INTERPOLATION AND APPROXIMATION BY FUNCTIONS ANALYTIC AND BOUNDED IN A GIVEN REGION

BY

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Introduction. Several new problems in the theory of approximation to an analytic function were suggested and discussed by Walsh in a recent paper(1). It is the purpose of the present paper to make additional contributions to the solution of these problems and to indicate the extent to which the results carry over to the theory of harmonic functions.

1. **Representation of a special harmonic function.** Let R be a finite sum of disjoint regions in the complex plane and let S be a closed set interior to R. When suitable restrictions are placed upon the boundaries C_0 and C_1 of S and R, respectively, there exists a function $\phi(z)$ equal to zero on C_0 and unity on C_1, continuous in the extended complex plane, harmonic except upon C_0 and C_1.

It is convenient for the present to assume that R and S satisfy the following set of conditions:

A1. *The boundary of each component region T of R consists of a finite number of disjoint analytic Jordan curves.*

A2. *For each component T, the boundary of TS consists of a finite number of analytic Jordan curves such that those constituting the boundary of each component region of $T-TS$ are disjoint.*

A3. *Each component T contains at least one point of S.*

A4. *No point of $T-TS$ is separated from S by the boundary of T.*

Under these conditions, the function $\phi(z)$ exists, is necessarily equal to zero on S and unity exterior to R.

Let $\psi(z)$ be a function conjugate to $\phi(z)$ in $R-S$. The limit of $\psi(z)$, as z, remaining within a component region of $R-S$, approaches a curve of C_0 or C_1, exists in the small and forms on the boundary of this component region a continuous periodic function of the arc length. Of course, if a component curve of C_0 or C_1 is common to the boundaries of two such component regions of $R-S$, the limits of the function $\psi(z)$ from the two sides of this curve are, in general, not equal.

THEOREM 1. *Suppose R is a finite sum of mutually disjoint regions and S a closed set interior to R such that R and S satisfy conditions* (A). *Then, for z not on C_0+C_1, we have*

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(1) J. L. Walsh, Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 477–486.

$$\phi(z) = \phi(\infty) - \frac{1}{2\pi}\int_{C_0+C_1} \log|z-\zeta|\, d\psi(\zeta), \tag{1}$$

where ζ traces C_0+C_1 in the positive sense with respect to $R-S$.

In the integration over C_0+C_1, each component region of $R-S$ is considered in turn and $\psi(z)$ is taken as defined by approach to C_0+C_1 from within this component. Thus in the integration in (1) some of the curves of C_0+C_1 may be traversed twice and in opposite directions.

Theorem 1 is a direct consequence of Green's third identity for z finite. Equation (1) is valid for $z=\infty$ in the sense that $\lim_{z\to\infty}(1/2\pi)\int_{C_0+C_1}\log|z-\zeta|\, d\psi(\zeta)=0$.

The normal derivative $(\partial\phi/\partial n)$ of $\phi(z)$ on C_0 (normal directed outward from $R-S$) is negative; on C_1 it is positive. From the well known property of harmonic functions that $\int_{C_0+C_1}(\partial\phi/\partial n)ds=0$, it follows that

$$\int_{C_1} d\psi = \int_{C_1}(\partial\phi/\partial n)ds = -\int_{C_0}(\partial\phi/\partial n)ds = -\int_{C_0} d\psi = \tau > 0.$$

Fix the range of ψ on C_1 as $0\leqq\psi<\tau$, on C_0 as $\tau\leqq\psi<2\tau$. (For each component of C_0+C_1 will appear one or two subintervals according as this component appears once or twice in the integration in (1).)

THEOREM 2. *Let R be a finite sum of disjoint regions and S a closed set interior to R such that R and S satisfy conditions* (A). *Let the points $\alpha_{nk}(k=1, 2, \cdots, n)$ and $\beta_{nk}(k=1, 2, \cdots, n+1)$ for $n=1, 2, \cdots$ be uniformly distributed*(2) *with respect to ψ on C_1 and C_0, respectively. Then for any point $z\neq\infty$ not on C_0+C_1 we have*

$$\lim_{n\to\infty}\left|\frac{(z-\beta_{n1})(z-\beta_{n2})\cdots(z-\beta_{n\,n+1})}{(z-\alpha_{n1})(z-\alpha_{n2})\cdots(z-\alpha_{nn})}\right|^{1/n} = \exp\left[-\frac{2\pi}{\tau}(\phi(\infty)-\phi(z))\right], \tag{2}$$

the convergence being uniform on any closed bounded set disjoint from C_0+C_1.

In the event that a curve C is traversed twice in the integration in equation (1), there are two methods of effecting the distribution of points. If T_1 and T_2 are the two component regions involved, and if ψ_1 and ψ_2 are the corresponding limit functions of ψ on C, then there may be a double distribution of points α_{nk} (or β_{nk}) on C or there may be a single distribution on C made with respect to $\psi_1-\psi_2$.

(2) J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1935, pp. 164–165.

Equation (2) follows from equation (1) and the relationship

$$\frac{1}{2\pi}\int_{C_0+C_1} \log|z-\zeta|\,d\psi(\zeta) = \frac{\tau}{2\pi}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\log|z-\alpha_{nk}|}{n} - \frac{\tau}{2\pi}\lim_{n\to\infty}\sum_{k=1}^{n+1}\frac{\log|z-\beta_{nk}|}{n},$$

which is a consequence of the definition of uniform distribution.

Under special circumstances, it may occur that in some subregion of $R-S$ the function $\phi(z)$ is given by

$$\sum_{k=1}^{m}\log|z-\zeta_k| = A(\phi(z)-B),$$

where A and B are constants, and the points ζ_k are not interior to $R-S$. Here it is not necessary to use the points α_{nk} and β_{nk} to approximate the function $\phi(z)$.

