)} < \frac{t}{12}, \qquad \theta < t < \pi.$$

Hence

$$\left|\int_0^{\theta} \sin (\kappa + 1/2)t\left\{\frac{1}{2 \sin (t/2)} - \frac{1}{t}\right\} dt\right| < \frac{1}{12}\int_0^{\theta} t dt = \frac{1}{24}\theta^2,$$

and

$$S_\kappa(\theta_\kappa) = -\frac{\theta_\kappa}{2} + \int_0^{(\kappa+1/2)\theta_\kappa} \frac{\sin t}{t} dt + \delta_\kappa \theta_\kappa^2$$
$$= \int_0^{\alpha} \frac{\sin t}{t} dt + \int_\alpha^{(\kappa+1/2)\theta_\kappa} \frac{\sin t}{t} dt - \frac{\theta_\kappa}{2} + \delta_\kappa \theta_\kappa^2,$$

where $|\delta_\kappa| < 1/24$. Thus, for $0 < \theta_\kappa < 1$,

$$\left|S_\kappa(\theta_\kappa) - \int_0^{\alpha} \frac{\sin t}{t} dt\right| < |(\kappa + 1/2)\theta_\kappa - \alpha| + \theta_\kappa < |\kappa\theta_\kappa - \alpha| + 2\theta_\kappa,$$

and from (3.3)

$$\left|\frac{1}{n}\sum_{1}^{n} S_\kappa(\theta_\kappa) - \int_0^{\alpha} \frac{\sin t}{t} dt\right| < \frac{1}{n}\sum_{1}^{n} |\kappa\theta_\kappa - \alpha| + \frac{2}{n}\sum_{1}^{n} \theta_\kappa \to 0;$$

which proves Theorem 3. For the case of $(C, 2)$ summability we prove the following theorem.

THEOREM 4. *If* $(C, 2) \lim \tau_n = \tau$,

$$\sum_1^n \nu^2 |\theta_\nu - \theta_{\nu+1}| = O(n), \quad \text{as} \quad n \to \infty, \tag{3.6}$$

and if (3.3) *holds, then* (3.4) *holds.*

Evidently (3.3) implies $\sum_1^n \nu\theta_\nu = O(n)$; furthermore

$$\sum_1^n \nu^2(\theta_\nu - \theta_{\nu+1}) = \sum_1^n (2\nu - 1)\theta_\nu - n^2\theta_{n+1} = O(n) - n^2\theta_{n+1}, \tag{3.7}$$

hence from (3.6)

$$n\theta_n = O(1) \quad \text{as} \quad n \to \infty. \tag{3.8}$$

We have from (3.1)

$$nH_n = \sum_1^{n-2} \tau_\nu''\Delta^2 C_{n\nu} + \tau_{n-1}''(C_{n,n-1} - 2C_{nn}) + \tau_n''C_{nn},$$

where $nC_{nn} = \sin n\theta_n = O(1)$, and

$$n(C_{n,n-1} - 2C_{nn}) = O(1) + (n/(n-1))\{\sin (n-1)\theta_{n-1} + \sin (n-1)\theta_n\} = O(1).$$

