MARKOFF CHAINS—DENUMERABLE CASE

BY

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Introduction. Let $\{p_{ij}(t)\}$, $i, j=1, 2, \cdots$, $0<t<\infty$, be functions satisfying the conditions

$$(0.1)\qquad p_{ij}(t) \geqq 0, \qquad i, j = 1, 2, \cdots,$$

$$(0.2)\qquad \sum_j p_{ij}(t) = 1, \qquad i = 1, 2, \cdots,$$

$$(0.3)\qquad \sum_j p_{ij}(s)p_{jk}(t) = p_{ik}(s+t), \qquad i, k = 1, 2, \cdots; 0 < s; t < \infty.$$

Then $\{p_{ij}(t)\}$ can be considered the transition probability functions of a Markoff chain: some system can assume various numbered states and $p_{ij}(t)$ is the probability that the system is in the jth state at the end of a time interval of length t if it was in the ith state at the beginning of the interval, whatever the initial point of the interval and the states assumed before the initial point. Conversely a Markoff chain corresponding to a system which can assume denumerably many states, with the property that the probability of a transition from state i at time t_1 to state j at time t_2 is a function of i, j, t_2-t_1, determines functions satisfying (0.1), (0.2), (0.3).

Let $x(t)$ be the number assigned to the state assumed by the system at time t. Then $x(t)$ is a chance variable for each value of t, and the study of the probability properties of the process is simply the study of the one parameter family of chance variables $\{x(t)\}$. If for each value of t a sample value of $x(t)$ is obtained, the totality of these values defines a sample function usually also denoted suggestively and confusingly by $x(t)$. The sample functions are integral-valued functions and the probability theory of the process when studied rigorously is the study of the probability measure defined for certain sets in the space of all sample functions.

For reasons detailed below, it is always assumed that

$$(0.4)\qquad \lim_{t\to 0} p_{ij}(t) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Assuming further regularity conditions discussed below, the $p_{ij}(t)$ must satisfy the differential equations

$$(0.5)\qquad p'_{ik}(t) = -q_i p_{ik}(t) + \sum_{j\neq i} q_{ij}p_{jk}(t),$$

$$(0.6)\qquad p'_{ik}(t) = -p_{ik}(t)q_k + \sum_{j\neq i} p_{ij}(t)q_{jk}, \qquad t \geqq 0;\ i, k = 1, 2, \cdots,$$

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where

$$(0.7) \qquad q_i = \lim_{t\to 0} \frac{1 - p_{ii}(t)}{t} = - p'_{ii}(0), \qquad q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)}{t} = p'_{ij}(0).$$

The $p_{ij}(t)$ can be studied from the purely probability point of view, in which case the guiding influence is the succession of transitions of the system, that is, the discontinuities of the sample functions, or from the point of view of classical analysis, in which case the study is simply that of the functional equations (0.3) and the differential equations (0.5) and (0.6)(1). In the present paper it will be shown how the relation between the differential equations (0.5) and (0.6) and the functional equations (0.3) can be analyzed more completely than hitherto by combining the above two points of view. It will be seen that although (0.5) is true under the usual very weak hypotheses, (0.6) is not true in an important class of processes for which (0.5) is true. The q_i, q_{ij} corresponding to a process belonging to this class have the property that there are infinitely many solutions to (0.5), satisfying (0.1), (0.2), (0.3) and the initial conditions (0.4).

There are two problems to be considered in analyzing the relations between the probability process defined by the $p_{ij}(t)$ and the differential equations (0.5) and (0.6).

1. Functions $p_{ij}(t)$ are supposed given, satisfying (0.1), (0.2), (0.3) and suitable regularity conditions. The conditions necessary and sufficient for the validity of the differential equations (0.5) and (0.6) are to be derived.

2. The q_i, q_{ij} are supposed given. Functions $p_{ij}(t)$ are to be found satisfying (0.1), (0.2), (0.3), (0.4) and (0.7).

Partial answers to both problems are given below. A review of the subject from the point of view of this paper is presented in the next section. More complete references to previous work can be found in [1] and [2](2). The exact (measure) meaning of the somewhat careless sounding statements on continuity, jumps, and so on, of almost all sample functions is discussed in [1].

1. **Review and analysis of known results.** The following considerations justify the selection of (0.4) as the minimum regularity conditions to be imposed on the $p_{ij}(t)$. Let α be any positive integer. The hypothesis that the system is initially in the state α and that the $p_{ij}(t)$ (satisfying (0.1), (0.2), (0.3)) are the transition functions fully determines the probability relations of the process. The continuity of the $p_{ij}(t)$ and the existence of $\lim_{t\to 0} p_{ij}(t)$ for all i, j are implied by the measurability of the $p_{ij}(t)$ [1, p. 38]. If the process itself is to have any reasonable measurability properties, (0.4) muse also be true, at least for the states i, j which may occur, that is, those for which $p_{\alpha i}(t), p_{\alpha j}(t)$ do not vanish identically [1, pp. 42, 43]. There is no essential further loss in generality involved in the assumption that (0.4) is true with no restriction

(1) There is of course no precise line of demarcation between these points of view.

(2) Numbers in brackets refer to the Bibliography at the end of the paper.

on i, j. Conversely (0.4) implies the continuity of the $p_{ij}(t)$ and the measurability of the process [1, pp. 43, 51].

The role of the q_i, q_{ij} will now be analyzed. Suppose that (0.4) is true. Then the limit defining q_i in (0.7) always exists, but q_i may be infinite [1, p. 52]. Moreover the following limits also exist:

$$\text{(1.1)}\qquad \begin{aligned} &\text{if } q_i = \infty: \quad \lim_{t\to 0} \frac{p_{ij}(t)}{1 - p_{ii}(t)} = \lim_{t\to 0} \frac{p_{ji}(t)}{1 - p_{ii}(t)} = 0, \\ &\text{if } q_i < \infty: \quad \lim_{t\to 0} \frac{p_{ij}(t)}{t} = q_{ij} < \infty, \qquad \lim_{t\to 0} \frac{p_{ji}(t)}{t} = q_{ji} < \infty\,(^3). \end{aligned}$$

It follows trivially from (0.2) that if $q_i < \infty$,

$$\text{(1.2)}\qquad \sum_j q_{ij} \leqq q_i.$$

The limits defining q_i, q_{ij} thus can always be assumed to exist without losing any interesting probability processes, but it will be seen below that further restrictions, finiteness of the q_i, equality in (1.2), will ordinarily be imposed on these numbers. The reason for this can be seen best from the role of the q_i, q_{ij} in the transitions of the probability system under discussion but it can also be seen at once that these restrictions are essential to the validity of the differential equations (0.5). In fact if (0.5) is to be significant, the q_i must be finite, and if (0.5) is to be true, summing over k and integrating (using (0.2)) shows that there must be equality in (1.2).

The significance of the q_i, q_{ij} can be described very simply in terms of the sample functions $\{x(t)\}$. Let t_0 be some fixed value of t.

(i) If $q_i = \infty$, $\limsup_{t\to t_0} x(t) = \infty$ whenever $x(t_0) = i$, neglecting zero probabilities. More than this, i and ∞ are the only limiting values, neglecting zero probabilities [1, p. 59]. Thus $x(t)$ can be continuous at t_0 only when $x(t_0) \neq i$, neglecting zero probabilities.