The boundary conditions (A) can be considerably relaxed. The conclusions of Theorems 1 and 2 are valid if the boundary curves are merely Jordan curves and if the disjointness provision in A2 and A3 is omitted. We shall designate the revised conditions by (A′).

2. **Interpolation and approximation by rational functions.** Let C_0, C_1, $\phi(z)$, $\psi(z)$, and τ bear the same relation to R and S as in §1. Denote generically by C_ν, $0<\nu<1$, the locus $\phi(z)=\nu$. Let R_ν, $0<\nu\leqq 1$, be the point set where $0\leqq\phi(z)<\nu$ and let $\overline{R}_\nu$ be the closure of R_ν ($\overline{R}_0=S$). Let the points α_{nk} and β_{nk} be uniformly distributed with respect to $\psi(z)$ on C_1 and C_0, respectively. For a given function $f(z)$ analytic on S, let $r_n(z)$ be the unique rational function of degree n[3] having poles at the n points α_{nk} and interpolating to $f(z)$ in the $n+1$ points β_{nk}.

THEOREM 3. *Let R be a finite sum of disjoint regions and S a closed set interior to R such that R and S satisfy conditions* (A′). *If the function $f(z)$ is analytic throughout R_ρ, $0<\rho<1$, then the sequence $\{r_n(z)\}$ converges uniformly to $f(z)$ on any closed subset of R_ρ, and for any σ satisfying $0\leqq\sigma<\rho$ we have*

$$\limsup_{n\to\infty}\,[\max|f(z)-r_n(z)|,\ z \text{ in } \overline{R}_\sigma]^{1/n} \leqq e^{-2\pi(\rho-\sigma)/\tau}; \tag{3}$$

if $\rho\leqq\mu<1$, we have

$$\limsup_{n\to\infty}\,[\max|r_n(z)|,\ z \text{ in } \overline{R}_\mu]^{1/n} \leqq e^{-2\pi(\rho-\mu)/\tau}. \tag{4}$$

If $f(z)$ is analytic throughout R_ρ but coincides on S with no function analytic throughout R_{ρ_1} for any $\rho_1>\rho$, then the equality holds in (3) *and* (4).

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(3) J. L. Walsh, op. cit., pp. 184 ff.

Suppose σ is given, $0 \leqq \sigma < \rho$. Choose ρ', $\sigma < \rho' < \rho$, such that $R_\rho - R_{\rho'}$ contains neither critical points of $\phi(z)$ nor the point at infinity. The function $f(z)$ is analytic for z in $\overline{R}_{\rho'}$ and(4)

$$(5) \qquad f(z) - r_n(z) = \frac{1}{2\pi i} \int_{C_{\rho'}} \frac{(z-\beta_{n1}) \cdots (z-\beta_{n\,n+1})(t-\alpha_{n1}) \cdots (t-\alpha_{nn}) f(t)\,dt}{(t-\beta_{n1}) \cdots (t-\beta_{n\,n+1})(z-\alpha_{n1}) \cdots (z-\alpha_{nn})(t-z)}.$$

Choose σ', $\sigma < \sigma' < \rho'$, such that $\sigma' \neq \phi(\infty)$. Then the convergence in (2) is uniform on the point set $C_{\sigma'} + C_{\rho'}$. The locus $C_{\rho'}$ is rectifiable and $|z-t|$, for z on $C_{\sigma'}$ and t on $C_{\rho'}$, has a positive lower bound. The integrand in (5) is an analytic function of z throughout $\overline{R}_{\sigma'}$ (if properly defined at $z = \infty$ in the event that $R_{\sigma'}$ is not bounded). Therefore it follows, by the principle of maximum, that

$$\limsup_{n \to \infty} [\max |f(z) - r_n(z)|, z \text{ in } \overline{R}_{\sigma'}]^{1/n} \leqq e^{-2\pi(\rho'-\sigma')/\tau}.$$

The left-hand member is not affected if $\rho' \to \rho$. Also, σ' can be replaced by σ, and the proof of inequality (3) is complete. For (4), we make use of the relationship(5)

$$(6) \qquad r_n(z) = \frac{1}{2\pi i} \int_{C_{\rho'}} \left[1 - \frac{(z-\beta_{n1}) \cdots (z-\beta_{n\,n+1})(t-\alpha_{n1}) \cdots (t-\alpha_{nn})}{(t-\beta_{n1}) \cdots (t-\beta_{n\,n+1})(z-\alpha_{n1}) \cdots (z-\alpha_{nn})} \right] \frac{f(t)\,dt}{(t-z)}.$$

Suppose now that $f(z)$ is analytic throughout R_ρ but not throughout R_{ρ_1} for any $\rho_1 > \rho$. The assumption that the inequality hold in either (3) or (4) will lead immediately to a contradiction(6).

Either the requirement in Theorem 3 that no point of $R - S$ be separated from C_1 by S or some other related assumption is necessary. This is seen from a consideration of the following situation: C_1 is the unit circle $|z| = 1$ and R is its interior; $S = C_0$ is the point set $|z| = 1/2$; $f(z)$ is the function $1/z$.

A special case of Theorem 3 is that in which the points α_{nk} and β_{nk} are independent of n. Here the sequence of rational functions $\{r_n(z)\}$ is replaced by the ordinary series of interpolation(7).

The dual of Theorem 3 is of some interest. Instead of the sum of regions R, we can consider the sum R^* of regions comprising the exterior points of S and having $C_1^* = C_0$ as its boundary: thus R^* is the point set upon which $\phi(z)$ is positive. For the closed point set S^* choose the set where $\phi(z)$ is unity: thus $C_0^* = C_1$ is the boundary of S^*. That conditions (A′) are satisfied for R^* and S^* follows immediately from the corresponding assumptions upon R and S. The dual function $\phi^*(z)$ is $1 - \phi(z)$. Equation (1) becomes

(4) J. L. Walsh, ibid.

