GALOIS CONNEXIONS

BY
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This paper contains in the main a section of my Colloquium lectures on the theory of *Mathematical relations* given in 1941 at the Summer Meeting of the American Mathematical Society at the University of Chicago. (A brief review of these lectures can be found in the report of the Summer meeting at Chicago, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 169–182.) Due to various causes it has been necessary for me to postpone the book on the subject in the Colloquium Series, probably until after the war. I have found it desirable, however, to publish certain parts of this theory at the present time. A contributing reason for this decision is the fact that I have at various times discussed aspects of the theory with others who have become interested in these problems to the extent of wishing to publish contributions of their own.

The object of this paper is to discuss a general type of correspondence between structures which I have called Galois connexions. These correspondences occur in a great variety of mathematical theories and in several instances in the theory of relations. It seemed desirable therefore to give a separate exposition of their main properties and interpretations. The name is taken from the ordinary Galois theory of equations where the correspondence between subgroups and subfields represents a special correspondence of this type.

After some introductory remarks on closure relations the general properties of Galois connexions are discussed. Next it is shown that every Galois connexion can be conceived of as being defined by means of a continuous mapping and conversely every mapping of a closure relation defines a Galois connexion. A different interpretation can be given by means of binary relations. It has already been pointed out by Garrett Birkhoff that any binary relation defines a correspondence of the type of a Galois connexion between the subsets of two sets and it is easily seen that conversely every Galois connexion can be constructed in this manner. As an illustrative example all binary relations with a perfect Galois connexion are determined. The Galois connexion defines a pair of dual topologies so that such topologies can be defined by means of binary relations. The construction of self-dual topologies is discussed. The possibility of a general Galois theory for relations is indicated briefly and the case of an equivalence relation is solved as an example. There are some final remarks on the Galois connexion defined by a permutation group.

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1. **Closure operations.** Let P be a partially ordered set. Any correspondence

$$a \to \bar{a} \tag{1}$$

which associates with each element a some other element $\bar{a}$ in P shall be called a *closure operation* provided it satisfies the three conditions:

(1) $\bar{\bar{a}} = \bar{a}$,
(2) $\bar{a} \supset a$,
(3) $a \supset b$ implies $\bar{a} \supset \bar{b}$.

Any image element $\bar{a}$ in the correspondence (1) is called a *closed element*. One sees that an element b is closed if and only if $\bar{b} = b$. It is also seen from the axioms that the *closure* $\bar{a}$ of an element a is the least closed element containing a.

We shall assume next that $P = \Sigma$ is a structure with a universal element u and a zero element o. The definition of a closure relation implies that u is closed, $\bar{u} = u$. When a zero element exists in the partially ordered set it is customary to make the fourth axiomatic assumption:

(4) The zero element is closed: $\bar{o} = o$.

Next let $\{a_i\}$ be a set of elements in Σ. The union of these elements shall be denoted by $\bigvee a_i$. One can then show:

$$\overline{\bigvee \bar{a}_i} = \overline{\bigvee a_i} \supseteq \bigvee \bar{a}_i. \tag{2}$$

To prove this relation we observe that for every i

$$\bigvee \bar{a}_i \supseteq \bigvee a_i \supseteq a_i.$$

By taking the closures one obtains

$$\overline{\bigvee \bar{a}_i} \supseteq \overline{\bigvee a_i} \supseteq \bigvee \bar{a}_i$$

and when the closure operation is applied a second time (2) follows.

Similarly one shows for the crosscut,

$$\overline{\bigwedge \bar{a}_i} = \bigwedge \bar{a}_i \supseteq \overline{\bigwedge a_i}. \tag{3}$$

To prove this result we notice that

$$\bar{a}_i \supseteq a_i \supseteq \bigwedge a_i,$$

hence when the closures are taken

$$\bigwedge \bar{a}_i = \overline{\bigwedge a_i}.$$

This may be applied to the elements $\bar{a}_i$ instead of the a_i so that

$$\bigwedge \bar{a}_i \supseteq \overline{\bigwedge \bar{a}_i}$$

and (3) follows.

When one assumes that Σ is a complete structure the relations (2) and

(3) hold for the union and crosscut of arbitrary sets of elements. According to (3) the crosscuts of closed sets are closed.

We shall say that a structure Σ_1 is *order contained* in another structure Σ when Σ_1 is a subset of Σ such that when $a_1 \supset b_1$ for two elements in Σ_1 then $a_1 \supset b_1$ also in Σ. Furthermore Σ_1 is *order contained over* Σ when the universal element and the zero element of Σ and Σ_1 are the same. Also Σ_1 is a *substructure of* Σ *with respect to crosscut* when the crosscut operation in Σ_1 coincides with the crosscut operation in Σ both for finite and infinite sets. From the previous remarks one obtains without difficulty:

THEOREM 1. *Let Σ be a complete structure in which a closure operation is defined. The closed elements in Σ form a complete structure Σ_1 which is a substructure over Σ with respect to crosscut. Conversely when Σ_1 is a given substructure over Σ with respect to crosscut one obtains a closure operation in Σ by associating with each element a in Σ, as its closure $\bar{a}$, the least element in Σ_1 containing a.*

This theorem in a slightly different formulation is due to Morgan Ward [1](1). It should be mentioned that this theorem can be expressed in other ways which connect it with the theory of closure relations in sets in general (Ore [1]).

Any complete structure Σ can be conceived of as being the structure of closed sets for some closure relation Γ defined in any subset S of Σ which constitutes a basis for Σ in the sense that all other elements in Σ can be obtained by forming unions of elements in S.

