# REPRESENTATIONS OF GROUPS AS QUOTIENT GROUPS. III. INVARIANTS OF CLASSES OF RELATED REPRESENTATIONS

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In the first part of this investigation we divided the representations of a group $G$ as a quotient group $H/M$ into classes of related representations, and we showed the existence of a vast system of structural invariants of these classes of related representations.

The system of all these invariants is, however, a much more complicated structure than the original group which actually occurs in this system of invariants. For this reason, and for the obvious reason that our knowledge of nonabelian groups is rather limited, we shall restrict ourselves usually to the consideration of the abelian invariants. It is natural now to wonder whether the structure of the original group $G$ is determined by these abelian invariants, whether they determine the classes of related representations and to what extent they determine each other. It is our object in this note to start an investigation of these problems.

In order to apply the most powerful tools developed in the first two parts of this theory we shall restrict ourselves usually to the consideration of representations of the form $H/M$ where $M$ is part of the commutator subgroup of $H$, and to the consideration of invariants derived from the lower central series of $H$ with regard to $M$.

In the first two sections of the present note we derive criteria for two representations of a group to be related or to be equivalent, and in the last two sections we obtain criteria for a homomorphism to be an isomorphism. In §4 we show that the invariants of the classes of related representations are much more interrelated than one would expect at first glance. §3 finally is devoted to the study of a new fully invariant subgroup which we need in §§4 and 5, and which seems to be of independent interest.

Results, definitions and notations of the first two parts of this theory are used throughout. Reference will be made to Chapters I to III of the first part [1], and thus it will be convenient to refer to the second part[2] as to Chapter IV.

1. **Equivalence and relatedness of representations.** If the representations $H/M$ and $K/N$ of the group $G$ are *related*[3], then there exists a homo-

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(1) Baer [4]. Numbers in brackets refer to the Bibliography at the end of the paper.

(2) Baer [5].

(3) See II.1 for the fundamental concepts used in classifying representations.

morphism $\eta$ of $H$ into $K$ and a homomorphism $\kappa$ of $K$ into $H$ which induce reciprocal isomorphisms between $H/M$ and $K/N$. Any such pair $\eta, \kappa$ of homomorphisms shall be termed *a pair (of homomorphisms) relating the representations $H/M$ and $K/N$*, and $\eta$ as well as $\kappa$ will be called *a homomorphism relating the representations $H/M$ and $K/N$*. We note that the existence of a homomorphism relating the representations $H/M$ and $K/N$ of $G$ is equivalent to the relatedness of the representations $H/M$ and $K/N$ of $G$.

THEOREM 1. *If $\eta, \kappa$ is a pair of homomorphisms relating the representations $H/M$ and $K/N$ of the group $G$, and if there exists some $p$ such that the $p$-$M$-series of $H$ is strictly interlocking*[4], *then:*

(i) *$\eta$ and $\eta\kappa$ are isomorphisms of $H$;*

(ii) *$H^\eta \cap W = 1$ for $W$ the kernel of $\kappa$.*

**Proof.** Since $\eta$ induces an isomorphism of $H/M$ upon $K/N$, and since $\eta\kappa$ is an endomorphism of $H$ which induces the identity in $H/M$, we deduce from (c) of Theorem 2 of II.2 that $\eta$ and $\eta\kappa$ are isomorphisms of $H$, proving (i). This implies in particular that $\kappa$ induces an isomorphism of $H^\eta \leqq K$ into $H$, a fact that is equivalent to (ii).

COROLLARY 1. *If the $p$-$M$-series of $H$ is, for some $p$, strictly interlocking, then the following condition is necessary and sufficient for the relatedness of the representations $H/M$ and $K/N$ of the group $G$.*

(r) *There exists a subgroup $L$ of $K$ such that $H/M$ and $L/(L\cap N)$ are equivalent representations of $G$ and such that $L/(L\cap N)$ and $K/N$ are related representations of $G$.*

**Proof.** We note first that equivalent representations are related and the relatedness of representations is a transitive relation. This implies immediately the sufficiency of condition (r). If there exists conversely a pair $\eta, \kappa$ of homomorphisms relating the representations $H/M$ and $K/N$ of $G$, then we infer from Theorem 1 that $\eta$ is an isomorphism of $H$ upon $L = H^\eta$. Since $\eta$ induces an isomorphism of $H/M$ upon $K/N$, no element outside $M$ is mapped by $\eta$ upon an element in $N$ whereas $M$ is mapped by $\eta$ upon part of $N$, and thus we have $M^\eta = L\cap N$. The isomorphism $\eta$ of $H$ upon $L$ effects therefore an equivalence between the representations $H/M$ and $L/(L\cap N)$. But $K/N$ and $H/M$ are related representations of $G$, and hence it follows from the transitivity of relatedness that $K/N$ and $L/(L\cap N)$ are related representations of $G$ too, completing the proof.

Two groups $S$ and $T$ have been termed[5] *related* if isomorphic quotient groups of $S$ and $T$ respectively always constitute related representations.

COROLLARY 2. *If the $p$-$M$-series of $H$ is, for some $p$, strictly interlocking,*

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(4) See II.2 for an explanation of these concepts.

(5) This term has been introduced in II.4, Remark.

*and if the group K is related to each of its subgroups, then the following condition is necessary and sufficient for the relatedness of the representations $H/M$ and $K/N$ of $G$:*

(r′) *$H$ is isomorphic to a subgroup of $K$.*

This is an immediate inference from Corollary 1.

*Remark* 1. The most interesting example of a group related to all its subgroups is constituted by the free groups, as may be deduced from Schreier's Theorem and Theorem 1 of III.1.

