CLEFT RINGS

BY

BERNARD VINOGRADE

A non-nilpotent associative ring \( R \) with minimum condition on left ideals (ML) has a nilpotent radical and is semisimple modulo the radical, that is, the minimum condition insures that \( R \) be semi-primary\(^{(1)} \). In this paper the assumption is made that \( R \) is an associative ring with ML, and the main purpose is to launch an investigation of \( R \) when it is a cleft ring. \( R \) will be called cleft when it contains a subring \( R^* \cong R/N \) such that \( R = R^* + N \), a group direct sum.

Perhaps the most widely known results pertaining to cleft rings are those concerning linear associative algebras. Wedderburn proved that every algebra over a field of characteristic zero is cleft [14, p. 158]. More recently this theorem has been generalized to the theorem that an algebra is cleft if it is separable modulo its radical [7, p. 24]. Cleft algebras also suggest themselves as having simpler multiplication tables than uncleft algebras [1, p. 172].

Not only does the study of cleft rings appear to be a natural step toward the study of general rings with ML, but the cleavage property (property of being cleft) also permits the exploitation of methods used to prove the general structure theorems on semisimple rings [2]. The cleavage of \( R \) is equivalent to the cleavage of the completely primary rings defined by the primitive idempotents of \( R \). This property is the key to the theory developed here.

If \( R \) is a cleft ring with a unit \( e \), then a commutative group \( V = eV \) with \( R \) as left operator set and a finite admissible composition series is shown to be a sum of vector spaces (composite module) over certain division rings derived from \( R \). In particular \( R \) furnishes a composite representation module for itself. This representation is developed and finally applied to the case of algebras.

Regarding the notation used: When \( R \) has a unit \( e \) and is a left operator set for the module \( V = eV \), then the inverse \( R \)-homomorphism ring of \( V \) will be denoted by \( R' \), and the application of an element \( r' \) of \( R' \) to \( V \) will be indicated by a right multiplication \( Vr' \). Then \( V \) is an \( R \)-left and \( R' \)-right

---

\(^{(1)} \) See [3] or [10]. Adopting the notation of many writers on ring theory (Artin, Kothe, Deuring, and so on), semi-primary will mean that \( R/N \) is semisimple; primary, that \( R/N \) is simple; completely primary, that \( R/N \) is a division ring. In the latter two cases it is assumed that \( R \) has a unit. For conditions equivalent to the semi-primary property see [12]. Numbers in brackets refer to the Bibliography at the end of the paper.
module, or briefly, an \((R, R')\) module. \(R''\) will denote the \(R'\)-homomorphism ring of \(V\), and is applied to the left of \(V\). Since it will be assumed that \(r \neq 0\), \(R\) will be isomorphic to a subring of \(R''\).

Part I is devoted mainly to the properties of \(R\) as an abstract ring. Part II deals with the representation theory of \(R\) when it has a unit, and without loss of generality only rings which are indecomposable two-sided ideals are considered.

I

1. \(R\) is a ring with \(ML\). If \(R\) can be expressed as a group direct sum \(R = R^* + N\), where \(R^*\) is a semisimple ring and \(N\) is the radical, then \(R\) is a cleft ring.

**Theorem 1.** If \(R^*\) is any semisimple subring of \(R\), then \(R^*\) is isomorphic to a subring of \(\overline{R} = R/N\) and defines a cleft subring \(R^* + N\) of \(R\).

**Proof.** Let \(r^*\) and \(s^*\) be distinct elements of \(R^*\). Then \(r^* - s^*\) is in \(R^*\) and hence not in \(N\). Therefore \(r^*\) and \(s^*\) are distinct elements of \(\overline{R}\). If \(\overline{R}^*\) is the subset of \(\overline{R}\) which can be represented by \(R^*\), then \(\overline{R}^* \cong R^*\), and if \(P\) is the totality of elements of \(R\) which define the elements of \(\overline{R}^*\) then \(P = R^* + N\) is a cleft ring.

If \(R^* \cong \overline{R}^*\), and if \(R\) is an algebra over a field \(F\) and \(R^*\) a subalgebra (over \(F\)), then \(R = R^* + N\); the converse is obvious. If \(r\) is an element of \(R\) which has an inverse, then \(rRr^{-1}\) gives another cleavage of \(R\), for \(R = rRr^{-1} = rR^*r^{-1} + rNr^{-1}\) and \(rNr^{-1} = N\).

**Lemma 1.** If \(e^*\) is the unit of \(R^*\) in the cleft ring \(R\), then \(e^*\) is a principal idempotent of \(R\). If \(R\) has a unit, then \(e^*\) is that unit.

**Proof.** If there were an idempotent \(e' = r^* + n \neq 0\) such that \(e'e^* = e^*e' = 0\) then \(e^*e' = e^*r^* + e^*n = 0\) implies \(e^*r^* = r^* = 0\). Therefore \(e' = n\), which is impossible unless \(n = 0\). Hence \(e = 0\). If \(R\) has a unit \(e\), then \((e - e^*)e^* = e^*(e - e^*) = 0\), hence the idempotent \(e - e^* = 0\), that is, \(e = e^*\).

Let \(R = \sum Re_{ij} + N_0\) be a decomposition of \(R\) where \(Re_{ij}\) is an indecomposable left ideal and \(N_0\) is a subset of \(N\) [7, p. 13]. The \(e_{ij}\) are primitive orthogonal idempotents, \(e = \sum e_{ij}\) is a principal idempotent, and the notation is chosen so that \(\overline{R} = \sum_{i=1}^t \overline{R}^i = \sum_{i=1}^t \sum_{j=1}^s \overline{R} e_{ij}\) is the corresponding decomposition of the semisimple ring \(\overline{R}\) into simple rings and sets of isomorphic left ideals respectively.

**Lemma 2.** The number of primitive orthogonal idempotents into which any principal idempotent of \(R\) can be decomposed is constant.

**Proof.** \(n_i\) and \(s\) are unique for \(\overline{R}\). Hence \(\sum_{i=1}^t n_i\) is constant.

Let \(C_{ij}\) denote the completely primary ring \(e_{ij}Re_{ij}\). Let \((C_{ij})\) and \((C_{ij}')\) be the sets of completely primary rings corresponding to two arbitrary prin-
principal idempotents $e = \sum e_{ij}$ and $e' = \sum e'_{ij}$ of $R$.

