THE THEOREM OF BERTINI ON THE VARIABLE SINGULAR POINTS OF A LINEAR SYSTEM OF VARIETIES

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1. **Pure transcendental extensions of the ground field.** Let V/k be an irreducible r-dimensional algebraic variety over a given ground field k. We assume that V/k is immersed in an n-dimensional projective space and we denote by $x_1, x_2, \cdots, x_n$ the nonhomogeneous coördinates of the general point of V/k. Let $u_1, u_2, \cdots, u_m$ be indeterminates with respect to the field $k(x)$ $[=k(x_1, x_2, \cdots, x_n)]$ of rational functions on V/k. We adjoin these indeterminates to the field $k(x)$ and we denote by K the field $k(u)$ $[=k(u_1, u_2, \cdots, u_m)]$. This subfield K of the field $k(x, u)$ we take as new ground field, and over this new ground field we consider the variety V/K defined by the same general point $(x_1, x_2, \cdots, x_n)$ as V/k. The varieties V/k and V/K are of the same dimension r over their respective ground fields k and K. We shall say that the variety V/K is *the extension of the variety* V/k under the ground field extension $k \rightarrow K$.

By precisely the same argument, every irreducible subvariety W/k of V/k has as extension an irreducible subvariety W/K of V/K, of the same dimension as W/k. If $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ are the nonhomogeneous coördinates of the general point of W/k, then $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ are also the nonhomogeneous coördinates of the general point of W/K. Moreover, $u_1, u_2, \cdots, u_m$ are indeterminates with respect to the field $k(\bar{x})$, that is, they are algebraically independent over this field.

Not every irreducible subvariety of V/K is the extension of a subvariety of V/k, but every such subvariety W^*/K defines an irreducible subvariety W/k of V/k, which we shall refer to as *the contraction of* W^*/K and which is obtained as follows. Let $x_1^*, x_2^*, \cdots, x_n^*$ be the nonhomogeneous coördinates of the general point of W^*/K. Since $W^*/K \subseteq V/K$, the ring $K[x_1^*, x_2^*, \cdots, x_n^*]$ is a homomorphic image of the ring $K[x_1, x_2, \cdots, x_n]$. Therefore, also the ring $k[x_1^*, x_2^*, \cdots, x_n^*]$ is a homomorphic image of the ring $k[x_1, x_2, \cdots, x_n]$. Therefore there is a unique irreducible subvariety W/k of V/k, whose general point $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ is defined by the condition that the rings $k[\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n]$ and $k[x_1^*, x_2^*, \cdots, x_n^*]$ be simply isomorphic and that $\bar{x}_i, x_i^*$ $(i=1, 2, \cdots, n)$ be corresponding elements in the isomorphism. This variety W/k shall be termed the contraction of W^*/K.

LEMMA 1. *Let W/k and W^*/K be irreducible subvariables of V/k and of V/K respectively, and let $(\bar{x})$, (x^*) $[(\bar{x})=(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n), (x^*)=(x_1^*, x_2^*, \cdots, x_n^*)]$ be their general points.*

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(a) *W/k is the contraction of W/K.*

(b) *If W/k is the contraction of W^*/K, then $W^*/K \subseteq W/K$.*

(c) *A necessary and sufficient condition that W^*/K be the extension of a subvariety of V/k is that $u_1, u_2, \cdots, u_m$ be algebraically independent over the field $k(x_1^*, x_2^*, \cdots, x_n^*)$.*

Proof. (a) Trivial.

(b) By hypothesis, the rings $k[\bar{x}]$ and $k[x^*]$ are simply isomorphic. Since $u_1, u_2, \cdots, u_m$ are algebraically independent over the field $k(\bar{x})$, it follows that $K[x^*]$ is a homomorphic image of $K[\bar{x}]$, that is, $W^*/K \subseteq W/K$.

(c) The condition is necessary in virtue of the very definition of W/K. Let us suppose that the condition is satisfied. Then every algebraic relation between the elements $x_1^*, x_2^*, \cdots, x_n^*$ over K is a consequence of algebraic relations between $x_1^*, x_2^*, \cdots, x_n^*$ over k. If then W/k is the contraction of W^*/K, then it follows in view of isomorphism $k[x^*] \simeq k[\bar{x}]$ that the two rings $K[x^*]$ and $K[\bar{x}]$ are simply isomorphic. This shows that $W^*/K = W/K$.

We denote the quotient ring of an irreducible subvariety W/k of V/k by $Q_V(W/k)$. Similarly, the quotient ring of W/K, regarded as a subvariety of V/K, shall be denoted by $Q_V(W/K)$. By definition, W/k is a simple subvariety of V/k if the ideal $\mathfrak{m}$ of non-units of $Q_V(W/k)$ has a base consisting of $r-s$ elements, where s is the dimension of W/k(1). The elements of such a base are referred to as *uniformizing parameters of* $W(V/k)$. If $t_1, t_2, \cdots, t_\rho$, $\rho = r-s$, are uniformizing parameters of $W(V/k)$, and if ω is any element of $\mathfrak{m}$ which is exactly divisible by $\mathfrak{m}^\nu$ [that is, $\omega \equiv 0(\mathfrak{m}^\nu)$, $\omega \not\equiv 0(\mathfrak{m}^{\nu+1})$], then $\omega = \phi_\nu(t_1, t_2, \cdots, t_\rho)$, where ϕ_ν is a form of degree ν with coefficients in $Q_V(W/k)$ but not all in $\mathfrak{m}$. If these coefficients are replaced by their residues mod $\mathfrak{m}$, we obtain a form of degree ν in ρ indeterminates, with coefficients in the residue field of W/k, not all zero. This is called the *leading form* of ω. It is easy to show that ρ elements of $Q_V(W/k)$ are uniformizing parameters of $W(V/k)$ if and only if their leading forms are linear and linearly independent(2).