2. **Galois connexions.** Let P and Q denote two partially ordered sets. We shall assume that there exists a correspondence from P to Q,

$$p \rightarrow \mathfrak{Q}(p), \tag{4}$$

and also a correspondence from Q to P,

$$q \rightarrow \mathfrak{P}(q). \tag{5}$$

These two correspondences (4) and (5) together shall be called a *Galois correspondence* between P and Q provided the two following conditions are fulfilled:

(α) When $p_1 \supset p_2$ are two elements in P or $q_1 \supset q_2$ two elements in Q then

$$\mathfrak{Q}(p_1) \subseteq \mathfrak{Q}(p_2), \qquad \mathfrak{P}(q_1) \subseteq \mathfrak{P}(q_2).$$

(β) For any element p in P or q in Q

$$\mathfrak{P}\mathfrak{Q}(p) \supseteq p, \qquad \mathfrak{Q}\mathfrak{P}(q) \supseteq q. \tag{6}$$

We shall also say that there exists a *Galois connexion* between P and Q when a pair of Galois correspondences (4) and (5) has been defined.

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

The condition (α) states that the two correspondences $\mathfrak{P}$ and $\mathfrak{Q}$ are order inverting. Through a combination of (α) and (β) one obtains directly

$$\mathfrak{Q}\mathfrak{P}\mathfrak{Q}(p) = \mathfrak{Q}(p), \qquad \mathfrak{P}\mathfrak{Q}\mathfrak{P}(q) = \mathfrak{P}(q). \tag{7}$$

The operations $\mathfrak{P}\mathfrak{Q}$ and $\mathfrak{Q}\mathfrak{P}$ represent correspondences of P to a subset and of Q to a subset. From (α) and (β) together with (7) it is easily shown that the correspondence $\mathfrak{P}\mathfrak{Q}$ has the properties:

(1) If $p_1 \supset p_2$ then $\mathfrak{P}\mathfrak{Q}(p_1) \supseteq \mathfrak{P}\mathfrak{Q}(p_2)$.

(2) $\mathfrak{P}\mathfrak{Q}(p) \supseteq p$.

(3) $\mathfrak{P}\mathfrak{Q}\mathfrak{P}\mathfrak{Q}(p) = \mathfrak{P}\mathfrak{Q}(p)$.

The analogous properties hold for $\mathfrak{Q}\mathfrak{P}$ and they show that the two operations

$$p \to \bar{p} = \mathfrak{P}\mathfrak{Q}(p), \qquad q \to \bar{q} = \mathfrak{Q}\mathfrak{P}(q) \tag{8}$$

are closure operations in P and Q respectively. The following facts about Galois connexions are easily derived.

THEOREM 2. *Let P and Q be partially ordered sets connected through the Galois correspondences* (4) *and* (5). *The application of one of these correspondences after the other defines a closure operation in P and in Q. The closed elements are the image elements $\mathfrak{P}(q)$ and $\mathfrak{Q}(p)$. Under the Galois correspondence each element has the same image as its closure*:

$$\mathfrak{Q}(p) = \mathfrak{Q}(\bar{p}), \qquad \mathfrak{P}(q) = \mathfrak{P}(\bar{q}), \tag{9}$$

and the Galois correspondences represent a one-to-one order inverting correspondence between the closed elements in P and Q.

It shall be assumed from now on that the two partially ordered sets P and Q are complete structures. The universal elements in P and Q shall be denoted by u_P and u_Q respectively and the zero elements are o_P and o_Q. Subsequently also the following additional condition for the Galois correspondences shall be postulated:

(γ) $o_Q = \mathfrak{Q}(u_P)$, $o_P = \mathfrak{P}(u_Q)$.

This assures that in the corresponding closure relations the zero elements are closed:

$$\bar{o}_P = o_P, \qquad \bar{o}_Q = o_Q.$$

In this case the closed elements in P and Q form complete structures P_1 and Q_1 which are substructures with respect to crosscut over P and Q respectively. According to Theorem 2 there exists a one-to-one order inverting correspondence between the structures P_1 and Q_1. Consequently this correspondence must be a dual isomorphism which takes unions in P_1 into crosscuts in Q_1 and crosscuts in P_1 into unions in Q_1. Thus if

$$\bar{p}_1 = \mathfrak{P}(\bar{q}_1), \qquad \bar{p}_2 = \mathfrak{P}(\bar{q}_2),$$

it follows that

$$
\mathfrak{Q}(\bar{p}_1 \cap \bar{p}_2) = \overline{\bar{q}_1 \cup \bar{q}_2},
\qquad (10)
$$
$$
\mathfrak{Q}(\bar{p}_1 \cap \bar{p}_2) = \bar{q}_1 \cap \bar{q}_2,
$$

and similarly for the correspondence from Q_1 to P_1. The rules (10) hold also for the unions and crosscuts of arbitrary sets of closed elements. In terms of arbitrary elements in Q the formulas (10) may be expressed in the following way

$$
\mathfrak{Q}(\mathfrak{P}(q_1) \cap \mathfrak{P}(q_2)) = \mathfrak{Q}\mathfrak{P}(q_1 \cup q_2),
$$
$$
\mathfrak{Q}(\mathfrak{P}(q_1) \cup \mathfrak{P}(q_2)) = \mathfrak{Q}\mathfrak{P}(q_1) \cap \mathfrak{Q}\mathfrak{P}(q_2),
$$

and analogously for the elements in P.

The preceding results show that one can obtain all Galois connexions between the two complete structures P and Q by the following construction: All pairs of complete structures P_1 and Q_1 must be determined for which P_1 is a substructure over P with respect to crosscut and Q_1 a substructure over Q with respect to crosscut such that P_1 and Q_1 are dually isomorphic. With each element p in P one associates as its closure $\bar{p}$ the smallest element in P_1 containing it and analogously in Q. The Galois correspondences are then defined by

$$
p \to \mathfrak{Q}(\bar{p}), \qquad q \to \mathfrak{P}(\bar{q})
$$

where $\mathfrak{P}$ denotes the dual isomorphism from Q_1 to P_1 and $\mathfrak{Q}$ its inverse.

