A NEW CRITERION FOR COMPLETELY MONOTONIC FUNCTIONS

BY

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A function \( f(x) \) is completely monotonic (c.m.) in \( 0 < x < \infty \) if it belongs to \( C^\infty \) and

\[
(-1)^{k} f^{(k)}(x) \geq 0 \quad (k \geq 0, x > 0).
\]

If \( f(x) \) can be extended to be continuous at \( x = 0 \) it is said to be c.m. in \( 0 \leq x < \infty \).

Various conditions are known under which a function is c.m. [4](4). Bernstein proved that if

\[
(-1)^{k} \Delta^{k} f(x) = \sum_{n=0}^{\infty} C_{k,n} (-1)^{n} f(x + nh) \geq 0 \quad (k \geq 0, x > 0, h > 0)
\]

then \( f(x) \) is c.m. in \( 0 < x < \infty \). It is known, though apparently not stated explicitly in the literature, that if we assume the continuity of \( f(x) \), then we need require (2) only for some infinite sequence of integers \( k \). (This may be obtained, for example, by use of the results of [1].) A fundamental theorem of Bernstein and Widder states that a function is c.m. in \( 0 < x < \infty \) if and only if it admits the representation

\[
f(x) = \int_{0}^{\infty} e^{-x} t dF(t), \quad x > 0, F(t) \text{ increasing}.
\]

A new difference criterion which includes the above is suggested by the following considerations. If \( h_{k} = o(1/k^2) \) then (3) is inverted by [3, Theorem 4.2],

\[
F(t) - F(0) = f(\infty) + \lim_{k \to \infty} d_{k} \int_{k/t}^{\infty} x^{k-1} \Delta_{h} f(x) dx,
\]

where

\[
d_{k} = (- h_{k})^{-k}/(k - 1)!.\]

This suggests the following theorem, which is the principal result of this paper.

**Theorem.** Let \( f(x) \) be continuous for \( x \geq 0 \) and have a limit at infinity. Suppose it satisfies the inequalities

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(4) Numbers in brackets refer to the references listed at the end of the paper.
for an infinite sequence of integers \( k \), where

\[ h_k > 0, \quad h_k = o(1/k^2) \quad (k \to \infty). \]

Then \( f(x) \) is c.m. in \( 0 \leq x < \infty \).

This says essentially that in the difference criteria it is sufficient that the inequalities hold for one (suitable) value of \( h \) for each \( k \), rather than all \( h > 0 \).

Before proceeding to the proof we make some observations about the theorem.

(i) The conditions are trivially necessary.

(ii) The continuity of \( f(x) \) is not a redundant condition. For let \( \phi(x) \) be a discontinuous solution of the functional equation \( \phi(x+y) = \phi(x) + \phi(y) \) [2, p. 96]. Both \( \pm \phi(x) \) are convex and hence unbounded in every interval [2, pp. 91–92]. Then (5) is satisfied for arbitrarily small \( \{ h_k \} \) by the function \( f(x) = e^{\phi(x)} \).

1. **Lemmas**. The following identity is known [4, p. 303].

**Lemma 1.** For \( k > 0, x \geq 0, u > 0 \),

\[
\frac{\partial^k}{\partial u^k} \left[ e^{-kz/u} u^{k-1} \right] = \frac{1}{u} \left( \frac{kx}{u} \right)^k e^{-kz/u}.
\]

For fixed \( x \), these functions are increasing in \( 0 \leq u \leq x/2 \).

**Lemma 2.** If \( g(x) \) and \( r(x) \) are any functions of \( x \) then

\[
\Delta_h^k [g(x) r(x)] = \sum_{0}^{k} C_{k,n} \Delta_h^n g(x + k - n h) \Delta_h^{k-n} r(x).
\]

This is the analogue of Leibniz' rule for the differentiation of a product and can be established by induction.

**Lemma 3 (Generalized Rolle's theorem).** If \( f(x) \in C^k \) then

\[
\Delta_h^k f(x) = h^k f^{(k)}(X)
\]

where \( X \) lies between \( x \) and \( x + kh \).

**Lemma 4.** Suppose \( k \geq 1, x > 0, h > 0 \) are fixed, and \( f(u) \) is continuous for \( u \geq kh, f(\infty) = 0 \). Then

\[
\lim_{\rightarrow 0} \int_{kh}^{\infty} \Delta_{-h}^k [e^{-u - kx/u} u^{k-1}] f(u) du = \int_{kh}^{\infty} \Delta_{-h}^k [e^{-kz/u} u^{k-1}] f(u) du,
\]

where we difference with respect to \( u \).
Proof. The existence of the integral on the left is guaranteed by the presence of the factor \( e^{-\epsilon u} \). The integral on the right converges since, by virtue of Lemmas 3 and 1, its integrand is dominated for large \( u \) by \( (u - kh - k^{-1} |f(u)| \).