We use the notation of the preceding lemma and we assume that W/k

(1) See O. Zariski, *Algebraic varieties over ground fields of characteristic zero*, Amer. J. Math. vol. 62 (1940) p. 199.

(2) *Proof.* If $\omega_1, \omega_2, \cdots, \omega_\rho$ are non-units in $Q_V(W/k)$, then $\omega_i = \sum_{j=1}^{\rho} A_{ij} t_j$, where the A_{ij} are elements of $Q_V(W/k)$. Let A be the matrix $\|A_{ij}\|$ and $\bar{A}$ be the matrix $\|\bar{A}_{ij}\|$, where $\bar{A}_{ij}$ is the $\mathfrak{m}$-residue of A_{ij}. Suppose that the leading forms of $\omega_1, \omega_2, \cdots, \omega_\rho$ are linear and linearly independent. Then $|\bar{A}| \neq 0$, that is, $|A| \not\equiv 0(\mathfrak{m})$, and therefore $t_1, t_2, \cdots, t_\rho$ can be expressed as linear forms in $\omega_1, \omega_2, \cdots, \omega_\rho$, with coefficients which are elements of the matrix A^{-1}. These elements are in $Q_V(W/k)$, and hence $\omega_1, \omega_2, \cdots, \omega_\rho$ form a basis for $\mathfrak{m}$.

Conversely, assume that $\omega_1, \omega_2, \cdots, \omega_\rho$ are uniformizing parameters of $W(V/k)$. We can then write: $t_i = \sum_{j=1}^{\rho} B_{ij}\omega_j$, and hence $t_i = \sum_{j=1}^{\rho} C_{ij} t_j$ where $C = BA$, $C = \|C_{ij}\|$, $B = \|B_{ij}\|$. Since every element of $Q_V(W/k)$ has a unique leading form, it follows from the relations $t_i = \sum_{j=1}^{\rho} C_{ij} t_j$ that modulo $\mathfrak{m}$ the matrix C is the unit matrix. Hence $|B| \cdot |A| \not\equiv 0(\mathfrak{m})$, whence the leading forms of $\omega_1, \omega_2, \cdots, \omega_\rho$ are linear and linearly independent.

is the contraction of W^*/K. If W/k is of dimension s, then W^*/K is of dimension $s-\sigma$, $\sigma\geqq 0$, by part (b) of Lemma 1, that is, the field $K(x^*)$ is of degree of transcendency $s-\sigma$ over $K[=k(u)]$. Since $k(x^*)\simeq k(\bar{x})$ and since $k(\bar{x})$ is of degree of transcendency s over k, it follows that the field $k(u, x^*)$ is of degree of transcendency $m-\sigma$ over $k(x^*)$. If then $U_1, U_2, \cdots, U_m$ are indeterminates over $k(x^*)$, the polynomials in the polynomial ring $k(x^*)[U_1, U_2, \cdots, U_m]$ which vanish after the specialization $U_i\rightarrow u_i$ form a prime ideal $\mathfrak{P}$ of dimension $m-\sigma$. The ideal of non-units in the quotient ring of this polynomial ideal has a base of σ elements[3]. Let $G_1(U), G_2(U), \cdots, G_\sigma(U)$ be such a base, where we may assume, without loss of generality, that the $G_i(U)$ are polynomials in the U's (with coefficients in $k(x^*)$): $G_i(U)=G_i(U, x^*)$. We put

$$\tau_i = G_i(u, x), \qquad i = 1, 2, \cdots, \sigma. \tag{1}$$

LEMMA 2. (a) *If W/K, of dimension s, is simple for V/K, then W/k is simple for V/k, and if $t_1, t_2, \cdots, t_\rho$ $(\rho=r-s)$ are uniformizing parameters for $W(V/k)$, they are also uniformizing parameters for $W(V/K)$.*

(b) *If W/k, of dimension s, is the contraction of W^*/K which is of dimension $s-\sigma$, $\sigma\geqq 0$, and if W/k is simple for V/k, with $t_1, t_2, \cdots, t_\rho$ as uniformizing parameters, then also W^*/K is simple for V/K, and the elements $t_1, t_2, \cdots, t_\rho, \tau_1, \tau_2, \cdots, \tau_\sigma$ are uniformizing parameters of $W^*(V/K)$.*