(ii) If $q_i < \infty$, $x(t) = i$ in some neighborhood of t_0 whenever $x(t_0) = i$, neglecting zero probabilities. The probability that if $x(t_0) = i$ then $x(t) \equiv i$ for $t_0 \leqq t \leqq t_0 + h$ is $\exp(-q_i h)$ [1, pp. 52–54]. Thus if $x(t_0) = i$ the probability density of distribution of the time s to the first discontinuity beyond t_0 is $q_i \exp(-q_i s)$. If $x(t_0) = i$ and if there is a discontinuity in the interval $t_0 \leqq t \leqq t_0 + h$, then the probability that the first discontinuity is a jump(4) to the value j is q_{ij}/q_i [1, p. 58].

In spite of the impossibility (except in the trivial case $p_{\alpha\alpha}(t) \equiv 1$ when

(3) [1, p. 52].

(4) An integral-valued function has a jump at a point t_0 if it is constant on some open interval with left-hand end point t_0 and is constant (with a different value) on some open interval with right-hand end point t_0. The jumps of an integral-valued function are isolated discontinuities.

$x(t)\equiv$const.) that the sample functions $x(t)$ be everywhere continuous functions of t with positive probability, there is always, if (0.4) is true, a kind of average continuity, sometimes called stochastic continuity: if $0<\epsilon<1$, and if $x(0)=\alpha$,

$$
\begin{aligned}
\lim_{h\downarrow 0} P\{\,|\,x(t+h)-x(t)\,|<\epsilon\} &= \lim_{h\downarrow 0}\sum_i p_{\alpha i}(t)p_{ii}(h)\\
&= \sum_i p_{\alpha i}(t) = 1,\\
\lim_{h\downarrow 0} P\{\,|\,x(t-h)-x(t)\,|<\epsilon\} &= \lim_{h\downarrow 0}\sum_i p_{\alpha i}(t-h)p_{ii}(h)\\
&= \sum_i p_{\alpha i}(t) = 1.
\end{aligned}
\tag{1.3}
$$

If the q_i are all finite the type of continuity is stronger. In that case, as discussed in the preceding paragraph, at each t_0, $x(t)$ is continuous (that is, remains constant) in some neighborhood of t_0, with probability 1. The strongest conceivable statement on the continuity of the individual sample functions (excluding the trivial case of constancy) is that all their discontinuities are jumps, that is, that the discontinuities have no finite limit points. It is clear from the preceding paragraph that if this is true the q_i with i corresponding to states of positive probability must be finite, and for those values of i there must be equality in (1.2). It will be shown below that the converse is not true: finiteness of the q_i and equality in (1.2) for all i are not sufficient to insure that the sample functions $\{x(t)\}$ vary only by jumps.

The following definitions will be useful below. If $q_i<\infty$ define $\Pi_{ij}^{(n)}$ by

$$
\Pi_{ij}^{(1)} = q_{ij}/q_i,\quad j\neq i,\qquad \Pi_{ij}^{(n+1)} = \sum_k \Pi_{ik}^{(n)}\Pi_{kj}^{(1)},\quad n\geqq 1(^5),\qquad \Pi_{ii}^{(1)}=0.
\tag{1.4}
$$

If the q_i are finite and if there is equality in (1.2), then there is, neglecting zero probabilities, a first discontinuity, if any, to the right of any given t-value t_0, a second, a third, and so on, all jumps [1, p. 56]. Moreover it is easily verified that $\Pi_{ij}^{(n)}$ is the probability that if $x(t_0)=i$ and if $x(t)$ has at least n discontinuities in the interval $t_0\leqq t\leqq t_0+h$, then the nth jump is a jump to the value j. If $q_i<\infty$, and $q_k<\infty$, define ${}_np_{ik}(t)$ by

$$
\begin{aligned}
{}_0p_{ik}(t) &= \delta_{ik}\exp(-q_it),\\
{}_{n+1}p_{ik}(t) &= \begin{cases}\displaystyle\sum_{j\neq i}\int_0^t \exp(-q_i(t-s))q_{ij}\,{}_np_{jk}(s)ds,\\[2ex] \displaystyle\sum_{j\neq k}\int_0^t {}_np_{ij}(s)q_{jk}\exp(-q_k(t-s))ds,\end{cases}\qquad n\geqq 0.
\end{aligned}
\tag{1.5}
$$

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(5) Whenever a q_i in the denominator vanishes, the fraction is to be interpreted as 0.

It can be shown(6) that the series in the integrand converges, and that ${}_np_{ij}(t)$ is the probability that if $x(t_0)=i$, then $x(t_0+t)=k$ and the transition to k is by way of exactly n jumps. The probability $\bar{p}_{ij}(t)$ of going from i to k through a finite number of discontinuities (necessarily all jumps) is given by

$$\bar{p}_{ik}(t) = \sum_{n=0}^{\infty} {}_np_{ik}(t), \tag{1.6}$$

and the probability of going from i to k through infinitely many discontinuities is then $p_{ik}(t)-\bar{p}_{ik}(t)$.

The fundamental analytic properties of the $\bar{p}_{ij}(t)$ are almost identical with those of the $p_{ij}(t)$. These will be listed point by point below. The comparison is of fundamental importance, since one standard method (cf. [2]) of solving (0.1), (0.2), (0.3) for specified q_i, q_{ij} is to find $\bar{p}_{ij}(t)$, and impose conditions on the q_i, q_{ij} strong enough to make $\bar{p}_{ij}(t)=p_{ij}(t)$. It will be seen that the differential equations (0.5) and (0.6) are satisfied by $\bar{p}_{ij}(t)$ but not necessarily by $p_{ij}(t)$.

(i) It is clear that

$$0 \leqq \bar{p}_{ij}(t) \leqq p_{ij}(t) \tag{1.7}$$

so that (0.1) is satisfied by $\bar{p}_{ij}(t)$.

(ii) The equality (0.2) must in general be replaced by the inequality

$$\sum_j \bar{p}_{ij}(t) \leqq 1. \tag{1.8}$$

There is equality in (1.8) for a given i only when $\bar{p}_{ij}(t)=p_{ij}(t)$ for all j, that is when the discontinuities of $x(t)$ following the assumption of the value i are all jumps.

(iii) The probability significance of $\bar{p}_{ij}(t)$ shows that $\bar{p}_{ij}(t)$ satisfies (0.3).