(5) J. L. Walsh, ibid.

(6) Cf. J. L. Walsh, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 293–304, esp. p. 298, Theorem 3.

(7) J. L. Walsh, op. cit., pp. 188–189.

$$\phi^*(z) = \phi^*(\infty) - \frac{1}{2\pi}\int_{C_0^*+C_1^*} \log \left| z - \zeta \right| d\psi^*(\zeta).$$

For the analogue of (2), take $\alpha_{nk}^* = \beta_{n-1k}$ $(k=1, 2, \cdots, n)$ and $\beta_{nk}^* = \alpha_{n+1k}$ $(k=1, 2, \cdots, n+1)$. In order to approximate a function $F(z)$ analytic on S^*, choose rational functions $r_n^*(z)$ of degree n having poles at the points α_{nk}^* and interpolating to $F(z)$ in the points β_{nk}^*. If $F(z)$ is analytic throughout R_ρ^* but not throughout $R_{\rho_1}^*$ if $\rho_1 > \rho$, and if $0 \leqq \sigma < \rho$, we have

$$\limsup_{n\to\infty} [\max \left| F(z) - r_n^*(z) \right|, z \text{ in } \overline{R}_\sigma^*]^{1/n} = e^{-2\pi(\rho-\sigma)/\tau};$$

if $\rho \leqq \mu < 1$, we have

$$\limsup_{n\to\infty} [\max \left| r_n^*(z) \right|, z \text{ in } \overline{R}_\mu^*]^{1/n} = e^{-2\pi(\rho-\mu)/\tau}.$$

3. **Best approximation by functions of given norm.** Suppose now that R is a single region. Let $f(z)$ be a function analytic on the closed set S but let there exist no function analytic throughout R which coincides with $f(z)$ on S. For a given positive quantity M, there exists, among the functions analytic and of modulus not greater than M in R, a function $f_M(z)$ which has the property that $m_M = [\max \left| f(z) - f_M(z) \right|, z \text{ on } S]$ is least([8]).

Let us assume here that the region R and the point set S satisfy the conditions:

B1. *The boundaries of R and S consist of a finite number of continua, each continuum not a single point.*

B2. *No point of $R-S$ is separated by S from the boundary of R.*

Then we have the following theorem:

THEOREM 4. *Let the region R and the closed set S interior to R satisfy conditions* (B). *If the function $f(z)$ is analytic throughout R_ρ, but not throughout R_{ρ_1} for any $\rho_1 > \rho$, then as M is allowed to increase without bound the functions $\{f_M(z)\}$ converge uniformly to $f(z)$ on every closed subset of R_ρ. Indeed, for any σ, $0 \leqq \sigma < \rho$, we have*

$$\limsup_{M\to\infty} [\max \left| f(z) - f_M(z) \right|, z \text{ in } \overline{R}_\sigma]^{1/\log M} = e^{(\sigma-\rho)/(1-\rho)}. \tag{7}$$

Moreover, if $\rho \leqq \mu \leqq 1$, we have

$$\limsup_{M\to\infty} [\text{l.u.b. } \left| f_M(z) \right|, z \text{ in } R_\mu]^{1/\log M} = e^{(\mu-\rho)/(1-\rho)}. \tag{8}$$

This theorem has been proved([9]) for the case in which $R-S$ is connected. Here we lighten the restrictions upon R and S and make use of a new method of proof.

([8]) J. L. Walsh, Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) p. 477.

([9]) J. L. Walsh, ibid.

We shall assume that R is finite and bounded by a finite number of disjoint analytic Jordan curves: this assumption entails no loss in generality since the theorem is invariant under a one-one conformal map. It is now possible to extend the function $\phi(z)$ harmonically across C_1. Choose η, $0<\eta<\rho$, so small that there are no critical points of $\phi(z)$ on the point sets where $1\leqq\phi(z)\leqq 1+\eta$ and $0<\phi(z)\leqq\eta$. Let R' be that region bounded by $C_1'=C_{1+\eta}$ and containing R. Let S' be the set S together with all points z where $0<\phi(z)\leqq\eta$. If Theorem 3 is applied to R' and S', we obtain

$$\limsup_{n\to\infty}\,[\max\,|f(z)-r_n(z)|,\ z \text{ on } C_0']^{1/n}\leqq e^{-2\pi(\rho-\eta)/\tau}$$

and

$$\limsup_{n\to\infty}\,[\max\,|r_n(z)|,\ z \text{ on } C_1]^{1/n}\leqq e^{-2\pi(\rho-1)/\tau},$$

where the poles of $r_n(z)$ are on C_1' and the points of interpolation on C_0'. The quantity τ is independent of η. For arbitrary $\epsilon>0$, there exist constants M' and M'' such that for all n we have

$$|f(z)-r_n(z)|\leqq M'e^{-2\pi n(\rho-\eta-\epsilon)/\tau},\ z \text{ on } C_0, \tag{9}$$

and

$$|r_n(z)|\leqq M''e^{-2\pi n(\rho-1-\epsilon)/\tau},\ z \text{ on } C_1. \tag{10}$$

When M is given not less than the right-hand member of (10), the function $r_n(z)$ is itself analytic and of modulus not greater than M in R. Thus it follows that

$$|f(z)-f_M(z)|\leqq M'e^{-2\pi n(\rho-\eta-\epsilon)/\tau},\ z \text{ on } C_0$$

and, consequently, that

$$\limsup_{M\to\infty}\,[\max\,|f(z)-f_M(z)|,\ z \text{ on } S]^{1/\log M}\leqq e^{\rho/(\rho-1)}. \tag{11}$$

On the other hand, the definition of $f_M(z)$ yields the inequality

$$\limsup_{M\to\infty}\,[\text{l.u.b. }|f_M(z)|,\ z \text{ in } R]^{1/\log M}\leqq e. \tag{12}$$

The remainder of the proof of Theorem 4 consists of a simple application of the following three lemmas. These lemmas are extensions of a recent theorem by Walsh[10] to which reference has been made previously.