**Lemma 3.** The rings in the sets $(C_{ij})$ and $(C'_{ij})$ are isomorphic, in the order for instance of the subscripts.

**Proof.** From the uniqueness of $\bar{R}^i$ it follows that $\bar{R}^i = \sum_{k=1}^{n_i} \bar{R}e_{ik} = \sum_{l=1}^{n'_i} \bar{R}e'_{il}$, where $\bar{R}e_{ik} \cong \bar{R}e'_{il}$ for every $k$ and $l$. Now, it is known that an isomorphism between two indecomposable left ideals $Re_i$ and $Re'_j$ as $R$ left spaces ($e_i$ and $e'_j$ are primitive idempotents) is equivalent to the isomorphism of $\bar{R}e_i$ and $\bar{R}e'_j$ as $R$ left spaces [12, p. 182]. But $\bar{R}e_{ik} \cong \bar{R}e'_{il}$ not only as an $\bar{R}$ space but as an $\bar{R}$ space, because $\bar{R}\bar{R}^i = \delta_{ij}\bar{R}^i$. Hence $Re_{ik} \cong Re'_{il}$ as $R$ left spaces. Further, the inverse homomorphism rings of these two ideals are isomorphic to $C_{ik}$ and $C'_{il}$ respectively; for if $r \in Re_{ik}$ and $\sigma$ is a mapping in the inverse homomorphism ring, $\sigma(r) = r\sigma(e_{ij}) = r_{ij}\sigma(e_{ij})e_{ij}$; hence $C_{ik} \cong C'_{il}$.

**Lemma 4.** If each $C_{ij}$ of $(C_{ij})$ is cleft, then each $C'_{ij}$ of $(C'_{ij})$ is cleft.

**Proof.** The proof follows from Lemma 3.

According to a known decomposition theorem [7, p. 17] $R$ can be written as $R = \sum_{j=1}^{n_i} P^j + N_0$ corresponding to the decomposition of $r \in R$ in the expression $r = ere + (r - ere)$, where $e$ is a principal idempotent. $P^i = e^i_Re^i$, where $e^i = \sum_{j=1}^{n_i} e_{ij}$, is a primary ring such that $P^i \cong P^i/e^iNe^i \cong \bar{R}^i$. Since the elements of $N_0$ are of the form $\sum_{j=1}^{n_i} e_{ij}e^j + r - ere$, then $e^iNe^i = e_{ij}N_0e_{ij} = 0$.

Let $e_i$ be an arbitrary idempotent of the set $e_{ij}, j = 1, \ldots, n_i$; let $C_i$ be the corresponding ring of the set $(C_{ij})$; let $N^i = e^iNe^i$; let $(Q)_n$ be a total matric set of degree $n$ over a set $Q$, $e_{ij}$ the matric units such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and $q_{ij} = e_{ij}q$ for $q \in Q$, hence $(Q)_n = \sum Qe_{kl}$.

**Lemma 5.** If $R$ is cleft, then every $C_{ij}$ is cleft.

**Proof.** Choose, by Lemma 1, the unit of $R^*$ for $e$ and decompose it in $R^*$. Then from $R^* = R + N$ it follows that $C_i = e_iR^*e_i + e_iNe_i$. Now $e_iR^*e_i$ is a division ring $K_i \cong \bar{C}_i = C_i/e_iNe_i$ and $e_iNe_i$ is the radical of $C_i$. Hence $C_i$ is cleft.

**Lemma 6.** The radical of $\sum P^j + N_0$ is $\sum N^i + N_0$.

**Proof.** $N_0$ and the $N^i$ are contained in $N$ because of their definition. On the other hand, suppose $n \in N_i, n = \sum p^i + n_0$ where $p^i \in P^i$ and $n_0 \in N_0$. Then $e^iNe^i = p^i \in N^i$ for every $i$. Hence $n \in \sum N^i + N_0$.

**Lemma 7.** $P^i \cong (C_i)_{n_i}$.

**Proof.** $P^i = \sum_{k=1}^{n_i} e_{ik}P^i$, where the $e_{ik}P^i$ are isomorphic indecomposable right ideals. $P^i$ is its own homomorphism ring as a $P^i$ right space, and $C_{ik}$ is the homomorphism ring of $e_{ik}P^i$ as a $P^i$ right space. Hence, by a known theorem [7, p. 18], $P^i \cong (C_i)_{n_i}$.

**Lemma 8.** $e_iNe_i = e_{kl}$ is the intersection of $C_i e_{kl}$ and the radical of $(C_i)_{n_i}$. 


Proof. If \( e \neq 0 \) then \( e \neq 0 \) for some \( n \in N \). Suppose there is no \( n \) such that \( e \nequiv 0 \) then \( e \nequiv 0 \). Then there exists \( \rho \in C_i \) such that \( e \nequiv \rho \nequiv 0 \). Therefore \( e \nequiv \rho \nequiv 0 \) is not nilpotent. Therefore \( e \nequiv \rho \nequiv 0 \) is not in the radical of \( (C_i)_n \). On the other hand, \( e \nequiv \rho \nequiv 0 \) is properly nilpotent in \( (C_i)_n \).

Lemma 9. \( (e \nequiv Ne_i)_n \) is the radical of \( (C_i)_n \).

Proof. By Lemma 8, \( (e \nequiv Ne_i)_n \) is contained in the radical of \( (C_i)_n \). Now let \( n \) be an element of the radical of \( (C_i)_n \). Then \( n = \sum e_i^N e_i \) and so \( e_i^N e_i e_i^N e_i = e_i^N e_i \). Hence \( e_i^N e_i \) is an element of the radical of \( (C_i)_n \). But \( e_i^N \in C_i \), and therefore, by Lemma 8, \( e_i^N \) is an element of \( e_i Ne_i \).

Lemma 10. If every \( C_{ij} \) is cleft, then \( R \) is cleft.