Furthermore, using (2.4) and (3.8),

$$\begin{aligned}
&\sum_1^{n-2} \nu^2 |\Delta^2 C_{n\nu}| \\
&= \sum_1^{n-2} \nu^2 \left| \frac{1}{\nu} \sum_{\kappa=\nu}^n \sin \nu\theta_\kappa - \frac{2}{\nu+1} \sum_{\kappa=\nu+1}^n \sin (\nu+1)\theta_\kappa + \frac{1}{\nu+2} \sum_{\kappa=\nu+2}^n \sin (\nu+2)\theta_\kappa \right| \\
&\leqq \sum_1^{n-2} \nu^2 \left| \frac{1}{\nu} (\sin \nu\theta_\nu + \sin \nu\theta_{\nu+1}) - \frac{2}{\nu+1} \sin (\nu+1)\theta_{\nu+1} \right| \\
&\qquad + \sum_1^{n-2} \nu^2 \left| \sum_{\kappa=\nu+2}^n \left(\frac{\sin \nu\theta_\kappa}{\nu} - \frac{2 \sin (\nu+1)\theta_\kappa}{\nu+1} + \frac{\sin (\nu+2)\theta_\kappa}{\nu+2} \right) \right| \\
&< O(n) + \sum_1^{n-1} \nu |\sin \nu\theta_\nu + \sin \nu\theta_{\nu+1} - 2 \sin (\nu+1)\theta_{\nu+1}| \\
&\qquad + \sum_1^{n-2} \nu^2 \sum_{\kappa=\nu+2}^n |\Delta^2 \nu^{-1} \sin \nu\theta_\kappa| \\
&< O(n) + \sum_1^{n-1} \nu(\theta_{\nu+1} + \theta_{\nu+1} + \nu |\theta_\nu - \theta_{\nu+1}|) + \sum_1^n \nu^2 \left(\sum_{\kappa=\nu}^n \theta_\kappa^3 \right) \\
&= O(n) + O\left(\sum_1^n \nu^3\theta_\nu^3 \right) = O(n).
\end{aligned}$$

Now (3.4) follows as in Theorem 3. This proves Theorem 4. In this connection the following lemma is of interest:

LEMMA. *If for some* $c>0$

$$0 \leqq \theta_{n+1} \leqq (1 + c/n)\theta_n, \qquad n = 1, 2, 3, \cdots, \tag{3.9}$$

and

$$\sum_1^n \nu\theta_\nu = O(n), \qquad \text{as } n \to \infty, \tag{3.10}$$

then (3.6) *and* (3.8) *hold.*

We write $\sum_1^n \nu^2|\theta_\nu - \theta_{\nu+1}| = \sum' + \sum''$, where $\sum'$ contains all terms with $\theta_\nu \leqq \theta_{\nu+1}$, and $\sum''$ the rest. Now, using (3.9) and (3.10),

$$\sum{}' \leqq c\sum_1^n \nu\theta_\nu = O(n),$$

and from (3.7)

$$\sum{}'' - \sum{}' < \sum_1^n (2\nu - 1)\theta_\nu = O(n),$$

hence $\sum'' = O(n)$ and (3.6) is proved. This and (3.10) yield (3.8). In particular in Theorem 4 assumption (3.6) can be replaced by $\theta_n \downarrow$.

4. **The integral mean** $(1/\theta_n)\int_0^{\theta_n}(\sum_1^n \tau_\nu \nu^{-1}\sin \nu t)dt$. To a given sequence $\{\tau_n\}$ we consider the transform

$$S_n = \frac{1}{\theta_n}\int_0^{\theta_n}\left(\sum_1^n \tau_\nu \sin \nu t\right)dt = \sum_1^n \frac{1 - \cos \nu\theta_n}{\nu^2\theta_n}\tau_\nu = \sum_1^n a_{n\nu}\tau_\nu, \ \theta_n \to 0; \tag{4.1}$$

so that now

$$a_{n\nu} = (1 - \cos \nu\theta_n)/\nu^2\theta_n, \qquad \nu = 1, 2, \cdots, n. \tag{4.2}$$

Evidently $0 < a_{n\nu} \to 0$ as $n \to \infty$; hence a necessary and sufficient condition that for every convergent sequence $\tau_n \to \tau$ the sequence S_n has a limit $\sigma\tau$ is that $\lim \sum_1^n A_{n\nu} = \sigma$ exists (from Theorem A). Now