LEMMA 1. *Under the conditions of Theorem 4 on R, S, and $f(z)$, suppose $\{f_\gamma(z)\}$ is a continuous family of functions analytic and bounded in R such that*

$$\limsup_{\gamma\to\infty}\,[\max\,|f(z)-f_\gamma(z)|,\ z \text{ on } S]^{1/\gamma}\leqq e^{\beta}<1 \tag{13}$$

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[10] J. L. Walsh, Trans. Amer. Math. Soc. vol. 47 (1940) p. 298.

and

$$\limsup_{\gamma\to\infty} [\text{l.u.b. } f_\gamma(z),\ z \text{ in } R]^{1/\gamma} \leqq e^\alpha,\ e^\alpha > 1. \tag{14}$$

If $\alpha\rho+\beta-\beta\rho=0$ *and if* $\{\gamma_n\}$ *is any monotone sequence of values of* γ *approaching infinity, then*

$$\limsup_{n\to\infty} [\max |f(z) - f_{\gamma_n}(z)|,\ z \text{ in } \overline{R}_\sigma]^{1/\gamma_n} \leqq e^{(\sigma-\rho)(\alpha-\beta)} \tag{15}$$

whenever $0\leqq\sigma<\rho$; *if* $\rho\leqq\mu\leqq 1$, *we have*

$$\limsup_{n\to\infty} [\text{l.u.b. } |f_{\gamma_n}(z)|,\ z \text{ in } R_\mu]^{1/\gamma_n} \leqq e^{(\mu-\rho)(\alpha-\beta)}. \tag{16}$$

LEMMA 2. *Under the conditions of Theorem* 4 *on* R, S, *and* $f(z)$, *suppose* $\{f_{\gamma_n}(z)\}$ *is any sequence of functions analytic and bounded in* R *such that*

$$\limsup_{n\to\infty} [\max |f(z) - f_{\gamma_n}(z)|,\ z \text{ on } S]^{1/\gamma_n} \leqq e^\beta < 1 \tag{17}$$

and

$$\limsup_{n\to\infty} [\text{l.u.b. } |f_{\gamma_n}(z)|,\ z \text{ in } R]^{1/\gamma_n} \leqq e^\alpha,\ e^\alpha > 1, \tag{18}$$

where the sequence $\{\gamma_n\}$ *is monotone increasing and satisfies*

$$0 < \liminf_{n\to\infty} (\gamma_{n+1} - \gamma_n) \leqq \limsup_{n\to\infty} (\gamma_{n+1} - \gamma_n) < \infty. \tag{19}$$

Then we have

$$\alpha\rho + \beta - \beta\rho \geqq 0. \tag{20}$$

LEMMA 3. *With the hypotheses and notation of Lemma* 2, *if* $\alpha\rho+\beta-\beta\rho=0$, *we have, whenever* $0\leqq\sigma<\rho$,

$$\limsup_{n\to\infty} [\max |f(z) - f_{\gamma_n}(z)|,\ z \text{ in } \overline{R}_\sigma]^{1/\gamma_n} = e^{(\sigma-\rho)(\alpha-\beta)}, \tag{21}$$

and, whenever $\rho\leqq\mu\leqq 1$,

$$\limsup_{n\to\infty} [\text{l.u.b. } |f_{\gamma_n}(z)|,\ z \text{ in } R_\mu]^{1/\gamma_n} = e^{(\mu-\rho)(\alpha-\beta)}. \tag{22}$$

The proof of Lemma 1 is by contradiction. Assume for some σ and some $\epsilon>0$ that

$$\limsup_{n\to\infty} [\max |f(z) - f_{\gamma_n}(z)|,\ z \text{ in } \overline{R}_\sigma]^{1/\gamma_n} \geqq e^{(\sigma-\rho)(\alpha-\beta-\epsilon)}. \tag{23}$$

The sequence $\{\gamma_n\}$ can be altered so as to give a new sequence $\{\gamma_n''\}$ for which (23) is still true but for which

$$\lim_{n\to\infty} \gamma_n''/n = 1.$$

Indeed, choose a subsequence $\{\gamma_n'\}$ of $\{\gamma_n\}$ such that

$$[\max |f(z) - f_{\gamma_n'}(z)|, z \text{ in } \overline{R}_\sigma]^{1/\gamma_n'} \geqq e^{(\sigma-\rho)(\alpha-\beta-\epsilon)}$$

and such that $\gamma_{n+1}'-\gamma_n' \geqq 1$. Take $\gamma_k''=\gamma_n'$ if $k \leqq \gamma_n' < k+1$; take $\gamma_k''=k$ if there is no γ_n' satisfying $k \leqq \gamma_n' < k+1$. We now have

$$\limsup_{n\to\infty} [\max |f(z) - f_{\gamma_n''}(z)|, z \text{ in } \overline{R}_\sigma]^{1/n} \geqq e^{(\sigma-\rho)(\alpha-\beta-\epsilon)},$$

and we obtain a contradiction to the theorem by Walsh referred to above. A similar proof is used to obtain (16).

