AN EXTENSION OF METRIC DISTRIBUTIVE LATTICES WITH AN APPLICATION IN GENERAL ANALYSIS

BY

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Introduction. This paper shows how to embed a metric distributive lattice(1) [1, pp. 41 and 74] whose modular functional is bounded into a field of sets with a completely additive measure and applies this result to an abstract example of the second general analysis theory of E. H. Moore [10, vol. 1, pp. 14–15]. Moore devised the example as a generalization of Radon's generalization [11] of Hellinger integration [4]. Our investigation started with a lattice-theoretic interpretation of Moore's example. We were led to the extension problem in an effort to decide whether Moore's system was really more general than Radon's. Our result may be interpreted as stating that this is not the case, provided we assume bounded measure. We have been unable to answer this question in general. In connection with recent efforts to supply a very general basis for integration theory via lattice theory [2, 12, 3], it is interesting to point out that Moore's assumption of distributivity is essential in order that his basic matrix be positive.

Even without the application we have in mind, our result on the extension of metric distributive lattices would seem to be interesting and important. The method of proof also has interest in that it combines the results of Stone [13] and of Wallman [14] with methods of MacNeille [8] and of Kakutani [6].

Our paper falls naturally into two parts. In Part I we present our process of extending a metric distributive lattice. Here we shall use the standard notations of set theory and those special notations of lattice theory which are appropriate in distributive lattices and Boolean rings [1, pp. 96–97] with the following exception. We find occasion to employ a concept which, although used in many parts of mathematics, does not seem to have received a standard name. Roughly, this concept is that of a "*set of elements some of which are alike.*" Precisely, it is the concept of a maximal class of equivalent (non-null) sequences $(p_\alpha | \alpha \in A)$ of elements of a class $\mathfrak{P} \equiv [p]$ where two sequences $(p_\alpha | \alpha \in A)$ and $(p_\beta | \beta \in B)$ are equivalent if there is a one-to-one correspondence $\alpha \rightleftarrows \beta$ of A and B such that $\alpha \rightleftarrows \beta$ implies $p_\alpha = p_\beta$. The terminology

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(1) References to the bibliography at the end of the paper will be enclosed in brackets.

collection has recently been used for this concept [7, 9]. For lack of a better word, we shall adopt this terminology.

In Part II we discuss the example of Moore. It seems appropriate to us not to forsake the notational style of Moore when discussing his work. We make an attempt in Part II to employ notations in keeping with the spirit of the system Moore developed [10, vol. 1]. In fact, we shall adopt the precise system of notation of Moore in the concluding section of Part II. On the other hand, we should find it inconvenient not to employ the terminology of lattice theory, and we retain this terminology throughout the paper.

PART I. AN EXTENSION OF METRIC DISTRIBUTIVE LATTICES

In this part of our paper, $L \equiv [a, b, c, \cdots]$ will denote a metric(2) distributive lattice whose modular functional is $\mu(a)$. We assume that L has a least element(3) O and that(4) $\mu(O)=0$. In §1 we extend the definition of $\mu(a)$ to the set of all finite collections of elements of L and derive the properties of the extended functional. Here we follow closely the method of MacNeille [8]. The field of sets containing L is constructed, with the aid of Stone's result [13], in §2. A method due to Kakutani [6, p. 533] is employed in §3 to derive a completely additive measure under the additional assumption that $\mu(a)$ is bounded on L.

1. **The modular functional on collections.** Let $H \equiv [h, k, \cdots]$ be the set of all finite collections of elements of L. We may represent the elements of H as *formal sums*

$$h = a_1 + \cdots + a_n, \qquad k = b_1 + \cdots + b_m. \tag{1}$$

For convenience we shall use the notation h_q for the formal sum (and also for the collection which it represents) obtained by suppressing the first q elements of h. Thus, if h is as in (1), then

$$h_q = a_{q+1} + \cdots + a_n \qquad (q < n).$$

We define the product of two elements $h, k \in H$ as the formal sum

$$hk = (\sum a_i b_j \mid i = 1, \cdots, n; j = 1, \cdots, m)$$

or, equivalently, as the collection represented by the displayed formal sum. Further, if $h \in H$, we define

$$\mu(h) \equiv \sum_{i=1}^{n} (-2)^{i-1} \sigma_{\mu i}(a_1, \cdots, a_n),$$

(2) It would suffice to require that L be merely *quasi-metric*. For the sake of simplicity, we suppose that the identification which gives rise to a metric has already been made.

(3) Here it would suffice to assume that g.l.b. $[\mu(a) \mid a \in L] > -\infty$. Again we suppose that the preliminary introduction of a least element O has already been accomplished.

(4) Our discussion is not affected by the transformation $\mu(a) \rightarrow \mu(a) + \lambda$ where λ is a real number. This is, then, only an assumption of convenience.

where

$$\sigma_{\mu i} \equiv \sum \mu(a_{j_1} a_{j_2} \cdots a_{j_i})$$

and the summation extends over all properly monotone sequences $j_1, j_2, \cdots, j_i$ of the integers $1, 2, \cdots, n$.