If we subtract the right-hand side the problem is then to show that

\[
H(\epsilon) = \int_{kh}^{\infty} \Delta_{-h}[g(u)r(u)]f(u)\,du = o(1) \quad (\epsilon \to 0 +),
\]

where

\[
g(u) = e^{-kz/u}u^{k-1}, \quad r(u) = 1 - e^{-\epsilon u}.
\]

Using Lemma 2 with \( h \) replaced by \(-h\) and separating out the term for which \( n = k \) we obtain

\[
H(\epsilon) = \sum_{n=0}^{k-1} C_{k,n} \int_{kh}^{\infty} \Delta_{-h}^{n}g(u - \bar{k} - n h) \cdot \Delta_{-h}^{k-n}r(u) \cdot f(u)\,du
\]

\[
+ \int_{kh}^{\infty} \Delta_{-h}^{k}g(u) \cdot r(u) \cdot f(u)\,du
\]

\[
= K(\epsilon) + L(\epsilon).
\]

By Lemmas 3 and 1 the integral \( \int_{kh}^{\infty} \Delta_{-h}^{k}g(u) \cdot f(u)\,du \) exists. Hence

\[
L(\epsilon) = \int_{kh}^{\infty} (1 - e^{-\epsilon u})\Delta_{-h}^{k}g(u)f(u)\,du = o(1) \quad (\epsilon \to 0 +).
\]

We turn now to the expression \( K(\epsilon) \). First by Lemma 3

\[
\Delta_{-h}^{k-n}r(u) = (-h)^{k-n} r^{(k-n)}(U) = (he)^{k-n} e^{-U}, \quad u - (k-n)h \leq U \leq u,
\]

so that for \( 0 < \epsilon < 1 \)

\[
| \Delta_{-h}^{k-n}r(u) | \leq K_1 \epsilon^{k-n} e^{-\epsilon u},
\]

where \( K_1 \) does not depend on \( u \) or \( \epsilon \). Also

\[
\Delta_{-h}^{n}g(u - \bar{k} - n h) = (-h)^{n} g^{(n)}(U), \quad u - kh \leq U \leq u - (k-n)h.
\]

But

\[
g^{(n)}(u) = \sum_{j=0}^{n} C_{n,j} \frac{\partial^j}{\partial u^j} \left[ e^{-kz/u} \right][u^{k-1}]^{(n-j)}.
\]

If \( x > 0 \)

\[
\frac{\partial^j}{\partial u^j} \left[ e^{-kz/u} \right] = O(u^{-j}) \quad (u \to \infty),
\]

so that

\[
g^{(n)}(u) = \sum O(u^{-j}u^{k-1-n+j}) = O(u^{k-n-1}), \quad 0 \leq n \leq k - 1.
\]
If \( x = 0 \) this result is obvious. Then by (9)

\[
\left| \Delta^n_{-h} g(u - k - nh) \right| \leq K_2 u^{k-n-1} \quad (0 \leq n \leq k - 1),
\]

where \( K_2 \) does not depend on \( u \). Then by (7), (8), (10)

\[
| K(\epsilon) | \leq K_1 K_2 \sum_{n=0}^{k-1} C_{k,n} \int_{k\epsilon}^{k\epsilon + e} e^{k-n} e^{-\epsilon u} u^{k-n-1} |f(u)| \, du \quad (0 < \epsilon < 1).
\]

Since \( f(\infty) = 0 \) a simple Abelian argument proves that each term of the sum approaches zero with \( \epsilon \). Hence \( K(0+) = 0 \), so \( H(0+) = 0 \); this establishes (6).

**Lemma 5.** Let \( k \geq 1, \ h > 0, \ x \geq 0 \) be fixed, \( f(u) \) continuous, \( 0 \leq u < \infty \), \( f(\infty) = 0 \). If \( \Delta_k f(u) \) does not change sign in \( 0 \leq u < \infty \), then

\[
\int_0^{\infty} e^{-x/u} u^{k-1} \Delta_k f(u) \, du = \sum_{n=0}^{k-1} C_{k,n} (-1)^{k-n} \int_{k\epsilon}^{k\epsilon + e} e^{-x/(u-nh)} (u - nh)^{k-1} f(u) \, du
\]

\[
+ \int_{k\epsilon}^{k\epsilon + e} \Delta_k \left[ e^{-x/u} u^{k-1} \right] f(u) \, du,
\]

where we difference with respect to \( u \).