To prove Lemma 2 we note that (17), (18), and (19) imply the existence, for each ϵ $(0<\epsilon<-\beta)$, of constants A, B, and $\overline{M}$ such that for all n sufficiently large we have

$$|f(z) - f_{\gamma_n}(z)| \leqq \overline{M} e^{(\beta+\epsilon)\gamma_n}, \qquad z \text{ on } S, \tag{24}$$

$$|f_{\gamma_n}(z)| \leqq \overline{M} e^{(\alpha+\epsilon)\gamma_n}, \qquad z \text{ in } R, \tag{25}$$

and

$$0 < A < \gamma_{n+1} - \gamma_n < B < \infty. \tag{26}$$

Thus

$$|f_{\gamma_{n+1}}(z) - f_{\gamma_n}(z)| \leqq \overline{M} e^{(\beta+\epsilon)\gamma_n}[1 + e^{(\beta+\epsilon)A}], \qquad z \text{ on } S,$$

and

$$|f_{\gamma_{n+1}}(z) - f_{\gamma_n}(z)| \leqq \overline{M} e^{(\alpha+\epsilon)\gamma_n}[1 + e^{(\alpha+\epsilon)B}], \qquad z \text{ in } R.$$

From the Two-Constant Theorem[11] we can conclude for z on C_ν, $0<\nu<1$, and for all n sufficiently large, that

$$|f_{\gamma_{n+1}}(z) - f_{\gamma_n}(z)| \leqq \overline{M} e^{\gamma_n(\alpha\nu+\beta-\beta\nu+\epsilon)}[1 + e^{(\beta+\epsilon)A}]^{1-\nu}[1 + e^{(\alpha+\epsilon)B}]^{\nu}. \tag{27}$$

Thus the sequence $\{f_{\gamma_n}(z)\}$ converges uniformly in $\overline{R}_\nu$ provided $\alpha\nu+\beta-\beta\nu<0$: that is, provided $\nu<-\beta/(\alpha-\beta)$. Since this sequence converges to $f(z)$ on S, it follows that $\rho \geqq -\beta/(\alpha-\beta)$.

The inequality (27) suffices to prove that the left-hand member of (21) or (22) is not greater than the corresponding right-hand member. For, if z is in $\overline{R}_\sigma$, $0 \leqq \sigma < \rho$,

$$f(z) = f_{\gamma_n}(z) + [f_{\gamma_{n+1}}(z) - f_{\gamma_n}(z)] + [f_{\gamma_{n+2}}(z) - f_{\gamma_{n+1}}(z)] + \cdots,$$

while, for z in R_μ, $\rho \leqq \mu \leqq 1$,

$$f_{\gamma_{N+n}}(z) = f_{\gamma_N}(z) + [f_{\gamma_{N+1}}(z) - f_{\gamma_N}(z)] + \cdots + [f_{\gamma_{N+n}}(z) - f_{\gamma_{N+n-1}}(z)].$$

[11] R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936, p. 42.

That actual equality must hold in (21) and (22) is seen from Lemma 2.

There are many further consequences of Lemmas 1, 2, and 3. Some of these are included in the corollaries which follow.

COROLLARY 1. *For any continuous family of functions* $\{g_M(z)\}$ *such that* $g_M(z)$ *is analytic and of modulus not greater than* M *in* R, *we have*

$$\limsup_{M\to\infty} [\max |f(z) - g_M(z)|, z \text{ in } \overline{R}_\sigma]^{1/\log M} \geqq e^{(\sigma-\rho)/(1-\rho)}$$

for all σ, $0 \leqq \sigma < \rho$.

COROLLARY 2. *Suppose* $\{M_n\}$ *is a monotone sequence of positive quantities such that*

$$1 < \liminf_{n\to\infty} M_{n+1}/M_n \leqq \limsup_{n\to\infty} M_{n+1}/M_n < \infty. \tag{28}$$

Then for $0 \leqq \sigma < \rho$, *we have*

$$\limsup_{n\to\infty} [\max |f(z) - f_{M_n}(z)|, z \text{ in } \overline{R}_\sigma]^{1/\log M} = e^{(\sigma-\rho)/(1-\rho)}, \tag{29}$$

while for $\rho \leqq \mu \leqq 1$, *we have*

$$\limsup_{n\to\infty} [\text{l.u.b. } |f_{M_n}(z)|, z \text{ in } R_\mu]^{1/\log M} = e^{(\mu-\rho)/(1-\rho)}. \tag{30}$$

COROLLARY 3. *If* $\{M_n\}$ *is a monotone sequence of positive quantities satisfying* (28) *and if* $\{g_{M_n}(z)\}$ *is any sequence of functions such that* $g_{M_n}(z)$ *is analytic and of modulus not greater than* M *in* R, *then for* $0 \leqq \sigma < \rho$ *we have*

$$\limsup_{n\to\infty} [\max |f(z) - g_{M_n}(z)|, z \text{ in } \overline{R}_\sigma]^{1/\log M_n} \geqq e^{(\sigma-\rho)/(1-\rho)}.$$

The equality in (29) and (30) is valid for many sequences of functions other than an extremal sequence $\{f_{M_n}(z)\}$. An example is the sequence of functions $\{r_n(z)\}$ with poles on C_1' introduced in the proof of Theorem 4 but with C_1' approaching C_1 as n increases indefinitely. Compare Theorem 3.

Essentially, the right-hand member of (7) is not dependent upon S but rather upon $\overline{R}_\sigma$ itself. If S is replaced by $S^* = R_\alpha$, $0 < \alpha < \sigma$, the harmonic function $\phi^*(z)$ for $R - S^*$ is $(\phi(z) - \alpha)/(1-\alpha)$ and

$$e^{(\sigma^*-\rho^*)/(1-\rho^*)} = e^{(\sigma-\rho)/(1-\rho)}.$$

A similar property for (8) and for the corollaries is immediately suggested.