Let P and Q be two partially ordered sets with a Galois connexion and P_1 and Q_1 the subsets of their closed elements. The Galois connexion shall be said to be *perfect in* P when every element in P is closed, hence when $P = P_1$. In terms of the Galois correspondences (4) and (5) the connexion is perfect in P when

$$
\mathfrak{P}\mathfrak{Q}(p) = p
$$

for every element p in P. Similarly the connexion is *perfect in* Q when $Q = Q_1$, so that every element in Q is closed and

$$
\mathfrak{Q}\mathfrak{P}(q) = q
$$

for every element q in Q. Finally the Galois connexion is *perfect* when it is perfect in both P and Q.

It is an important problem in the application of the theory of Galois connexions to determine when a given Galois connexion is perfect. This for instance represents the main content of the ordinary Galois theory of equations. The following criterion for a Galois connexion to be perfect is sometimes useful.

THEOREM 3. *A Galois connexion between two structures P and Q is perfect in P if and only if any two distinct elements $p_1 \supset p_2$ in P always have distinct images*

$$
\mathfrak{Q}(p_1) \subset \mathfrak{Q}(p_2).
$$

Proof. When the Galois connexion is not perfect in P there exists an element p different from its closure $\bar{p}$, and p and $\bar{p}$ have the same image in Q. Conversely when the Galois connexion is perfect in P every element p in P is closed and, since there is a one-to-one correspondence between the elements in P_1 and Q_1, distinct elements in P must have distinct images under $\mathfrak{Q}$.

3. **Mappings and Galois connexions.** It is possible to express the theory of Galois connexions in many forms. Later we shall give a formulation by means of binary relations. In this section we shall indicate an interpretation of the theory in terms of continuous mappings. This presentation is based upon certain investigations on closure relations and mappings which are as yet unpublished. However, since they will be prepared for publication shortly, I have taken the liberty of applying certain consequences in this section in order to give a more complete form to the present exposition.

We have already mentioned in §1 that any complete structure P can be considered to be the structure of closed sets for a suitable topological space. Any substructure P_1 of P with respect to crosscut corresponds to a mapping of this space onto a space whose structure of closed sets is isomorphic to P_1. Furthermore it shall be recalled that any substructure P_2 over P with respect to union can be considered to be the structure of closed sets of some dense relative space of a space with P as its structure of closed sets (Ore [2, p. 778]). Next let Q be the structure of closed sets of some other space and Q_1 a substructure over Q with respect to crosscut. To Q there exists a dual structure Q^* with the same elements as Q but with the operations of union and crosscut interchanged. The dual Q_1^* of Q_1 becomes a substructure over Q^* with respect to union. In the case of a Galois connexion between the structures P and Q the substructures P_1 and Q_1 of closed elements are dually isomorphic. But then P_1 and Q_1^* are isomorphic so that any space whose structure of closed sets is P can be mapped onto a space whose structure of closed sets is Q_1^*. On the other hand Q_1^* is the structure of closed sets of some dense relative space of a space for which Q^* is the structure of closed sets. It follows therefore that any Galois connexion between P and Q corresponds to a mapping of a space with a structure of closed sets isomorphic to P onto a dense subspace of a space whose structure of closed sets is isomorphic to the dual structure Q^* of Q.

It may be of interest to go somewhat further into details about this relation between mappings and Galois connexions. Let S and T be two spaces with the structures of closed sets P and Q respectively. Furthermore α shall be a mapping of S onto the relative space T_o of T. Each closed set q in T defines a closed set

$$q_o = q \cdot T_o \tag{11}$$

in T_o. The closed sets q_o in T_o form a complete structure which we shall denote by Q_o. Among all closed sets q in T giving rise to the same closed q_o

in T_o by means of (11) there is a unique minimal one which we shall denote by q_1. It is the intersection of all those sets q for which (11) holds. Thus to each closed set q_o in T_o there is associated in a one-to-one manner a closed set q_1 in T. The sets q_1 are seen to form a substructure Q_1 of Q with respect to union and because the correspondence between Q_o and Q_1 is order preserving these two structures are isomorphic. Furthermore since we shall suppose that T_o is a dense subspace of T the structure Q_1 is a substructure over Q.

By the mapping α of the space S onto the space T_o the inverse images

$$p_1 = q_o^{\alpha^{-1}}$$

of the closed sets in T_o are closed in S according to the general mapping theory and they form a substructure P_1 with respect to crosscut over the structure P of closed sets in S. This leads to an order preserving correspondence from Q to P_1 when one defines

$$q \to q \cdot T_o \to (q \cdot T_o)^{\alpha^{-1}}. \tag{12}$$

Conversely a correspondence from P to Q_1 is obtained by putting

$$p \to \bar{p}^{\alpha} \cdot T_o = q_o \to q_1 \tag{13}$$

where $\bar{p}^{\alpha}$ is the closure of p^{α} in T and q_1 the element in Q_1 which corresponds to q_o. It is verified immediately that these two correspondences (12) and (13) define a Galois connexion between P and the dual structure Q^* of Q such that P_1 and Q_1^* form the structures of closed elements.

4. **Galois connexions within a structure.** It is possible to define Galois correspondences between a structure P and itself. For such *Galois connexions within a structure P* one deduces immediately:

THEOREM 4. *Let P denote a complete structure. Every Galois connexion within P is defined by means of a pair of substructures P_1 and Q_1 over P with respect to crosscut such that P_1 and Q_1 are dually isomorphic.*

The actual Galois correspondences which define the connexion within P may be obtained explicitly as follows: Let p be an element in P and $\bar{p}$ the least element in P_1 containing p. Similarly $\tilde{p}$ is the least element in Q_1 containing p. The Galois correspondences can then be written

$$p \to \bar{p}^{\alpha}, \qquad p \to \tilde{p}^{\alpha^{-1}}, \tag{14}$$

where α denotes the dual isomorphism from P_1 to Q_1.