Clearly the integral

\[
I(\epsilon) = \int_0^{\infty} e^{-x/u} u^{k-1} \Delta_k f(u) \, du
\]

exists for all \( \epsilon > 0 \). We have

\[
I(\epsilon) = \sum_{n=0}^{k} C_{k,n} (-1)^{k-n} \int_0^{\infty} e^{-x/u} e^{-x/(u-nh)} u^{k-1} f(u + nh) \, du \]

\[
= \sum_{n=0}^{k} C_{k,n} (-1)^{k-n} \left( \int_{nh}^{kh} + \int_{kh}^{\infty} \right) \exp \left[ -\epsilon(u - nh) - kx/(u - nh) \right] \cdot (u - nh)^{k-1} f(u) \, du,
\]

obtained by a change of variable. Then

\[
I(\epsilon) = A(\epsilon) + B(\epsilon),
\]

where

\[
A(\epsilon) = \sum_{n=0}^{k} C_{k,n} (-1)^{k-n} \int_{nh}^{kh} \exp \left[ -\epsilon(u - nh) - kx/(u - nh) \right] \cdot (u - nh)^{k-1} f(u) \, du,
\]

\[
B(\epsilon) = \int_{kh}^{\infty} \Delta_k \left[ e^{-x/u} u^{k-1} \right] f(u) \, du.
\]
By dominated convergence

\[(13)\quad A(0 +) = \sum_{n=0}^{\infty} C_{k,n}(-1)^{k-n} \int_{nh}^{kh} e^{-kz/(u-nh)}(u-nh)^{k-1}f(u)du,\]

and by Lemma 4

\[(14)\quad B(0 +) = \int_{kh}^{\infty} \Delta_{kh}^{-1} \left[ e^{-kz/u}u^{k-1} \right] f(u)du.\]

From (12), (13), (14) it follows that \(I(0+)\) exists and equals the expression on the right-hand side of (11). Since \(\Delta_{kh}^{k}f(u)\) does not change sign in \(0 \leq u < \infty\), a Tauberian theorem enables us to conclude that \(I(0+)\) is also equal to the left-hand side of (11) \([4, \text{p. 192}]\).

For the remainder of the paper it is assumed that \(k\) belongs to some sequence \(S\) of non-negative integers; \(k \to \infty\) means that \(k\) becomes infinite through the elements of \(S\).

**Lemma 6.** Let \(f(x)\) satisfy all the hypotheses of our theorem and suppose also that \(f(\infty) = 0\). Then for any fixed \(x > 0\) the quantities

\[I_{k} = d_{k} \int_{0}^{\infty} e^{-kz/u}u^{k-1} \Delta_{kh} f(u)du\]

approach \(f(x)\) as \(k \to \infty\). The \(d_{k}\) are defined as in (4).

**Proof.** By Lemma 5 we have \(I_{k} = A_{k} + B_{k}\) where

\[A_{k} = d_{k} \sum_{n=0}^{\infty} C_{k,n}(-1)^{k-n} \int_{nh}^{kh} e^{-kz/(u-nh)}(u-nh)^{k-1}f(u)du,\]

\[B_{k} = d_{k} \int_{kh}^{\infty} \Delta_{kh}^{-1} \left[ e^{-kz/u}u^{k-1} \right] f(u)du.\]

We show first that \(A_{k}\) vanishes with \(1/k\). Let \(M\) be the maximum of \(|f(x)|\) in \((0, \infty)\). Then

\[\left| A_{k} \right| \leq M \left| d_{k} \right| \sum_{n=0}^{k} C_{k,n} \int_{nh}^{kh} e^{-kz/(u-nh)}(u-nh)^{k-1}du\]

\[\leq M \left| d_{k} \right| \sum_{n=0}^{k} C_{k,n} \int_{0}^{(k-n)kh} e^{-kz/u}u^{k-1}du\]

\[\leq M \left| d_{k} \right| \sum_{n=0}^{k} C_{k,n} \int_{0}^{kh} e^{-kz/u}u^{k-1}du\]

\[\leq M \left| d_{k} \right| e^{-z/kh} \sum_{n=0}^{k} C_{k,n} \int_{0}^{kh} u^{k-1}du\]

\[= Me^{-z/kh}2^{k}k^{k}/k!.\]
Choose \( k_0 \) so that \( kh_k < x/2 \) for \( k > k_0 \). Then for \( k > k_0 \)
\[
|A_k| \leq Me^{-2k^2/k}/k!.
\]

By the test-ratio test this last is the general term of a convergent series, so that \( A_k = o(1), k \to \infty \).