Proof. (a) We put $\mathfrak{J}=Q_V(W/k)$, $\bar{\mathfrak{J}}=Q_V(W/K)$ and we denote by $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ respectively the ideals of non-units in $\mathfrak{J}$ and in $\bar{\mathfrak{J}}$. The elements of $\bar{\mathfrak{J}}$ are all of the form $\phi(u)/\psi(u)$, where $\phi(u), \psi(u)\in\mathfrak{J}[u_1, u_2, \cdots, u_m]$ and where not all the coefficients of $\psi(u)$ are in $\mathfrak{m}$. The following relations are therefore obvious:

$$\bar{\mathfrak{m}} = \bar{\mathfrak{J}}\cdot\mathfrak{m}, \tag{2}$$

$$\mathfrak{m} = \bar{\mathfrak{m}}\cap\mathfrak{J}. \tag{2'}$$

Let W/K be simple for V/K. It follows then directly from (2) that already the ideal $\mathfrak{m}$ must contain ρ elements which—regarded as elements of $Q_V(W/K)$—have leading forms which are linear and linearly independent. There exists therefore a set of uniformizing parameters of $W(V/K)$ which consists of elements of $\mathfrak{m}$. Let $t_1, t_2, \cdots, t_\rho$ be such a set. If ω is any element of $\mathfrak{m}$, we have by (2'):

$$\omega\cdot\psi(u) = \phi_1(u)\cdot t_1 + \phi_2(u)\cdot t_2 + \cdots + \phi_\rho(u)\cdot t_\rho, \tag{3}$$

where $\psi(u), \phi_i(u)\in\mathfrak{J}[u]$ and where at least one coefficient of $\psi(u)$ does not belong to $\mathfrak{m}$. The presence of such a coefficient and the fact that $u_1, u_2, \cdots, u_m$ are algebraically independent over $\mathfrak{J}$ have, by (3), the consequence that ω

(3) See O. Zariski, *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc. vol. 53 (1943) p. 541.

belongs to ideal $\mathfrak{J}\cdot(t_1, t_2, \cdots, t_\rho)$. Hence $\mathfrak{m}=\mathfrak{J}(t_1, t_2, \cdots, t_\rho)$, and W/k is simple for V/k, with $t_1, t_2, \cdots, t_\rho$ as uniformizing parameters. This and relation (2) complete the proof of part (a) of the lemma.

(b) Let $\mathfrak{J}^*$ denote the quotient ring $Q_V(W^*/K)$ and let $\mathfrak{m}^*$ be the ideal of non-units in $\mathfrak{J}^*$. We assume that W/k, the contraction of W^*/K, is simple for V/k, with $t_1, t_2, \cdots, t_\rho$ as uniformizing parameters. The elements of $\mathfrak{J}^*$ are all of the form $\phi(x, u)/\psi(x, u)$, where ϕ and ψ are polynomials with coefficients in k and where $\psi(x^*, U)\not\equiv 0 \pmod{\mathfrak{P}}$. Since $W^*/K\subseteq W/K$, the variety W/K is defined locally at W^*/K by a prime ideal $\mathfrak{p}^*$ in the quotient ring $\mathfrak{J}^*$. This ideal consists of those quotients $\phi(x, u)/\psi(x, u)$ for which $\phi(x^*, U)=0$ (or in equivalent form: $\phi(\bar{x}, u)=0$). It is therefore clear that the residue class ring $\mathfrak{J}^*/\mathfrak{p}^*$ coincides with the quotient ring of the prime ideal $\mathfrak{P}$. Hence we have by (1):

$$\mathfrak{J}^*\cdot\mathfrak{m}^* = \mathfrak{J}^*\cdot(\mathfrak{p}^*, \tau_1, \tau_2, \cdots, \tau_\sigma). \tag{4}$$

Now if $\phi(x^*, U)=0$, that is, if $\phi(\bar{x}, u)=0$, then $\phi(x, u)\equiv 0(\overline{\mathfrak{m}})$, whence $\phi(x, u)\equiv 0(\overline{\mathfrak{J}}\cdot\mathfrak{m})$. Consequently $\mathfrak{p}^*=\mathfrak{J}^*\cdot(t_1, t_2, \cdots, t_\rho)$, and therefore, by (4),

$$\mathfrak{J}^*\cdot\mathfrak{m}^* = \mathfrak{J}^*\cdot(t_1, t_2, \cdots, t_\rho, \tau_1, \tau_2, \cdots, \tau_\sigma). \tag{5}$$

This completes the proof of part (b) of the lemma.

The above lemma implies that the *singular locus of V/K is the extension of the singular locus of V/k*, in the sense that the irreducible components of the singular locus of V/K are extensions of the irreducible components of the singular locus of V/k.