We shall now prove a series of lemmas whose goal is to establish the modular character of the extended functional μ on a certain partition of H.

LEMMA 1. *Let $h \in H$, $b \in L$, then $\mu(h+b) = \mu(h) + \mu(b) - 2\mu(hb)$.*

Proof. Consider elements $h \in H$, $b \in L$. It is clear that

$$\begin{aligned} &\sigma_{\mu 1}(a_1, \cdots, a_n, b) = \sigma_{\mu 1}(b) + \sigma_{\mu 1}(a_1, \cdots, a_n), \\ (2) \quad &\sigma_{\mu i}(a_1, \cdots, a_n, b) = \sigma_{\mu i}(a_1, \cdots, a_n) + \sigma_{\mu i-1}(a_1 b, \cdots, a_n b) \quad (1 < i \leqq n), \\ &\sigma_{\mu n+1}(a_1, \cdots, a_n, b) = \sigma_{\mu n}(a_1 b, \cdots, a_n b). \end{aligned}$$

Lemma 1 is now an immediate consequence of (2) and the definition of μ on H.

COROLLARY. *If $b \in L$, then $\mu(b+b) = 0$.*

LEMMA 2. *Let $h \in H$, $b \in L$, then $\mu(h+b+b) = \mu(h)$.*

Proof. This is clear by two applications of Lemma 1.

COROLLARY. *If $h \in H$, then $\mu(h+h) = 0$.*

LEMMA 3. *Let $h = c_1 + \cdots + c_n \in H$; a, $b \in L$, then*

$$\mu(h + a + b) = \mu(h + a \cup b + ab). \tag{3}$$

Proof. We shall use induction on the number n. The relation (3) is easily verified if $n = 1$ by means of the modular character of μ and the distributive law in L. Now assume that (3) holds for all $h \in H$ for which $n < q$. Consider elements $h \in H$, a, $b \in L$ for which $n = q$. Using Lemma 1, distributivity in L, and the hypothesis of induction, we find that

$$\begin{aligned} \mu(h + a + b) &= \mu(h_1 + a + b) + \mu(c_1) - 2\mu(c_1 h_1 + c_1 a + c_1 b) \\ &= \mu(h_1 + a \cup b + ab) + \mu(c_1) - 2\mu(c_1 h_1 + c_1(a \cup b) + c_1 ab) \\ &= \mu(h + a \cup b + ab). \end{aligned}$$

COROLLARY. *If h, $k \in H$, a, $b \in L$, then*

$$\mu(h + (a + b)k) = \mu(h + (a \cup b + ab)k).$$

Proof. This follows easily by successive applications of Lemma 3.

LEMMA 4. *If h, $k \in H$ are as in (1), then there are decreasing sequences $(a_i' \mid i = 1, \cdots, n)$ and $(b_j' \mid j = 1, \cdots, m)$ of elements of L representing elements h', $k' \in H$ and such that*

$$\mu(h') = \mu(h), \qquad \mu(k') = \mu(k), \qquad \mu(h' + k') = \mu(h + k), \qquad \mu(h'k') = \mu(hk).$$

Proof. Successive applications of Lemma 3 [cf. 8, p. 454] to $\mu(h)$ and to $\mu(k)$ yield the desired decreasing sequences. The *same* applications of Lemma 3 to $\mu(h+k)$ show that $\mu(h+k)=\mu(h'+k')$. To secure $\mu(hk)=\mu(h'k')$, it is convenient to show first that $\mu(hk)=\mu(h'k)$. By the corollary to Lemma 3, this may be done by the *same* applications of Lemma 3 which led to h'. Using this result gives $\mu(hk)=\mu(h'k)=\mu(h'k')$ as desired.

LEMMA 5. *Let $h=c_1+\cdots+c_n\in H$ be such that the sequence $(c_i|i=1,\cdots,n)$ is decreasing, then*

$$\mu(h) = (\textstyle\sum (-1)^{i-1}\mu(c_i) \mid i = 1, \cdots, n).$$

Proof. This is easy to prove by induction and the use of Lemma 1.

COROLLARY. *For $h\in H$, we have $\mu(h)\geqq 0$.*

LEMMA 6. *Let the sequences $(a_i|i=1,\cdots,n)$ and $(b_j|j=1,\cdots,m)$ representing h, $k\in H$, respectively, be decreasing. Then $\mu(hk)=\mu(hb_1)-\mu(hk_1)$.*

Proof. For elements h, $k\in H$ as in the lemma, Lemma 1 gives

$$\begin{aligned}\mu(hk) &= \mu(b_1h + k_1h) = \mu(b_1a_1 + b_1h_1 + k_1h)\\ &= \mu(b_1a_1) - \mu(b_1h_1 + k_1h).\end{aligned}\tag{4}$$