$$\begin{aligned}\sum_1^n \frac{1 - \cos \nu\theta_n}{\nu^2} &= \int_0^{\theta_n}\left(\sum_1^n \frac{\sin \nu t}{\nu}\right)dt \\ &= \int_0^{\theta_n}\left\{-\frac{t}{2} + \int_0^t \frac{\sin (n + 1/2)^u}{2 \sin (u/2)}d\kappa\right\}dt \\ &= -\frac{\theta_n^2}{4} + \int_0^{\theta_n}(\theta_n - t)\frac{\sin (n + 1/2)t}{2 \sin (t/2)}dt,\end{aligned}$$

thus

$$\sum_1^n a_{n\nu} = -\frac{\theta_n^2}{4} + \theta_n^{-1}\int_0^{\theta_n}(\theta_n - t)\frac{\sin (n+1/2)t}{t}dt$$

$$+ \theta_n^{-1}\int_0^{\theta_n}(\theta_n - t)\left\{\frac{1}{2\sin(t/2)} - \frac{1}{t}\right\}\sin (n+1/2)t dt,$$

and using (3.5)

$$\theta_n^{-1}\left|\int_0^{\theta_n}(\theta_n - t)\left\{\frac{1}{2\sin(t/2)} - \frac{1}{t}\right\}\sin (2n+1)\frac{t}{2}dt\right|$$

$$< \theta_n^{-1}\int_0^{\theta_n}(\theta_n - t)\frac{tdt}{12} < \frac{1}{24}\theta_n^2 \to 0.$$

Hence

$$\sum_1^n a_{n\nu} = o(1) + \theta_n^{-1}\int_0^{(n+1/2)\theta_n}\left(\theta_n - \frac{t}{n+1/2}\right)\frac{\sin t}{t}dt$$

$$= o(1) + \int_0^{(n+1/2)\theta_n}\frac{\sin t}{t}dt - \frac{2}{(2n+1)\theta_n}[1 - \cos (n+1/2)\theta_n].$$

If $\beta \leqq \infty$ is a limit point of the sequence $\{n\theta_n\}$, then for a subsequence of integers n

$$\sum_1^n a_{n\nu} \to \int_0^{\beta}\frac{\sin t}{t}dt - \frac{1-\cos\beta}{\beta} \equiv g(\beta).$$

Now $g'(\beta) = (1-\cos\beta)/\beta^2 \geqq 0$, hence $g(\beta) \uparrow$, as β increases, to $g(\infty) = \pi/2$. We have thus proved:

THEOREM 5. *In order that for every convergent sequence $\tau_n \to \tau$ the transform $S_n = \sum_1^n \tau_\nu(1-\cos \nu\theta_n)/\nu^2\theta_n$ has a limit, it is necessary and sufficient that* $\lim_{n\to\infty}\theta_n = \beta$, $\beta \leqq \infty$, *exists. We then have $S_n \to \tau g(\beta)$, where $g(\beta) = \int_0^\beta (t^{-1}\sin t)dt - (1-\cos\beta)/\beta \leqq \pi/2$.*

We now assume only $n^{-1}\sum_1^n \tau_\nu \to \tau$, or $n^{-1}\tau_n' \to \tau$. We have $S_n = \sum_1^{n-1}(a_{n\nu} - a_{n,\nu+1})\tau_\nu' + a_{nn}\tau_n'$; and the additional condition for the existence of lim S_n is by Theorem A:

$$\sum_1^{n-1}\nu\left|\frac{1-\cos\nu\theta_n}{\nu^2\theta_n} - \frac{1-\cos(\nu+1)\theta_n}{(\nu+1)^2\theta_n}\right| + \frac{1-\cos n\theta_n}{n\theta_n} = O(1),$$

or

$$(4.3)\qquad \theta_n^{-1}\sum_1^{n-1}\nu^{-3}\left|(\nu+1)^2(1-\cos\nu\theta_n) - \nu^2(1-\cos(\nu+1)\theta_n)\right| = O(1).$$

But

$$\theta_n^{-1} \sum_1^n \nu^{-3}(2\nu + 1)(1 - \cos \nu\theta_n) < 3 \sum_1^n \frac{1 - \cos \nu\theta_n}{\nu^2\theta_n} = 3 \sum_1^n a_{n\nu} = O(1),$$

hence (4.3) reduces to $\theta_n^{-1}\sum_1^{n-1}\nu^{-1}|\cos \nu\theta_n - \cos (\nu+1)\theta_n| = O(1)$, as $n\to\infty$, or $\sum_1^n \nu^{-1}|\sin (\nu+1/2)\theta_n| = O(1)$.