Proof. Since \( C_i = K_i + e_i Ne_i \), then \( (C_i)_n = (K_i)_n + (e_i Ne_i)_n \). It is known that \( (K_i)_n \) is simple. Also, by Lemma 9, \( (e_i Ne_i)_n \) is the radical of \( (C_i)_n \). And by Lemma 7, \( P_i \cong (C_i)_n \), so \( P_i \) is cleft, that is, \( P_i = S_i + N_i \) where \( S_i \cong (K_i)_n \) and \( N_i \cong (e_i Ne_i)_n \). Then from \( R = \sum P_i + N_0 \) it follows that \( R = \sum S_i + \sum N_i + N_0 \). Since \( P_i \neq P_i = \delta_{ij} P_i \), then \( R = \sum S_i \) is a semisimple ring. Moreover, by Lemma 6, \( N = \sum N_i + N_0 \). Hence \( R \) is a cleft ring \( R^* + N \).

Theorem II. The condition that \( R \) be cleft is equivalent to the condition that every \( C_{ij} \) of an arbitrary set \( (C_{ij}) \) be cleft.

Proof. The proof follows from Lemmas 5 and 10.

Lemma 11. If \( R \) contains a division ring \( K \) and any idempotent \( e \) such that \( eK = Ke \neq 0 \), then \( eKe \neq 0 \).

Proof. \( eK = Ke \neq 0 \) implies that \( ek = k'e \) is an automorphism of \( K \). It also implies that \( ek = ke \). Hence \( ke = ek \). Therefore \( ke = ek \) is an isomorphic mapping of \( K \) onto \( eKe \).

In particular, if \( e = \sum e_{ij} \) is a principal idempotent such that \( e_{ij} K = Ke_{ij} \neq 0 \) for every \( i, j \), then each \( e_{ij} Ke_{ij} \neq 0 \).

Theorem III. If \( R \) is an algebra over a field \( F \) and contains division subalgebras (over \( F \)) \( G_i \), \( i = 1, \ldots, s \), such that \( e_i G_i = G_i e_i \neq 0 \) and \( G_i \cong C_{ij} \) for at least one \( j \) for every \( i \), then \( R \) is cleft, and conversely.

Proof. By Lemma 11, \( C_i \supseteq e_i G_i e_i \cong G_i \cong C_{ij} \), \( i = 1, \ldots, s \). Hence \( C_i = e_i G_i e_i + e_i Ne_i \) shows that \( C_i \) is cleft. Therefore every \( C_{ij} \) is cleft and, by Theorem II, \( R \) is cleft. On the other hand, if \( R \) is cleft, then take \( G_i = K_i \).

Lemma 12. Let \( A \) be an algebra over a field \( F \) and let \( G_i \equiv Fe_i \) the set of scalar multiples of the idempotent \( e_i \). Then \( e_i G_i e_i = G_i \cong F \).

Corollary 1. If \( A = A/N \) is a ring direct sum of total matric rings over \( F \), then \( A \) is cleft [1, p. 47].
Proof. \( P^i \cong (C_i)_{n_i} \) by Lemma 7, and \( \overline{P^i} \cong \overline{A^i} \). Hence \( \overline{A^i} \cong (\overline{C_i})_{n_i} \) and \( \overline{C_i} \cong F \). Therefore, by Lemma 12, \( \overline{C_i} \cong e_i F e_i \equiv G_i \). By Theorem III it now follows that \( A \) is cleft.

Corollary 2. If \( K \) is a splitting field of \( \overline{A} \) over \( F \), then \( A_K \) is cleft. (If \( F \) is algebraically closed take \( K \equiv F \).

Proof. \( (\overline{A})_K \) is a ring direct sum of total matric rings over \( F \), hence, by Corollary 1, \( A_K \) is cleft.

2. The following discussion about algebras is of interest in light of the known theorem [7, p. 24] that the separability of \( A/N \) implies that \( A \) is cleft. Let \( Z(R) \) denote the center of \( R \).

Theorem IV. The separability of \( A/N \) is equivalent to the separability of each \( C_{ij} \) in the set \( (C_{ij}) \).

Proof. For any ring, \( \overline{R^i} \cong (\overline{C_i})_{n_i} \) and \( Z((\overline{C_i})_{n_i}) \cong Z(\overline{C_i}) \). Hence if \( R \) is an algebra \( A \) then the separability of \( Z(\overline{A^i}) \) is equivalent to the separability of \( Z(\overline{C_i}) \). Now, it is known that the separability of a semisimple algebra is equivalent to the separability of its center [1, p. 44]. Therefore the separability of \( A/N \) is equivalent to the separability of \( Z(\overline{A^i}) \), and the separability of \( \overline{C_i} \) is equivalent to the separability of \( Z(\overline{C_i}) \). This implies the theorem.

If \( A \) is a simple algebra over \( F \), then the separability of \( A \) is implied by the centrality of \( A \) [1, p. 43]. If \( A \) is a primary (or completely primary) algebra over \( F \), then \( A/N \) is separable if \( F \) is isomorphic to the center of \( A/N \), which implies (besides the fact that \( A \) is cleft) that \( A \) is an algebra over the center of \( A/N \). On the other hand, if \( A \) is cleft, then \( A/N \) need not be separable.

3. The expression \( R = \sum P^i + N_0 \) yields at least two interesting decompositions of \( R \) in which the cleavage of \( P^i \) has a role. When \( R \) is not cleft, it may be said to be partially cleft if the \( R^* \) of the following theorem is not vacuous.

Theorem V A. \( R = R^* + (R^{*.'} \cup N) \), where \( R^* \) is a semisimple ring (unique within isomorphisms) and \( R^{*.'} \) is a ring with unit whose radical is \( R^{*.'} \cap N \).

Proof. Rearrange the \( P^i \) in \( \sum P^i + N_0 \) so that the cleft ones come first, that is, \( P^i = S^i + N^i \) for \( i = 1, \ldots, t \) only. Then \( R^* = \sum_{i=1}^t S^i \) is semisimple and unique within isomorphism because the \( P^i \) are. Let \( N^* = \sum_{i=1}^t N^i \) and \( R^{*.'} = \sum_{i=t+1}^s P^i \). Then \( R^{*.'} + N^* + N_0 = R^{*.'} \cup N \). Hence the theorem.

Theorem V B. The \( P^i \) for which \( e_i N = Ne_i = 0 \) form a unique semisimple ring.