2. **The general member of a linear system of V_{r-1}'s on V/k.** Let $f_0(x), f_1(x), \cdots, f_m(x)$ be $m+1$ elements in $k[x]$ which are linearly independent over *the field of constants*, that is, over the algebraic closure of the ground field k in the field $k(x)$ of rational functions on V/k. The $m+1$ polynomials $f_i(x)$ determine uniquely a set of $m+1$ integral divisors $\mathfrak{A}_i$ of the field $k(x)$ with the following properties: (1) each $\mathfrak{A}_i$ is a divisor of the first kind with respect to a derived normal model V' of V/k(4); (2) $f_i/f_j=\mathfrak{A}_i/\mathfrak{A}_j$; (3) $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m$ are relatively prime. Each divisor $\mathfrak{A}_i$ determines on V/k (and—if V/k is locally normal(5)—is determined by) a pure $(r-1)$-dimensional subvariety F_i/k, whose irreducible components are counted to well-defined multiplicities. The irreducible components of F_i correspond to the prime factors of $\mathfrak{A}_i$, but if V/k is not locally normal, two distinct prime factors of $\mathfrak{A}_i$ may very well correspond to one and the same irreducible component of F_i/k. However, we agree to keep separate the identity of the irre-

(4) For the definition of a derived normal model, see O. Zariski, *Some results in the arithmetic theory of algebraic varieties*, Amer. J. Math. vol. 61 (1939) p. 292. For a discussion of divisors of the first kind, see O. Zariski, *Pencils on an algebraic variety and a new proof of a theorem of Bertini*, Trans. Amer. Math. Soc. vol. 50 (1941) p. 49.

(5) For a definition of locally normal varieties, see (3), p. 512.

ducible components of F_i/k relative to distinct prime factors of $\mathfrak{A}_i$. This means that we actually regard F_i as a subvariety of the normal derived variety V' rather than of V/k. In this connection we may add the remark that an $(r-1)$-dimensional irreducible subvariety of V/k may arise from more than one prime divisor only if it is singular for V/k.

If $\lambda_0, \lambda_1, \cdots, \lambda_m$ are arbitrary elements of k, then there exists a unique divisor $\mathfrak{A}(\lambda)$, of the first kind with respect to the normal variety V', such that

$$(6) \qquad [\lambda_0 f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x)]/f_0(x) = \mathfrak{A}(\lambda)/\mathfrak{A}_0.$$

Let $F(\lambda)$ be the $(r-1)$-dimensional subvariety of V/k (or—better—of V') defined by $\mathfrak{A}(\lambda)$. As the λ's vary in k, $F(\lambda)$ varies in a *linear system* $|F(\lambda)|$. The varieties $F_0, F_1, \cdots, F_m$ are particular members of the system.

We proceed to associate with the linear system $|F(\lambda)|$ an irreducible $(r-1)$-dimensional subvariety F^*/K of the variety V/K considered in the preceding section. We define F^* by the following conditions:

(a) The nonhomogeneous coördinates $\eta_1, \eta_2, \cdots, \eta_n$ of the general point of F^* shall satisfy the relation:

$$(7) \qquad f_0(\eta) + u_1 f_1(\eta) + \cdots + u_m f_m(\eta) = 0.$$

(b) The rings $k[x]$ and $k[\eta]$ shall be isomorphic, and in the isomorphism the elements x_i and η_i shall correspond to each other.

We have to show that: (1) there exists an F^*/K, of dimension $r-1$, satisfying conditions (a) and (b); (2) F^*/K is uniquely determined by these two conditions and by the condition that it be of dimension $r-1$; (3) F^*/K is a subvariety of V/K.

We start by introducing another copy $k(\eta)$ of the field $k(x)$, that is, we assume that $k(\eta) \simeq k(x)$, $\eta_i \to x_i$. We then adjoin to $k(\eta)$ a set of $m-1$ indeterminates $v_1, v_2, \cdots, v_{m-1}$ and we put(6)

$$v_m = -[f_0(\eta) + v_1 f_1(\eta) + \cdots + v_{m-1} f_{m-1}(\eta)]/f_m(\eta).$$

LEMMA 3. *The elements $v_1, v_2, \cdots, v_m$ are algebraically independent over k.*

Proof. The elements $v_1, v_2, \cdots, v_m$ are elements of the polynomial ring $k(\eta)[v_1, v_2, \cdots, v_{m-1}]$. If these elements were algebraically dependent over k, the specialization $v_1 \to 1$, $v_j \to 0$, $j = 2, 3, \cdots, m-1$, would lead to a true relation(7) of algebraic dependence of $[f_0(\eta) + f_1(\eta)]/f_m(\eta)$ over k. This contradicts our hypothesis that $f_0(\eta), f_1(\eta), \cdots, f_m(\eta)$ are linearly independent over the algebraic closure of k in $k(\eta)$.

In view of Lemma 3 we can identify $v_1, v_2, \cdots, v_m$ with $u_1, u_2, \cdots, u_m$ respectively. We have then an algebraic variety F^*/K, with general point $(\eta_1, \eta_2, \cdots, \eta_n)$, and conditions (a) and (b) are satisfied.

(6) Note that $f_m(\eta) \neq 0$ since the elements $f_0(\eta), f_1(\eta), \cdots, f_m(\eta)$ were assumed to be linearly independent.

(7) See B. L. van der Waerden, *Moderne Algebra*, vol. 2, Hilfsatz on p. 17.