But this condition is equivalent to $n\theta_n = O(1)$ (cf. [3, §2]). Thus we have the following:

THEOREM 6. *The transform* $S_n = \sum_1^n \tau_\nu(1 - \cos \nu\theta_n)/\nu^2\theta_n)$ *has a limit for every sequence* $\{\tau_n\}$ *which is summable* $(C, 1)$ *to* τ, *if and only if* $\lim n\theta_n = \beta < \infty$ *exists. We then have* $S_n \to \tau g(\beta)$.

We finally assume $(C, 2) \lim \tau_n = \tau$; using the formula

$$\sum_1^n a_\nu \tau_\nu = \sum_1^{n-2} \overset{2}{\tau}_\nu \Delta^2 a_\nu + \overset{2}{\tau}_{n-1}(a_{n-1} - 2a_n) + \overset{2}{\tau}_n a_n,$$

we now get for the existence of $\lim S_n$ the additional conditions

$$(4.4)\quad \theta_n^{-1} \sum_1^{n-2} \overset{2}{\nu} \left| \frac{1 - \cos \nu\theta_n}{\nu^2} - 2\,\frac{1 - \cos (\nu + 1)\theta_n}{(\nu + 1)^2} + \frac{1 - \cos (\nu + 2)\theta_n}{(\nu + 2)^2} \right| = O(1),$$

$$(4.5)\quad \theta_n^{-1} \overset{2}{n} \left| \frac{1 - \cos (n - 1)\theta_n}{(n - 1)^2} - 2\,\frac{1 - \cos n\theta_n}{n^2} \right| + \frac{1 - \cos n\theta_n}{\theta_n} = O(1).$$

In particular $\sin^2 n\theta_n/2 = O(\theta_n) = o(1)$, hence $n\theta_n \to \beta = 2\lambda\pi$, λ an integer. We assume $\lambda > 0$; on putting $n\theta_n = 2\lambda\pi + 2\epsilon_n$, $\epsilon_n \to 0$, we must have

$$\sin^2 (\lambda\pi + \epsilon_n) = \sin^2 \epsilon_n = O((\lambda\pi + \epsilon_n)/n) = O(1/n),$$

or $\epsilon_n = O(1/n^{1/2})$. To satisfy (4.5) the additional condition is $\sin^2 (n-1)\theta_n = O(\theta_n)$, or $(\sin n\theta_n \cos \theta_n - \cos n\theta_n \sin \theta_n)^2 = O(1/n)$, which reduces to our previous condition. We finally show that now (4.4) is also satisfied. We have

$$\frac{1 - \cos \nu\theta}{\nu^2} = \int_0^\theta dt \int_0^t \cos \nu u du = \int_0^\theta (\theta - t) \cos \nu t dt,$$

hence

$$\Delta^2 \frac{1 - \cos \nu\theta}{\nu^2} = \int_0^\theta (\theta - t)\Delta^2 \cos \nu t dt = R \int_0^\theta (\theta - t) z^\nu (1 - z)^2 dt, \quad z = e^{it},$$

and

$$\left| \Delta^2 \frac{1 - \cos \nu\theta}{\nu^2} \right| < \int_0^\theta (\theta - t) \mid 1 - e^{it} \mid^2 dt < \int_0^\theta (\theta - t) t^2 dt = \frac{\theta^4}{12} \cdot$$

Using this inequality, the left side of (4.4) is less than $\sum_1^{n-2}\nu^2\theta_n^3 < n^3\theta_n^3 = O(1)$. This proves the following theorem.