COROLLARY 3. *The group $V$ is a free group if, and only if, there exists a normal subgroup $N$ of $V$ and an admissible $p$ such that:*

(i) *The $p$-$N$-series of $V$ is strictly interlocking.*

(ii) *The representation $V/N$ of the group $G$ is related to a representation of $G$ as a quotient group of a free group.*

**Proof.** If conditions (i) and (ii) are satisfied by $V$, then we infer from Corollary 2 and Remark 1 that $V$ is isomorphic to a subgroup of a free group, and we deduce from Schreier's Theorem that $V$ is a free group. If conversely $V$ is a free group, then the lower central series of $V$ with regard to $(V, V)$ is strictly interlocking(6), since it consists of the subgroups ${}^iV$ and ${}_i(V, V) = {}^{i+1}V$, showing that conditions (i) and (ii) are satisfied by free groups.

*Remark* 2. Corollary 3 constitutes an improvement upon Theorem 1 of III.1.

*Remark* 3. The method of the proof of Corollary 3 may be extended. For we may substitute for the class of free groups any class $\Delta$ of groups meeting the following requirements:

(1) *If $G$ is in $\Delta$, and if $S$ is a subgroup of $G$, then $S$ is in $\Delta$.*

(2) *To every $G$ in $\Delta$ there exists a normal subgroup $N$ of $G$ and an admissible $p$ such that the $p$-$N$-series of $G$ is strictly interlocking.*

THEOREM 2. *If $H/M$ and $K/N$ are representations of $G$, if the $p$-$M$-series of $H$ is, for suitable $p$, strictly interlocking, and if $\eta$ is a homomorphism of $H$ into $K$, then the following conditions are necessary and sufficient for the existence of a homomorphism $\kappa$ of $K$ upon $H$ such that $H = H^{\eta\kappa}$ and such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$.*

(i) *$\eta$ effects an equivalence between the representations $H/M$ and $H^\eta/(N\cap H^\eta)$ of $G$.*

(ii)(7) *There exists an idempotent endomorphism $\epsilon$ of $K$ upon $H^\eta$ which induces an isomorphism of $K/N$ upon $H^\eta/(N\cap H^\eta)$.*

**Proof.** Suppose first that conditions (i) and (ii) are satisfied by $\eta$. It is a consequence of (i) that $\eta$ is an isomorphism of $H$ upon $L = H^\eta$ and that

(6) Magnus [1] has shown that 1 is the cross cut of the groups ${}^iV$, if $V$ is a free group.

(7) This condition (ii) is equivalent to the hypothesis that the kernel of $\epsilon$ be part of $N$.

$M^\eta = N \cap L$. There exists by (ii) an idempotent endomorphism $\epsilon$ of $K$ which maps $K$ upon $L$ and which induces an isomorphism of $K/N$ upon $L/(N \cap L)$. This implies in particular that the kernel $E$ of $\epsilon$ is part of $N$. If $x$ is any element in $K$, then $x = x^\epsilon x'$ for a uniquely determined $x'$ in $K$. But $x^\epsilon = x^{\epsilon^2} x'^\epsilon = x^\epsilon x'^\epsilon$ implies $x'^\epsilon = 1$. Thus $x'$ belongs to $E$ and therefore to $N$. Hence $K = K^\epsilon E = LN = H^\eta N$; and we have shown that the isomorphism $\eta$ of $H$ into $K$ induces an isomorphism of $H/M$ upon $K/N$, since $M = (N \cap L)^{\eta^{-1}}$. Since $\eta$ is an isomorphism of $H$ upon $H^\eta = K^\epsilon$, it follows that $\kappa = \epsilon\eta^{-1}$ is a well determined homomorphism of $K$ *upon* $H$. From $E \leqq N$ it follows that $\epsilon$ induces the identity in $K/N$, and from $\kappa\eta = \epsilon$ it follows that $\kappa$ and $\eta$ induce reciprocal isomorphisms between $K/N$ and $H/M$. Thus conditions (i) and (ii) have been shown to be sufficient conditions, since $\eta\kappa = \eta\epsilon\eta^{-1} = \eta\eta^{-1} = 1$ because of $H^\eta = K^\epsilon$.

To prove the necessity of conditions (i) and (ii) assume the existence of a homomorphism $\kappa$ of $K$ upon $H$ such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$ and such that $\eta\kappa$ maps $H$ upon $H$. It is a consequence of (i) of Theorem 1 that $\eta$ and $\eta\kappa$ are isomorphisms of $H$. If we denote by $W$ the kernel of the homomorphism $\kappa$, then we deduce $H^\eta \cap W = 1$ from (ii) of Theorem 1. If $x$ is an element in $K$, then $x^\kappa$ belongs to $H = H^{\eta\kappa}$. There exists therefore an element $y$ in $H$ such that $y^{\eta\kappa} = x^\kappa$. Since $W$ is the kernel of $\kappa$, this implies $x \equiv y^\eta$ modulo $W$, and we have shown $H^\eta W = K$. Every coset of $K/W$ contains consequently one and only one element in $H^\eta$. If $x$ is any element in $K$, then we denote by $x^\epsilon$ the uniquely determined element in the cross cut of $H^\eta$ and $Wx$. Clearly $\epsilon$ is an idempotent endomorphism of $K$ satisfying $K^\epsilon = H^\eta$, and the kernel of $\epsilon$ is $W$. But $W$ is part of $N$, since $W$ is the kernel of $\kappa$, and since $\kappa$ induces an isomorphism in $K/N$. Thus $x \equiv x^\epsilon$ modulo $W$ implies $x \equiv x^\epsilon$ modulo $N$, proving that $\epsilon$ induces the identity in $K/N$. From $K^\epsilon = H^\eta$ we infer now that $\epsilon$ induces an isomorphism of $K/N$ upon $H^\eta/(N \cap H^\eta)$. Finally we note that $M^\eta \leqq H^\eta \cap N$. But $\eta$ induces an isomorphism of $H/M$ upon $K/N$ so that no element outside $M$ is mapped by $\eta$ upon an element in $N$. Hence $M^\eta = H^\eta \cap N$. Since $\eta$ is, as has been mentioned before, an isomorphism of $H$, it follows now that $\eta$ effects an equivalence between the representations $H/M$ and $H^\eta/(N \cap H^\eta)$ of $G$. Thus we have shown that conditions (i) and (ii) are satisfied by $\eta$, completing the proof.