A special case of a Galois connexion within a structure occurs when the two defining substructures P_1 and Q_1 are identical. In this case we shall say that the Galois connexion is *structure self-dual.* To express the condition for such a Galois connexion we shall call a one-to-one correspondence α of a structure P to itself a *dual automorphism* when

$$(a \cup b)^\alpha = a^\alpha \cap b^\alpha, \qquad (a \cap b)^\alpha = a^\alpha \cup b^\alpha.$$

From Theorem 4 one concludes:

THEOREM 5. *All structure self-dual Galois connexions within a complete structure P are defined by some substructure P_1 over P with respect to crosscut which has a dual automorphism α. The Galois correspondences defining the connexion within P are of the form*

$$(15) \qquad p \rightarrow \bar{p}^{\alpha}, \qquad p \rightarrow \bar{p}^{\alpha^{-1}},$$

where $\bar{p}$ is the least element in P_1 containing the element p in P.

A Galois connexion within a structure P shall be called *self-dual* when the two defining correspondences (14) are identical. It follows immediately that a self-dual Galois connexion is structure self-dual. From Theorem 5 and (15) one derives:

THEOREM 6. *All self-dual Galois connexions within a complete structure P are defined by those substructures P_1 over P with respect to crosscut which possess a dual automorphism α such that*

$$\alpha = \alpha^{-1}.$$

A dual automorphism of this kind may be called an *involution*.

A still more special type of structures are those in which there exists an *orthogonality* or *polarity*. A polarity shall be defined to be an involution

$$a \rightleftarrows a^*$$

with the additional property that a^* is the complement of a,

$$a \cap a^* = o, \qquad a \cup a^* = u.$$

Because of the importance of the applications such polarities are of particular interest.

We shall consider briefly the Galois connexions between a complete structure P and its dual P^*. It follows from the general theory that any such Galois connexion is defined by a pair of isomorphic structures P_1 and Q_1, where P_1 is a substructure with respect to crosscut and Q_1 a substructure with respect to union both over P. To each element p in P one associates the least element $\bar{p}$ in P_1 containing p and the greatest element p^* in Q_1 contained in p. The Galois correspondences become

$$(16) \qquad p \rightarrow (\bar{p})^{\alpha}, \qquad p \rightarrow (p^*)^{\alpha^{-1}},$$

where α is a fixed isomorphism between P_1 and Q_1. In the special case where $P_1 = Q_1$ it follows that any substructure P_1 over P defines a Galois connexion between P and P^*. The correspondences in this case have the same form

(16), where $\bar{p}$ is the least element in P_1 containing p and p^* the greatest element in P_1 contained in p and α some automorphism of P_1. The following special type of such a Galois connexion may be of sufficient interest to be mentioned separately.

THEOREM 7. *Let P be a complete structure, P^* its dual and α a complete structure homomorphism of P. Then for any element p in P there exists a least element p_o and a greatest element p_1 such that*

$$p^{\alpha} = p_o^{\alpha} = p_1^{\alpha}$$

and the correspondences

$$p \rightarrow p_o, \qquad p \rightarrow p_1$$

define a Galois connexion between P and P^*.

5. **Binary relations and Galois connexions.** There are a great number of applications of the theory of Galois connexions. The term Galois connexion is of course chosen with the Galois theory of equations in mind. In this case one is concerned with a Galois connexion between the subfields of an algebraic extension K of a given field K_o and the subgroups of the group of all those automorphisms of K which leave K_o elementwise fixed. The basic result of the Galois theory of equations is that when K is a finite separable algebraic extension of K_o this Galois connexion is perfect. In the more general case where the extension K of K_o is not finite the Galois connexion is not perfect and as Krull [1] has shown there can be established a one-to-one correspondence only between the subfields and certain closed subgroups (see also Krasner [1]).

We shall not enter into this theory, nor into various other well known examples of Galois connexions. For the remaining part of this paper we shall consider the interrelation between Galois connexions and the theory of binary relations along the lines indicated in the Colloquium lectures in Chicago in 1941.

Let S and S' be two sets which under circumstances may be permitted to be identical. A *binary relation* R from S to S' is an association

$$a \rightarrow R_a = R(a) \tag{17}$$

of a subset R_a of S' to each element a of S. When a' is an element belonging to R_a one writes

$$a'Ra \tag{18}$$

and says that *a' is in the relation R to a*. The sets R_a in (17) shall be called the *basic sets* for R.

Let us assume next that the sets R_a *cover* S', that is, each element in S'

belongs to at least one set R_a. Then one can define a converse relation R^* from S' to S such that

$$aR^*a' \tag{19}$$

holds if and only if (18) is satisfied. The basic sets $R^*_{a'}$ for R^* consist of all elements a for which (18) holds for the fixed element a'.

The relational notation (18) can be extended to arbitrary subsets by writing

$$A'RA$$

whenever the elements a' in the subset A' of S' satisfy the relation (18) for every a in A. To each set A in S one can define a set $R(A)$ consisting of all elements a' for which

$$a'RA.$$

Clearly one must have

$$R(A) = \prod_{a \subset A} R(a). \tag{20}$$

This shows that the sets $R(A)$ form a complete intersection ring of sets in S'. If one adjoins the whole set S' and the void set O' to the family of sets $R(A)$ a closure relation $\Gamma' = \Gamma_R'$ is defined in S'. We shall call Γ' the *closure relation induced by R in S'*. It is often convenient to assume that the void set O' already belongs to the family (20). This is the case if and only if there exists no element a' in S' such that the relation (18) holds for every a in S since then $R(S) = O'$. The analogous observations may be made in connexion with the converse relation R^*. To each subset A' of S' there exists a set $R^*(A')$ consisting of all elements a in S for which

$$aR^*A'.$$

Corresponding to (20) one finds

$$R^*(A') = \prod_{a' \subset A'} R^*(a'). \tag{21}$$

These sets $R^*(A')$ form a complete intersection ring of sets in S, hence they induce a closure relation $\Gamma = \Gamma_{R^*}$ in S. The void set O belongs to the family (21) if and only if there exists no element a in S such that (18) holds for every a' in S'.