THEOREM 7. *The transform* S_n *has a limit* $\sigma\tau$ (*with* $\sigma\neq 0$) *for every sequence* $\{\tau_n\}$ *which is summable* $(C, 2)$ *to* τ *if and only if* $n\theta_n = 2(\lambda\pi + \epsilon_n)$, $\epsilon_n = O(n^{-1/2})$, λ *an integer greater than* 0. *We then have* $S_n \to \tau\int_0^{2\lambda\pi} t^{-1}\sin t\,dt = \tau\sigma(2\lambda\pi)$.

5. $(C, 2)$ **summability of** $\{nb_n\}$ **for sine series.** We shall prove the following theorem.

THEOREM 8. *If* $f(\theta) \sim \sum_1^\infty b_\nu \sin \nu\theta$, *and for some* j

$$\int_0^\theta |f(t) - j/2|\,dt = O(\theta),\quad \int_0^\theta \{f(t) - j/2\}\,dt = o(\theta),\ \text{as } \theta\downarrow 0, \tag{5.1}$$

then

$$(C, 2)\lim nb_n = j/\pi. \tag{5.2}$$

We have, for $\tau_n = nb_n$,

$$\tau_n^2 = \sum_1^n (n-\nu+1)\nu b_\nu = \frac{2}{\pi}\int_0^\pi f(t)\left(\sum_1^n (n-\nu+1)\nu\sin\nu t\right)dt,$$

and $\sum_1^n (n-\nu+1)\nu\sin\nu t = -(d/dt)\sum_1^n(n-\nu+1)\cos\nu t$. On putting

$$\sum_1^n \cos\nu t = -\frac{1}{2} + \frac{\sin(n+1/2)t}{2\sin(t/2)} \equiv \sigma_n(t),$$

we have

$$\begin{aligned}\sum_1^n (n-\nu+1)\cos\nu t = \sum_1^n \sigma_\nu(t) &= -\frac{n}{2} + \frac{1}{2\sin(t/2)}\sum_1^n \sin(\nu+1/2)t\\ &= -\frac{n}{2} + \frac{1}{2\sin(t/2)}\{\sin^2(n+1)(t/2) - \sin^2(t/2)\}\\ &= -\frac{n+1}{2} + \frac{1}{2}\cdot\frac{\sin^2(n+1)(t/2)}{\sin^2(t/2)},\end{aligned}$$

hence

$$\begin{aligned}&\sum_1^n (n-\nu+1)\nu\sin\nu t\\ &= \frac{1}{2}\,\frac{\sin(n+1)(t/2)}{\sin^3(t/2)}\left\{\cos\frac{t}{2}\sin(n+1)\frac{t}{2} - (n+1)\cos(n+1)\frac{t}{2}\sin\frac{t}{2}\right\}.\end{aligned} \tag{5.3}$$

Thus

$$(5.4)\qquad \tau_n^2=\frac{1}{\pi}\int_0^\pi f(t)\,\frac{\sin\,(n+1)(t/2)}{\sin^3\,(t/2)}\left\{\sin\,n\,\frac{t}{2}-n\cos\,(n+1)\,\frac{t}{2}\,\sin\,\frac{t}{2}\right\}dt;$$

furthermore

$$(5.5)\qquad \begin{aligned}&\int_0^\pi\left(\sum_1^n (n-\nu+1)\nu\sin\nu t\right)dt\\ &\qquad=\sum_1^n (n-\nu+1)\{1-(-1)^\nu\}=2\sum_{\lambda=0}^{(n-1)/2}(n-2\lambda)\\ &\qquad=4\sum_{\lambda=0}^{(n-1)/2}\left(\frac{n}{2}-\lambda\right)=h_n,\end{aligned}$$

say, where $2hn/n^2\rightarrow 1$ as $n\rightarrow\infty$. Our aim is to prove

$$(5.6)\qquad \tau_n^2/h_n\rightarrow j/\pi,\qquad\text{or}\qquad (\tau_n^2-jh_n/\pi)/h_n\rightarrow 0.$$