A limiting case of Theorem 4 is that in which $f(z)$ is analytic throughout R. For each ρ less than unity, inequalities (11) and (12) are valid. Thus by Lemma 1 we may conclude for each $\rho (0 < \rho < 1)$ that

$$\limsup_{M\to\infty} [\max |f(z) - f_M(z)|, z \text{ in } \overline{R}_\sigma]^{1/\log M} \leqq e^{(\sigma-\rho)/(1-\rho)}, \tag{31}$$

provided $0 \leqq \sigma < \rho$; moreover, if $\rho \leqq \mu \leqq 1$,

$$\limsup_{M\to\infty} [\text{l.u.b. } |f_M(z)|, z \text{ in } R_\mu]^{1/\log M} \leqq e^{(\mu-\rho)/(1-\rho)}. \tag{32}$$

The following corollaries now result:

COROLLARY 4. *If R and S satisfy the restrictions of Theorem 4 and if the function $f(z)$ is analytic throughout R but not bounded there, then*

$$\lim_{M\to\infty} [\max |f(z) - f_M(z)|, z \text{ in } \overline{R}_\sigma]^{1/\log M} = 0,$$

for $0 \leqq \sigma < 1$. If $0 \leqq \mu < 1$, then

$$\limsup_{M\to\infty} [\text{l.u.b. } |f_M(z)|, z \text{ in } R_\mu]^{1/\log M} = 1,$$

while

$$\limsup_{M\to\infty} [\text{l.u.b. } |f_M(z)|, z \text{ in } R]^{1/\log M} = e. \tag{33}$$

COROLLARY 5. *Under the hypotheses of Corollary 4 upon R, S, and $f(z)$, if $\{M_n\}$ is a monotone sequence of positive quantities satisfying (28), then*

$$\lim_{n\to\infty} [\max |f(z) - f_{M_n}(z)|, z \text{ in } \overline{R}_\sigma]^{1/\log M_n} = 0,$$

for $0 \leqq \sigma < 1$. If $0 \leqq \mu < 1$, then

$$\limsup_{n\to\infty} [\max |f_{M_n}(z)|, z \text{ in } \overline{R}_\mu]^{1/\log M_n} = 1,$$

while

$$\limsup_{n\to\infty} [\text{l.u.b. } |f_{M_n}(z)|, z \text{ in } R]^{1/\log M_n} = e. \tag{34}$$

4. **Approximation by functions of minimum norm.** Under the conditions of Theorem 4 upon R, S, and $f(z)$, there exists for each positive m a function $f_m(z)$ analytic and bounded in R such that $|f(z) - f_m(z)| \leqq m$ on S and such that the quantity

$$M_m = [\text{l.u.b. } |f_m(z)|, z \text{ in } R]$$

is a minimum. The existence of $f_m(z)$ is a direct consequence of Theorem 4 and well known properties of normal families of analytic functions.

THEOREM 5. *Let R, S, and $f(z)$ satisfy the conditions of Theorem 4. Then the family of functions $\{f_m(z)\}$ converges to $f(z)$ uniformly on each closed subset of R_ρ as $m \to 0$, and we have*

$$\limsup_{m\to 0} [\max |f(z) - f_m(z)|, z \text{ in } \overline{R}_\sigma]^{-1/\log m} = e^{(\sigma-\rho)/\rho}$$

whenever $0 \leqq \sigma < \rho$. Moreover, if $\rho \leqq \mu \leqq 1$,

$$\limsup_{m\to 0} [\text{l.u.b.} \mid f_m(z) \mid, z \text{ in } R_\mu]^{-1/\log m} = e^{(\mu-\rho)/\rho}.$$

This theorem also has appeared elsewhere[12] for the case in which $R-S$ is connected. It may be proved by a method analogous to that used for Theorem 4, or it may be demonstrated for the most part as a consequence of Theorem 4.

The analogues of Corollaries 1 to 3 of §3 are clear and need no further consideration. The analogue of Corollary 4 is the following:

COROLLARY. *Under the conditions of Theorem* 5 *upon* R *and* S, *if* $f(z)$ *is analytic throughout* R, *then*

$$\limsup_{m\to 0} [\max \mid f(z) - f_m(z) \mid, z \text{ on } S]^{-1/\log m} = e^{-1}.$$

If $0<\sigma<1$, *then*

$$\limsup_{m\to 0} [\max \mid f(z) - f_m(z) \mid, z \text{ in } \overline{R}_\sigma]^{-1/\log m} \leqq e^{\sigma-1},$$

while for all μ, $0\leqq\mu\leqq 1$, *we have*

$$\limsup_{m\to 0} [\text{l.u.b.} \mid f_m(z) \mid, z \text{ in } R_\mu]^{-1/\log m} = 1.$$

The corresponding result for sequences $\{f_{m_n}(z)\}$ of extremal functions is immediately suggested.

5. **Extension to regions of infinite connectivity.** The theorems of §§3 and 4 can be extended, under certain circumstances, to regions of infinite connectivity. Indeed, the conclusions of Theorems 4 and 5 are still valid if R and S satisfy the following conditions:

C1. *The boundaries of* R *and* S *consist of a denumerable number of continua, each continuum not a single point.*

C2. *Any point of* $R-S$, *which can be included in a Jordan curve of arbitrarily small diameter composed exclusively of points of* $R-S$, *also belongs to* $R-S$.

C3. *No point of* $R-S$ *is separated by* S *from the boundary of* R.

These conditions are sufficient to insure the existence of the harmonic function equal to zero on the boundary of S and unity on the boundary of R[13].

The extended forms of Theorems 4 and 5 for these new conditions (C) on R and S result from a consideration of limiting cases of the results of §§3 and 4; the proofs are routine affairs and need not be discussed in detail.

6. **Further relaxation of boundary restrictions.** Theorems 1, 2, and 3 can be extended to cover situations in which much less restrictive conditions are imposed upon the boundary of $R-S$ than those contained in the set (A′).

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(12) J. L. Walsh, Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 477–486.

(13) H. Lebesgue, Rend. Circ. Mat. Palermo vol. 24 (1907) pp. 371–402, especially §16.