(iv) One simple solution of (1.7), (1.8) and (0.3) (for $\bar{p}_{ij}(t)$) is $\bar{p}_{ij}(t)\equiv 0$. This solution can actually be the correct one, since if $q_i=\infty$, $\bar{p}_{ij}(t)$ certainly vanished identically for all j. Thus (0.4) is not necessarily satisfied by the $\bar{p}_{ij}(t)$, even if it is satisfied by the $p_{ij}(t)$. On the other hand suppose that (0.4) is satisfied by $p_{ij}(t)$ and that for some subscript i, $q_i<\infty$. Then using (1.7) and the fact that $\exp(-q_ih)$ is the probability that if $x(t_0)=i$, $x(t)\equiv i$ for $t_0\leqq t\leqq t_0+h$, it follows that

$$\exp(-q_ih) \leqq \bar{p}_{ii}(h) \leqq p_{ii}(h). \tag{1.9}$$

Hence

$$\frac{1-\exp(-q_ih)}{h} \geqq \frac{1-\bar{p}_{ii}(h)}{h} \geqq \frac{1-p_{ii}(h)}{h} \tag{1.10}$$

(6) Cf. the development of similar results in [1, pp. 55, 63].

so that

$$\lim_{h\to 0}\frac{1-\bar{p}_{ii}(h)}{h}=q_i. \tag{1.11}$$

Thus if $q_i<\infty$, $\bar{p}_{ij}(t)$ satisfies (0.4) and even the first limit equation in (0.7) for that subscript i.

(v) It will now be seen that both equations in (0.7) are satisfied by the $\bar{p}_{ij}(t)$ if q_i and q_j are finite. The first has been checked in (iv). According to (1.5),

$$ {}_1p_{ij}(t)=\int_0^t \exp\left(-q_is\right)q_{ij}\exp\left(-q_j(t-s)\right)ds. \tag{1.12}$$

Hence if $j\neq i$,

$$\frac{1}{t}\int_0^t \exp\left(-q_is\right)q_{ij}\exp\left(-q_j(t-s)\right)ds\leqq\frac{\bar{p}_{ij}(t)}{t}\leqq\frac{p_{ij}(t)}{t}, \tag{1.13}$$

which implies the second limit equation in (0.7). Thus *if* $p_{ij}(t)$ *satisfies* (0.1), (0.2), (0.3) *and* (0.4), *and if all the* q_i *are finite,* $\bar{p}_{ij}(t)$ *will satisfy the same equations, except that* (0.2) *may be weakened to* (1.8). It now is evident that it will be exceedingly difficult to distinguish between the two solutions of the listed equations, when they are not identical. This difficulty is exhibited when the differential equations (0.5) and (0.6) are analyzed in (vi), (vii) and (viii).

(vi) Suppose now that the $p_{ij}(t)$ satisfy (0.1), (0.2), (0.3) and (0.4). Then the probability $p_{ik}(t)$ of going from i to k in time t in a finite number of jumps can be evaluated as follows. If $q_i=\infty$ or if $q_k=\infty$, then $\bar{p}_{ik}(t)=0$. If $q_i<\infty$, and if $q_k<\infty$, then q_{ij} and q_{jk} are finite (see (1.1)) and

$$\bar{p}_{ik}(t)=\delta_{ik}\exp\left(-q_it\right)+\int_0^t\sum_{j\neq i}\exp\left(-q_i(t-s)\right)q_{ij}\bar{p}_{jk}(s)ds, \tag{1.14}$$

$$\bar{p}_{ik}(t)=\delta_{ik}\exp\left(-q_kt\right)+\int_0^t\sum_{j\neq k}\bar{p}_{ij}(s)q_{jk}\exp\left(-q_k(t-s)\right)ds. \tag{1.15}$$

These two equations are integrated versions of (0.5) and (0.6) respectively. They can be derived from (1.5) and (1.6) or directly from the probability significance of the quantities concerned. Thus *if the* $p_{ij}(t)$ *satisfy* (0,1), (0.2), (0.3) *and* (0.4), *then the* $\bar{p}_{ij}(t)$ *satisfy* (0.1), *the weakened form* (1.8) *of* (0.2), *and* (0.3), *and the derivatives* $\bar{p}_{ij}'(t)$ *exist and are continuous. If, in addition, the* q_i (*defined in terms of the* $p_{ij}(t)$) *are finite, the* $\bar{p}_{ij}(t)$ *also satisfy* (0.4), (0.5), (0.6) *and* (0.7). It is thus clear that the $\bar{p}_{ij}(t)$ satisfy the differential equations (0.5) and (0.6) whenever these equations have meaning, that is, whenever the constant coefficients are finite-valued. It will be seen below that the $p_{ij}(t)$ satisfy these equations only when further restrictions are imposed.

(vii) Suppose that the $p_{ij}(t)$ satisfy (0.1), (0.2), (0.3) and (0.4) and that q_i is finite. Using (0.3) it follows that

$$
(1.16)\quad
\begin{aligned}
\frac{p_{ik}(t+h)-p_{ik}(t)}{h} &= -p_{ik}(t)\frac{1-p_{ii}(h)}{h} + \sum_{j\neq i}\frac{p_{ij}(h)}{h}p_{jk}(t),\\
\frac{p_{ik}(t)-p_{ik}(t-h)}{h} &= -p_{ik}(t-h)\frac{1-p_{ii}(h)}{h} + \sum_{j\neq i}\frac{p_{ij}(h)}{h}p_{jk}(t-h),
\end{aligned}
\qquad h > 0.
$$

These equations imply that

$$
(1.17)\qquad \liminf_{h\to 0}\frac{p_{ik}(t+h)-p_{ik}(t)}{h} \geqq -q_i p_{ik}(t) + \sum_{j\neq i} q_{ij}p_{jk}(t).
$$

Now suppose that there is equality in (1.2). Then if $N>i$,

$$
(1.18)\qquad \frac{1}{h}\sum_{j>N} p_{ij}(h) = \frac{1-\sum_{j=1}^{N}p_{ij}(h)}{h} \sim q_i - \sum_{j\leqq N,\, j\neq i} q_{ij} = \sum_{j>N} q_{ij}\quad (h\to 0),
$$

so that the series in (1.16) converge uniformly in h. Thus in this case, (1.16) implies that $p'_{ik}(t)$ exists and that (0.5) is satisfied. Conversely it has already been noted that if (0.5) is true, summing over k and integrating yields (1.2) with equality. Thus *if the* $p_{ij}(t)$ *satisfy* (0.1), (0.2), (0.3) *and* (0.4), *and if the* q_i *are finite, there is equality in* (1.2) *for all i if and only if the* $p_{ij}(t)$ *have continuous derivatives which satisfy the differential equations* (0.5). It follows from (vi) and (vii) that the solution to the differential equations (0.5) cannot be uniquely determined by the initial conditions unless the hypotheses already made imply the identity of $p_{ij}(t)$, $\bar{p}_{ij}(t)$—and this is not so (see §2).