From (5.3), (5.4) and (5.5)

$$\tau_n^2-\frac{1}{\pi}jh_n=\frac{1}{\pi}\int_0^\pi\{f(t)-j/2\}\,\frac{\sin\,(n+1)(t/2)}{\sin^3\,(t/2)}\,K_n(t)dt,$$

where

$$(5.7)\qquad \begin{aligned}K_n(t)&=\sin\,(nt/2)-n\cos\,((n+1)t/2)\sin\,(t/2)\\ &=n\sin\,(t/2)(1-\cos\,((n+1)t/2))-(n\sin\,(t/2)-\sin\,(nt/2)).\end{aligned}$$

Now from (3.5)

$$\begin{aligned}0<\frac{1}{\sin^3\,(t/2)}-\frac{1}{(t/2)^3}&=\left(\frac{1}{\sin\,(t/2)}-\frac{2}{t}\right)\left(\frac{1}{\sin^2\,(t/2)}+\frac{2}{t\sin\,(t/2)}+\frac{4}{t^2}\right)\\ &<\frac{t}{6}\cdot\frac{3}{\sin^2\,(t/2)}<\frac{\pi}{2\sin\,(t/2)},\end{aligned}$$

and

$$|K_n(t)|<n\sin\,(t/2)+|\sin\,n(t/2)|.$$

Hence

$$\begin{aligned}&\left|\int_0^\pi\{f(t)-j/2\}\left(\frac{1}{\sin^3\,(t/2)}-\frac{1}{(t/2)^3}\right)\sin\,(n+1)\,\frac{t}{2}\,K_n(t)dt\right|\\ &\qquad<\frac{\pi}{2}\int_0^\pi|f(t)-j/2|\left(n+\frac{|\sin\,(nt/2)|}{\sin\,(t/2)}\right)dt=O(n)=o(h_n);\end{aligned}$$

thus (5.6) will follow if we prove

$$\int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3} K_n(t)dt = o(n^2).$$

In view of (5.7) this will follow from

$$(5.8) \quad \int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3} \sin\frac{t}{2}\left(1 - \cos(n+1)\frac{t}{2}\right) dt = o(n),$$

and

$$(5.9) \quad \int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3}\left(n \sin\frac{t}{2} - \sin n\frac{t}{2}\right) dt = o(n^2).$$

Again, using (3.5),

$$\left|\int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3}\left(\frac{t}{2} - \sin\frac{t}{2}\right)\left(1 - \cos(n+1)\frac{t}{2}\right) dt\right|$$
$$< \int_0^\pi |f(t) - j/2|\, dt = O(1),$$

thus (5.8) is equivalent to

$$\int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^2}\left(1 - \cos(n+1)\frac{t}{2}\right) dt = o(n),$$

or

$$I_n \equiv \int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2) - 2^{-1}\sin(n+1)t}{t^2} dt = o(n).$$

We write $I_n = \int_0^{c/n} + \int_{c/n}^\pi = L_1 + L_2$, say, where c is a constant, arbitrarily large. On putting $\int_0^\theta \{f(t) - j/2\}dt = F(\theta)$, and using integration by parts, we get from (5.1)

$$L_1 = \frac{F(t)}{t} \frac{\sin(n+1)(t/2) - 2^{-1}\sin(n+1)t}{t}\Bigg]_0^{c/n}$$
$$- (n+1)\int_0^{c/n} F(t) \frac{\cos((n+1)t/2) - \cos(n+1)t}{2t^2} dt$$
$$+ 2\int_0^{c/n} F(t) \frac{\sin((n+1)t/2) - 2^{-1}\sin(n+1)t}{t^3} dt$$
$$= o(n) + o(n)\cdot \int_0^{c/n} t^{-1} |\sin(3(n+1)t/4)\sin((n+1)t/4)|\, dt$$
$$+ o(1)\int_0^{c/n} t^{-2}\sin^2((n+1)t/4)dt = o(n),$$