Proof. From the properties of \( R \) it follows that \( \overline{R^i} = \overline{P^i \cup N} = \sum_{i=1}^t \overline{R \varepsilon_{iy}} = \sum_{j=1}^n \overline{e_{iy} R} \), and that the isomorphism as \( R \) left spaces of the \( \overline{R \varepsilon_{ij}} \) is equiva-
lent to the isomorphism as left ideals of the $Re_{ij}$ for $j = 1, \cdots, n_i$, and similarly for the $e_{ij}R$ and the $e_{ij}R$ (see proof of Lemma 3). Now consider $e_{ik}$ and $e_{it}$ for a fixed $i$. Suppose $Ne_{ik} = 0$ and in an isomorphic mapping as $R$ left spaces of $Re_{ik}$ on $Re_{it}$ that $nre_{ik} = e_{it}$. Then $nre_{ik} = ne_{it}$ for $n \in N$. But $nr \in N$, hence $nre_{ik} = 0$ and $ne_{it} = 0$. Therefore $Ne_{ii} = 0$. Similarly, $e_{ik}N = 0$ implies $e_{ii}N = 0$; and $e_{ik}N = Ne_{it} = 0$ implies $e_{ii}N = Ne_{it} = 0$. Now define the following rings:

$R^* = \sum P^i$, where $e^iN = Ne^i = 0$,

$R_1 = \sum P^i$, where $e^iN \neq Ne^i$,

$R_2 = \sum P^i$, where $e^iN \neq 0 = Ne^i$,

$R_3 = \sum P^i$, summed over the remaining $P^i$,

$P = R_1 + R_2 + R_3 + N_0 = R_1 + R_2 + (R_3 \cup N)$.

Then $R = R^* + P$ and $R^*P = PR^* = 0$. $R_1$ and $R_2$ are semisimple, $R_3$ has a unit, its radical is $R_3 \cap N$, and $R_3 \cup N$ is a ring, $R_1$, $R_2$, and $R_3$ are orthogonal. Indeed, so far this decomposition seems to be like that obtained by M. Hall [9]. To show that it actually is identical it is only necessary to prove that $P$ is a bound ring, that is, the only elements of $P$ which are two-sided annihilators of $N$ are in $N$. Consider an element $p$ of one of the $P^i$ which appear in $P$ such that $pN = Np = 0$. $P^iP^i$ is a two-sided ideal of $P^i$ and the elements in the classes of $(P^iP^i \cap N^i)/N^i$ must coincide with the elements of the classes of a two-sided ideal of $P^i/N^i$. Therefore the non-radical elements of $P^iP^i \cup N^i$ and therefore of $P^iP^i$ must include an idempotent $e$ such that $eN = 0$ and $Ne = 0$ could not both be true, which contradicts the fact that the elements of $P^iP^i$ are two-sided annihilators of $N$. Therefore $P^iP^i$ has no non-radical elements, in particular $e^iP^i = p \in N^i$. This shows that $P$ is bound. The fact that the decomposition $R = R^* + P$ is unique can now be easily proved exactly as Hall did it [9, p. 393].

II

1. Although many of the results below can be proved in greater generality, the following conditions will be assumed throughout: $R$ is a cleft ring with unit $e$; $V = eV = Vf$, where $f$ is the unit of $R'$, is an $R$ left module with a finite composition series as an $R$ left module and a finite composition series as an $R'$ right module; $rV = 0$ implies $r = 0$; $R'$ is cleft and has MR (minimum condition on right ideals); and finally, $R'' = R$. These conditions are known to hold if $R$ is a semisimple ring with ML and if $V = eV$ has minimum condition on $R$ left sub-modules [11, p. 70].

Consider a particular cleavage $R^* + N$. By Lemma 1 the unit of $R^*$ is the unit $e$ of $R$. Hereafter $e$ will be decomposed in $R^*$: $e = \sum e_{ij}$ with $e_{ij}$ from $R^* = \sum R^*e_{ij}$. From the meaning of $R'$ and $R''$ it follows that $e_{ij}V$ and $Vf_{ij}$ are not zero for any $i, j$, where $f = \sum f_{ij}$.

Let $V = V_1 \supset V_2 \supset \cdots \supset V_{n+1} = 0$ be a composition series of $V$ as an $R$ left module. Hence $V_i/V_{i+1} \equiv \bar{V}_i$ is irreducible, that is, there exists an $x \neq 0$
in $V_t$ such that $R\bar{x} = V_t$. Or, since $R = R^* + N$ and $N V_t = 0$, it follows that $R^* \bar{x} = V_t$.

Let $W = RW$ be any irreducible $R$ left module. Then $W = R^* W = \sum R^* e_{ij} W$. Each $R^* e_{ij} W = 0$ or $W$. Hence $W \cong R^* e_{ij}$ as $R^*$ left module for some $i$ and for $j = 1, \ldots, n_i$ by the correspondence $e_{ij} \mapsto e_{ij} w_j$ where $w_j \in W$.

Suppose $V_t$ is not isomorphic to $R^* e_{ij}$. Consider $e_{ij} V_t$. Since $V_t$ properly contains $V_{t+1}$, then $e_{ij} V_t \supseteq e_{ij} V_{t+1}$. If $e_{ij} V_t$ properly contains $e_{ij} V_{t+1}$, then there exists a nonzero $x = e_{ij} \bar{x}$ in $e_{ij} V_t$, and $R^* e_{ij} \bar{x} = V_t$. This implies that $V_t \cong R^* e_{ij}$, contrary to assumption. So $e_{ij} V_t = e_{ij} V_{t+1}$. If necessary, treat $V_{t+1}$ in the same way, continuing until a module $V_{t+n}$ is finally reached such that $V_{t+n} \cong R^* e_{ij}$ or $V_{t+n} = V_{n+1} = 0$.

Now, for each $V_t \neq 0$ there exists an $i$ such that $V_t \cong R^* e_{ij}, j = 1, \ldots, n_i$ for a fixed $i$. Let $\sigma_i$ equal the number of $V_t$ for which this is true. $R^* e_{ij} V_t = V_t, j = 1, \ldots, n_i$.