The probability significance of equality in (1.2) has already been noted—that the first discontinuity of $x(t)$ following a specified t_0 is a jump, with probability 1. The probability significance of the differential equations (0.5) is directly linked to this property of $x(t)$. In fact if the q_i are finite, the probability $p_{ik}(t)$ of going from i to k in time t is at least equal to the probability of this transition when the first discontinuity is a jump:

$$
(1.19)\qquad p_{ik}(t) \geqq \sum_{j\neq i}\int_0^t \exp(-q_i(t-s))q_{ij}p_{jk}(s)ds + \delta_{ik}\exp(-q_i t)\text{(}^7\text{)}.
$$

Equality means that the first discontinuity is a jump, with probability 1, and equality gives an integral form of (0.5). To see the exact significance of inequality ($>$) in (0.5) or (1.17), note that the probability of a transition from i at time t_0 to k at time t_0+t is at least equal to that of a transition from i to k for which the first discontinuity, if any, before time t_0+h is a jump:

(7) Cf. the development of similar integral expressions in [1].

$$p_{ik}(t) \geqq \exp(-q_i h) p_{ik}(t-h) + \sum_{j \neq i} \int_0^h \exp(-q_i(h-\tau)) q_{ij} p_{jk}(t-h+\tau) d\tau. \tag{1.20}$$

There is equality if and only if when $x(t_0)=i$ and $x(t_0+t)=k$ the first discontinuity before time t_0+h is a jump, with probability 1, and we have already seen that the condition for this is equality in (1.2). Now (1.20) can also be put in the form

$$\frac{p_{ik}(t) - p_{ik}(t-h)}{h} \geqq -q_i p_{ik}(t) + \sum_{j \neq i} q_{ij} p_{jk}(t) + o(1) \tag{1.21}$$

so that (1.17), or (0.5) if there is equality, is really simply an evaluation of the rate of change of $p_{ik}(t)$ with t, when t increases, as related to the changes of $x(t)$ in jumps. This makes it clear why the differential equations (0.5) hold for $\bar{p}_{ik}(t)$ as well as for $p_{ik}(t)$. In fact the argument in terms of $p_{ik}(t)$ leading to (1.19) leads to the same expression with equality if the $p_{ij}(t)$ are replaced by the $\bar{p}_{ij}(t)$.

(viii) Suppose again that the $p_{ij}(t)$ satisfy (0.1), (0.2), (0.3) and (0.4) and that the q_i are finite. Using (0.3) it follows that

$$\begin{aligned} \frac{p_{ik}(t+h) - p_{ik}(t)}{h} &= -p_{ik}(t) \frac{1 - p_{kk}(h)}{h} \\ &\quad + \sum_{j \neq k} p_{ij}(t) \frac{p_{jk}(h)}{h}, \\ \frac{p_{ik}(t) - p_{ik}(t-h)}{h} &= -p_{ik}(t-h) \frac{1 - p_{kk}(h)}{h} \\ &\quad + \sum_{j \neq k} p_{ij}(t-h) \frac{p_{jk}(h)}{h}, \end{aligned} \qquad h > 0. \tag{1.22}$$

These equations imply that

$$\liminf_{h \to 0} \frac{p_{ik}(t+h) - p_{ik}(t)}{h} \geqq -p_{ik}(t) q_k + \sum_{j \neq k} p_{ij}(t) q_{jk}. \tag{1.23}$$

If there is equality in (1.2), $p'_{ik}(t)$ exists, by (vii), so that (1.23) then becomes

$$p'_{ik}(t) \geqq -p_{ik}(t) q_k + \sum_{j \neq k} p_{ij}(t) q_{jk}. \tag{1.24}$$

It will be seen in the next section that the inequality (1.23), or (1.24) if $p'_{ik}(t)$ exists, cannot in general be replaced by equality. The development in (vii) guided by the study of $x(t)$ for increasing t corresponds exactly to a somewhat more complicated development guided by the study of $x(t)$ for decreasing t.

The probability, if $x(0)=i$, if $x(t_0)=k$ and if there is a discontinuity of $x(t)$ in the interval (t_0-t, t_0), that the last discontinuity before t_0 is a jump is

$$\frac{\sum_{j\neq k}\int_0^t p_{ij}(t_0-s)q_{jk}\exp(-q_k s)ds}{p_{ik}(t_0)-p_{ik}(t_0-t)\exp(-q_k t)} \leqq 1. \tag{1.25}$$

The inequality (1.25) has the same significance for the discontinuities of $x(t)$ when t decreases as (1.2) or (1.19) has when t increases. There is equality if and only if the last discontinuity before t_0 is a jump. Equality implies the existence of the derivative $p'_{ik}(t)$ and also implies equality in (1.24). Thus *if the $p_{ij}(t)$ satisfy* (0.1), (0.2), (0.3), (0.4) *and if the q_i are finite, the existence of the $p'_{ij}(t)$ and the truth of* (0.5) *are equivalent to the fact that the first discontinuity of $x(t)$ after any specified t_0 is a jump, and the existence of the $p'_{ij}(t)$ and the truth of* (0.6) *is equivalent to the fact that the last discontinuity before any specified t_0 is a jump.*

The probability of going from i to k in time t is at least equal to that of going from i to k in time t in such a way that the last discontinuity, if any, in the interval of length h ending at t is a jump:

$$p_{ik}(t) \geqq p_{ik}(t-h)\exp(-q_k h) + \sum_{j\neq k}\int_0^h p_{ij}(t-\tau)q_{jk}\exp(-q_k\tau)d\tau. \tag{1.26}$$

There is equality if and only if when $x(0)=i$ and $x(t)=k$ the last discontinuity, if any, after time $t-h$ is a jump, with probability 1. The inequality (1.26) leads to

$$\frac{p_{ik}(t)-p_{ik}(t-h)}{h} \geqq q_k p_{ik}(t) + \sum_{j\neq k} p_{ij}(t)q_{jk} + o(1) \qquad (h\to 0), \tag{1.27}$$

so that (1.23), or (1.24) if the derivative exists, is really an evaluation of the rate of change of $p_{ik}(t)$ with t, when t decreases, as related to the changes of $x(t)$ in jumps. The argument leading to (1.25) leads to the same expression with equality when the $p_{ij}(t)$ are replaced by the $\bar{p}_{ij}(t)$.

In the next section a general method is given for constructing examples in which the $p_{ij}(t)$ satisfy (0.1)–(0.4), the q_i are all finite, and there is equality in (1.2), and in which (0.5) but not (0.6) is satisfied. Thus under the usual hypotheses made in studies of this sort—which hypotheses are exactly those just stated—(0.6) must be replaced by (1.24). No example of functions $p_{ij}(t)$ satisfying (0.1)–(0.4) is known to the author in which the derivatives $\{p'_{ij}(t)\}$ do not exist, or in which there is inequality in (1.2) and so in (0.5), or in which (0.5) and (0.6) are both satisfied by the $p_{ij}(t)$ even though the $p_{ij}(t)$ and $\bar{p}_{ij}(t)$ are not identical.

2. **Processes whose transitions are well ordered in time.** The analysis in

the previous section presented no new results on the Markoff chains under consideration, but merely organized known results to indicate the present state of knowledge concerning the usual basic hypotheses and their immediate implications. This section will be devoted to a general type of Markoff chain which includes the types studied heretofore (in which the $p_{ij}(t)$ and $\bar{p}_{ij}(t)$ are identical) as a special case.

The following theorem, which is a special case of a very general theorem on Markoff processes, will be useful below. The measure ideas involved are somewhat glossed over in the proof, in order to avoid complexities out of keeping with the rest of this paper.