*Remark* 4. In the first part of the proof we have shown that conditions (i) and (ii) imply the existence of a homomorphism $\kappa$ of $K$ upon $H$ such that $\eta\kappa = 1$ and such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$.

A group $G$ has been termed(8) an *S-group*, if no subgroup, different from $G$, is isomorphic to $G$.

COROLLARY 4. *If $H/M$ and $K/N$ are representations of $G$, if the p-M-series*

(8) See Baer [2, p. 273 ff.] where a detailed analysis of this concept may be found.

*of $H$ is, for suitable $p$, strictly interlocking, and if $H$ is an $S$-group, then the conditions* (i) *and* (ii) *of Theorem* 2 *are necessary and sufficient for the homomorphism $\eta$ of $H$ into $K$ to relate the representations $H/M$ and $K/N$ of $G$.*

**Proof.** If $\eta$ relates the representations $H/M$ and $K/N$ of $G$, then there exists a homomorphism $\kappa$ of $K$ into $H$ such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$. We deduce from (i) of Theorem 1 that $\eta\kappa$ is an isomorphism of $H$ into $H$. But $H$ is an $S$-group. Hence $H = H^{\eta\kappa}$. Thus we may deduce from Theorem 2 that its conditions (i) and (ii) are satisfied by $\eta$. The sufficiency of these conditions (i) and (ii) is an immediate consequence of Theorem 2.

COROLLARY 5. *If $H/M$ and $K/N$ are related representations of the same group $G$, if the $p$-$M$-series of $H$ and the $p$-$N$-series of $K$ are, for suitable $p$, both strictly interlocking, and if at least one of the groups $H$ and $K$ is an $S$-group, then $H/M$ and $K/N$ are equivalent representations of $G$.*

**Proof.** We assume without loss in generality that $H$ is an $S$-group. There exists a homomorphism $\eta$ of $H$ into $K$ which relates the representations $H/M$ and $K/N$. We deduce from Corollary 4 that $\eta$ effects an equivalence between the representations $H/M$ and $H^\eta/(N \cap H^\eta)$ of $G$, and that there exists an idempotent endomorphism $\epsilon$ of $K$ which maps $K$ upon $H^\eta$ and which induces an isomorphism of $K/N$ upon $H^\eta/(N \cap H^\eta)$. But it follows from (i) of Theorem 1 that $\epsilon$ is an isomorphism of $K$, and this implies $\epsilon = 1$ so that $\eta$ effects an equivalence between the representations $H/M$ and $K/N$ of $G$.

THEOREM 3. *If the $p$-$M$-series of the reduced free group*[9] *$R$ and the $q$-$N$-series of the reduced free group $S$ are both strictly interlocking, and if $R/M$ and $S/N$ are representations of the same group $G$, then $R \sim S$ is a necessary and sufficient condition for relatedness* (*similarity*) *of the representations $R/M$ and $S/N$ of $G$.*

**Proof.** Since the $p$-$M$-series of $R$ is strictly interlocking, we have $M \leqq (R, R)$ so that $G/(G, G) \sim R/(R, R)$, and likewise we have $G/(G, G) \sim S/(S, S)$. Thus $R$ and $S$ are reduced free groups with isomorphic commutator quotient groups. Hence we deduce from Theorem 2 of III.3 that $r(R) = r(S)$.

If the representations $R/M$ and $S/N$ of $G$ are related, then there exists a homomorphism $\eta$ of $R$ into $S$ and a homomorphism $\kappa$ of $S$ into $R$ such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $R/M$ and $S/N$. We deduce from Theorem 1 that $\eta$ as well as $\kappa$ is an isomorphism. Hence $R \sim S$ is a consequence of (iv) of Theorem 3 of III.4.

If conversely $R \sim S$, then we deduce from Theorem 1 of III.3 the similarity of the groups $R$ and $S$ and from Theorem 1 of II.4 the similarity—and there-

(9) This concept has been introduced in III. 3.

fore the relatedness—of the representations $R/M$ and $S/N$ of $G$.

2. **Representations with isomorphic invariants.** A group $G$ has been termed a *Q-group*(10) if $N=1$ is the only normal subgroup $N$ of $G$ satisfying $G\sim G/N$.

THEOREM 1. *If the $p$-$M$-series of $H$ is strictly interlocking, if*(11) *$M(i-1, p)/M(i, p)$ is, for every positive $i$, a Q-group, if $W\leqq M$ is a normal subgroup of $H$, and if $H^*=H/W$, $M^*=M/W$, then the following properties of $W$ imply each other*:

(a)(12) *$H/M$ and $H^*/M^*$ are related representations of the same group $G$.*

(b) *$M(i-1, p)/M(i, p)\sim M^*(i-1, p)/M^*(i, p)$ for $0<i$.*

(c) *$W=1$.*

**Proof.** From the easily verified relations $H^*(j, p)=H(j, p)W/W$ and $M^*(j, p)=M(j, p)W/W$ one deduces immediately that the $p$-$M^*$-series of $H^*$ is interlocking, since the $p$-$M$-series of $H$ is interlocking.

It is obvious that (a) is a consequence of (c). That (a) implies (b) may be inferred from (a) of Theorem 2 of II.2, since both the $p$-$M$-series of $H$ and the $p$-$M^*$-series of $H^*$ are interlocking.

Assume finally that (b) holds true. Then we have for every positive $i$ the following isomorphies:

$$M(i-1, p)/M(i, p) \sim M^*(i-1, p)/M^*(i, p) \sim M(i-1, p)W/M(i, p)W$$
$$\sim M(i-1, p)/[M(i-1, p) \cap M(i, p)W],$$

since $M(i, p)\leqq M(i-1, p)$. But $M(i-1, p)/M(i, p)$ is a $Q$-group and $M(i, p) \leqq M(i-1, p)\cap M(i, p)W$. Hence we obtain for every positive $i$

$$M(i, p) = M(i-1, p) \cap M(i, p)W = M(i, p)[M(i-1, p) \cap W]$$

by Dedekind's law, and this implies $M(i-1, p)\cap W\leqq M(i, p)$. Hence we deduce from $M(i, p)\leqq M(i-1, p)$ that

$$M(i-1, p) \cap W = M(i, p) \cap W \qquad \text{for } 0 < i.$$

But $W\leqq M=M(0, p)$, and hence it follows by complete induction that $W$ is part of every $M(i, p)$ (cp. Theorem 2 of I.2). Since the $p$-$M$-series of $H$ is strictly interlocking, we deduce $W=1$, completing the proof.