Now Lemma 2 gives

$$\begin{aligned}\mu(b_1h_1 + k_1h) &= \mu(b_1a_2 + b_1h_2 + k_1h)\\ &= \mu(b_1a_2) + \mu(b_1h_2 + k_1h) - 2\mu(b_1h_2 + k_1h_2).\end{aligned}$$

The same manipulation yields

$$\begin{aligned}\mu(b_1h_2 + k_1h) &= \mu(b_1a_3 + b_1h_3 + k_1h)\\ &= \mu(b_1a_3) + \mu(b_1h_3 + k_1h) - 2\mu(b_1h_3 + k_1h_2).\end{aligned}$$

Again,

$$\begin{aligned}\mu(b_1h_2 + k_1h_2) &= \mu(b_1a_3 + b_1h_3 + k_1h_2)\\ &= \mu(b_1a_3) - \mu(b_1h_3 + k_1h_2).\end{aligned}$$

Substituting these results in (4) we have

$$\mu(hk) = \mu(b_1a_1) - \mu(b_1a_2) + \mu(b_1a_3) - \mu(b_1h_3 + k_1h).$$

Repetition of this process leads, by Lemma 5, to the lemma if n is odd. If n is even, we obtain

$$\mu(hk) = \mu(b_1a_1) - \mu(b_1a_2) + \cdots + \mu(b_1a_{n-1}) - \mu(b_1a_n + k_1h).$$

Using Lemma 1 and its corollary, we then find that

$$\mu(hk) = \mu(b_1h) - \mu(k_1h).$$

The proof is complete.

LEMMA 7. *If h, $k \in H$, then* $\mu(h+k)=\mu(h)+\mu(k)-2\mu(hk)$.

Proof. Let h and k be as in (1). We shall use induction on the integer m. For $m=1$, we proved the lemma as our Lemma 1. Now assume that the lemma holds for all $k \in H$ for which $m<q$. Consider an element $k \in H$ as in (1) for which $m=q$, and an element $h \in H$ as in (1). By Lemma 4, we may assume, without loss of generality, that the sequences $(a_i | i=1, \cdots, n)$ and $(b_j | j=1, \cdots, m)$ representing h and k are decreasing. By Lemma 1 and the hypothesis of induction we have

$$\begin{aligned}\mu(h + k_1 + b_1) &= \mu(h + k_1) + \mu(b_1) - 2\mu(hb_1 + k_1) \\ &= \mu(h) + \mu(k_1) - 2\mu(hk_1) + \mu(b_1) \\ &\quad - 2\mu(hb_1) - 2\mu(k_1) + 4\mu(hk_1) \\ &= \mu(h) + \mu(b_1) - \mu(k_1) - 2[\mu(hb_1) - \mu(hk_1)].\end{aligned}$$

Using Lemmas 5 and 6, we then find that

$$\mu(h + k) = \mu(h) + \mu(k) - 2\mu(hk).$$

The induction is complete.

In concluding this section we note the following properties of μ on H. For h, k, $l \in H$, we have

(i) $\mu(h)-\mu(hk)=\mu(h+hk)\geqq 0$,
(ii) $\mu(h+k)=\mu(h)+\mu(k)-2\mu(hk)\geqq |\mu(h)-\mu(k)|$,
(iii) $\mu(k+l)-\mu(hk+hl)\geqq 0$,
(iv) $|\mu(h+k)-\mu(h+l)| \leqq \mu(k+l)$,
(v) $\mu(h+k+h+l)=\mu(k+l)$.

Using the preceding lemmas, each of these properties is easily proved.

2. **The extension to a metric Boolean ring.** Let us now introduce the quasi-metric $\delta(h, k) \equiv \mu(h+k)$ on H. That $\delta(h, k)$ *is* a quasi-metric is a consequence of the corollaries to Lemmas 2 and 5 and (iv). The properties (ii), (iii), and (v) then state that the functional μ and the operations of multiplication and formal addition are uniformly continuous in the quasi-metric δ. Hence in the partition $\mathfrak{B}$ of H by the equivalence relation $\delta(h, k)=0$, we may introduce a functional μ and operations of multiplication and addition by the following definition.

DEFINITION. *For A, $B \in \mathfrak{B}$, we define $A+B$ as the unique equivalence class represented by $h+k$ for $h \in A$, $k \in B$. Under the same hypothesis, we define AB to be the unique equivalence class represented by hk, and we define $\mu(A)$ to be the unique value of $\mu(h)$ for $h \in A$.*

THEOREM 1. *The system* $(\mathfrak{B}, +, \cdot, \mu)$ *is a metric Boolean ring.*

Proof. The associative, commutative, and distributive laws for $+$ and $\cdot$ are immediate. The element $O \in L$, considered as a collection $(O) \in H$, represents an element $\Theta \in \mathfrak{B}$ for which $A + \Theta = A$ by Lemma 1. Further $A + A = \Theta$ and $AA = A$ for $A \in \mathfrak{B}$ follow by the corollary to Lemma 2. Thus $(\mathfrak{B}, +, \cdot)$ is a Boolean ring. For elements $A, B \in \mathfrak{B}$, Lemma 7 gives

$$\mu(A + B) = \mu(A) + \mu(B) - 2\mu(AB).$$

Introducing the lattice operation

$$A \cup B \equiv A + B + AB$$

into $\mathfrak{B}$ yields

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(AB).$$

This means that μ is a modular functional on $\mathfrak{B}$. Noting that

$$\mu(A \cup B) - \mu(AB) = \mu(A + B)$$

we conclude that μ is sharply positive on $\mathfrak{B}$. Therefore $(\mathfrak{B}, +, \cdot, \mu)$ is a metric Boolean ring.