THEOREM 2.1. *Suppose that the $p_{ij}(t)$ satisfy* (0.1)–(0.4) *and that the q_i are finite. Let τ be a non-negative chance variable depending on the $x(t)$ in such a way that for each $s>0$ the inequalities $\tau<s$, $\tau>s$ impose conditions on the $x(t)$ only for $t<s$. Suppose that $x(\tau+0)$ exists with probability* 1, *that is, that $x(t)$ is constant in some interval with left-hand end point τ (the value depending on the sample function) with probability* 1. *Then for each $t>0$, $x(\tau+t)$ is a chance variable, and the stochastic process corresponding to $y(t)=x(\tau+t)$ is a Markoff process with the same transition probabilities as the $x(t)$ process.*

The hypotheses imply the truth of the point set equality

$$\{x(\tau+0)=j\} = \lim_{n\to\infty} \sum_{r=0}^{\infty} \left\{\frac{r}{2^n} \leqq \tau < \frac{r+1}{2^n},\ x\left(\frac{r+1}{2^n}\right) = j\right\} \tag{2.1}$$

neglecting sets of zero probability, so that $x(\tau+0)$ is a chance variable, and

$$P\{x(\tau+0)=j\} = \lim_{n\to\infty} \sum_{r=0}^{\infty} P\left\{\frac{r}{2^n} \leqq \tau < \frac{r+1}{2^n},\ x\left(\frac{r+1}{2^n}\right) = j\right\}. \tag{2.1'}$$

Similarly, $x(\tau+t)$ is a chance variable, with

$$\begin{aligned} P\{x(\tau+t)=k\} &= \lim_{\epsilon\to 0}\lim_{n\to\infty} \sum_j \sum_{r=0}^{\infty} P\left\{\frac{r}{2^n} \leqq \tau < \frac{r+1}{2^n},\ x\left(\frac{r+1}{2^n}\right) = j,\right. \\ &\qquad\qquad \left. x\left(\frac{r+1}{2^n}+s\right) = k,\ |s-t|<\epsilon\right\} \\ &= \lim_{\epsilon\to 0} \sum_j P\{x(\tau+0)=j\} p_{jk}(t-\epsilon) \exp(-q_k 2\epsilon) \\ &= \sum_j P\{x(\tau+0)=j\} p_{jk}(t), \end{aligned} \tag{2.2}$$

and more generally, if $t_1 < \cdots < t_l$

$$\begin{aligned} &P\{x(\tau+t_i)=k_i,\ i=1,\cdots,l\} \\ &\quad = \sum_j P\{x(\tau+0)=j\} p_{jk_1}(t_1) p_{k_1k_2}(t_2-t_1) \cdots p_{k_{l-1}k_l}(t_l-t_{l-1}). \end{aligned} \tag{2.2'}$$

This equation shows that the $y(t)=x(\tau+t)$ process is a Markoff process with the same transition probabilities as the $x(t)$ process.

In this section those Markoff chains will be studied in which the discontinuities of the function $x(t)$, ordered in terms of increasing t, form a well ordered series, with probability 1. This includes the case of isolated discontinuities, in which the series has order type not greater than ω. An analysis of processes of this general type will be made which shows how the most general example can be constructed.

Suppose then that the discontinuities of $x(t)$ are well ordered. It follows that if $x(0)$ is specified, there is a first discontinuity, a second discontinuity, and so on, with probability 1. Hence, according to the preceding section, the q_i are finite, there is equality in (1.2), and the differential equations (0.5) are satisfied. If $x(0)=i_0$, $x(t)$ remains at i_0 for a time τ_0 determined by the probability density of distribution $\phi(q_{i_0}, \tau_0)$, where

$$\phi(q, s) = \begin{cases} q \exp(-qs), & s \geqq 0, \\ 0, & s < 0(^8). \end{cases}$$

At τ_0 there is a jump to i_1, where

$$P\{i_0; i_1 = j\} = q_{i_0 j}/q_{i_0}(^9).$$

If $u>0$, the condition $\tau_0<u$ is equivalent to the condition that $x(t)\neq i_0$ for some $t<u$. The condition $\tau_0>u$ is equivalent to the condition that $x(t)\equiv i_0$ for $t<u$, neglecting zero probabilities. Hence the theorem is applicable: the process with variables $\{y(t)\}=\{x(\tau_0+t)\}$ has precisely the same probability properties as the original process, except that the initial value is i_1 (determined as described above) instead of i_0. Hence

$$P\{x(0) = i_0; x(\tau_0 + t) = j\} = \sum_i \Pi_{i_0 i}^{(1)} p_{ij}(t). \tag{2.3}$$

Applying the same argument to the y process, the time τ_1 to the next discontinuity of $x(t)$ is determined by the probability density of distribution $\phi(q_{i_1}, \tau_1)$ and at $\tau_0+\tau_1$ there is a jump to i_2, where

$$P\{i_0, i_1; i_2 = j\} = \Pi_{i_1 j}^{(1)} \tag{2.4}$$

and

$$P\{x(0) = i_0; x(\tau_0 + \tau_1 + t) = j\} = \sum_i \Pi_{i_0 i}^{(2)} p_{ij}(t). \tag{2.5}$$

Continuing the argument, sequences

---

(8) The following discussion is subject to trivial modifications if a q_i vanishes, the essential formal idea being that $\phi(0, s)$ symbolizes a distribution which is concentrated at ∞.

(9) $P\{z, \cdots; C\}$ will be used to denote the conditional probability of C for specified values of the variables $z, \cdots$.

$$\tau_0, \tau_1, \cdots, \qquad i_0, i_1, i_2, \cdots$$

are determined. Define τ_ν' by

$$\tau_0' = 0, \qquad \tau_\nu' = \sum_{j<\nu} \tau_j, \qquad \nu > 0. \tag{2.6}$$

Then τ_ν' is the point where $x(t)$ jumps from $i_{\nu-1}$ to i_ν, a value maintained for time τ_ν. It is easily verified that for specified i_0 the probability $\psi_\nu(s, j)$ that $i_\nu = x(\tau_\nu'+) = j$ and that $\tau_\nu' = s$ (density in s) is determined by the equations

$$\begin{aligned} \psi_1(s, j) &= \Pi_{i_0 j}^{(1)} \phi(q_{i_0}, s), \\ \psi_{\nu+1}(s, j) &= \sum_i \int_0^s \psi_\nu(s - \tau, i) \Pi_{ij}^{(1)} \phi(q_j, \tau) d\tau. \end{aligned} \tag{2.7}$$

The probability density of distribution $\psi_\nu(\tau_\nu')$ of τ_ν' for specified i_0 is given by

$$\psi_\nu(s) = \sum_j \psi_\nu(s, j) \tag{2.8}$$

and finally the probability that $i_\nu = x(\tau_\nu'+) = j$ for specified i_0 is given by

$$P\{i_0; i_\nu = x(\tau_\nu' +) = j) = \Pi_{i_0 j}^{(\nu)}. \tag{2.9}$$

For each ν the variables $\{x(\tau_\nu' + t)\}$ $(t > 0)$ have the same probability properties as the variables $\{x(t)\}$ $(t > 0)$ except that the initial value is not necessarily i_0. For example, generalizing (2.3), we obtain

$$P\{i_0; x(\tau_\nu' + t) = j\} = \sum_i \Pi_{i_0 i}^{(\nu)} p_{ij}(t). \tag{2.10}$$

If

$$\tau_\omega' = \lim_{\nu\to\infty} \tau_\nu' = \sum_0^\infty \tau_j = \infty$$

with probability 1, the discontinuities of $x(t)$ are all jumps. If this holds for all i_0, the differential equations (0.6), as well as (0.5), are satisfied, and the $p_{ij}(t)$, $\bar{p}_{ij}(t)$ are identical. The interesting case for present purposes, however, is when $\tau_\omega' < \infty$ with positive probability. In this case the discontinuities of $x(t)$ may have a finite limit point τ_ω' which is a discontinuity more complicated than a jump. Feller[10] has given the conditions necessary and sufficient that this case arise, under the hypotheses that the q_i are finite and that there is equality in (1.2) for all i. Up to the present stage of the analysis the hypothesis that the discontinuities of $x(t)$ are well ordered has only been used to insure that the q_i are finite and that there is equality in (1.2) for all i.