For different idempotents $e_{ij} V_t \neq e_{kl} V_t$ if $e_{ij} V_t \neq 0$. If $V_t \cong R^* e_{ij}$ then $e_{ij} V_t \neq 0$ and there exists a nonzero $x_{ij} \in V_t$ such that $R^* e_{ij} x_{ij} = V_t$, or $V_t = R^* e_{ij} x_{ij} + V_{t+1}$. Hence $e_{ij} V_t = K_{ij} x_{ij} + e_{ij} V_{t+1}$, where $K_{ij} = e_{ij} R^* e_{ij}$. This proves the following lemma:

**Lemma 13.** $e_{ij} V = \sum e_{ij} V_t = \sum K_{ij} x_{ij}$ (first summation over $V_t \cong R^* e_{ij}$, $j = 1, \ldots, n_i$). Similarly, $V f_{ij} = \sum L_{ij} y_{ij} L_{ij}$, where $L_{ij}$ and $y_{ij}$ are the counterparts in $R'$ and $V$ of $K_{ij}$ and $x_{ij}$.

**Lemma 14.** $\sum K_{ij} x_{ij}$ is a group direct sum.

**Proof.** Suppose $\sum K_{ij} x_{ij} = 0$. Then if $k'_{ij}$ is the first nonzero coefficient, $x_{ij}$ is expressible as a linear combination of elements from sub-modules of the composition series for $V$ which are below the sub-module from which $x_{ij}$ was chosen. This is impossible, hence $k'_{ij} = 0$.

**Theorem VI.** $V = \sum e_{ij} V_t = \sum K_{ij} x_{ij}$ is a group direct sum.

**Proof.** $V = e V = \sum e_{ij} V_t$. Then the theorem follows by Lemmas 13 and 14.

**Theorem VII.** $e_{ij} V$ is an indecomposable $R'$ right module and $C_{ij} = e_{ij} R e_{ij}$ is its homomorphism ring. $V f_{ij} = e_{ij} V$ is an indecomposable $R$ left module and $C_{ij}' = f_{ij} R' f_{ij}$ is its homomorphism ring (see [8, p. 533]).

**Proof.** Any mapping $\sigma$ of $e_{ij} V$ into itself as an $R'$ right module is completely specified by $\sigma(e_{ij} v) = r''(e_{ij} v) = e_{ij} r'' e_{ij} v$ for some $r'' \in R''$. But $R = R''$ insures that $r'' \in R$. Conversely every element of $C_{ij}$ is easily seen to give a mapping $\sigma$.

Also, every mapping $\sigma$ of $V f_{ij}$ into itself is by definition of $R'$ given by some $r' \in R'$. Hence $\sigma(v f_{ij}) = v f_{ij} r' = v f_{ij} r' f_{ij}$.

Now suppose $e_{ij} V = e_{ij} V_t + e_{ij} V_2$ is a decomposition of $e_{ij} V$. Then $C_{ij}$, the homomorphism ring of $e_{ij} V$, would contain two orthogonal idempotents, one
which leaves $e_{ij}V_1$ undisturbed but sends $e_{ij}V_2$ into zero and one which leaves $e_{ij}V_2$ undisturbed but sends $e_{ij}V_1$ into zero. This is a contradiction. A similar argument holds for $Vf_{ij}$.

**Theorem VIII.** An isomorphism between $e_{ij}V$ and $e_{kl}V$ as $R'$ right modules is equivalent to an isomorphism between $e_{ij}R$ and $e_{kl}R$ as right ideals of $R$. An isomorphism between $Vf_{ij}$ and $Vf_{kl}$ as $R$ left modules is equivalent to an isomorphism between $R'f_{ij}$ and $R'f_{kl}$ as left ideals of $R'$. (See [8, p. 530] for a similar theorem for groups.)

**Proof.** Any mapping $e_{ij}V \cong e_{kl}V$ as $R'$ right modules is given by an element of $e_{ij}Re_{kl}$. Suppose $e_{ij}V \cong e_{kl}V$. Then there exist elements $r_{ik}$ and $r_{ki}$ of $e_{ij}Re_{kl}$ and $e_{ij}Re_{ij}$ respectively such that $r_{ik}V = e_{ij}V$ and $r_{ik}r_{ki}V = e_{ij}V$ and $r_{ik}r_{ki} = e_{ij}$. Now consider the mapping $e_{ij}V \cong e_{kl}V$, $r \in R$. $r_{ik}r = 0$ would imply $r_{ik}r_{ki} = e_{ij} = 0$. Hence the mapping defines an isomorphism between $e_{ij}R$ and $e_{kl}R$ as right ideals. On the other hand, suppose $e_{ij}R \cong e_{kl}R$. Then there exists an element $r_{ki}$ which will effect the mapping, and an element $r_{ik}$ such that $r_{ik}r_{ki} = e_{ij}$. Then the mapping $e_{ij}V \cong e_{kl}V$, $v \in V$, maps $e_{ij}V$ onto $e_{kl}V$ isomorphically as $R'$ modules.

**Corollary.** For a particular decomposition of $e$, the number of isomorphic indecomposable modules $e_{ij}V$ equals the number of isomorphic right ideals $e_{ij}R$ or left ideals $Re_{ij}$, and hence equals the number of minimal right or left ideals in $R^*$.

Now any irreducible $R^*;i$ left module which is isomorphic to $R^*e_{ij}$ has a $K_{ij}$ right dimension equal to the number $n_i$ of minimal left or right ideals of $R^*$, and $n_i$ is the dimension of the Wedderburn representation ($\tau$) of $R^*$ [2].

**Theorem IX.** The results of this section can be summarized in a table as follows:

<table>
<thead>
<tr>
<th>Spaces of $V$</th>
<th>Space</th>
<th>Multiplicity</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isomorphic indecomposable $R$ left modules</td>
<td>$Vf_{ij}$</td>
<td>$n_i$</td>
<td>$\tau_i$ over $L_{ij}$</td>
</tr>
<tr>
<td>Irreducible $R'$ right modules</td>
<td>$V'<em>i \cong f</em>{ij}R'^*$</td>
<td>$\tau_i$</td>
<td>$n'<em>i$ over $K</em>{ij}$</td>
</tr>
<tr>
<td>Isomorphic indecomposable $R'$ right modules</td>
<td>$e_{ij}V$</td>
<td>$n_i$</td>
<td>$\sigma_i$ over $K_{ij}$</td>
</tr>
<tr>
<td>Irreducible $R$ left modules</td>
<td>$V_i \cong R^*e_{ij}$</td>
<td>$\sigma_i$</td>
<td>$n_i$ over $L_{ij}$</td>
</tr>
</tbody>
</table>

As an illustration, let $R$ be a matric algebra over an algebraically closed field $F$, and suppose $R = R''$. Then $R$ is cleft, as is its commuting algebra $R'$.