COROLLARY 1. *If $H/M$ and $K/N$ are representations of the same group $G$, if the $p$-$M$-series of $H$ is strictly interlocking, if $M(i-1, p)/M(i, p)$ is, for every positive $i$, a Q-group, and if the homomorphism $\eta$ of $H$ upon $K$ induces an isomorphism of $H/M$ upon $K/N$, then the following properties of $\eta$ imply each other*:

(a) *$H/M$ and $K/N$ are related representations of $G$.*

(10) Baer [2, p. 267].

(11) For notations cp. I.2.

(12) The isomorphy of $H/M$ and $H^*/M^*$ is a consequence of the definition of $H^*$ and $M^*$.

(b) $M(i-1, p)/M(i, p) \sim N(i-1, p)/N(i, p)$ *for* $0<i$.

(c) $\eta$ *effects an equivalence between the representations* $H/M$ *and* $K/N$ *of* $G$.

**Proof.** Denote by $W$ the kernel of $\eta$. Then $W \leqq M$, since $\eta$ induces an isomorphism in $H/M$. Let $H^* = H/W$ and $M^* = M/W$. Since $\eta$ induces an isomorphism of $H/M$ upon $K/N$, and since $H^\eta = K$, it is readily seen that $M^\eta = N$ and that $M$ is the inverse image of $N$ under $\eta$. If we denote by $\eta^*$ the homomorphism induced by $\eta$ in $H^*$, then it is clear that $\eta^*$ effects an equivalence between the representations $H^*/M^*$ and $K/N$ of $G$.

Now it is readily seen that (a) is equivalent to the relatedness of the representations $H/M$ and $H^*/M^*$, that (b) is equivalent to the isomorphy of $M(i-1, p)/M(i, p)$ and $M^*(i-1, p)/M^*(i, p)$ for $0<i$, and that (c) is equivalent to $W=1$. Comparing this with Theorem 1, the equivalence of the present properties (a), (b), (c) is immediately evident.

*Remark* 1. The condition that $\eta$ be a homomorphism of $H$ *upon* $K$ which induces an isomorphism of $H/M$ upon $K/N$ is certainly necessary for the validity of (c). But with its omission even (a) and (b) would cease to be equivalent, as may be seen from the following *example*.

Denote by $H$ the group generated by four elements $q, r, s, t$, subject to the relations:

$$qs = sq, \qquad qt = tq, \qquad rs = sr, \qquad rt = tr,$$
$$x(q, r) = (q, r)x \text{ and } x(s, t) = (s, t)x \text{ for every } x \text{ in } H.$$

Let $M=(H, H)$. Then $H/M$ is a free abelian group of rank 4 and $M$ is part of the center of $H$ so that $(M, H)=1$. Furthermore $M=M/(M, H)$ is a free abelian group of rank 2.

Next denote by $K$ the group generated by four elements $u, v, w, z$, subject to the relations:

$$uz = zu, \qquad vz = zv, \qquad wz = zw, \qquad uw = wu,$$
$$x(u, v) = (u, v)x \text{ and } x(v, w) = (v, w)x \text{ for every } x \text{ in } K.$$

Let $N=(K, K)$. Then $K/N$ is a free abelian group of rank 4 and $N$ is part of the center of $K$ so that $(N, K)=1$. Furthermore $N=N/(N, K)$ is a free abelian group of rank 2.

The only $p$-$M$-series consists of the pairs $H$, $M$ and $(H, H)$, 1 and the only $p$-$N$-series consists of the pairs $K$, $N$ and $(K, K)$, 1, and it is clear that the two representations $H/M$ and $K/N$ of the free abelian group of rank 4 lead to the same invariants, and all the series are strictly interlocking. But $H$ and $K$ are not isomorphic, since $K$ possesses a direct factor which is an infinite cyclic group whereas $H$ does not possess such a direct factor. In particular the representations $H/M$ and $K/N$ are not equivalent. Consequently they are not related either, since their relatedness would imply their equivalence, as may be inferred from Corollary 2 of II.2.

*Remark* 2. The impossibility of omitting the condition that the groups $M(i-1, p)/M(i, p)$ be $Q$-groups may be seen from the following *example.*

Denote by $F$ the free group of rank 2, and put $H=F/{}^3F$, $M={}^1F/{}^3F$. Then $H/M$ is the free abelian group of rank 2, $M/{}_1M$ is an infinite cyclic group, ${}_1M/{}_2M$ is a free abelian group of countably infinite rank and ${}_2M=1$.

Denote by $W$ any subgroup of ${}_1M$ such that ${}_1M/W$ is still a free abelian group of countably infinite rank, though $W\neq 1$. Since ${}_1M$ is part of the center of $H$, $W$ is a normal subgroup of $H$. Thus condition (b) of Theorem 1 is satisfied by $W$, but neither (a) nor (c). This example may be modified in an obvious fashion so as to apply to Corollary 1 too.