as $n\to\infty$. Furthermore $|L_2| < \int_{c/n}^{\pi}|f(t)-j/2|\,2\sin^2((n+1)t/4)/t^2dt$; we use here Fejér's inequality, $\sin^2 x < (2x/(1+x))^2$ for $x>0$. Then

$$|L_2| < \frac{(n+1)^2}{2}\int_{c/n}^{\pi}|f(t)-j/2|\frac{dt}{(1+((n+1)t/4))^2}$$
$$= \frac{(n+1)^2}{2}\frac{\phi(t)}{(1+((n+1)t/4))^2}\Bigg]_{c/n}^{\pi}$$
$$+\frac{(n+1)^2}{2}\int_{c/n}^{\pi}\frac{(n+1)\phi(t)dt}{4(1+((n+1)t/4))^3},$$

where $\phi(\theta)=\int_0^\theta|f(t)-j/2|dt$. But from (5.1), $\phi(\theta)<\gamma\theta$, γ an absolute constant, hence

$$|L_2| < O(1)+\frac{(n+1)^3}{8}\gamma\int_{c/n}^{\pi}\frac{tdt}{(1+((n+1)t/4))^3}$$
$$< O(1)+\frac{(n+1)^2}{2}\gamma\int_{c/n}^{\pi}\frac{dt}{(1+((n+1)t/4))^2} < \frac{2\gamma(n+1)}{1+c/4}+O(1),$$

and $\limsup_{n\to\infty} n^{-1}L_2 \leqq 8\gamma/(4+c)$. Thus $\limsup_{n\to\infty} n^{-1}|I_n| \leqq 8\gamma/(4+c)$; but c being arbitrary, now (5.8) follows.

Similarly we decompose the left side of (5.9) into $\int_0^{c/n}+\int_{c/n}^{\pi}=M_1+M_2$ say, where

$$M_1=\int_0^{c/n}\{f(t)-j/2\}t^{-3}\left[\frac{n}{2}\left(\cos n\frac{t}{2}-\cos(n+2)\frac{t}{2}\right)\right.$$
$$\left.+\frac{1}{2}\left(\cos(2n+1)\frac{t}{2}-\cos\frac{t}{2}\right)\right]dt$$
$$\equiv\int_0^{c/n}\{f(t)-j/2\}t^{-3}Q_n(t)dt$$
$$=F(t)t^{-3}Q_n(t)\Bigg]_0^{c/n}-\int_0^{c/n}F(t)t^{-3}Q_n'(t)dt$$
$$+3\int_0^{c/n}t^{-4}Q_n(t)F(t)dt.$$

The mean value theorem yields

$$\frac{Q_n(t)}{n^4t^4}\sim\frac{Q_n'(t)}{4n^4t^3}\sim\frac{Q_n''(t)}{12n^4t^2}\sim\frac{Q'''(t)}{24n^4t}\sim\frac{1}{96}$$

for $t\to 0$, hence, using (5.1), we find that $M_1=o(n^2)$, as $n\to\infty$.

To estimate M_2 we use the formula

$$n \sin x - \sin nx = \sin x \sum_{\nu=1}^{n} \{1 - \cos (n - 2\nu + 1)x\} > 0, \qquad 0 < x < \pi,$$

which follows from

$$\frac{\sin nx}{\sin x} = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} = \sum_{1}^{n} e^{i(n-2\nu+1)x} = \sum_{1}^{n} \cos (n - 2\nu + 1)x.$$