($\tau$) We refer to the representation of a simple ring $R$ by a total matric set over a division ring $K$ as the Wedderburn representation of $R$, and $K$ is called a Wedderburn division ring for $R$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let $V$ be a column vector space with coefficients from $F$. The modules of Theorem IX will reflect the indecomposable and irreducible constituents of $R$ and $R'$. Furthermore $F$ is the Wedderburn division ring of all the irreducible constituents. Hence we have the following relations: To every indecomposable constituent $I_i$ of $R$ corresponds an irreducible constituent $G_i$ of $R'$; to every indecomposable constituent $J_i$ of $R'$ corresponds an irreducible constituent $F_i$ of $R$. There are $n'_i$ constituents $I_i$ and $n'_i$ is the degree of $G_i$; there are $n_i$ constituents $J_i$ and $n_i$ is the degree of $F_i(3)$.

2. Hereafter let the ring $R$ be the module $V$. Assume that $R$ has MLR(4); then it must have MaLR(4) [10, p. 726; 11, p. 71]. Hence it has a finite composition series of ideals, left or right [11, p. 71]. The elements of $R$, $R'$, and $R''$ may be identified. Without loss of generality $R$ will always be assumed an indecomposable two-sided ideal.

When speaking of a module let $(K, R)$ denote a set of left operators $K$ and right operators $R$.

$e_{ij}V$ and $Vf_{ij}$ are now $e_{ij}R$ and $Re_{ij}$ respectively.

**Lemma 15.** $e_{ij}N$ and $Ne_{ij}$ are the unique maximal sub-ideals of $e_{ij}R$ and $Re_{ij}$ respectively [13, p. 362].

By Theorems VI and VII, $e_{ij}R = \sum K_{ij}x_{ij}$ is an indecomposable $(K_{ij}, R)$ space. Because it is a right ideal of $R$ it has a finite composition series of $(K_{ij}, R)$ modules, and by Lemma 15 such a series will start out as $e_{ij}R \supset e_{ij}N \equiv V_2 \supset \cdots \cdots$. Each composition factor module will contain a finite number of $K_{ij}$ left independent elements in terms of which a residue system can be expressed as linear combinations over $K_{ij}$. In fact a set of residue systems exists which will serve in all composition series [6, p. 505]. Hence, if a column vector $X_{ij}$, formed from $K_{ij}$ left independent elements of all the composition factor modules is right multiplied by $R$: $X_{ij}R = I_{ij}X_{ij}$, then the representation $I_{ij}$ is similar to that due to any other composition series. Schematically,

$$I_{ij} = \begin{pmatrix} F_{ij}^1 & * \\ \vdots & \dots \\ F_{ij}^p \\ 0 \end{pmatrix}, \text{ of degree } \sigma_i.$$  

If $X_{ij}$ is inverted then the zero and the star reverse.

**Lemma 16.** For a suitable basis $X_{ij}$, $I_{ij}$ contains

\(^{(4)}\) This proves a theorem due to Brauer [4].

\(^{(4)}\) MLR $\equiv$ minimum condition on left and right ideals; MaLR $\equiv$ maximum condition on left and right ideals.
Proof. Form the composition series from a refinement of the Loewy series $e_{ij}R^* + e_{ij}N \supset e_{ij}N \supset e_{ij}N^2 \supset \cdots$. Each factor module of this Loewy series is a $(K_{ij}, R^*)$ module and is right annihilated by $N$, and a refinement of it to a composition series of $(K_{ij}, R^*)$ modules will give the desired result.

**Corollary.** For $X_{ij}$ chosen this way, $I_{ij}$ is a group direct sum of

$$I_{ij}(R^*) = \begin{pmatrix} F_{ij}^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F_{ij}^p \end{pmatrix} \quad \text{and} \quad I_{ij}(N) = \begin{pmatrix} 0 & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

**Lemma 17.** The irreducible constituent $F_{ij}^1$ is a Wedderburn representation of $R^*^i$, that is, $F_{ij}^1 = (K_{ij})_{n_i}$.

**Proof.** $e_{ij}R^* = e_{ij}R^*$ is a $K_{ij}$ left module of dimension $n_i$ over $K_{ij}$, where $n_i$ is the number of minimal left or right ideals of $R^*^i$. In the composition series for $e_{ij}R$, choose a basis $X_{ij}^*$ of $e_{ij}R^*$ as the residue system of $e_{ij}R/e_{ij}N$. Then $X_{ij}^*R^* = F_{ij}^1X_{ij}^*$. But this gives a Wedderburn representation of $R^*^i$ [2].

**Corollary.** For every $i = 1, \cdots, s$ there exists an $F_{ij}^1 \cong R^*^i$.

**Proof.** $e_{ij}R \neq 0$ for any $i$.

**Lemma 18.** $F_{ij}^1$ is an irreducible representation over $K_{ij}$ of $R^*^k$ for some $k$.

**Proof.** Any irreducible $(K_{ij}, R^*)$ module is annihilated by all except one of the $R^*^i$ and hence gives a representation of one of them.

Hereafter $X_{ij}$ will be chosen from the composition series indicated in Lemma 16.

If the whole composite representation module $R = \sum e_{ij}R = \sum_{ij} K_{ij} x_{ij}$ be considered with $X_{ij}$ for each $e_{ij}R$ chosen as agreed upon, then the first composite representation is generated:

$$R = \begin{pmatrix} I_{n_1} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_s} \end{pmatrix}.$$

A second representation.
is generated by the ideals $Re_{ij}: RY_{ij} = Y_{ij}J_{ij}$, where

$$J_{ij} = \begin{pmatrix}
G_{ij}^1 & * \\
& \ddots & \ddots \\
& & G_{ij}^p & \\
0 & \cdots & & 
\end{pmatrix}, \text{ of degree } \tau_i.$$

Because of the choice of the columns $X_{ij}$, $R$ can be written as $R^* + R$ where $R^*$ is a diagonal representation of $R^*$. Similarly for $\mathcal{G}$ because of the choice of the rows $Y_{ij}$.