DEFINITION. *For $k \in L$, we define*

$$B(k) \equiv [h \mid \mu(h + k) = 0],$$
$$\mathfrak{B}_L \equiv [B(a) \mid a \in L].$$

THEOREM 2. *The correspondence $a \rightleftarrows B(a)$ between L and the subset $\mathfrak{B}_L$ of $\mathfrak{B}$ is an isomorphism.*

Proof. If $a, b \in L$, then $B(a) = B(b)$ implies

$$\mu(a + b) = \mu(a) + \mu(b) - 2\mu(ab) = \delta(a, b) = 0$$

and consequently that $a = b$. Thus the correspondence is one-to-one. Now $\mu(a) = \mu(B(a))$ by the definition of μ on $\mathfrak{B}$. Further, $B(a)B(b) = B(ab)$, $B(a) + B(b) = B(a + b)$ by the definition of multiplication and addition on $\mathfrak{B}$. Finally,

$$\begin{aligned} B(a) \cup B(b) &= B(a) + B(b) + B(a)B(b) \\ &= B(a + b) + B(ab) \\ &= B(a + b + ab) = B(a \cup b). \end{aligned}$$

The last equality follows from Lemma 3 and the corollary to Lemma 2 as follows:

$$\mu(a + b + ab + a \cup b) = \mu(a + b + a + b) = 0.$$

The proof is complete.

Remark. Our extension is now complete. However, if it is desired to represent the abstract Boolean ring $\mathfrak{B}$ as a more concrete system, we may employ the result of Stone [13]. No complicated procedure such as we have used in §1 is required for this process. It suffices merely to attach the same real number $\mu(A)$ to the subset which represents A in Stone's construction.

3. A bounded modular functional yields a completely additive measure. In this section we shall make the additional hypothesis that

$$\text{l.u.b. } [\mu(a) \mid a \in L] < +\infty \tag{5}$$

and we shall prove that we may then embed L into a field of sets with a completely additive measure.

If L has a unit element, we shall denote it by 1. Even if L has no unit element we may introduce one, when (5) holds, setting

$$\mu(1) = \text{l.u.b. } [\mu(a) \mid a \in L].$$

We note that this extension does not affect the modular nor the sharply positive character of the functional μ. Application of the procedure of §§1 and 2 then yields a Boolean ring $\mathfrak{B}$ with unit, I, determined by 1. For, if $A \in \mathfrak{B}$, $h \in A$, then $1 \cdot h = h$, and $AI = A$. Now the result of Wallman [14] assures us that we may represent $\mathfrak{B}$ as the set K of all simultaneously open and closed sets of a totally disconnected bicompact T_1-space Ω. It is clear that μ is finitely additive on K, and as Kakutani points out [6, p. 533], it follows that μ is *completely additive* on K, since no element of K can be the union of an *infinite* disjoint sequence of non-null elements of K. Again following Kakutani, we may apply the result of Hopf [5, p. 2] to extend μ to the least Borel field containing K. Thus the hypothesis (5) permits us to embed L into a field of sets on which is defined a completely additive measure which agrees with μ on the subsystem corresponding to $\mathfrak{B}_L$.

Part II. A lattice-theoretic interpretation of E. H. Moore's last instance of his second general analysis theory

In this part of our paper we shall discuss the example of Moore [10, vol. 1, pp. 14–15]. We first introduce certain well known generalities in §1. This prepares us for the description of Moore's instance in lattice-theoretic terms in §2. The relation of distributivity to the positive character of Moore's basic matrix is discussed in §3. In §4 we show that the identification we propose does not affect the value of Moore's J-integral as applied to *accordant* vectors.

1. Generalities. Most of the material presented in this section is well known, and we shall, therefore, offer no proofs of the statements contained herein. The main purpose of this section is to introduce the proper notation and terminology with which to state and prove the results of the following sections.