From (20) one finds that if $A_1 \supset A_2$ are two sets in S then

$$R(A_1) \subseteq R(A_2)$$

and similarly for R^*. One also verifies that

$$R^*R(A) \supset A, \qquad RR^*(A') \supset A'.$$

These results lead immediately to the following result formulated first by

Garrett Birkhoff [1] in a slightly different manner:

THEOREM 8. *Every binary relation R and its converse R^* between the two sets S and S' define a Galois connexion between the subsets of these sets through the correspondences*

$$A \to R(A), \qquad A' \to R^*(A').$$

All the consequences of the previous theory of Galois connexions follow. One has

$$RR^*R(A) = R(A), \qquad R^*RR^*(A) = R^*(A)$$

and also the identities corresponding to the rules (10) which express the dual isomorphism between the two families of closed sets.

It has already been observed that in the generality in which Theorem 8 has been stated the sets $R(A)$ and $R^*(A')$ do not necessarily form a closure relation in S and S' since the sets O, O', S, S' are not always closed. To bring Theorem 8 in complete agreement with our previous theory of Galois connexions between structures we shall restate Theorem 8 in the restricted form:

THEOREM 9. *Let R and R^* denote a binary relation and its converse defined between the two sets S and S'. These relations shall be subject to only the restriction that there shall be no element a' in S' such that*

$$a'Ra$$

for every a in S, nor any a in S such that this relation holds for every a' in S'. Then the correspondences

$$A \to R(A), \qquad A' \to R^*(A')$$

supplemented with

$$O \rightleftarrows S', \qquad O' \rightleftarrows S$$

define a Galois connexion between S and S' such that there exists a dual isomorphism between the closed sets under the two induced closure relations Γ_{R^*} in S and Γ'_R in S'.

We have just observed that any binary relation R defines a Galois connexion between the structures of all subsets of the two sets S and S'. It is not difficult to show that the converse is also true (see Everett's paper in the present issue of these Transactions).

THEOREM 10. *Any Galois connexion between the structures of all subsets of the two sets S and S' can be defined by means of a binary relation R between the two sets.*

Proof. We denote by P and Q the structures of all subsets of the two sets S and S' and by P_1 and Q_1 the substructures with respect to crosscuts defining the Galois connexion. The Galois correspondences are

$$A \to \mathfrak{Q}(A), \qquad A' \to \mathfrak{P}(A')$$

for the subsets A and A' in S and S'. We also write

$$\overline{A} = \mathfrak{P}\mathfrak{Q}(A), \qquad \overline{A}' = \mathfrak{Q}\mathfrak{P}(A')$$

for the two closures defined in the sets by the connexion. Between the structures P_1 and Q_1 there exists a dual isomorphism

$$(22)\qquad \begin{aligned} \overline{A} \to \overline{A}^{\alpha} &= \mathfrak{Q}(\overline{A}) = \mathfrak{Q}(A), \\ \overline{A}' \to \overline{A}'^{\alpha^{-1}} &= \mathfrak{P}(\overline{A}') = \mathfrak{P}(A'). \end{aligned}$$

To define the Galois connexion we introduce the two binary relations

$$(23)\qquad a'Ra, \qquad aR_1a'$$

which shall hold respectively when

$$(24)\qquad a' \subset a^{\alpha} = \bar{a}^{\alpha}, \qquad a \subset (a')^{\alpha^{-1}} = \overline{a}'^{\alpha^{-}}.$$

By the Galois connexion one of these conditions implies the other so that in (23) $R_1 = R^*$ is the converse of R. Furthermore one has

$$R(A) = \prod_{a \subset A} R(a) = \prod_{a \subset A} \bar{a}^{\alpha}$$

and by the dual structure isomorphism between P_1 and Q_1 this gives

$$R(A) = (\bigvee_{a \subset A} \bar{a})^{\alpha} = \overline{A}^{\alpha}$$

so that a comparison with (22) shows that the relation R defines the Galois connexion.

6. **Relations with perfect Galois connexion.** Let us investigate when the Galois connexion defined between the subsets of two sets by a binary relation can be perfect. We shall use the previous notations. The binary relation R defines a Galois connexion between the structures P and Q of all subsets of the sets S and S'. When the Galois connexion defined by R shall be perfect in Q every subset of S' must be the intersection of sets $R(a)$. This can be the case only when every maximal subset of S',

$$M'_{a'} = S' - a',$$

containing all but a single element a' of S', belongs to the family of sets R_a. By selecting one element a_1 in S for each element a' in S' such that

$$R_a = S' - a',$$

a one-to-one correspondence

$$a_1 \rightleftarrows a' = a_1^{\alpha}$$

is defined between S' and the set S_1 of all elements a_1. This leads to the result:

THEOREM 11. *The Galois connexion between the subsets of two sets S and S' defined by a binary relation R is perfect in the subsets of S' if and only if R can be constructed as follows: Let α be a one-to-one correspondence from a subset S_1 of S to S'. For an element a_1 in S_1 the relation*

$$a'Ra$$

shall hold if and only if

$$a' \neq a_1^{\alpha}.$$

For the elements in $S-S_1$ the relation R may be defined arbitrarily.