The field of rational functions on F^*/K is the field $k(\eta, u_1, u_2, \cdots, u_{m-1})$, of degree of transcendency $m-1$ over $k(\eta)$, and the field $k(\eta)$ is of degree of transcendency r over k. Hence the field $k(\eta, u_1, u_2, \cdots, u_{m-1})$ is of degree of transcendency $r+m-1$ over k. The subfield K is of degree of transcendency m over k, and hence $k(\eta, u_1, u_2, \cdots, u_{m-1})$ is of degree of transcendency $r-1$ over K. This shows that F^*/K is of dimension $r-1$.

The above construction of F^*/K shows that if F'^*/K is another irreducible $(r-1)$-dimensional variety, with general point $(\eta_1', \eta_2', \cdots, \eta_n')$ satisfying conditions (a) and (b), then the rings $K[\eta]$ and $K[\eta']$ are necessarily simply isomorphic. Hence F^*/K is uniquely determined.

Finally, every algebraic relation between $x_1, x_2, \cdots, x_n$, with coefficients in K, is a consequence of algebraic relations between $x_1, x_2, \cdots, x_n$ with coefficients in k (since $u_1, u_2, \cdots, u_m$ are algebraically independent over $k(x)$). Hence every such relation remains a true relation after the specialization $x_i \rightarrow \eta_i$, since $k(\eta) \simeq k(x)$. Hence the ring $K[\eta]$ is a homomorphic image of the ring $K[x]$, that is, F^*/K lies on V/K. Thus our three assertions are established.

We shall call the $(r-1)$-dimensional variety F^*/K—*the general member of the linear system* $|F(\lambda)|$.

We note that *the contraction of* F^*/K *is the entire variety* V/k: this follows from the isomorphism $k[\eta] \simeq k[x]$. Also, as a consequence of part (b) of Lemma 2, F^*/K is simple for V/K.

3. **The base locus of the linear system** $|F(\lambda)|$. An irreducible subvariety W/k of V/k is a base variety of the linear system $|F(\lambda)|$ if it lies on every $F(\lambda)$. It is known([8]) that W/k is a base variety if it lies on each F_i, $i=0, 1, 2, \cdots, m$.

THEOREM 1. *If* W/k *is a base variety of the linear system* $|F(\lambda)|$, *then* W/K *lies on the general member* F^*/K *of the system; and conversely.*

Proof. Without loss of generality we may assume that W/k is at finite distance with respect to the nonhomogeneous coördinates $x_1, x_2, \cdots, x_n$ of the general point of V/k. We denote by $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ the nonhomogeneous coördinates of the general point of W/k. Let

$$f_i(x) = \mathfrak{M}\mathfrak{A}_i, \qquad i = 0, 1, \cdots, m, \tag{8}$$

be the divisor decomposition of $f_i(x)$ on the derived normal model V' of V/k. Here $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m$ are the integral divisors considered in the preceding section, while $\mathfrak{M}$ is a fractional divisor whose denominator consists entirely of divisors at infinity. If $\mathfrak{o}$ denotes the ring of nonhomogeneous coördinates of the normal variety V', then the divisors $\mathfrak{A}_i$ may be identified with certain pure $(r-1)$-dimensional ideals in $\mathfrak{o}$, ideals which we shall continue to denote by $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m$.

([8]) See ([3]), p. 528.

Let W/k be a base variety of the linear system $|F(\lambda)|$. That means that W/k lies on the subvariety of V' defined by the ideal $(\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m)$. To prove that W/K lies on F^*/K we have to show the following: *if* $H(u, \eta)=0$, *where* H *is a polynomial with coefficients in* k, *then* $H(u, \bar{x})=0$. Actually we shall prove the following stronger result: *if* $H(u, \eta)=0$, *then all the coefficients of* $H(u, x)$, *regarded as a polynomial in* $u_1, u_2, \cdots, u_m$, *belong to the ideal* $(\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m)$.

Let $X_1, X_2, \cdots, X_n$ denote indeterminates and let

$$f(u, X) = f_0(X) + u_1 f_1(X) + \cdots + u_m f_m(X).$$

The elimination of u_m between $H(u, X)$ and $f(u, X)$ leads to an identity of the form:

$$(9) \qquad [f_m(X)]^q \cdot H(u, X) = Q(u, X)\cdot f(u, X) + R(u_1, u_2, \cdots, u_{m-1}, X),$$

where all polynomials have coefficients in k. By hypothesis, we have $H(u, \eta)=0$, hence $R(u_1, u_2, \cdots, u_{m-1}, \eta)=0$, since $f(u, \eta)=0$. But since $u_1, u_2, \cdots, u_{m-1}$ are algebraically independent over $k(\eta)$ (by definition of F^*/K) and since $k(\eta)\simeq k(x)$, we conclude that $R(u, x)=0$. The specialization $X\rightarrow x$ in the above identity yields therefore the following relation:

$$(10) \qquad H(u, x) = f(u, x)\cdot Q(u, x)/[f_m(x)]^q.$$

On the left we have a polynomial in the indeterminates $u_1, u_2, \cdots, u_m$, with coefficients in $k[x]$, hence in $\mathfrak{o}$. On the right we have a product of two polynomials in the same indeterminates, with coefficients respectively in $\mathfrak{o}$ and in the quotient field of $\mathfrak{o}$. Since $\mathfrak{o}$ is integrally closed, the generalized Kronecker lemma(9) is applicable. Since the coefficients of $f(u, x)$ are $f_0(x), f_1(x), \cdots, f_m(x)$, it follows from this lemma of Kronecker and from (8) that the coefficients of $H(u, x)$ must indeed all belong to the ideal $(\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m)$.