If $\mathfrak{P} \equiv [p]$ is a general range, we shall say that a relation R on $\mathfrak{P}\mathfrak{P}$ is a *quasi-order* relation (R^{qo}) in case R is *reflexive* and *transitive*,

$$R^{qo} \cdot \equiv \cdot pRp : p_1Rp_2 \cdot p_2Rp_3 \supset p_1Rp_3.$$

If, in addition, R is *symmetric*, we call R an *equivalence* relation (R^e),

$$R^e \cdot \equiv \cdot R^{qo} : p_1 R p_2 \supset p_2 R p_1.$$

On the other hand, those quasi-order relations R which are *anti-symmetric* are called (partial) *order* relations (R^o),

$$R^o \cdot \equiv \cdot R^{qo} : p_1 R p_2 \cdot p_2 R p_1 \supset p_1 = p_2.$$

Each quasi-order relation R^{qo} gives rise to an equivalence relation E_R by means of the following definition:

$$R^{qo} : \supset : p_1 E_R p_2 \cdot \equiv \cdot p_1 R p_2 \cdot p_2 R p_1.$$

We note the implication

$$R^{qo} \cdot \supset \cdot p_1 E_R p_2 \cdot p_2 R p_3 \cdot p_3 E_R p_4 \supset p_1 R p_4.$$

It follows that the partition $\mathfrak{P}_R \equiv [p_R] \equiv [[p_1 | p_1 E_R p] | p]$ of $\mathfrak{P}$ by E_R may be ordered by the relation

$$p_{R1} \mathcal{R} p_{R2} \equiv p_1 R p_2 \qquad (p_1 \in p_{R1},\ p_2 \in p_{R2}).$$

We may then show that $\mathcal{R}$ on $\mathfrak{P}_R \mathfrak{P}_R$ is an order relation,

$$R^{qo} \supset \mathcal{R}^o.$$

If each two-element subset (p_1, p_2) of $\mathfrak{P}$ has a greatest lower bound $p_1 \otimes p_2$ and a least upper bound $p_1 \oplus p_2$ in the sense of a relation R^{qo} on $\mathfrak{P}\mathfrak{P}$ we say that $(\mathfrak{P}, R)$ is a *quasi-lattice*. The functions $(p_1 \otimes p_2 | p_1 p_2)$ and $(p_1 \oplus p_2 | p_1 p_2)$ need not be singled-valued, but the subclasses $[p_1 \otimes p_2 | p_1 p_2]$ and $[p_1 \oplus p_2 | p_1 p_2]$ of $\mathfrak{P}$ are elements of $\mathfrak{P}_R$. When $(\mathfrak{P}, R)$ is a quasi-lattice, then $(\mathfrak{P}_R, \mathcal{R})$ is a lattice in which

$$p_{R1} \cap p_{R2} = [p_1 \otimes p_2 | p_1 \in p_{R1},\ p_2 \in p_{R2}],$$
$$p_{R1} \cup p_{R2} = [p_1 \oplus p_2 | p_1 \in p_{R1},\ p_2 \in p_{R2}].$$

A real-valued function μ defined on $\mathfrak{P}$ will be called R-monotone (with respect to R^{qo} on $\mathfrak{P}\mathfrak{P}$) in case $p_1 R p_2$ implies $\mu(p_1) \leqq \mu(p_2)$. In this case E_R preserves μ in the sense that

$$p_1 E_R p_2 \supset \mu(p_1) = \mu(p_2) \tag{1.1}$$

and we may define $\mu(p_R) \equiv \mu(p)$ $(p \in p_R)$. The resulting function $(\mu(p_R) | p_R)$ on $\mathfrak{P}_R$ to the real numbers is then $\mathcal{R}$-monotone. If, further, $(\mathfrak{P}, R)$ is a quasi-lattice, we may consider (by (1.1)) the additive-multiplicative property[5]

$$\mu(p_1 \oplus p_2) + \mu(p_1 \otimes p_2) = \mu(p_1) + \mu(p_2). \tag{1.2}$$

If $(\mathfrak{P}, R)$ is a quasi-lattice and μ is an R-monotone real valued function de-

[5] This is Moore's terminology. Garrett Birkhoff calls a functional satisfying (1.2) *modular*.

fined on $\mathfrak{P}$ and satisfying (1.2), we call the system $(\mathfrak{P}, R, \mu)$ a *quasi-metric quasi-lattice* because we may introduce the quasi-metric [1, p. 41]

$$\delta_{\mu R}(p_1, p_2) \equiv \mu(p_1 \oplus p_2) - \mu(p_1 \otimes p_2).$$

We are thus led to consider a second equivalence relation

$$p_1 E_{\mu R} p_2 \equiv \delta_{\mu R}(p_1, p_2) = 0.$$

We note that $p_1 E_R p_2$ implies $p_1 E_{\mu R} p_2$, and hence that the partition $\mathfrak{P}_{\mu R}$ of $\mathfrak{P}$ by $E_{\mu R}$ is coarser than $\mathfrak{P}_R$. It may be shown that the partition of $\mathfrak{P}_R$ by $E_{\mu R}$ and the partition of $\mathfrak{P}$ by $E_{\mu R}$ are isomorphic metric lattices.

2. **Moore's instance.** Let us consider, following E. H. Moore, a general range $\mathfrak{P} \equiv [p]$ on which are defined two binary, associative, idempotent, and commutative operations $(p_1 + p_2 | p_1 p_2)$ and $(p_1 \cdot p_2 | p_1 p_2)$ which satisfy the distributive law.