Suppose, instead, that R and S satisfy the following:

B′1. *The boundaries of R and S consist of a finite number of mutually disjoint bounded continua, each continuum not a single point.*

B′2. *Each component region of R contains at least one point of S.*

B′3. *No point of $R-S$ is separated by S from the boundary of R.*

Denote by R_k $(k=1, 2, \cdots, m)$ the component regions of $R-S$. Each region R_k can be mapped one-one and conformally onto a region R_k' whose boundary $C_{k0}'+C_{k1}'$ consists of disjoint analytic Jordan curves: C_{k0}' and C_{k1}' correspond to C_{k0} and C_{k1} of the boundary of R_k, and R_k' has the same connectivity as R_k. If $z=F_k(z')$ is the mapping function of R_k' onto R_k, and if ζ' is a point on the boundary $C_{k0}'+C_{k1}'$, then for normal approach to this boundary

$$\lim_{z'\to\zeta'} F_k(z'), \qquad z' \text{ in } R_k',$$

exists for all ζ' on $C_{k0}'+C_{k1}'$ except possibly for a set of (Lebesgue) measure zero.

The function $\phi(z)$, equal to zero on S and to unity exterior to R, harmonic (although not necessarily monogenic) in $R-S$, exists here also. Let T_σ, $0\leqq\sigma<1/2$, be the set of all points z in $R-S$ for which $\sigma<\phi(z)<1-\sigma$. Choose $\sigma_0>0$ so small that T_{σ_0} contains all the critical points of $\phi(z)$ in $R-S$ and also the point at infinity in case some R_k is not bounded. For $0<\sigma\leqq\sigma_0$, the increase τ of $\psi(z)$ on $C_{1-\sigma}$, as the latter is traced in the positive sense with respect to T_σ, is independent of σ; the constant τ is also equal to the decrease of the conjugate function $\psi(z)$ along C_σ as the latter is traced in the positive sense (with respect to T_σ). Fix the range of $\psi(z)$ on $C_{1-\sigma}$ $(0<\sigma\leqq\sigma_0)$ as $0\leqq\psi(z)<\tau$, on C_σ as $\tau\leqq\psi(z)<2\tau$.

The variable z, expressed as a function $z=Z(\phi, \psi)$ of ϕ and ψ, is a single-valued function in $(R-S)-T_\sigma$ when the range of ψ is thus defined. Moreover, the limits

$$\lim_{\phi\to 1} Z(\phi, \psi), \qquad \lim_{\phi\to 0} Z(\phi, \psi), \qquad \psi \text{ fixed},$$

exist for almost all ψ in $0\leqq\psi<\tau$ resp. $\tau\leqq\psi<2\tau$. Denote the limit functions, where defined, by $Z_1(\psi)$ and $Z_0(\psi)$, respectively.

THEOREM 6. *Let R be a finite sum of disjoint regions and S a closed set interior to R such that R and S satisfy conditions* (B′). *Then for all finite z not on the boundary C_0+C_1 of $R-S$ we have*

$$\phi(z) = \phi(\infty) - \frac{1}{2\pi}\int_0^\tau \log\left|Z_1(\psi) - z\right| d\psi - \frac{1}{2\pi}\int_{2\tau}^\tau \log\left|Z_0(\psi) - z\right| d\psi. \tag{35}$$

Suppose first that the point z is in $R-S$. Choose σ_1, $0<\sigma_1\leqq\sigma_0$, such that T_{σ_1} contains z. Define $\phi_\sigma(z)=(\phi(z)-\sigma)/(1-2\sigma)$ for $0<\sigma<\sigma_1$. By Theorem 1,

$$\phi_\sigma(z) = \phi_\sigma(\infty) - \frac{1}{2\pi}\int_{C_\sigma} \log \left| Z(\sigma, \psi) - z \right| d\psi$$
$$- \frac{1}{2\pi}\int_{C_{1-\sigma}} \log \left| Z(1-\sigma, \psi) - z \right| d\psi.$$

Equation (35) results if σ is allowed to approach zero. A similar proof suffices for z interior to S or exterior to R.

THEOREM 7. *Under the conditions of Theorem* 6, *there exist sets of points* $\{\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}\}$, $n=1, 2, \cdots$, and $\{\beta_{n1}, \beta_{n2}, \cdots, \beta_{n\,n+1}\}$, $n=1, 2, \cdots$, *on* C_1 resp. C_0 *such that for* $z \neq \infty$ *not on* C_0+C_1 *we have*

$$\lim_{n\to\infty} \left| \frac{(z-\beta_{n1})\cdots(z-\beta_{n\,n+1})}{(z-\alpha_{n1})\cdots(z-\alpha_{nn})} \right|^{1/n} = \exp\left[-\frac{2\pi}{\tau}(\phi(\infty) - \phi(z))\right], \tag{36}$$

the convergence being uniform on any closed bounded set disjoint from C_0+C_1.

To prove the theorem, let us extend the definition of the functions $Z_1(\psi)$ and $Z_0(\psi)$. Heretofore, $Z_1(\psi)$ has been defined only when $\lim_{\phi\to 1} Z(\phi, \psi)$ exists. For the remaining values of ψ in the interval $0 \leqq \psi < \tau$ define

$$\arg Z_1(\psi) = \limsup_{\phi\to 1} \arg Z(\phi, \psi), \qquad -\pi < \arg Z(\phi, \psi) \leqq \pi,$$

and

$$\left| Z_1(\psi) \right| = \limsup_{\phi\to 1} \left| Z(\phi, \psi) \right|.$$

In a similar fashion extend the definition of $Z_0(\psi)$.