**Theorem X.** The following table summarizes some of the properties of $R$ and $\mathcal{G}$.

<table>
<thead>
<tr>
<th></th>
<th>$I_{ij}$</th>
<th>$F_{ij}^p$</th>
<th>$J_{ij}$</th>
<th>$G_{ij}^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of distinct</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
</tr>
<tr>
<td>Number of isomorphic</td>
<td>$n_i$</td>
<td>$\leq \tau_i$</td>
<td>$n_i$</td>
<td>$\leq \sigma_i$</td>
</tr>
<tr>
<td>Degree (for $p = 1$)</td>
<td>$\sigma_i$</td>
<td>$n_i$</td>
<td>$\tau_i$</td>
<td>$n_i$</td>
</tr>
<tr>
<td>Total of all kinds</td>
<td>$\sum n_i$</td>
<td>$\leq \sum \tau_i$</td>
<td>$\sum n_i$</td>
<td>$\leq \sum \sigma_i$</td>
</tr>
</tbody>
</table>

**Remarks.**

1. $e_{kl}Re_{ij}$ is a $K_{ij}$ right module of finite dimension.

**Proof.** It is a $K_{ij}$ sub-module of $Re_{ij}$.

2. The number of times an irreducible representation of $R^{*i}$ occurs in $I_{kl}$ is less than or equal to the dimension of $e_{kl}Re_{ij}$ as a $K_{ij}$ right module. Similarly, the representation of $R^{*i}$ occurs in $J_{kl}$ with multiplicity less than or equal to the dimension of $e_{ij}Re_{kl}$ as a $K_{ij}$ left module [13, p. 364].

**Proof.** Apply the procedure of §1 but with $V = e_{kl}R$.

3. Any two division rings $K_{ij}$ and $K_{kl}$ have representations over one another.

**Proof.** Every $F_{ij}^p$ is isomorphic to some $R^{*i}$ and hence to some $F_{ij}^1$. The existence of $F_{ij}^p$ implies $e_{kl}Re_{ij} \neq 0$ by Remark 2. Also $e_{kl}Re_{ij}$ is a $K_{ij}$ right and $K_{kl}$ left module of finite dimension both ways by Remark 1. Hence it is a representation module for $K_{ij}$ over $K_{kl}$ and vice versa. Now, by use of a known theorem [13, p. 364] every $K_{ij}$ and $K_{kl}$ can be chained by a sequence.
Lemma 19. \( e_{ij}R^*e_{ik} \) does not vanish and is of left dimension one over \( K_{ij} \).

Proof. In the decomposition \( R^* = \sum_{i=1}^{n_i} e_{ii}R^* \), \( e_{ij}R^* \cong e_{ik}R^* \) for every \( j, k \); and in a mapping \( \sigma \) of the two ideals the image of \( e_{ik} \) is \( \sigma(e_{ik}) = \sigma(e_{ik})e_{ik} = e_{ij}\sigma(e_{ik})e_{ik} = \sigma(e_{ik})e_{ij} \) in \( R^* \), hence \( e_{ij}R^*e_{ik} \neq 0 \). Now consider the sum \( e_{ij}R^* = \sum_{i=1}^{n_i} e_{ij}R^*e_{ii} \). None of the summands vanishes and each is a \( K_{ij} \) left module of dimension one, for the dimension of \( e_{ij}R^* \) over \( K_{ij} \) is exactly \( n_i \).

Because of this lemma a basis \( X_{ij}^* \) of \( e_{ij}R^* \) can be chosen as follows: \( X_{ij}^* = e_{ij1}, x_{ij2}e_{ij2}, \ldots, x_{ijr}e_{ijr-1}, e_{ijr}, x_{ijr+1}e_{ijr+1}, \ldots, x_{ijn}e_{ijn} \). Extend this to a basis \( X_{ij} \) of \( e_{ij}R \) in accordance with the stipulations of §2. This is the basis that will be used hereafter.

Let \( \rho_{ij} \) denote the top \( n_i \) rows of \( I_{ij} \).

Lemma 20. There exist elements of \( R \) with arbitrary coefficients in \( \rho_{ij} \) for any \( i, j \).

Proof. The \( j \)th basis element in \( X_{ij} \) is \( e_{ij} \). Hence the \( j \)th line of coefficients in \( I_{ij} \) is arbitrary over \( K_{ij} \). Furthermore, \( F_{ij}^1 \) is complete (a total matric set) over \( K_{ij} \). Therefore there is an element \( A \) in \( R^* \) with zeros everywhere in \( F_{ij}^1 \) except for \( e_{ij} \) in the \( j, j \)th place. Let \( M \) be an element of \( R \) with arbitrary coefficients in the \( j \)th line, then \( AM \) will have these same coefficients in the \( j \)th line again but zeros elsewhere in \( \rho_{ij} \). Now, there is an element \( B \) in \( R^* \) with zeros everywhere in \( F_{ij}^1 \) except for \( e_{ij} \) in the \( t, j \)th place. Then \( BAM \) will have zeros everywhere in \( \rho_{ij} \) except for arbitrary coefficients in the \( t \)th line.

Lemma 21. \( \rho_{ij} \) is independent of \( \rho_{ik} \) if \( k \neq i \).

Proof. \( X_{ij}^*R = X_{ij}^*\sum_{s=1}^{s_i} e_{is}R = X_{ij}^*\sum_{s=1}^{s_i} e_{is}R \), and \( X_{ij}^*\sum_{s=1}^{s_i} e_{is}R = 0 \) for \( k \neq i \).

Lemma 22. No nonzero element of \( R \) vanishes in every \( \rho_{ij} \) simultaneously.

Proof. If \( r \) vanished in all \( \rho_{ij} \) then \( e_{ij}r = 0 \) for all \( i, j \); hence \( \sum_i e_{ij}r = er = r = 0 \).

Corollary. Every element of \( R \) has some nonzero coefficient in some \( \rho_{ij} \).

Lemma 23. If \( e_{ii}R^*r = 0 \), where \( r \subseteq R \), then \( e_{ii}R^*r = 0 \) for \( j = 1, \ldots, n_i \).