(MI) $$p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3, \qquad p_1(p_2 p_3) = (p_1 p_2) p_3,$$
$$p + p = p = pp, \qquad p_1 + p_2 = p_2 + p_1, \qquad p_1 p_2 = p_2 p_1,$$

(MII) $$p_1(p_2 + p_3) = p_1 p_2 + p_1 p_3.$$

It is also assumed that there is a real valued function $(\beta(p) | p)$ defined on $\mathfrak{P}$ and satisfying

(MIII) $$\beta(p_1 + p_2) + \beta(p_1 p_2) = \beta(p_1) + \beta(p_2),$$

(MIV) $$\beta(p_1 + p_2) \geqq \beta(p_1).$$

Our primary purpose in this section is to show that (MI)–(MIV) imply that there is a relation R on $\mathfrak{P}\mathfrak{P}$ which is such that $(\mathfrak{P}, R, \beta)$ is a distributive quasi-metric quasi-lattice in which $E_R = E_{\beta R}$ and $p_1 + p_2 \in p_1 \oplus p_2$, $p_1 \cdot p_2 \in p_1 \otimes p_2$.

We define $p_1 R p_2 \equiv \beta(p_1 + p_2) = \beta(p_2) \sim \beta(p_1 p_2) = \beta(p_1)$.

LEMMA 1. $\beta(p_1 p_2) \leqq \beta(p_1)$.

Proof. We have, by (MIII) and (MIV),

$$\beta(p_1 p_2) = \beta(p_1) + \beta(p_2) - \beta(p_1 + p_2) \leqq \beta(p_1).$$

LEMMA 2. R^{qo}.

Proof. The condition (MI) insures that R is reflexive. To prove that R is transitive, consider elements $p_1, p_2, p_3 \in \mathfrak{P}$ such that $p_1 R p_2$ and $p_2 R p_3$. Using Lemma 1, (MI)–(MIV), we find that

$$\begin{aligned} \beta(p_1 + p_2 + p_3) &= \beta(p_1) + \beta(p_2 + p_3) - \beta(p_1 p_2 + p_1 p_3) \\ &= \beta(p_1) + \beta(p_3) - \beta(p_1 p_2) - \beta(p_1 p_3) + \beta(p_1 p_2 p_3) \\ &\leqq \beta(p_3) \leqq \beta(p_3 + p_1) \leqq \beta(p_1 + p_2 + p_3). \end{aligned}$$

It follows that $\beta(p_1+p_3)=\beta(p_3)$ and that p_1Rp_3. Consequently, R is transitive and we have R^{qo}.

LEMMA 3. *The system* $(\mathfrak{P}, R)$ *is a quasi-lattice in which each two-element subset* (p_1, p_2) *has* p_1+p_2 *as a least upper bound and has* $p_1\cdot p_2$ *as a greatest lower bound in the sense of the relation* R.

Proof. Consider a two-element subset $\sigma\equiv(p_1, p_2)$ of $\mathfrak{P}$. We show first that p_1+p_2 is effective as a least upper bound of σ. We have $p_1R(p_1+p_2)$ and $p_2R(p_1+p_2)$ by the definition of R and (MI). Let us consider an element p such that p_1Rp and p_2Rp. Using (MI)–(MIV), we find that

$$\begin{aligned}\beta(p + p_1 + p_2) &= \beta(p + p_1) + \beta(p_2) - \beta(pp_2 + p_1p_2)\\ &= \beta(p) + \beta(pp_2) - \beta(pp_2 + p_1p_2)\\ &\leqq \beta(p) \leqq \beta(p + p_1 + p_2).\end{aligned}$$

It follows that $(p_1+p_2)Rp$ and that p_1+p_2 is a least upper bound of σ. We next prove that p_1p_2 is effective as a greatest lower bound of σ. The relations $(p_1p_2)Rp_1$ and $(p_1p_2)Rp_2$ follow immediately from (MI), (MIII), and the definition of R. For each element p such that pRp_1 and pRp_2, we find, using Lemma 1 and (MI)–(MIV), that

$$\begin{aligned}\beta(pp_1p_2) &= \beta(pp_1) + \beta(p_2) - \beta(p + p_1p_2)\\ &= \beta(p) + \beta(p + p_2) - \beta(p + p_1p_2)\\ &\geqq \beta(p) + \beta(p + p_2) - \beta((p + p_1)(p + p_2))\\ &\geqq \beta(p) \geqq \beta(pp_1p_2).\end{aligned}$$

Hence we have $pR(p_1p_2)$ and consequently $p_1\cdot p_2$ is a greatest lower bound of σ. The proof is complete.

LEMMA 4. *The function* β *is* R-*monotone.*

Proof. If p_1Rp_2, we conclude, by (MIV) and the definition of R, that $\beta(p_1)\leqq\beta(p_1+p_2)=\beta(p_2)$.