Choose a denumerable set $\{z_m\}$ of points ($z_m \neq \infty$, not on C_0+C_1) everywhere dense in the complex plane. Corresponding to each z_m, form the sum

$$\frac{1}{2\pi}\int_0^\tau \log \left| Z_1(\psi) - z_m \right| d\psi + \frac{1}{2\pi}\int_{2\tau}^\tau \log \left| Z_0(\psi) - z_m \right| d\psi. \tag{37}$$

By a theorem of Hahn[14], sets of numbers $\psi_{11}^{(n)}, \psi_{12}^{(n)}, \cdots, \psi_{1n}^{(n)}$, and $\psi_{01}^{(n)}, \psi_{02}^{(n)}, \cdots, \psi_{0\,n+1}^{(n)}$, can be found ($(i-1)\tau/n \leqq \psi_{1i}^{(n)} \leqq i\tau/n$ and $\tau+(i-1)\tau/(n+1) \leqq \psi_{0i}^{(n)} \leqq \tau + i\tau/(n+1)$) such that for all m

$$\lim_{n\to\infty} \frac{\tau}{2\pi n} \sum_{i=1}^{n} \log \left| Z_1(\psi_{1i}^{(n)}) - z_m \right| = \frac{1}{2\pi}\int_0^\tau \log \left| Z_1(\psi) - z_m \right| d\psi$$

and

$$\lim_{n\to\infty} \frac{\tau}{2\pi n} \sum_{i=1}^{n+1} \log \left| Z_0(\psi_{0i}^{(n)}) - z_m \right| = \frac{1}{2\pi}\int_{2\tau}^\tau \log \left| Z_0(\psi) - z_m \right| d\psi.$$

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(14) H. Hahn, Sitzungsberichte der mathematisch-naturwissenschaftliche klasse, K. Akademie der Wissenschaften, Vienna, vol. 123 II a 1 (1914) pp. 713–743. Cf. B. Jessen, Ann. of Math. vol. 35 (1934) pp. 248–251.

The functions

$$(38)\qquad \frac{\tau}{2\pi n}\sum_{i=1}^{n}\log\left|Z_1(\psi_{1i}^{(n)})-z\right|+\frac{\tau}{2\pi n}\sum_{i=1}^{n+1}\log\left|Z_0(\psi_{0i}^{(n)})-z\right|,\qquad n=1,2,\cdots,$$

are harmonic for finite z not on C_0+C_1. In any closed bounded set T disjoint from C_0+C_1, these functions are uniformly bounded and therefore equicontinuous. Convergence on the everywhere dense set $\{z_m\}$ implies uniformity of convergence on T to a function harmonic for all finite z not on C_0+C_1.

The function

$$(39)\qquad \frac{1}{2\pi}\int_{2\tau}^{\tau}\log\left|Z_0(\psi)-z\right|d\psi+\frac{1}{2\pi}\int_{0}^{\tau}\log\left|Z_1(\psi)-z\right|d\psi$$

is harmonic in z for the values of z under consideration and coincides on the everywhere dense set $\{z_m\}$ with the limiting function of the sequence (38). Thus these two harmonic functions must coincide. All that remains to be done for the proof of (36) is to choose $\alpha_{nk}=Z_1(\psi_{1k}^{(n)})$ and $\beta_{nk}=Z_0(\psi_{0k}^{(n)})$. Moreover, we have at once the extension of Theorem 3:

THEOREM 8. *Let R be a finite sum of disjoint regions and S a closed set interior to R such that R and S satisfy conditions* (B′). *Let $f(z)$ be a function analytic throughout R_ρ, $0<\rho<1$, but not throughout R_{ρ_1} for any $\rho_1>\rho$. Let $r_n(z)$ be the rational function of degree n with poles at the n points α_{nk} $(k=1,2,\cdots,n)$ defined above which interpolates to $f(z)$ in the $n+1$ points β_{nk} $(k=1,2,\cdots,n+1)$. Then for any σ, $0\leqq\sigma<\rho$, we have*

$$\limsup_{n\to\infty}\left[\max\left|f(z)-r_n(z)\right|,\ z\text{ in }\overline{R}_\sigma\right]^{1/n}=e^{-2\pi(\rho-\sigma)/\tau};$$

also, if $\rho\leqq\mu<1$, we have

$$\limsup_{n\to\infty}\left[\max\left|r_n(z)\right|,\ z\text{ in }\overline{R}_\mu\right]^{1/n}=e^{-2\pi(\rho-\mu)/\tau}.$$

7. **Application to harmonic functions.** Many of the results in the preceding paragraphs carry over to the theory of approximation to harmonic functions. The three lemmas of §3 are valid for harmonic functions if R and S satisfy the following conditions:

D1. *The boundary of R consists of a finite number of disjoint continua, each not a single point.*

D2. *The set of interior points of S forms a finite sum of regions, each of finite connectivity.*

D3. *The set S is the closure of the set of its interior points.*

D4. *No point of $R-S$ is separated by S from the boundary of R.*

The methods used are essentially those introduced in an earlier paper by Walsh([15]).

Theorems 4 and 5 with their corollaries carry over to the present situation. In general, the problem of approximation to a harmonic function $u(x, y)$ is referred back to the corresponding problem of approximation to the analytic function $f(z)$ of which $u(x, y)$ is the real part. For the most part, the difficulty arising from the possibility of multiple-valuedness of the conjugate function $v(x, y)$ of $u(x, y)$ can be avoided. Obtaining the analogue of equation (11) is the only point offering serious difficulty. Here it can be shown that there exists a finite set of points (a_i, b_i), $i=1, 2, \cdots, n$, exterior to R and corresponding constants τ_i such that

$$v(x, y) - \sum_{i=1}^{n} \frac{\tau_i}{2\pi} \arctan \frac{y - b_i}{x - a_i}$$

is single-valued in R. Theorem 4 can be applied to the function

$$\left\{u(x, y) - \sum_{i=1}^{n} \frac{\tau_i}{2\pi} \log \left[(x - a_i)^2 + (y - b_i)^2\right]^{1/2}\right\} + i\left\{v(x, y) - \sum_{i=1}^{n} \frac{\tau_i}{2\pi} \arctan \frac{y - b_i}{x - a_i}\right\}.$$

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([15]) J. L. Walsh, Ann. of Math. vol. 38 (1937) pp. 321–354.