To prove the second part of our theorem, we show that if W/k is not a base variety of the linear system $|F(\lambda)|$ then W/K does not lie on F^*/K. Without loss of generality we may assume then that W/k does not lie on the variety defined by the ideal $\mathfrak{A}_0$. Under this assumption, the quotients $f_i(x)/f_0(x)$, $i=1, 2, \cdots, m$, belong to the quotient ring $Q_V(W/k)$. Hence we may write: $f_i(x)/f_0(x)=\phi_i(x)/\phi_0(x)$, where $\phi_0(\bar{x})\neq 0$. By (7) we have the following true relation between $\eta_1, \eta_2, \cdots, \eta_n$:

$$\phi_0(\eta) + u_1\phi_1(\eta) + \cdots + u_m\phi_m(\eta) = 0.$$

This relation, however, is destroyed by the specialization $\eta\rightarrow\bar{x}$, since $\phi_0(\bar{x})\neq 0$ and since $u_1, u_2, \cdots, u_m$ are indeterminates over $k(\bar{x})$. Hence W/K does not lie on F^*/K. This completes the proof of the theorem.

(9) W. Krull, *Idealtheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 3, p. 125.

We consider an irreducible *simple* subvariety W/k of V/k. If Φ is a pure (not necessarily irreducible) $(r-1)$-dimensional subvariety of V/k containing W/k, Φ is given in the quotient ring $Q_V(W/k)$ by a principal ideal (ω). We say that W/k is simple for Φ if the leading form of ω in $Q_V(W/k)$ is of degree 1. It is well known(10) that if Φ is irreducible, this definition is equivalent to our usual definition of simple subvarieties of Φ.

Suppose now that the simple variety W/k is a base variety of the linear system $|F(\lambda)|$. We then say that W/k is *a singular base variety of* $|F(\lambda)|$ if it is singular for each $F(\lambda)$.

LEMMA 4. *In order that W/k be a singular base variety of the linear system $|F(\lambda)|$ it is sufficient that it be singular for $F_0, F_1, \cdots, F_m$.*

Proof. Let Δ denote the partial power product of those prime factors of $\mathfrak{M}$ which represent $(r-1)$-dimensional varieties containing W/k, and let us write $\mathfrak{M}=\Delta\mathfrak{N}$. Since Δ is defined in $Q_V(W/k)$ by a principal ideal, we can find a polynomial $g(x)$ with coefficients in k such that $g(x)=\Delta\cdot\mathfrak{N}'$, where each prime divisor which occurs in $\mathfrak{N}'$ represents an $(r-1)$-dimensional subvariety of V/k which does *not* pass through W/k. We have $f_i(x)/g(x)=\mathfrak{N}\mathfrak{A}_i/\mathfrak{N}'$, and hence $f_i(x)/g(x)$ *belongs to the quotient ring* $Q_V(W/k)$. Moreover, $F(\lambda)$ will be defined in this quotient ring by the principal ideal

$$\left(\frac{f_0(x)+\lambda_1 f_1(x)+\cdots+\lambda_m f_m(x)}{g(x)}\right).$$

Now if W/k is singular for F_i, $i=0, 1, \cdots, m$, then the leading form of $f_i(x)/g(x)$ is of degree greater than 1. Therefore also the leading form of $[f_0(x)+\lambda_1 f_1(x)+\cdots+\lambda_m f_m(x)]/g(x)$ will be greater than 1 for all λ, q.e.d.

4. **The theorem of Bertini.** From the preceding considerations we can now derive a well known theorem of Bertini. For reasons explained in the next section, our formulation of this theorem is different from the classical formulation. We first prove the following theorem:

THEOREM 2. *A base variety W/k of $|F(\lambda)|$ which is simple for V/k is a singular base variety if and only if W/K is singular for the general member F^*/K of the linear system $|F(\lambda)|$.*

Proof. The proof of Lemma 4 shows that we can write $f_i(x)/g(x)=\bar{f}_i(x)/\bar{g}(x)$, where $\bar{g}(x)\neq 0$ on W, $\bar{f}_i(x)=\overline{\mathfrak{M}}\mathfrak{A}_i$, and where no prime factor of $\overline{\mathfrak{M}}$ represents a variety passing through W/k. Since the polynomials $\bar{f}_i(x)$ are proportional to the polynomials $f_i(x)$, the linear system $|F(\lambda)|$ and the general member F^*/K are equally well defined by the $\bar{f}_i(x)$ as by the $f_i(x)$. Hence we may assume that no prime factor of $\mathfrak{M}$ represents a variety passing