LEMMA 5. $p_1E_Rp_2\sim\beta(p_1+p_2)=\beta(p_1p_2)$.

Proof. The forward implication is trivial by the definitions of R and E_R. To prove the reverse implication it suffices to note that, by (MIV) and Lemma 1, we have $\beta(p_1+p_2)\geqq\beta(p_1)\geqq\beta(p_1p_2)$ and $\beta(p_1+p_2)\geqq\beta(p_2)\geqq\beta(p_1p_2)$.

COROLLARY. $E_R=E_{\beta R}$.

THEOREM 1. *The partition* $\mathfrak{P}_R$ *of* $\mathfrak{P}$ *by* E_R *forms with* β *and* R *a system* $(\mathfrak{P}, R, \beta)$ *which is a distributive metric lattice.*

Proof. This is clear.

THEOREM 2. *If the function* β *is bounded, then we may embed the system* $(\mathfrak{P}, +, \cdot, \beta)$ *isomorphically into a field of sets with a completely additive measure.*

Proof. This is clear by Theorem 1 of Part II and the results of Part I.

THEOREM 3. *If* $(\mathfrak{P}, R, \beta)$ *is a quasi-metric distributive lattice. then* $(\mathfrak{P}, \cup, \cap, \beta)$ *satisfies* (MI)–(MIV).

Proof. This is clear.

3. **The relation between distributivity and the positive character of Moore's basic matrix.** On the basis (MI)–(MIV), Moore defined a matrix $\epsilon(p_1p_2) \equiv \beta(p_1 \cdot p_2)$. He showed that ϵ is *positive* whenever β is non-negative. This property of ϵ is essential in the construction of an integration process J_ϵ based on ϵ. We shall show in this section that, in a certain sense, the assumption of distributivity is necessary in order that ϵ be positive.

THEOREM 4. *Let* $(\mathfrak{P}, R, \beta)$ *be a quasi-metric lattice for which* $\epsilon(p_1p_2) \equiv \beta(p_1 \cap p_2)$ *is a positive matrix. Then the partition of* $\mathfrak{P}$ *by* $E_{\beta R}$ *is a distributive metric lattice.*

Proof. We note first that $\beta(p) \geqq 0$ by the positive character of ϵ. It suffices to show that in the partition of $\mathfrak{P}$ by $E_{\beta R}$ the distributive law holds. Suppose that the distributive law fails to hold in this partition. The Dedekind (modular) law *is* valid [1, p. 43], and hence the simplest modular nondistributive lattice [1, p. 75] is a sublattice of the partition of $\mathfrak{P}$ by $E_{\beta R}$. Let p_i $(i=0, \cdots, 4)$ be the five elements of this sublattice with p_0 the least element and p_4 the greatest element in the sense of the order relation of $\mathfrak{P}_{\beta R}$. Define $y \equiv \beta(p_0) \geqq 0$, and $x \equiv \beta(p_1) - y > 0$. We have, setting $\beta(p_i) = \beta_i$ $(i=0, \cdots, 4)$,

$$x = \beta_4 - \beta_3 = \beta_2 - \beta_0 = \beta_4 - \beta_1 = \beta_3 - \beta_0 = \beta_4 - \beta_2.$$

The section $\epsilon(\sigma, \sigma)$ with $\sigma \equiv (p_0, \cdots, p_4)$ is

$$\left\| \begin{matrix} y & y & y & y & y \\ y & x+y & y & y & x+y \\ y & y & x+y & y & x+y \\ y & y & y & x+y & x+y \\ y & x+y & x+y & x+y & 2x+y \end{matrix} \right\|.$$

The determinant of this matrix is $-x^4y$ which is *negative* unless $y=0$. However, if $y=0$, the principal minor formed by deleting the first row and the first column of $\epsilon(\sigma, \sigma)$ has the value $-x^4<0$. But this is contrary to the hypothesis that ϵ is a positive matrix. It follows that the partition of $\mathfrak{P}$ by $E_{\beta R}$ is a distributive lattice.

4. **Effect of identification on Moore's J-integral.** It remains to consider the application of Moore's general theory to the partition $\mathfrak{P}_R$ of $\mathfrak{P}$ by E_R.

We shall prove in this section that the J-integral, at least as applied to *accordant* vectors, suffers no alteration in this process. Since the existence of the J-integral is proved, in the general theory, only for vectors which are *at least* accordant, we are satisfied with this result.

LEMMA 6. *If* $p_1E_Rp_2$ *then* $\beta(pp_1)=\beta(pp_2)$ *for every* p.

Proof. This is clear by Lemma 4 and the corollary to Lemma 5.

LEMMA 7. *If* ξ *is a vector on* $\mathfrak{P}$ *which is accordant as to* ϵ *and if* $p_1E_Rp_2$, *then* $\xi(p_1)=\xi(p_2)$.