We must prove then that
\[
\lim_{k \to \infty} B_k = f(x).
\]

By Lemmas 3 and 1, and equation (4),
\[
B_k = \frac{1}{(k - 1)!} \left( \int_{z/2}^{\infty} + \int_{kh_k}^{z/2} \right) \left[ \frac{1}{u} \left( \frac{kx}{u} \right)^k e^{-kx/u} \right] \int_0^{z/2} f(v) dv
\]
\[= C_k + D_k,
\]
where
\[
0 \leq \phi_k \leq kh_k.
\]

By the second part of Lemma 1
\[
|D_k| \leq \frac{1}{(k - 1)!} \left[ \frac{1}{u} \left( \frac{kx}{u} \right)^k e^{-kx/u} \right] \int_0^{z/2} |f(v)| dv
\]
\[= o(1) \quad (k \to \infty).
\]

Our problem then is to show that \( \lim_{k \to \infty} C_k = f(x) \). But it is known that \( [4, p. 283] \)
\[
J_k = \frac{1}{(k - 1)!} \int_{z/2}^{\infty} \frac{1}{u} \left( \frac{kx}{u} \right)^k e^{-kx/u} f(u) du \to f(x), \quad k \to \infty.
\]
It therefore remains only to show that \( \lim_{k \to \infty} (J_k - C_k) = 0 \) and the proof of the lemma will be complete.

Now
\[
J_k - C_k = \frac{1}{(k - 1)!} \int_{z/2}^{\infty} \frac{1}{u} \left( \frac{kx}{u} \right)^k e^{-kx/u} f(u) P_k(u) du,
\]
where
\[
P_k(u) = 1 - \left( \frac{u}{u - \phi_k} \right)^{k+1} \exp \left( - \frac{kx\phi_k}{u(u-\phi_k)} \right) \quad (u \geq x/2).
\]
We have
\[
\log (1 - P_k(u)) = (k + 1) \int_{u-\phi_k}^u \frac{dv}{v} - \frac{kx\phi_k}{u(u-\phi_k)},
\]
\[
|\log (1 - P_k(u))| \leq (k + 1)\phi_k + \frac{kx\phi_k}{u(u-\phi_k)}.
\]
By (15) we have $0 \leq \phi_k \leq kh_k$. Since $h_k = o(1/k^2)$, $\phi_k < x/4$ for $k > k_0$. It follows that for $u \geq x/2$, $k > k_0$,

$$| \log (1 - P_k(u)) | \leq \frac{(k + 1)kh_k}{x/2 - x/4} + \frac{kx \cdot kh_k}{(x/2)(x/2 - x/4)} = o(1)$$

uniformly for $u \geq x/2$.

Let $\epsilon > 0$ be arbitrary. Then for $k > k_1$,

$$| P_k(u) | < \epsilon$$

By (16)

$$| J_k - C_k | < \frac{\epsilon}{(k - 1)!} \int_{x/2}^{\infty} \frac{1}{u} \left( \frac{kx}{u} \right)^k e^{-kz/u} | f(u) | du.$$

As $k \to \infty$ the right-hand side approaches $\epsilon | f(x) |$ [4, p. 283]. Hence

$$\lim \sup_{k \to \infty} | J_k - C_k | \leq \epsilon | f(x) |,$$

and this completes the proof.

2. Proof of the theorem. We may assume $f(\infty) = 0$ (otherwise consider $f(x) - f(\infty)$). By hypothesis

$$L_{k,t}[f] = ((-h_k)^{-k}/k!) \left[ x^{k+1} \Delta_h^k f(x) \right]_{x^{-k/t} \geq 0}$$

for an infinite sequence of integers $k$. By Lemma 6 with a change of variable we have

$$f(x) = \lim_{k \to \infty} \int_0^\infty e^{-zL_{k,t}[f]} dz$$

It remains only to show that the integrals

$$(17) \quad L_k = \int_0^\infty L_{k,t}[f] dt$$

exist and are bounded. For then it will follow by a familiar argument that $f(x)$ has the representation (3) [4, p. 307]. From Lemma 5 with $x = 0$ it follows that

$$L_k = d_k \sum_{n=0}^k C_{k,n} (-1)^{k-n} \int_{nh_k}^{kh_k} (u - nh_k)^{k-1} f(u) du$$

$$= (-h_k)^{-k} \Delta_h^k F_k(0)$$

where

$$F_k(x) = \frac{1}{(k - 1)!} \int_x^{kh_k} (u - x)^{k-1} f(u) du.$$
By Lemma 3

\[ L_k = (-1)^k F_k^{(k)}(X_k), \quad 0 \leq X_k \leq kh_k. \]

But \( F_k^{(k)}(x) = (-1)^k f(x) \), so that \( L_k = f(X_k) \) and \( \lim_{k \to \infty} L_k = f(0) \). Then \( L_k \) is bounded.

References


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