COROLLARY 2. *If $H/M$ and $K/N$ are representations of the same group $G$, if* 1 *is the cross cut of the subgroups*(13) *${}_iM$, if $M\leqq(H, H)$, if both $H/(H, H)$ and $M/(M, H)$ are generated by a finite number of elements, and if the homomorphism $\eta$ of $H$ upon $K$ induces an isomorphism of $H/M$ upon $K/N$, then the following properties of $\eta$ imply each other:*

(a) *$H/M$ and $K/N$ are related representations of $G$.*

(b) *${}_{i-1}M/{}_iM\sim{}_{i-1}N/{}_iN$ for $0<i$.*

(c) *$\eta$ effects an equivalence between the representations $H/M$ and $K/N$ of $G$.*

**Proof.** It is a consequence of Theorem 3 of I.3 and of $M\leqq(H, H)$ that the lower central series of $H$ relative to $M$ is interlocking. The series is therefore strictly interlocking, since 1 is supposed to be the cross cut of the subgroups ${}_iM$. Since $H/(H, H)$ and $M/(M, H)$ are both generated by a finite number of elements, we deduce from (a) of Theorem 1 of IV.5 that ${}_{i-1}M/{}_iM$ is generated by a finite number of elements. But ${}_{i-1}M/{}_iM$ is an abelian group, and abelian groups generated by a finite number of elements are $Q$-groups. Thus all the general hypotheses of Corollary 1 are satisfied, and Corollary 2 is now an immediate consequence of Corollary 1.

THEOREM 2. *If $H/M$ and $K/N$ are representations of the same group $G$, if the $p$-$M$-series of $H$ is strictly interlocking, if $M(i-1, p)/M(i, p)$ is, for every positive $i$, a $Q$-group, and if the homomorphism $\eta$ of $H$ into $K$ induces an isomorphism of $H/M$ upon $K/N$, then the following conditions are necessary and sufficient for the existence of a homomorphism $\kappa$ of $K$ upon $H$ such that $H=H^{\eta\kappa}$ and such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$.*

(i) *$M(i-1, p)/M(i, p)\sim[N(i-1, p)\cap K(i, p)]N(i, p)/N(i, p)$ for $0<i$.*

(ii) *There exists an idempotent endomorphism $\epsilon$ of $K$ inducing an isomorphism in $K/N$ and satisfying $K^\epsilon=H^\eta$.*

**Proof.** Assume first the existence of a homomorphism $\kappa$ of $K$ upon $H$ such that $H=H^{\eta\kappa}$ and such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$. Then we deduce from Theorem 2 of §1 the necessity of the

(13) For the definition of the lower central series ${}_iM$, ${}^iH$ of the group $H$ relative to the normal subgroup $M$, see I.3.

present condition (ii). Furthermore the representations $H/M$ and $K/N$ are clearly related representations of $G$. Hence we deduce from Corollary 2 of II.2 the isomorphy of the groups $[N(i-1, p)\cap K(i, p)]N(i, p)/N(i, p)$ and $[M(i-1, p)\cap H(i, p)]M(i, p)/M(i, p)$ for positive $i$. But the latter of these groups is equal to $M(i-1, p)/M(i, p)$, since the $p$-$M$-series of $H$ is interlocking, and since therefore $M(i, p)\leqq M(i-1, p)\leqq H(i, p)$. This proves the necessity of condition (i).

Assume conversely that the conditions (i) and (ii) are satisfied by $\eta$. Put $L=H^\eta$ and $P=L\cap N$. Denote by $E$ the kernel of the idempotent endomorphism $\epsilon$ of condition (ii). Then $K^\epsilon=L$ and $E\leqq N$, since $\epsilon$ induces an isomorphism in $K/N$. But $x\equiv x^\epsilon$ modulo $E$ implies now that $\epsilon$ induces the identity in $K/N$; and thus it is clear that $\epsilon$ relates the representations $K/N$ and $L/P$ of $G$. Hence we may deduce from Corollary 2 of II.2 the isomorphy of the groups $[N(i-1, p)\cap K(i, p)]N(i, p)/N(i, p)$ and $[P(i-1, p)\cap L(i, p)]P(i, p)/P(i, p)$ for positive $i$. Since $\eta$ induces an isomorphism of $H/M$ upon $K/N$, it follows that the kernel of $\eta$ is part of $M$ and that $\eta$ induces an isomorphism of $H/M$ upon $L/P$. Since $\eta$ maps $H$ upon $L$, and since the $p$-$M$-series of $H$ is interlocking, it follows that the $p$-$P$-series of $L$ is interlocking. Consequently $[P(i-1, p)\cap L(i, p)]P(i, p/P(i, p)$ $=P(i-1, p)/P(i, p)$, and we infer from (i) and an isomorphy just established that $M(i-1, p)/M(i, p)\sim P(i-1, p)/P(i, p)$ for $0<i$. This implies, by Corollary 1, that $\eta$ effects an equivalence between the representations $H/M$ and $L/P$ of $G$. Thus conditions (i) and (ii) of Theorem 2 of §1 are satisfied by $\eta$, proving the existence of the desired homomorphism $\kappa$, and completing the proof.

COROLLARY 3. *If $H/M$ and $K/N$ are representations of the same group $G$, if 1 is the cross cut of the subgroups ${}_iM$, if $M\leqq(H, H)$, if both $H/(H, H)$ and $M/(M, H)$ are generated by a finite number of elements, and if the homomorphism $\eta$ of $H$ into $K$ induces an isomorphism of $H/M$ upon $K/N$, then the following conditions are necessary and sufficient for the existence of a homomorphism $\kappa$ of $K$ upon $H$ such that $H=H^{\eta\kappa}$ and such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$.*

(i) ${}_{i-1}M/{}_iM\sim({}_{i-1}N\cap{}^iK)/{}_iN$ *for* $0<i$.

(ii) *There exists an idempotent endomorphism $\epsilon$ of $K$ inducing an isomorphism in $K/N$ and satisfying $K^\epsilon=H^\eta$.*

This is deduced from Theorem 2 in almost the same fashion in which we derived Corollary 2 from Corollary 1. A slight simplification in the formulation of (i) is due to the obvious inequality ${}_iN\leqq{}_{i-1}N\cap{}^iK$.

3. **The potence of a group.** The discussion of this section seems to be of interest independent of the applications we shall have to make in §§5 and 6. We define: The *potence* $P(G)$ of the group $G$ is the join group of all the subgroups $S$ of $G$ satisfying $S=(S, G)$. Clearly $P(G)$ is a characteristic subgroup of $G$ which is part of the commutator subgroup $(G, G)$ of $G$.