Proof. \( e_{ii}R^*r = 0 \) implies that \( e_{ii}s^*r = 0 \) where \( s^* \) is any element of \( R^* \). Now \( e_{ii}R \cong e_{ii}R \) as \( R \) right modules implies \( e_{ii}R \cong e_{ii}R^* \) and hence \( e_{ii}r \cong e_{ii}R^*r \). Therefore \( e_{ii}r = 0 \). Similarly, \( e_{ij}r = 0 \). Now a glance at the structure of \( X_{ij}^* \) shows that \( e_{ij}R^*r = 0 \), \( j = 1, \ldots, n_i \).

Corollary. \( \rho_{ij} \), \( j = 1, \ldots, n_i \), all vanish if any one does.

The elements of \( R \) can evidently be separated into \( s \) classes \( \rho_1, \ldots, \rho_s \).
where \( \rho_k \) consists of elements with zeros in every \( \rho_{ij} \) except those with \( i = k \).

(From the structure of \( X_{ij} \) it can be seen that \( \rho_k \) represents the right ideal \( e^kR \).) If \( E = \sum_{i=1}^n E_i \) is a decomposition of the unit of \( \mathcal{R}^* \), then \( \mathcal{R} = \sum \rho_i = E \sum \rho_i = \sum E_i \rho_i \) since \( E_i \rho_j = 0 \) if \( i \neq j \). So instead of \( \rho_k \), \( E_k \rho_k \) may be used for elements which vanish in all \( \rho_{ij} \) except for \( i = k \). Thus the revised \( \rho_k \) has coefficients only in rows which intersect the diagonal in a representation of \( R^{**} \).

Define further as the set \( \chi_{ii} \) all elements in \( \mathcal{R}^* \) (hence in \( \rho_i \)) which have \( \chi_{ii} \) for the \( \rho_i \) constituent. Obviously \( \chi_{ii} \equiv K_{ii} \).

Finally let \( T_{ui}^w \) be the element of \( \mathcal{R} \) with \( e_{ii} \) in the \( u, v \) place of \( \rho_{ii} \); \( u = 1, \cdots, n_i \); \( v = 1, \cdots, \sigma_i \).

**Theorem XI.** \( \rho_i = \sum_{u,v} \chi_{ii} T_{ui}^w \).

**Proof.** If any element of \( \rho_i \) has no coefficient in \( \rho_{ii} \) then it is zero by the corollary to Lemma 22.

**Corollary.** \( \mathcal{R} = \sum \rho_i \).

Consider now the rings \( C_{ij} \) and \( P^i \). The following results characterize all completely primary and primary rings which are cleft and have MLR.

First specializing to the case of \( C_{ij} \): There will be just one indecomposable representation, call it \( I \). And since \( C_{ij} = K_{ij} + e_{ij}Ne_{ij} = \sum K_{ij} X_{ij} \), the basis \( X \) will start out with \( X^* = e_{ij} \). Hence \( \rho_{ii} = \rho \) is a single line and is arbitrary over \( K_{ij} \) because \( e_{ij}C_{ij} = C_{ij} \). Each \( F_{ij}^p = F^p \) is a representation of \( K_{ij} \) with coefficients from \( K_{ij} \). Call the elements of \( \rho_i \equiv \rho \): \( T^1, \cdots, T^\sigma \). \( \chi_{ii} \equiv \chi \) is just the representation of \( K_{ij} \) and has \( K_{ij} \) itself in the \( F^1 \) place.

**Theorem XII.** \( C_{ij} \) is isomorphic to the indecomposable matric set \( I(C_{ij}) = \sum_1^n \chi T^i \) of degree \( \sigma \), where \( \chi \equiv K_{ij} \). \( F^1 \) is \( K_{ij} \) and the top row is arbitrary over \( K_{ij} \). Every element of \( I \) has a nonzero coefficient in the top row.

**Theorem XIII.** \( P^i \) is isomorphic to an indecomposable matrix set \( I(P^i) = \sum_1^n \chi T^w \) of degree \( n_i \sigma \), where \( \chi \equiv K_{ij} \) has \( n_i \times K_{ij} \) in the \( F^1 \) place. \( F^1 = (K_{ij})_{n_i} \) and the first \( n_i \) rows are arbitrary over \( K_{ij} \). Every element has a nonzero coefficient in one of the first \( n_i \) rows.

**Proof.** \( P^i \equiv (C_{ij})_{n_i} \) and the cleavage of \( P^i \) is equivalent to the cleavage of \( C_{ij} \). Hence \( I(C_{ij}) \) may be substituted for \( C_{ij} \) in \( (C_{ij})_{n_i} \). Then an easy coordinate adjustment gives the desired result.

4. When \( R \) is an algebra over a field \( F \), the usual regular representation of \( R \) is generated from a composition series whose factor modules are just \( R \) right admissible rather than \( (K_{ij}, R) \) admissible. The Loewy series for \( e_{ij}R \) is the same in both cases. But in general the two types of modules are not the same. For example consider the irreducible \((K, F)\) space \( V \equiv K \), where \( K = \sum_{i=1}^n u_i F \). As an \( F \) right space, \( V \) breaks up into irreducible spaces \( u_i F, \cdots, u_n F \).
Suppose $R$ is an algebra over $F$. Express each $K_{ij}$ over $F$: $K_{ij} = \sum h_i F u_i$. Then $e_{ij}R$ will have dimension $\sigma_i h_i$ over $F$. Now form a composition series of $R$ right modules for $e_{ij}R$. Then $I_{ij}$ will be of degree $\sigma_i h_i$. As an immediate consequence the multiplicity of all isomorphic $G_{ij}^p$ will be exactly the degree of $I_{ij}$ divided by the degree of $K_{ij}$ over $F$, that is, equal to $\sigma_i$. This sharpens Theorem X(5).

**Theorem XIV.** Theorem X holds for the usual representations of an algebra over a field $F$ if $n_i$, $\sigma_i$, and $\tau_i$ are replaced by $n_i h_i$, $\sigma_i h_i$, and $\tau_i h_i$ respectively, where $h_i$ is the dimension of $K_{ij}$ over $F$, and if the inequalities are replaced by equalities. A similar statement holds for Remark 2.

If $F$ is the Wedderburn division ring of every irreducible representation (for example if $F$ is algebraically closed) then the elementary module theory can be derived from the considerations of §3(6).

**Bibliography**


(5) Theorem XIV, which is derived for cleft algebras, corroborates properties summarized by Brauer [5].


University of Wisconsin, Madison, Wis.