We shall determine next when the Galois connexion is perfect both ways. In this case the two sets S and S' must have the same cardinal number according to Theorem 11. For each set $R^*_{a'}$ corresponding to an element a' in S' there is at least one element in S, namely

$$a_1 = (a')^{\alpha^{-1}},$$

which does not belong to it. But if the connexion shall be perfect in the subsets of S the converse sets $R^*_{a'}$ must include all maximal sets

$$M_a = S - a$$

in S. One concludes that this is possible only when the set S_1 is the whole set S and the sets $R^*_{a'}$ are the maximal subsets of S,

$$R^*_{a'} = S - (a').$$

This shows that α is a one-to-one correspondence between S and S' and the relation $a'Ra$ is equivalent to the statement

$$a' \neq a^{\alpha}.$$

For any relation R one can introduce a complementary relation R^c which holds,

$$a'R^c a,$$

if and only if the relation $a'Ra$ does not hold. One can then state our result as follows:

THEOREM 12. *The Galois connexion defined by a binary relation R between the structures of all subsets of two sets S and S' is perfect if and only if R is the complementary relation to a one-to-one correspondence,*

$$a' \rightleftarrows a^{\alpha},$$

between S and S'.

7. **Dual topologies and binary relations.** This section contains results es-

sentially due to Garrett Birkhoff [1], however in a somewhat different formulation and with certain supplementary results. Let S and S' be two spaces with the closure relations Γ and Γ' respectively. We shall say that S and S' are *dual spaces* (Ore [2]) when the structure of all closed sets in S is dually isomorphic to the structure of all closed sets in S'. From our Theorems 9 and 10 one obtains immediately:

THEOREM 13. *Any binary relation R between two sets S and S', such that*

$$a'Ra$$

cannot hold for a fixed a in S and all a' in S' and conversely, defines dual topologies in the two sets by means of the Galois connexion it induces. Conversely any pair of dual spaces can be obtained from some binary relation in this manner.

It may well happen that a space S can be topologized by two topologies Γ and Γ' which are dual. Through specialization of Theorem 13 one obtains immediately:

THEOREM 14. *Let R be a binary relation in a set S satisfying the condition that*

$$aRb$$

shall not hold for all b by a fixed a and conversely. Then R induces a Galois connexion within the family of all subsets of S and also two corresponding dual topologies in S. Conversely any space S with two dual closure relations can be obtained from some suitable binary relation in S in this way.

From the proof of Theorem 10 follows that the dual topologies may be defined by the two relations

$$aRb, \qquad bR^*a, \tag{25}$$

which hold if and only if

$$a \subset \bar{b}^{\alpha}, \qquad b \subset \tilde{a}^{\alpha^{-}}, \tag{26}$$

where $\bar{A}$ and $\tilde{A}$ denote the closure operation under the two topologies and α the dual isomorphism between their structures of closed sets.

A space is said to be *self-dual* when its structure of closed sets has a dual automorphism α. One can also define self-duality of a space by the property that there shall exist a one-to-one order preserving correspondence between the family of closed sets and the family of open sets. As a special case a space or topology shall be called *involutory* when there exists an involution in its structure of closed sets. It follows from the preceding that a self-dual topology in a space S is definable by a binary relation R in S, where R and its converse R^* in (25) hold respectively when

$$a \subseteq \bar{b}^{\alpha}, \qquad b \subseteq \bar{a}^{\alpha^{-1}}, \tag{27}$$

where α is the dual automorphism of the structure of closed sets.

For an involutory topology α is a correspondence for which $\alpha=\alpha^{-1}$ and the two relations (27) reduce to

$$a \subset \bar{b}^{\alpha}, \quad b \subset \bar{a}^{\alpha}.$$

This shows that the relation R is identical with its converse

$$R = R^{*}.$$

Such a relation is called a *symmetric relation.* Thus any involutory topology in a set can be induced by a symmetric binary relation in the set. Conversely any symmetric relation induces an involutory topology in the set, because when R is symmetric the Galois connexion induced by R within the structure of all subsets A of S is defined by the single correspondence

$$A \rightarrow R(A) = \prod_{a \subset A} R(a).$$

We state therefore:

THEOREM 15. *Any symmetric relation R in a set S subject to the condition that*

$$aRb$$

cannot hold for all b, for a fixed element a, defines an involutory Galois connexion and topology within the structure of all subsets of S. Conversely any complete involutory structure or topology can be obtained from a symmetric relation in this manner.

It is of interest to determine also how the self-dual spaces can be defined by means of binary relations. From (27) one sees that a self-dual space is defined by means of a binary relation R for which the basic sets

$$R_a = \bar{a}^{\alpha}$$

are closed under the closure relation Γ_{R^*} induced by the converse relation R^*, and analogously for R^*. Hence R and R^* generate the same closure relation in S. But on the other hand, when a relation R and its converse do define the same closure relation, the Galois connexion expresses that this topology is self-dual. As a consequence we can state:

THEOREM 16. *Any relation R in a set S (subject only to the condition in Theorem* 14) *such that R and R^* define the same closure relation induces a self-dual topology in S and every self-dual topology is obtainable in this manner.*

One may wish to express the condition for R and R^* to generate the same closure operation in S more directly in terms of relational properties. This may be done as follows. In order that the sets closed under Γ_R shall be closed under Γ_{R^*} it is necessary and sufficient that each set R_a be the intersection of sets R_b^*, hence that there exists a representation

$$R_a = \prod_{b \subset B_a} R_b^*$$

where the elements b run through some fixed set B_a associated with a. Analogously when every set closed under Γ_{R^*} is closed under Γ_R there must exist a set B_a^* such that

$$R_a^* = \prod_{b \subset B_a^*} R_b.$$

This leads to the criterion:

THEOREM 17. *The necessary and sufficient condition for a relation R and its converse R^* in a set S to define the same self-dual closure relation is that to each element a in S there exist two sets B_a and B_a^* such that the two relations*

$$xRa, \qquad aRx$$

hold if and only if

$$b_aRx, \qquad xRb_a^*$$

respectively for every b_a in B_a and b_a^* in B_a^*.

When the closure relation defined by R and R^* satisfies the finite chain condition or particularly when S is a finite set the condition for the two induced closure relations to be the same can be expressed more simply. In this case there exists for R a fundamental family of basic sets $\{R(a_0)\}$ consisting of those sets R_a which are not the intersection of others. Clearly R and R^* define the same closure relation when their fundamental families of sets are the same.