An immediate inference from the definition of the potence of a group is the following important formula.

(1) $P(G)=(G, P(G))$.

The lower central series ${}^vG$ of the group $G$ may be defined for transfinite $v$ by the following rules[14].

${}^0G=G$, ${}^{v+1}G=(G, {}^vG)$, ${}^\lambda G$ for $\lambda$ a limit ordinal is the cross cut of all the ${}^uG$ with $u<\lambda$. There exists clearly a uniquely determined smallest ordinal $\tau=\tau(G)$ such that ${}^\tau G={}^{\tau+1}G$.

(2) $P(G)={}^\tau G$.

**Proof.** That ${}^\tau G$ is part of $P(G)$ is a consequence of the definitions of $P(G)$ and of $\tau(G)$. Conversely one deduces by complete (transfinite) induction from (1) that $P(G)$ is part of every ${}^vG$ and is therefore part of ${}^\tau G$.

From (2) one infers readily the important fact that the potence $P(G)$ *is a fully invariant subgroup of G.*

This property (2) shows furthermore that the potence of a group may be considered as a dual to the hypercenter[15]. It shows finally that the property $P(G)=1$ is a generalization of the quality of nilpotency of a group, since for finite groups ${}^\tau G=1$ and nilpotency are equivalent assertions[16].

If $F$ is a nonabelian free group[17], then ${}^\omega F=1$ and consequently—by (2)—$P(F)=1$. But there exist homomorphic maps of $F$ whose potence is different from 1, showing that the property $P(G)=1$ is not invariant under homomorphisms. Not abelian free groups $F$ satisfy furthermore $Z(F)=1$, showing that both potence and hypercenter of a group may be equal to 1. Finally there exist[18] groups $G\neq 1$ which are equal to their (generalized) hypercenter though $(G, G)=P(G)\neq 1$.

(3) $P(G/P(G))=1$.

**Proof.** If $S$ is a subgroup of $G$ such that $P(G)\leqq S$ and $S/P(G) =(G/P(G), S/P(G))$, then $S=(G, S)P(G)$, and thus $S$ is a (normal) subgroup of $G$ which satisfies $(G, S)={}_2SP(G)$ by (1). Consequently we have $S=(G, S)P(G)={}_2SP(G)P(G)={}_2SP(G)=(G, S)$. We deduce $S\leqq P(G)$ now from the definition of $P(G)$, and thus $S=P(G)$ or $S/P(G)=1$, proving our contention (3).

(4) $P(G)$ *is the cross cut of all the normal subgroups $N$ of $G$ which satisfy* $P(G/N)=1$.

**Proof.** If $N$ is a normal subgroup of $G$ such that $P(G/N)=1$, then $[NP(G)]/N$ is a subgroup of $G/N$ which satisfies $(G/N, [NP(G)]/N) =[(G, P(G))N]/N=[NP(G)]/N$ by (1); hence $[NP(G)]/N=1$ is a consequence of the definition of the potence and of $P(G/N)=1$. Thus we have

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(14) Baer [2, p. 271].
(15) Remak [1, p. 3].
(16) Zassenhaus [1, p. 119].
(17) Magnus [1].
(18) Baer [1].

shown that $P(G) \leqq N$ is a consequence of $P(G/N) = 1$, and (4) is an immediate consequence of (3).

(5) *The subgroup S of G is the potence P(G) of G if, and only if,* $S = (G, S)$ *and* $P(G/S) = 1$.

**Proof.** The necessity of these conditions may be deduced from (1) and (3). If the conditions are satisfied by $S$, then $S \leqq P(G)$ is a consequence of the definition of $P(G)$ and of the first condition, and $P(G) \leqq S$ is a consequence of (4) and of the second condition.

It will be convenient to denote by $P(H, M)$ the uniquely determined subgroup of the group $H$ which contains the normal subgroup $M$ of $H$ and which satisfies $P(H/M) = P(H, M)/M$.

THEOREM 1. *If M is a normal subgroup of the group H, then* ${}_jM \leqq {}_jP(H, M)$ *and* ${}_jP(H, M)/{}_jM = P(H/{}_jM)$ *for every j.*

**Proof.** From $M \leqq P(H, M)$ one deduces by complete induction that ${}_jM \leqq {}_jP(H, M)$. Denote by $Q$ the uniquely determined subgroup of $H$ which contains ${}_1M$ and which satisfies $Q/{}_1M = P(H/{}_1M)$. Then we deduce from (1) that $Q/{}_1M = (H/{}_1M, Q/{}_1M)$ and that therefore $Q = {}_1M(H, Q)$. Hence $QM = M(H, Q) = M(H, MQ)$, since $(H, M) \leqq M$, and consequently $QM/M = [M(H, MQ)]/M = (H/M, MQ/M)$. We infer now from the definition of the potence of $H/M$ that $QM/M \leqq P(H/M)$ and that therefore $Q \leqq P(H, M)$. But this implies $(H, Q) \leqq {}_1P(H, M)$, and thus we obtain $Q = {}_1M(H, Q) \leqq {}_1M_1P(H, M) = {}_1P(H, M)$. From (1) we deduce, as before, that $P(H, M) = M(H, P(H, M))$; and it follows from the formulas (2), (3) of I.1 that ${}_1P(H, M) = {}_1M_2P(H, M)$. Hence ${}_1P(H, M)/{}_1M = (H/{}_1M, {}_1P(H, M)/{}_1M)$, and we infer from the definition of the potence of a group that ${}_1P(H, M)/{}_1M \leqq P(H/{}_1M)$ or ${}_1P(H, M) \leqq Q$. Thus we have shown that $Q = {}_1P(H, M)$ or

$${}_1P(H, M)_1M = P(H/{}_1M).$$

Suppose now that we have already verified ${}_iP(H, M)/{}_iM = P(H/{}_iM)$ for some $i$ (note that this formula is certainly true for $i = 0$ and 1). Then it follows from the formula just verified that

$$\begin{aligned} {}_{i+1}P(H, M)/{}_{i+1}M &= {}_1[{}_iP(H, M)]/{}_1[{}_iM] = {}_1P(H, {}_iM)/{}_1[{}_iM] \\ &= P(H/{}_1[{}_iM]) = P(H/{}_{i+1}M); \end{aligned}$$

and thus the validity of the theorem is verified by induction.