(10) If the leading form of ω is linear, then ω is a member of a set of uniformizing parameters $\omega_1(=\omega)$, $\omega_2, \cdots, \omega_\rho$ of $W(V/k)$ (see footnote 2). If $\bar{\omega}_2, \bar{\omega}_3, \cdots, \bar{\omega}_\rho$ are the Φ-residues of $\omega_2, \omega_3, \cdots, \omega_\rho$, then these $\rho-1$ elements are uniformizing parameters of $W(\Phi)$.

through W/k. Under this assumption the elements $f_0(x), f_1(x), \cdots, f_m(x)$ are relatively prime *locally* at W/k. By Theorem 1, and by our assumption that W/k is a base variety of $|F(\lambda)|$, it follows at any rate that W/K lies on F^*/K. We come back to equation (10); it is an identity in the polynomial ring $k(x)\ [u_1, u_2, \cdots, u_m]$. The coefficients of the polynomial $f(u, x) \cdot Q(u, x)$ are divisible by $[f_m(x)]^q$ in $k[x]$, hence a fortiori in $Q_V(W/k)$. But the coefficients of $f(u, x)$ are $f_0(x), f_1(x), \cdots, f_m(x)$, and these are relatively prime in $Q_V(W/k)$. Hence the coefficients of $Q(u, x)$ must be divisible by $[f_0(x)]^q$ in $Q_V(W/k)$. *Therefore $H(u, x)$ is divisible by $f(u, x)$ in $Q_V(W/k)$.* But $H(u, x)$ is an arbitrary polynomial which vanishes on F^*/K, that is, such that $F(u, \eta) = 0$. *We conclude therefore that F^*/K is defined in $Q_V(W/K)$ by the principal ideal $(f(u, x))$.*

If $t_1, t_2, \cdots, t_\rho$ are uniformizing parameters of $W(V/k)$, they are also uniformizing parameters of $W(V/K)$. By the results just obtained, W/K is simple for F^*/K if and only if the leading form of $f(u, x)$ is of degree 1. Since this degree is equal to the minimum of the degrees of the leading forms of $f_0(x), f_1(x), \cdots, f_m(x)$, our theorem follows from Lemma 4.

THEOREM OF BERTINI. *Let W^*/K be an irreducible subvariety of F^*/K and let W/k be its contraction. If W^*/K is singular for F^*/K, then W/k is either singular for V/k or is a base variety of the linear system $|F(\lambda)|$.*

Proof. We shall show that if W/k is simple for V/k and is not a base variety of $|F(\lambda)|$, then W^*/K is simple for F^*/K. We denote by $x_1^*, x_2^*, \cdots, x_n^*$ the nonhomogeneous coördinates of the general point of W^*/K. Since W/k is not a base variety of $|F(\lambda)|$, we may assume that the elements $f_i(x)$, $i=0, 1, \cdots, m$, are not all zero on W (see proof of Theorem 2), that is, that $f_i(\bar{x}) \neq 0$ for some i, where $(\bar{x})$ is the general point of W/k. Since $k[\bar{x}] \simeq k[x^*]$, also the $f_i(x^*)$ are not all zero. The polynomial $f(U, x^*)$ is zero for $U=u$, hence it belongs to the polynomial ideal $\mathfrak{P}$ considered in §1. It is linear in the U's and is not identically zero. Hence $f(U, x^*)$ can be taken as one of the σ polynomials $G_i(U, x^*)$, and $f(u, x)$ can be taken as one of the elements τ_i defined by (1). In other words, *$f(u, x)$ can be taken as one of a set of uniformizing parameters of $W^*(V/K)$.* Since F^*/K is defined in the quotient ring $Q_V(W^*/K)$ by the principal ideal $(f(u, x))$, it follows that W^*/K is simple for F^*/K, q.e.d.

5. **The geometric content of the theorem of Bertini.** In the classical formulation of Bertini's theorem there intervenes the notion of variable singular points of the varieties belonging to a linear system. The theorem is then stated in the following terms: *a variety V_{r-1} which varies in a linear system on a V_r cannot have variable singular points outside the singular locus of V_r and outside the base locus of the linear system.* We proceed to show that if the characteristic of the ground field k is equal to zero, then this classical formulation of Bertini's theorem is an easy consequence of our formulation of this

theorem as given in the preceding section; while if the characteristic of k is different from zero, then the classical formulation of Bertini's theorem cannot be maintained, for in that form the theorem is false.

With reference to the linear system $|F(\lambda)|$ introduced in §2, we must understand by a variable point P of $F(\lambda)$ the composite concept consisting of: (1) a point whose coördinates $x_1^*, x_2^*, \cdots, x_n^*$ are algebraic functions of $u_1, u_2, \cdots, u_m$ (over k) satisfying the relation:

$$f(u, x^*) = f_0(x^*) + u_1 f_1(x^*) + \cdots + u_m f_m(x^*) = 0 \tag{11}$$

and which are such that $k[x] \sim k[x^*]$.

(2) An *arbitrary* specialization $u_i \rightarrow \lambda_i$, $x_i^* \rightarrow x_i(\lambda) = x_i^\lambda$, where $\lambda_i \in k$ and where the $x_i(\lambda)$ are algebraic quantities over k. Since $k[x] \sim k[x^*]$, the point (x^λ) lies on V, and it is clear by (11) and (9) that (x^λ) belongs to $F(\lambda)$. If we wish to insist that P be actually a variable point in the set-theoretic sense, we must add the condition that the field $k(x_1^*, x_2^*, \cdots, x_n^*)$ be of degree of transcendency not less than 1 over k.