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[10] [2, pp. 506, 507]. In the notation of the present paper, Feller describes his conditions as conditions that the equations (0.2) hold for the $p_{ij}(t)$, under which circumstances the $p_{ij}(t)$ and $\bar{p}_{ij}(t)$ are identical, in view of (1.7).

In fact everything shown so far follows from these two hypotheses. Two cases are now conceivable: either τ_ω' is a limit point of discontinuities on the right as well as on the left or not. The first case is excluded from the present discussion by the hypothesis that the discontinuities are well ordered, but no example is yet known which shows that this case can actually arise when the well ordering of the discontinuities is not presupposed. We are thus supposing that τ_ω' is not a limit point of discontinuities on the right, that is, that τ_ω' is the left end point of an interval of constancy of $x(t)$. The probability $\Pi_{i_0j}^{(\omega)}$ that $x(\tau_\omega'+)=j$ is then easily evaluated:

$$\tag{2.11}\begin{aligned}\Pi_{i_0j}^{(\omega)} &= P\{i_0;\, x(\tau_\omega'+) = j\}\\ &= \lim_{\eta\to 0}\lim_{\epsilon\to 0}\lim_{\nu\to\infty}\sum_i \Pi_{i_0i}^{(\nu)}[p_{ij}(\epsilon) - \bar{p}_{ij}(\epsilon)]\exp(-q_j\eta)\\ &= \lim_{\epsilon\to 0}\lim_{\nu\to\infty}\sum_i \Pi_{i_0i}^{(\nu)}[p_{ij}(\epsilon) - \bar{p}_{ij}(\epsilon)].\end{aligned}$$

More generally,

$$\tag{2.12} P\{i_0;\, x(\tau_\omega' + t) = k\} = \sum_j \Pi_{i_0j}^{(\omega)} p_{jk}(t)$$

which extends (2.10). Theorem 2.1 can again be applied, to show that the process with variables $x(\tau_\omega'+t)$, $0<t<\infty$, has precisely the same properties as the original process, except that the initial value is now i_ω, which takes on the value j with probability $\Pi_{i_0j}^{(\omega)}$, instead of i_0. Equation (2.12) exhibits the conclusion of the theorem, in part. Thus the time τ_ω to the next discontinuity has probability density $\phi(q_{i_\omega}, \tau_\omega)$ and this discontinuity is a jump to $i_{\omega+1}=j$ with probability $\Pi_{i_\omega j}^{(1)}$. Continuing in this way sequences

$$\tag{2.13}\begin{gathered}\tau_0, \tau_1, \cdots, \tau_\omega, \cdots, \qquad i_0, i_1, \cdots, i_\omega, \cdots,\\ \tau_1', \tau_2', \cdots, \tau_\omega', \cdots\end{gathered}$$

are determined, where τ_ν', defined by (2.6) for both finite and transfinite ν, is the point at which $x(t)$ jumps from $i_{\nu-1}$ to i_ν, if ν is not a limit ordinal. The value i_ν is maintained for time τ_ν. The probabilities (2.7) are easily extended to the transfinite ordinals, and (2.9) defines $\Pi_{i_0j}^{(\nu)}$ for transfinite ν. The range of ν is limited by the condition that τ_ν' be finite with positive probability. Let N be the smallest ordinal number for which $\tau_N=\infty$ with probability 1, for every initial state i_0. Then N is evidently a denumerable order type. If we set $\Pi_{ij}^{(\nu)}=0$ whenever $\Pi_{ij}^{(\nu)}$ is not already defined (because τ_ν' is infinite), the matrix $\Pi^{(\nu)}: (\Pi_{ij}^{(\nu)})$ has the following properties:

$$\tag{2.14} \Pi^{(\mu+\nu)} = \Pi^{(\mu)}\cdot\Pi^{(\nu)}, \qquad 0 \leqq \Pi_{ij}^{(\nu)} \leqq 1, \qquad 0 \leqq \sum_j \Pi_{ij}^{(\nu)} \leqq 1.$$

If ν is finite, $\Pi^{(\nu)} = (\Pi^{(1)})^\nu$ is determined by the q_i, q_{ij} but if $N>\omega$ so that the transfinite ordinals are essential, it will be seen below that $\Pi^{(\nu)}$ is not determined uniquely by the q_i, q_{ij} when $\nu \geqq \omega$. Evidently the $\Pi^{(\nu)}$ and q_i determine the probability relations of the sequences (2.13) and therefore those of the $x(t)$ process completely. This suggests the possibility of constructing the most general process being considered here from any q_i, $\Pi^{(\nu)}$. This will now be done.

THEOREM 2.2. *Let* $\{x(t)\}$ *be the sample functions of a Markoff process which can assume denumerably infinitely many states. Suppose that whatever the initial state i, the discontinuities of* $x(t)$ *form well ordered series on the t axis, with probability* 1. *Then the limits in* (0.7) *exist, the* q_i, q_{ij} *are finite, and there is always equality in* (1.2). *Either of the two following cases may arise:*

(i) *Whatever the initial state i the discontinuities of* $x(t)$ *are all jumps, with probability* 1. *In this case the transition probability functions* $p_{ij}(t)$ *have continuous derivatives and* (0.5) *and* (0.6) *are satisfied.*

(ii) *For some initial state i, the probability that the discontinuities of* $x(t)$ *are all jumps is less than* 1. *In this case the transition functions* $p_{ij}(t)$ *have continuous derivatives, and* (0.5) *is satisfied, but the equality* (0.6) *must be replaced by inequality* ($\geqq$).

Conversely suppose that $\{q_i\}$, $\{q_{ij}\}$ *are any non-negative numbers satisfying* (1.2) *with equality for all i. Then one of the following two statements is true:*

(i) *There is exactly one Markoff process* (*that is, one set of* $p_{ij}(t)$ *satisfying* (0.1)–(0.4)) *for which the given* q_i, q_{ij} *satisfy* (0.7). *This process comes under case* (i) *above.*

(ii) *There are infinitely many Markoff processes for which the given* $q_i q_{ij}$ *satisfy* (0.7). *None of these processes come under case* (i) *above. The* $p_{ij}(t)$ *are the same for all these processes. There are even infinitely many such processes whose transitions are well ordered, with probability* 1, *whatever their initial states. These processes come under case* (ii) *above.*

Preliminary remarks. This theorem shows that there is always a solution to the differential equations (0.5), satisfying (0.1)–(0.4), but that this solution is not uniquely determined unless the q_i, q_{ij} satisfy Feller's condition that the $x(t)$ discontinuities are all jumps. It is not known whether the infinite class of processes corresponding to a given set of q_i, q_{ij}, as described under (ii) of the converse, may actually include processes whose discontinuities are not well ordered. In other words there is still no complete characterization of *all* the processes corresponding to a given set of q_i, q_{ij}, except in case (i). The reason the present paper has excluded the case of systems which can assume only finitely many states, that is, those in which the subscripts in (0.1)–(0.3) only range through finitely many values, is that case (ii) cannot then arise. This can be seen for example by the description of the possible limiting values of $x(t)$ at a discontinuity discussed in §1, where the hypothesis of infinite-dimensionality was never used. Feller's condition is also seen at once to be satisfied in the finite-dimensional case.