Now

$$| M_2 | < \int_{c/n}^{\pi} | f(t) - j/2 | t^{-3} \sin \frac{t}{2} \sum_{1}^{n} \left\{1 - \cos (n - 2\nu + 1) \frac{t}{2}\right\} dt$$

$$< \int_{c/n}^{\pi} | f(t) - j/2 | t^{-2} \sum_{2\nu \leqq n} \left\{1 - \cos (n - 2\nu + 1) \frac{t}{2}\right\} dt = \sum_{2\nu \leqq n} D_\nu$$

say, and, as in the estimate of L_2

$$D_\nu < \phi(\pi) \frac{(n - 2\nu + 1)^2}{2(1 + (n - 2\nu + 1)\pi/4)^2} + \frac{2\gamma(n - 2\nu + 1)}{1 + c/4}$$

$$< \phi(\pi) + \frac{8\gamma(n - 2\nu + 1)}{4 + c}.$$

Hence $|M_2| < n\phi(\pi) + 4\gamma n^2/(4+c)$, and $\limsup_{n\to\infty} |M_2|/n^2 \leqq 4\gamma/(4+c)$; c being arbitrarily large, we finally get (5.9), and Theorem 8 is proved.

6. **The jump of $f(\theta)$ and Gibbs' phenomenon.** The foregoing results enable us to give new formulae for the jump of $f(\theta)$ and to prove a Gibbs phenomenon.

THEOREM 9. *Under the assumptions* (5.1), *j is determined by any one of the formulae*

$$(C, 2) \lim nb_n = \frac{1}{\pi} j, \tag{6.1}$$

$$\sum_{1}^{n} b_\nu \sin \nu \frac{\pi}{n} \to \frac{1}{\pi} j \int_0^{\pi} \frac{\sin t}{t} dt, \tag{6.2}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} s_\nu\left(\frac{\pi}{\nu}\right) = \frac{1}{\pi} j \int_0^{\pi} \frac{\sin t}{t} dt, \tag{6.3}$$

$$\lim \frac{n}{2\pi} \int_0^{2\pi/n} s_n(t) dt = \frac{1}{\pi} j \int_0^{2\pi} \frac{\sin t}{t} dt, \tag{6.4}$$

where $s_n(t) = \sum_1^n b_\nu \sin \nu t$.

(6.1) is the statement of Theorem 8; (6.2) follows from Theorem 2 for $\theta_n \equiv 0$, $\phi_n = \pi/n$, $\tau_n = nb_n$, using (6.1). Similarly (6.3) follows from Theorem 4 for $\tau_n = nb_n$, $\theta_n = \pi/n$, and (6.4) follows from Theorem 7 for $\theta_n = 2\pi/n$.

To prove a Gibbs' phenomenon under the same assumptions, put, in Theorem 2, $\tau_n = nb_n$, $\theta_n \equiv 0$, $n\phi_n = \pi + O(1/n)$, then

$$s_n(\phi_n) \to \frac{1}{\pi} j \int_0^\pi t^{-1} \sin t dt,$$

hence

$$\limsup s_n(\theta_n) \geqq j/2 \times 1.08949 \cdots, \qquad \theta_n \downarrow 0.$$

Similarly from Theorem 4 for $\alpha = \pi$

$$\frac{1}{n} \sum_1^n s_\nu(\theta_\nu) \to \frac{1}{\pi} j \int_0^\pi t^{-1} \sin t dt,$$

hence

$$\limsup s_n(\theta_n) \geqq \frac{1}{\pi} j \int_0^\pi t^{-1} \sin t dt.$$

Theorem 7 also proves the presence of a Gibbs' phenomenon, however with a smaller constant.

We may also combine Theorem 8 with Theorem 3 of [3], putting there $\rho_n \equiv 1$, $\kappa = 2$, $n\theta_n = \pi + O(n^{-1})$. Summarizing, we have the following theorem.

THEOREM 10. *Under the assumptions* (5.1)

$$\limsup_{\theta_n \downarrow 0} \sum_1^n b_\nu \sin \nu\theta_n \geqq \frac{jg}{2},$$

where

$$g = \frac{2}{\pi} \int_0^\pi t^{-1} \sin t dt = 1.08949 \cdots.$$

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