A binary relation R in a set S is said to be *reflexive* when aRa for every a in S. Similarly we shall define R to be *anti-reflexive* when aRa holds for no element a in S. The converse relation R^* is reflexive or anti-reflexive at the same time as R. When R is anti-reflexive the set $R(a)$ does not contain a, hence in general for any set A

$$A \cdot R(A) = 0$$

and this property may also be used to define an anti-reflexive relation. From

$$R(A) \cdot RR(A) = 0$$

one concludes by the duality defined by the Galois connexion through R in the subsets of S

$$R^*(R(A) \cdot RR(A)) = R^*R(A) \cup R^*RR(A) = S$$

and similarly one obtains

$$RR^*(A) \cup RR^*R^*(A) = S.$$

Let us assume next that the relation R is both symmetric and anti-reflexive. In this case the previous results reduce to

$$R(A)\cdot RR(A) = 0, \qquad R(A) \cup RR(A) = S$$

for any subset A of S. This shows that the involution in the structure of closed sets defined by R is a polarity. Conversely let us assume that Σ is some complete structure with a polarity. Since a polarity α is a special case of an involution the symmetric relation aRb defining it will hold if and only if

$$a \subset \bar{b}^{\alpha}, \qquad \alpha^2 = 1.$$

But for polarity the two sets $\bar{a}$ and $\bar{a}^{\alpha}$ are disjoint, hence a is not contained in $\bar{a}^{\alpha}$ and R is an anti-reflexive relation. Thus it has been shown:

THEOREM. *Any symmetric anti-reflexive binary relation in a set S defines a Galois connexion within the structure of all subsets of S such that the structure of closed sets has a polarity. Conversely any complete structure with a polarity can be generated from some symmetric anti-reflexive relation in this manner.*

It should be noted that this result gives an actual method for the construction of all complete structures which have a polarity.

8. **Galois theory for relations.** To every binary relation R in a set S there is associated a *group of automorphisms* $\mathfrak{G}_R$. An *automorphism* α of the relation R is a one-to-one correspondence of the set S such that any relation

$$aRb$$

implies

$$a^{\alpha}Rb^{\alpha}, \qquad a^{\alpha^{-1}}Rb^{\alpha^{-1}}.$$

We shall not discuss the general theory of automorphisms of binary relations here. It may be observed only that through the group of automorphisms of a relation one can introduce a Galois theory with respect to this relation. To every subset A of S there exists a subgroup $\mathfrak{G}(A)$ of the group $\mathfrak{G}$ of all automorphisms of R consisting of those automorphisms which leave A elementwise fixed. Similarly to each subgroup $\mathfrak{A}$ of $\mathfrak{G}$ there exists a subset $S(\mathfrak{A})$ consisting of those elements in S which are left invariant by the automorphisms in $\mathfrak{A}$. One verifies immediately that in this manner a Galois connexion has been established between the subsets of S and the subgroups of $\mathfrak{G}$. (See also A. R. Richardson [1].)

We shall consider only as an example the case of an equivalence E in S. The equivalence relation defines a partition $\mathfrak{P}(B)$ of S into disjoint blocks B, each block a maximal set of equivalent elements. For each cardinal number κ let us assume that there exists a family

$$\mathfrak{F}_{\kappa} = \mathfrak{F}_{\kappa}(\mathfrak{B}_{\kappa}^{(i)}), \qquad F_{\kappa} = \sum \mathfrak{B}_{\kappa}^{(i)} \tag{28}$$

of n_{κ} blocks $B_{\kappa}^{(i)}$ each with κ elements. The automorphisms of E are seen to be those one-to-one correspondences of S which transform a block $B_\kappa^{(j)}$ into itself or takes it in its entirety into some other block in the same family $\mathfrak{F}_\kappa$. As a consequence one finds that the group of automorphisms of E is a direct product

$$\mathfrak{G} = \prod_\kappa \mathfrak{G}_\kappa \tag{29}$$

with one factor $\mathfrak{G}_\kappa$ for each family $\mathfrak{F}_\kappa$. Each factor is a complete monomial group of dimension n_κ over the symmetric group on κ elements (Ore [3, p. 610; 4])

$$\mathfrak{G}_\kappa = \Sigma_{n_\kappa}(\Sigma_\kappa) \tag{30}$$

defined as a permutation group over the set F_κ in (28).

Now let A be a subset of S and let us determine the subgroup $\mathfrak{G}(A)$ of those automorphisms of E which leave A elementwise invariant. From the representation (29) of the group of automorphisms one sees that $\mathfrak{G}(A)$ is also a direct product

$$\mathfrak{G}(A) = \prod_\kappa \mathfrak{G}_\kappa(A), \tag{31}$$

where the direct factor $\mathfrak{G}_\kappa(A)$ consists of all permutations in the group (30) leaving the intersection $A \cdot F_\kappa$ elementwise invariant.

In regard to the set A the blocks in the family $\mathfrak{F}_\kappa$ fall into two categories, namely, in the first category those blocks B_κ' without common elements with A, in the second those blocks B_κ'' which have a non-void intersection with A. The elements in a block B_κ' may be permuted arbitrarily or such a block may be transformed entirely into some other block of the same category. To leave the elements in A invariant the blocks B_κ'' of the second category must be transformed into themselves in such a manner that the intersections $A \cdot B_\kappa''$ remain elementwise fixed. These remarks show that in (31) each factor $\mathfrak{G}_\kappa(A)$ is itself the product of two direct factors

$$\mathfrak{G}_\kappa(A) = \mathfrak{G}_\kappa'(A) \times \mathfrak{G}_\kappa''(A). \tag{32}$$