Given a Markoff process whose functions can assume denumerably many values, it is well known that case (i) can arise (cf. for example the Poisson distribution below). It has already been shown in the previous section that in this case the q_i and q_{ij} are finite, there is equality in (1.2) for all i, the $p_{ij}(t)$ are continuous and the differential equations (0.5) and (0.6) are satisfied. No example of case (ii) has been exhibited previously, and even the possible existence of such an example has been doubted. The proof of the converse gives a method of constructing the most general process of this type, and the method will be used below to construct specific examples. If case (ii) does arise, it has already been seen that (0.1)–(0.4) are satisfied, the q_i, q_{ij} are finite, there is equality in (1.2) for all i, the $p'_{ij}(t)$ exist and (0.5) is satisfied. On the other hand although we have seen that (0.6) is then true if "$=$" is replaced by "$\geqq$," there cannot be equality for all pairs of subscripts. In fact if there were such equality, there would be a last discontinuity, which would be a jump, before any specified t_0, with probability 1. The last discontinuity before any rational value of t would then be a jump with probability 1 (the exceptional chance being independent of the rational value of t). This is obviously impossible in case (ii). Only the converse of Theorem 2.2 remains to be proved.

Proof of the converse. Suppose that $\{q_i, q_{ij}\}$ are any non-negative numbers satisfying (1.2) with equality for all i. Define $\Pi^{(\nu)}: (\Pi_{ij}^{(\nu)})$ as follows:

$$\Pi^{(1)} = (q_{ij}/q_i), \qquad \Pi^{(\nu)} = (\Pi^{(1)})^\nu, \tag{2.15}$$

for all finite ν. If ν is any transfinite ordinal, it can be written in the form $\nu=\lambda+\mu$, where λ is a limit ordinal and μ is finite. If we define

$$\Pi^{(\lambda+\mu)} = \Pi^{(\lambda)}\cdot\Pi^{(\nu)}, \tag{2.16}$$

only $\Pi^{(\lambda)}$ remains to be defined. Let $p_1, p_2, \cdots$ be any non-negative numbers whose sum is 1, and let Π be the matrix whose i, jth term is p_j. Then if λ is a limit ordinal define $\Pi^{(\lambda)}$ by

$$\Pi^{(\lambda)} = \Pi. \tag{2.17}$$

It is easy to verify that then $\Pi^{(\nu)}$, defined for all ordinal numbers ν, has non-negative terms, row sums 1, and satisfies the functional equation (2.16) for all ordinals λ, μ. This is a special type of solution of the functional equation, but it exhibits the fact that there are infinitely many solutions with the same $\Pi^{(1)}$. Let i be any positive integer. Define i_ν, τ_ν for ordinal numbers $\nu\geqq 0$ by induction as follows:

(i) $i_0=i$; τ_0 is a chance variable with density of distribution $\phi(q_{i_0}, s)$[11].

(ii) Suppose i_μ, τ_μ have been defined for $\mu\leqq\nu$. Then (for specified $i_0, \tau_0, \cdots, i_\nu, \tau_\nu$) $i_{\nu+1}$ is a chance variable which assumes the value j with probability $\Pi_{i_\nu j}^{(1)}$, and $\tau_{\nu+1}$ is a chance variable whose probability density of distribution is $\phi(q_{i_\nu}, s)$.

---

[11] In the following discussion, if any $q_j=0$ and if $i_\mu=j$, then $\tau_\mu=\infty$.

(iii) Suppose i_μ, τ_μ have been defined for $\mu<\lambda$ where λ is a limit ordinal. Then i_λ is a chance variable which for specified i_μ, τ_μ, $\mu\leqq\nu<\lambda$ (where ν is any ordinal preceding λ), assumes the value j with probability $\Pi_{i_\nu j}^{(\lambda-\nu)}$ and τ_λ is a chance variable whose probability density of distribution for specified i_μ, $\mu\leqq\lambda$, τ_μ, $\mu<\lambda$, is $\phi(q_{i_\lambda}, s)$. Note that because of the functional equation satisfied by $\Pi^{(\mu)}$ different choices of ν give mutually consistent results for the i_λ distribution. Define τ_ν' by (2.6). This chance variable is not necessarily finite-valued and in fact there is evidently a first ordinal number N (of denumerable order type) such that $\tau_N'=\infty$ with probability 1 for every initial value i_0. The chance variables (i_j, τ_j) will only be required in the following for $j<N$. The desired stochastic process is now defined as follows:

$$x(t) = i \quad \text{if} \quad \tau_j' \leqq t < \tau_{j+1}'. \tag{2.18}$$

It must be shown that the process so defined is a Markoff process with the given q_i, q_{ij} satisfying (0.7). Let $p_{\alpha ij}(s, t)$ be the probability that $x(t)=j$ if $x(s)=i$ and $x(0)=\alpha$. The following fact is fundamental: Let x be a chance variable with probability density of distribution $\phi(q, \xi)$. Then if $y>0$ the conditional probability density of distribution of $x-y$ under the hypothesis that $x>y$ is $\phi(q, \xi-y)$. Because of this fact, if $s>0$, $\Delta=\tau_{\nu+1}'-s$ is a chance variable which for given (i_j, τ_j'), $j\leqq\nu$, $\tau_\nu'<s$, and under the hypothesis that $\Delta>0$, has the probability density of distribution $\phi(i_\nu, \Delta)$. Hence the (i_j, τ_j') are not affected if the defining process is stopped when τ_j' exceeds s, thus defining $x(t)$ for $t\leqq s$, and if then the defining process is recommenced at $t=s$ in exactly the same way as at $t=0$, using however the $x(s)$ values as the initial values, with the appropriate corresponding probabilities. This implies the following facts:

(a) The conditional probability distribution of $x(t)$ for prescribed values of $x(\tau)$ when $\tau\leqq s$ depends only on $x(s)$. That is to say, the process is a Markoff process.

(b) The probability $p_{\alpha ij}(s, t)$ is independent of α and is a function of $t-s$. The conditional probability $p_{\alpha ij}(s, t)$ is defined if and only if there is a positive probability of a sequence of transitions from α to i. The α can always be chosen to make the probability positive, however—say by choosing $\alpha=i$. Thus if $p_{ij}(u)$ is defined by

$$p_{ij}(u) = p_{iij}(s, s+u), \tag{2.19}$$

$p_{ij}(u)$ satisfies (0.1), (0.2), (0.3). The fact that (0.7) is satisfied with the given q_i, q_{ij} follows at once from the fact that the q_i, q_{ij} in (0.7) determine the probability $\exp(-q_it)$, that if $x(0)=i$, then $x(s)\equiv i$ for $0\leqq s\leqq t$ and the probability q_{ij}/q_i that if $x(0)=i$, the first transition is a jump to j. In fact the process was defined in terms of exactly these two properties using the given q_i, q_{ij}.