The coördinates $x_1^*, x_2^*, \cdots, x_n^*$ define a general point of an irreducible zero-dimensional variety W^*/K which lies on F^*/K; and conversely, any irreducible zero-dimensional subvariety W^*/K of F^*/K defines a variable point of the variable member $F(\lambda)$ of the linear system $|F(\lambda)|$. The contraction W/k of W^*/K is the geometric locus of the variable point (x^λ).

We shall now assume that k is of characteristic zero. We suppose that the variety W/k is simple for V/k and that it is not a base variety of $|F(\lambda)|$. Then by the theorem of Bertini, as formulated in §4, W^*/K is simple for F^*/K. Let $H_1(u, X), H_2(u, X), \cdots, H_N(u, X)$ be a base of the prime ideal in the polynomial ring $K[X_1, X_2, \cdots, X_m]$ which defines the variety F^*/K. Then it is well known(11) that the Jacobian matrix

$$\|\partial H_i/\partial X_j\|, \qquad i = 1, 2, \cdots, N; j = 1, 2, \cdots, n, \tag{12}$$

must be of rank $n-r+1$ on W^*. We now use the identity (9). Since $R(u_1, u_2, \cdots, u_{m-1}, x) = 0$, it follows from (9) that if $h_1(X), h_2(X), \cdots, h_g(X)$ is a base of the prime ideal of V/k in $k[X]$, then $[f_m(X)]^q H(u, X)$ belongs to the ideal generated by $f(u, X), h_1(X), \cdots, h_g(X)$ in $K[X]$. Here the integer q depends on $H(u, X)$, and $H(u, X)$ is any polynomial which vanishes on F^*/K. For a suitable integer q it will then be true that the products

$$[f_m(X)]^q H_i(u, X), \qquad i = 1, 2, \cdots, N,$$

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(11) W. Schmeidler, *Über die Singularitäten algebraischer Gebilde*, Math. Ann. vol. 81 (1920) p. 227. That the definition of simple points by means of nonvanishing Jacobians is equivalent—in the case of characteristic zero—to our intrinsic definition by means of uniformizing parameters will be proved by us in a forthcoming paper in these Transactions. In the same paper we shall extend this equivalence to ground fields of characteristic $p \neq 0$ by using certain mixed Jacobians, which, in addition to partial derivatives with respect to the coördinates of the point, involve also partial derivatives with respect to certain parameters which appear in the coefficients of the equations of the variety.

belong to the ideal generated by $f(u, X)$, $h_1(X)$, $h_2(X)$, $\cdots$, $h_g(X)$ in the polynomial ring $K[X]$.

Since W/k is not a base variety and is simple for V/k, we may suppose without loss of generality that $f_m(x) \neq 0$ on W/k (see proof of Theorem 2). We therefore conclude, since the matrix (12) is of rank $n-r+1$ on W^*/K, that the matrix

$$J(u, X) = \|\partial f(u, X)/\partial X_j, \partial h_i(X)/\partial X_j\|, \qquad i = 1, 2, \cdots, g; j = 1, 2, \cdots, n, \tag{13}$$

is of rank not less than $n-r+1$ on W^*/K. But the matrix consisting of the last g columns of the matrix (13) is of rank not greater than $n-r$ on W/k, for V/k is of dimension r. *Hence the matrix* (13) *is exactly of rank* $n-r+1$ *on* W^*/K, that is, the matrix $J(u, x^*)$ is of rank $n-r+1$. From this it follows that in the m-dimensional linear space of $u_1, u_2, \cdots, u_m$ (over k) the points $(\lambda_1, \lambda_2, \cdots, \lambda_m)$, such that $J(\lambda, x(\lambda))$ is of rank less than $n-r+1$, lie on an algebraic variety L of dimension less than m. If then $F(\lambda)$ is any member of the linear system $|F(\lambda)|$ such that the point (λ) does not lie on the above algebraic variety L, the point x^λ is a simple point of $F(\lambda)$. We have therefore shown that if the locus W/k of a variable point x^λ of $|F(\lambda)|$ is a simple subvariety of V/k and is not a base variety of the linear system $|F(\lambda)|$, then x^λ cannot be a variable singular point of $F(\lambda)$. This is precisely the theorem of Bertini in its classical formulation.

If k is of characteristic p it is not difficult to construct any number of counterexamples. For instance, if k is an algebraically perfect field, then every member of the pencil of curves $x^p - \lambda y^p = 0$ in the (x, y)-plane consists entirely of p-fold points. In the following counterexample the curves of the pencil are absolutely irreducible: $x^p + y^2 - 2\lambda y = 0$. These curves have a variable double point $x = (\lambda^2)^{1/p}$, $y = \lambda$, for we have $x^p + y^2 - 2\lambda y = (x - (\lambda^2)^{1/p})^p + (y - \lambda)^2$.

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