Here the first factor is a complete monomial group

$$\mathfrak{G}_\kappa'(A) = \Sigma_{n'}(\Sigma_\kappa), \tag{33}$$

where the dimension $n' = n_\kappa'$ is the number of blocks B_κ' and Σ_κ the symmetric group over a set with κ elements. The second factor in (32) is a direct product of the symmetric groups corresponding to the various residual sets $B''-A$,

$$\mathfrak{G}_\kappa''(A) = \prod_{B''} \Sigma_{B''-A}. \tag{34}$$

In this manner the closed subgroups $\mathfrak{G}(A)$ in the Galois theory defined by the equivalence relation E have been determined. Conversely let us find the closed subsets $\bar{A}$. The closure $\bar{A}$ is the set of all elements in S left invariant by all automorphisms belonging to the group $\mathfrak{G}(A)$. It follows from the preceding that for any κ the elements in a block B'_κ cannot belong to $\bar{A}$ since there are permutations in $\mathfrak{G}(A)$ for which they are not invariant. The only exceptional case occurs when there is only a single set B'_κ and when this block consists of a single element, hence $\kappa=1$. Similarly the elements in a residual set $B''_\kappa - A$ are not left invariant by all permutations in $\mathfrak{G}(A)$ except in the case where it consists of a single element. We can summarize these results in the following way:

THEOREM 18. *In the Galois theory defined by an equivalence relation E in a set S the closed subgroups of the group of automorphisms of E are determined by* (31), (32), (33) *and* (34). *The closed subsets $\bar{A}$ of S are those which have the property that for each cardinal number $\kappa \neq 1$ and each block B_κ with κ elements the set $\bar{A}$ either contains B_κ entirely or omits at least two of its elements. For $\kappa=1$ the set A must either contain all blocks B_1 or omit at least two of them.*

It may be observed in general that when a group $\mathfrak{G}$ is represented as a permutation group on a set S there exists a Galois connexion between the subgroups of $\mathfrak{G}$ and the subsets of S. To each subset A corresponds the subgroup $\mathfrak{G}(A)$ of all permutations in $\mathfrak{G}$ leaving A elementwise invariant and to each subgroup $\mathfrak{H}$ corresponds the subset $S(\mathfrak{H})$ of the elements left invariant by $\mathfrak{H}$.

We consider first the case where $\mathfrak{G}$ is represented as a transitive group on S. Let $\mathfrak{H} = \mathfrak{H}_{a_0}$ be the subgroup leaving a single element a_0 in S fixed. The group which leaves an arbitrary element a of S invariant is then a conjugate group $\mathfrak{H}_a$ of $\mathfrak{H}$. According to the general representation theory the elements in S are in one-to-one correspondence with the cosets $m\mathfrak{H}$ of $\mathfrak{G}$ with respect to $\mathfrak{H}$,

$$\mathfrak{G} = \sum_m m\mathfrak{H},$$

and the permutation corresponding to an element x can be represented in the form

$$P_x = \begin{pmatrix} m\mathfrak{H} \\ xm\mathfrak{H} \end{pmatrix}.$$

We say that the representation is *effected* by the subgroup $\mathfrak{H}$. For any subset A of S the subgroup leaving A invariant is

$$\mathfrak{G}(A) = \prod_{a \subset A} \mathfrak{H}_a.$$

This leads to the statement:

THEOREM 19. *Let $\mathfrak{G}$ be a group represented transitively as a permutation* *group on a set S. By the Galois connexion between the subgroups of $\mathfrak{G}$ and the subsets of S the closed subgroups are the crosscuts of the conjugates of the group $\mathfrak{H}$ effecting the representation.*

The preceding argument can easily be extended to the case when $\mathfrak{G}$ is an intransitive group. We denote the transitive systems by S_i so that

$$S = \sum_i S_i$$

is a partition of S. In each set S_i the group $\mathfrak{G}$ is represented transitively by means of some subgroup $\mathfrak{H}_i$. The representation is isomorphic or homomorphic depending on whether $\mathfrak{H}_i$ contains or does not contain any normal subgroup of $\mathfrak{G}$. Since the group leaving an element a of S_i invariant is a conjugate $\mathfrak{H}_i^{(a)}$ of $\mathfrak{H}_i$ one obtains:

THEOREM 20. *Let $\mathfrak{G}$ be a group represented as a permutation group on a set S by means of a set of subgroups $\{\mathfrak{H}_i\}$. In the Galois connexion between $\mathfrak{G}$ and S the closed subgroups are those which are the intersection of conjugates of the groups $\{\mathfrak{H}_i\}$.*

This theorem shows that the closure relation defined in $\mathfrak{G}$ through the Galois connexion depends only upon the subgroups $\mathfrak{H}_i$ and their conjugates. A family $\mathfrak{F}(\mathfrak{H})$ of subgroups $\mathfrak{H}$ of $\mathfrak{G}$ may be called a *normal family* when it contains all conjugates of any of its groups. Every representation of a group $\mathfrak{G}$ defines a normal family of subgroups consisting of those subgroups which leave one element in the representation set S fixed. Conversely any normal family of subgroups $\mathfrak{F}(\mathfrak{H})$ can be used to construct representations of $\mathfrak{G}$. These representations are, however, not uniquely defined since the same group $\mathfrak{H}$ may be used to construct several transitive constituents in the representation. The representation is isomorphic only if the groups in the normal family have the unit group for their intersection. Theorem 20 may be restated:

THEOREM 21. *Let $\mathfrak{G}$ be a group whose isomorphic representation as a permutation group on a set S corresponds to the normal family of subgroups $\mathfrak{F}(\mathfrak{H}_i)$. In the corresponding Galois connexion the closed subgroups are the intersection of subgroups in $\mathfrak{F}$ and the Galois connexion is perfect in $\mathfrak{G}$ if and only if every subgroup of $\mathfrak{G}$ is such an intersection.*

This theorem reduces the question of finding those representations which give a perfect Galois connexion in the group to the problem of finding all sets of subgroups generating all subgroups of $\mathfrak{G}$ by forming conjugates and intersections.

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