There is thus at least one stochastic process corresponding to the given

q_i, q_{ij}. Now the ${}_n\bar{p}_{ik}(t)$ and $\bar{p}_{ik}(t)$ determined by (1.5) and (1.6) are determined uniquely by the q_i, q_{ij}. Thus the $\bar{p}_{ij}(t)$ are the same for all processes with the given q_i, q_{ij} (even without the hypothesis that the discontinuities of $x(t)$ are well ordered), and in every case

$$\bar{p}_{ij}(t) \leqq p_{ij}(t), \qquad \sum_j \bar{p}_{ij}(t) \leqq \sum_j p_{ij}(t) = 1.$$

If in particular there is one process for which the $\bar{p}_{ij}(t)$, $p_{ij}(t)$ are identical, that is, for which the discontinuities are only jumps, it follows that $\sum_j \bar{p}_{ij}(t) = 1$ for all i. Then there can be only one process with these q_i, q_{ij}: the process determined by the $\bar{p}_{ij}(t)$, and these functions satisfy (1.5) and (1.6). This case arises in the above discussion if $N=\omega$. The condition on the q_i, q_{ij} that this case occur has been given a simpler form by Feller [2, pp. 506, 507]. On the other hand if $\sum_j \bar{p}_{ij}(t)$ is not identically 1 for all i and t, the above method of construction leads to an $N>\omega$ and the $p_{ij}(t)$ finally obtained depend essentially on the choice of $\Pi_{ij}^{(\nu)}$. The question remains open whether in this case a different method of construction would lead to an entirely different type of process, the discontinuities of whose sample functions are not well ordered, and thus enlarge still further the class of processes with the given q_i, q_{ij}.

3. **Examples.** This section contains examples of processes of the type discussed in §2.

I. A simple example of case (i) of Theorem 2.2 is given by the Poisson distribution:

$$(3.1) \qquad \begin{aligned} &p_{jk}(t) = e^{-t}t^{k-j}/(k-j)!, && k \geqq j, \\ &p_{jk}(t) = 0, && k < j, \\ &q_{j\,j+1} = 1, \quad q_{jk} = 0, \quad q_j = 1, && k \neq j+1. \end{aligned}$$

In this example the sample function $x(t)$ increases monotonely from its initial value to ∞ as $t \to \infty$.

II. The following example will be illuminating. Suppose that

$$(3.2) \qquad q_{j\,j+1} = q_j, \qquad q_{jk} = 0, \qquad k \neq j+1.$$

Then if the q_i are all different, the explicit solution for $\bar{p}_{jk}(t)$ is (cf. [2, p. 513]):

$$(3.3) \qquad \begin{aligned} &\bar{p}_{jk}(t) = 0, \qquad \bar{p}_{jj}(t) = \exp(-q_j t), && k < j, \\ &\bar{p}_{jk}(t) = (-1)^{k-j} q_j \cdot q_{j+1} \cdots q_{k-1} \\ &\quad \cdot \sum_{\nu=j}^{k} \frac{\exp(-q_j t)}{(q_\nu - q_j)\cdot(q_\nu - q_{j+1}) \cdots (q_\nu - q_{\nu-1})\cdot(q_\nu - q_{\nu+1}) \cdots (q_\nu - q_k)}, && k > j. \end{aligned}$$

To find the conditions that this example come under case (i) or (ii) of Theorem 2.2, note that the τ_k (k finite) defined in the proof of that theorem are mutually independent in this example, and that the characteristic function of τ_k is (if $i_0=j$)

$$
(3.4)\qquad
\begin{aligned}
E\{x(0) = j;\, \exp(iz\tau_k)\} &= \int_0^\infty \exp(iz\lambda)\exp(-q_{j+k}\lambda)q_{j+k}\,d\lambda \\
&= \frac{q_{j+k}}{q_{j+k} - iz\lambda}\ \text{(12)}.
\end{aligned}
$$

Hence the characteristic function of τ_n' is

$$
(3.5)\qquad \prod_j^{j+n-1} \frac{q_r}{q_r - iz}.
$$

The condition of finiteness of τ_ω', that is, the condition that the example come under case (ii), is the condition that there be convergence in (3.5) as $n\to\infty$, that is, that $\sum_j 1/q_j$ converge. Thus this comes under case (i) if and only if $\sum_j 1/q_j = \infty$ [13].

III. A simple example under case (ii) of Theorem 2.2 will now be obtained by choosing the q_j differently in the preceding example. In fact suppose that

$$
(3.6)\qquad
\begin{gathered}
q_1 = 0, \qquad q_{j\,j+1} = q_j, \qquad q_{jk} = 0, \qquad k \neq j+1, \\
\sum_j 1/q_j < \infty.
\end{gathered}
$$

Then the $\bar{p}_{jk}(t)$ $(j>1)$ are given by (3.3) if the q_j are all different, and however the $p_{jk}(t)$ corresponding to these q_j, q_{jk} are defined,

$$
(3.7)\qquad \bar{p}_{11}(t) = p_{11}(t) \equiv 1, \qquad \bar{p}_{1j}(t) = p_{1j}(t) \equiv 0, \qquad j > 1.
$$

We define Π_{jk}^{ω} so that the sample functions will go monotonely to ∞ from their initial values, taking on the value 1 for t greater than τ_ω', the limit point of discontinuities:

$$
(3.8)\qquad \Pi_{j1}^{(\omega)} = 1; \qquad \Pi_{jk}^{(\omega)} = 0, \qquad k \neq 1.
$$

The argument of the preceding example gives the following evaluation of the characteristic function of τ_ω':

$$
(3.9)\qquad E\{x(0) = j;\, \exp(iz\tau_\omega')\} = \prod_j^\infty \frac{q_r}{q_r - iz}, \qquad j \neq 0.
$$

Evidently $p_{jk}(t) = \bar{p}_{jk}(t)$ for $k\neq 1$, and

$$
(3.10)\qquad
\begin{aligned}
p_{j1}(t) &= P\{x(0) = j;\, \tau_\omega' \leq t\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(izt) - 1}{iz} \prod_j^\infty \frac{q_r}{q_r - iz}\,dz, \qquad j \neq 1.
\end{aligned}
$$

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(12) In this section i will always be $(-1)^{1/2}$, and will not be used as a subscript.

(13) This condition was given by Feller [2, p. 514].

It is clear how this example can be modified very simply to derive more complicated examples in which the discontinuities form well ordered series corresponding to larger ordinal numbers.

BIBLIOGRAPHY

1. J. L. Doob, *Topics in the theory of Markoff chains*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 37–64.

2. W. Feller, *On the integro-differential equations of purely discontinuous Markoff processes*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 488–515. Errata, ibid. vol. 58 (1945) p. 474.

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