Applying an argument which we used twice in the course of this proof one deduces the following fact from Theorem 1.

COROLLARY 1. *If M is a normal subgroup of the group H, then*

$${}_jP(H, M) = {}_jM\,{}_{j+1}P(H, M) \qquad \textit{for every } j.$$

From Theorem 1 and Corollary 1 one deduces readily that

$$\begin{aligned} P(H/{}_jM) &= {}_jP(H, M)/{}_jM = {}_jM\,{}_{j+1}P(H, M)/{}_jM \\ &\sim {}_{j+1}P(H, M)/({}_{j+1}P(H, M) \cap {}_jM) \\ &\sim P(H/{}_{j+1}M)/[P(H/{}_{j+1}M) \cap {}_jM/{}_{j+1}M]. \end{aligned}$$

In order to show that all the groups $P(H/{}_jM)$ are isomorphic we would have to know that ${}_{j+1}M = {}_jM \cap {}_{j+1}P(H, M)$ for every $j$. Whether or not this formula is always true, the author has not been able to decide. The following special case will, therefore, be of interest.

COROLLARY 2. *The following properties of the normal subgroup $M$ of the group $H$ imply each other.*

(i) $P(H/{}_jM)=1$ *for every* $j$.

(ii) *There exists at least one* $j$ *such that* $P(H/{}_jM)=1$.

(iii) *There exists at least one* $j$ *such that* ${}_jP(H, M) \leqq M$.

**Proof.** It is obvious that (i) implies (ii), and that (ii) implies (iii) is an immediate consequence of $1=P(H/{}_jM)={}_jP(H, M)/{}_jM$ which we infer from Theorem 1. Suppose finally that ${}_jP(H, M) \leqq M$. Then it follows from Corollary 1 that ${}_{j-1}P(H, M)={}_{j-1}M\,{}_jP(H, M) \leqq M$; and applying induction we see that $P(H, M)={}_0P(H, M) \leqq M$. But this inequality implies $P(H, M)=M$. Hence it follows from Theorem 1 that $P(H/{}_jM)={}_jP(H, M)/{}_jM={}_jM/{}_jM=1$, showing that (i) is a consequence of (iii).

THEOREM 2. *If the homomorphism $\eta$ of the group $H$ into the group $K$ relates the representations $H/M$ and $K/N$ of the group $G$, then $\eta$ induces, for every $j$, an isomorphism of $P(H/{}_jM)$ upon $P(K/{}_jN)$.*

**Proof.** There exists a homomorphism $\kappa$ of $K$ into $H$ such that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/M$ and $K/N$. Since the potence of a group is a fully invariant subgroup, it follows that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $P(H/M)$ and $P(K/N)$. This implies in particular $P(H, M)^\eta \leqq P(K, N)$ and $P(K, N)^\kappa \leqq P(H, M)$. From this fact we deduce ${}_jP(H, M)^\eta \leqq {}_jP(K, N)$ and ${}_jP(K, N)^\kappa \leqq {}_jP(H, M)$. We deduce from Theorem 1 of I.4 that $\eta$ and $\kappa$ induce reciprocal isomorphisms between ${}^jH/{}_jM$ and ${}^jK/{}_jN$, and hence it follows from the lemma of I.4 that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $P(H/{}_jM)={}_jP(H, M)/{}_jM$ and ${}_jP(K/N)/{}_jN=P(K/{}_jN)$ where the equalities involved are consequences of Theorem 1.

THEOREM 3. *Suppose that the endomorphism $\epsilon$ of the group $G$ maps the normal subgroup $N$ of $G$ into itself. Then:*

(a) *$\epsilon$ induces the identity in $P(G/{}_{i-1}N)$, provided it induces the identity in $P(G/{}_iN)$.*

(b) *$\epsilon$ induces the identity in $P(G/{}_iN)$, provided it induces the identity in $P(G/{}_{i-1}N)$ and provided:*

(b′) ${}_iN={}_{i-1}N\cap{}_iP(G, N)$.

**Proof.** From $N^\epsilon \leqq N$ we deduce inductively ${}_jN^\epsilon \leqq {}_jN$, and we note that as a consequence of Theorem 1 and Corollary 1 we have:

$$P(G/{}_jN) = {}_jP(G, N)/{}_jN \quad\text{and}\quad {}_jP(G, N) = {}_jN\,{}_{j+1}P(G, N).$$

Assume now that $\epsilon$ induces the identity in $P(G/{}_iN)$ and that $x$ is an element in ${}_{i-1}P(G, N)$. Then $x=x'x''$ for $x'$ in ${}_{i-1}N$ and $x''$ in ${}_iP(G, N)$. Consequently $x'^\epsilon\equiv 1$ modulo ${}_{i-1}N$ and $x''^\epsilon\equiv x''$ modulo ${}_iN$. But ${}_iN\leqq{}_{i-1}N$, and hence we find $x^\epsilon\equiv x'^\epsilon x''^\epsilon\equiv x''^\epsilon\equiv x''\equiv x'x''\equiv x$ modulo ${}_{i-1}N$, proving (a).

Assume next that $\epsilon$ induces the identity in $P(G/{}_{i-1}N)$, and that (b′) is satisfied. If $x$ is an element in ${}_iP(G, N)$, then $x$ belongs to ${}_{i-1}P(G, N)$ so that $x^\epsilon\equiv x$ modulo ${}_{i-1}N$. Consequently $x^\epsilon x^{-1}$ belongs both to ${}_{i-1}N$ and to ${}_iP(G, N)$, since ${}_{i-1}P(G, N)^\epsilon\leqq{}_{i-1}P(G, N)$ implies ${}_iP(G, N)^\epsilon\leqq{}_iP(G, N)$. Hence it follows from (b′) that $x^\epsilon x^{-1}$ belongs to ${}_iN$ so that $x^\epsilon\equiv x$ modulo ${}_iN$, and $\epsilon$ has been shown to induce the identity in $P(G/{}_iN)$.