Proof. Consider a vector ξ on $\mathfrak{P}$ which is accordant as to ϵ, and two elements p_1, p_2 for which $p_1E_Rp_2$. Define $\sigma\equiv(p_1, p_2)$ and choose $\alpha(p_1)\neq 0$, $\alpha(p_2)=-\alpha(p_1)$. It is easy to verify, using Lemma 6, that $S_\sigma S_\sigma\alpha\epsilon\alpha=0$, while $S_\sigma\alpha\xi=\alpha(p_1)[\xi(p_1)-\xi(p_2)]$. Since ξ is accordant, we have $\xi(p_1)=\xi(p_2)$.

DEFINITION. *In the partition* $\mathfrak{P}_R$ *of* $\mathfrak{P}$ *by* E_R *we define*

$$\epsilon_R(p_{R1}, p_{R2}) \equiv \epsilon(p_1, p_2)(p_1 \in p_{R1}, p_2 \in p_{R2}),$$

$$\mathfrak{P}_0 \subset \mathfrak{P} \cdot\supset\cdot \mathfrak{P}_{0R} \equiv [[pE_Rp_0]|p_0],$$

$$\xi^{\text{on } \mathfrak{P}\cdot A} \supset \xi_R(p_R) \equiv \xi(p)(p \in p_R).$$

We note that ϵ_R is positive and hermitian. Thus, Moore's theory leads to a J_R-integral. To derive a relation between J_R and J, we prove the following lemma.

LEMMA 8. *Let* $\mathfrak{A}$ *be a number system of type* D, $\mathfrak{P}$ *be a general range, and* ϵ *on* $\mathfrak{P}\mathfrak{P}$ *to* $\mathfrak{A}$ *be a positive hermitian matrix. Then, if the vectors* ξ *and* η *are accordant as to* ϵ, *and if* σ *is a finite subset of* $\mathfrak{P}$, *we have* $J_\sigma\bar{\xi}\eta=J_{\sigma_1}\bar{\xi}\eta$, *where* $\sigma_1\subset\sigma$ *is such that* $\epsilon(\sigma_1, \sigma_1)$ *is a maximal nonsingular minor of* $\epsilon(\sigma, \sigma)$.

Proof. By Corollary 29.45(2) of Moore's work [10, vol. 1] as applied to ξ_σ and η_σ [10, vol. 2] we have

$$J_\sigma\bar{\xi}\eta = J_{\sigma_1}\overline{\xi_\sigma(\sigma_1)}\eta_\sigma(\sigma_1).$$

But Theorem 41.6(4) of the same work [10, vol. 2] then yields

$$\xi_\sigma(\sigma_1) = \xi(\sigma_1) \quad \text{and} \quad \eta_\sigma(\sigma_1) = \eta(\sigma_1)$$

since ξ and η are accordant vectors. It follows that $J_\sigma\bar{\xi}\eta=J_{\sigma_1}\bar{\xi}\eta$ as was to be proved.

Returning now to the question of J_R and J, we prove the following theorem.

THEOREM 5. *If* ξ *and* η *are vectors on* $\mathfrak{P}$ *which are accordant as to* ϵ, *then* ξ_R *and* η_R *are accordant as to* ϵ_R *and further*

$$J\xi\eta = J_R\xi_R\eta_R. \tag{4.1}$$

Proof. Let ξ be a vector on $\mathfrak{P}$ and accordant as to ϵ. Consider a vector α on $\mathfrak{P}_R$ and a finite subset $\sigma_R \subset \mathfrak{P}_R$ for which $S_{\sigma_R} S_{\sigma_R} \alpha \epsilon_R \alpha = 0$. Suppose that $S_{\sigma_R} \alpha \xi_R \neq 0$. Select in each $p_R \in \sigma_R$ a representative $p \in p_R$. Define $\alpha_1(p) \equiv \alpha(p_R)$ for the chosen representative, define $\alpha_1(p) \equiv 0$ for every other element of $\mathfrak{P}$, and define σ to be the finite subset of $\mathfrak{P}$ consisting of the chosen representatives. We have $S_\sigma S_\sigma \alpha_1 \epsilon \alpha_1 = 0$ and $S_\sigma \alpha_1 \xi = S_{\sigma_R} \alpha \xi_R \neq 0$. This is contrary to the assumption that ξ is accordant as to ϵ. Consequently, $S_{\sigma_R} \alpha \xi_R = 0$ and ξ_R is accordant as to ϵ_R.

We show next that if σ is a finite subset of $\mathfrak{P}$, then

$$J_\sigma \xi \eta = J_{\sigma_R} \xi_R \eta_R \tag{4.2}$$

from which (4.1) follows at once. Let $\sigma_1 \subset \sigma$ determine a maximal nonsingular minor of $\epsilon(\sigma, \sigma)$. Clearly, by Lemma 6, $\epsilon_R(\sigma_{1R}, \sigma_{1R})$ is also a maximal nonsingular minor of $\epsilon_R(\sigma_R, \sigma_R)$. Equation (4.2) is now easily proved by two applications of Lemma 8.

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