COROLLARY 3. *Suppose that the homomorphism $\eta$ of the group $H$ into the group $K$ maps the normal subgroup $M$ of $H$ into part of the normal subgroup $N$ of $K$, that the homomorphism $\kappa$ of $K$ into $H$ maps $N$ into part of $M$, and that $\eta$ and $\kappa$ induce reciprocal isomorphisms between $P(H/{}_iM)$ and $P(K/{}_iN)$. Then:*

(a) *$\eta$ and $\kappa$ induce reciprocal isomorphisms between $P(H/{}_{i-1}M)$ and $P(K/{}_{i-1}N)$, $0<i$.*

(b) *$\eta$ and $\kappa$ induce reciprocal isomorphisms between $P(H/{}_{i+1}M)$ and $P(K/{}_{i+1}N)$, provided ${}_{i+1}M={}_iM\cap{}_{i+1}P(H, M)$ and ${}_{i+1}N={}_iN\cap{}_{i+1}P(K, N)$.*

**Proof.** From $M^\eta\leqq N$ we deduce inductively that ${}_iM^\eta\leqq{}_iN$, and likewise we see that ${}_iN^\kappa\leqq{}_iM$ for every $i$. From Theorem 1 and Corollary 1 we deduce that

$$P(H/{}_jM) = {}_jP(H, M)/{}_jM,\quad {}_jP(H, M) = {}_jM\,{}_{j+1}P(H, M)$$

and

$$P(K/{}_jN) = {}_jP(K, N)/{}_jN,\quad {}_jP(K, N) = {}_jN\,{}_{j+1}P(K, N).$$

Consequently we have ${}_iP(H, M)^\eta\leqq{}_iP(K, N)$ and ${}_iP(K, N)^\kappa\leqq{}_iP(H, M)$, since $\eta$ and $\kappa$ induce reciprocal isomorphisms between $P(H/{}_iM)$ and $P(K/{}_iN)$. From these inequalities, the inequalities ${}_jM^\eta\leqq{}_jN$, ${}_jN^\kappa\leqq{}_jM$ and the above equations, we deduce now readily that ${}_jP(H, M)^\eta\leqq{}_jP(K, N)$ and ${}_jP(K, N)^\kappa \leqq{}_jP(H, N)$ for every $j$, proving that $\eta$ induces a homomorphism of $P(H/{}_jM)$ into $P(K/{}_jN)$, and that $\kappa$ induces a homomorphism of $P(K/{}_jN)$ into $P(H/{}_jM)$ for every $j$.

But $\eta\kappa$ induces the identity in $P(H/{}_iM)$ and $\kappa\eta$ induces the identity in $P(K/{}_iN)$. Hence we deduce from our hypotheses and Theorem 3 that $\eta\kappa$ induces the identity in $P(H/{}_{i-1}M)$ and in $P(H/{}_{i+1}M)$, and that likewise $\kappa\eta$ induces the identity in $P(K/{}_{i-1}N)$ and in $P(K/{}_{i+1}N)$. Thus $\eta$ and $\kappa$ induce homomorphisms between $P(H/{}_jM)$ and $P(K/{}_jN)$, for $j=i-1$ and $i+1$, whose products are the identity, and consequently they induce reciprocal isomorphisms between these groups.

THEOREM 4. *Suppose that the homomorphism* $\eta$ *of the group* $H$ *into the group* $K$ *maps the normal subgroup* $M$ *of* $H$ *into part of the normal subgroup* $N$ *of* $K$, *and that* $\eta$ *induces an isomorphism of* $P(H/{}_iM)$ *into* $P(K/{}_iN)$. *Then*:

(a) $\eta$ *induces an isomorphism of* $P(H/{}_{i-1}M)$ *into* $P(K/{}_{i-1}N)$, *provided* ${}_iN={}_iP(K, N)\cap{}_{i-1}N$, $0<i$.

(b) $\eta$ *induces an isomorphisms of* $P(H/{}_{i+1}M)$ *into* $P(K/{}_{i+1}N)$, *provided* ${}_{i+1}M={}_{i+1}P(H, M)\cap{}_iM$.

**Proof.** We deduce, as before, from Theorem 1 and Corollary 1 that $\eta$ induces a homomorphism of $P(H/{}_jM)$ into $P(K/{}_jN)$ for every $j$, since it does so for $j=i$. We note furthermore that ${}_jP(H, M)={}_jM{}_{j+1}P(H, M)$, and so on, are consequences of Corollary 1.

Assume now that the element $x$ in ${}_{i-1}P(H, M)$ is mapped by $\eta$ upon an element in ${}_{i-1}N$. Then $x=x'x''$ for $x'$ in ${}_iP(H, M)$ and $x''$ in ${}_{i-1}M$, and hence $x'^\eta$ belongs to ${}_iP(K, N)$, $x''^\eta$ belongs to ${}_{i-1}N$. But $x^\eta=x'^\eta x''^\eta$ belongs to ${}_{i-1}N$ too, showing that $x'^\eta$ belongs to the cross cut ${}_iN$ of ${}_iP(K, H)$ and ${}_{i-1}N$. Since $\eta$ induces an isomorphism of $P(H/{}_iM)$ into $P(K/{}_iN)$, this implies that $x'$ belongs to ${}_iM$ and $x$ belongs therefore to ${}_{i-1}M$. Hence $\eta$ induces an isomorphism of $P(H/{}_{i-1}M)$ into $P(K/{}_{i-1}N)$.

Assume next that the element $y$ in ${}_{i+1}P(H, M)$ is mapped by $\eta$ upon an element in ${}_{i+1}N$. Noting that ${}_{i+1}S\leqq{}_iS$ for any $S$, and that $\eta$ induces an isomorphism of $P(H/{}_iM)$ into $P(K/{}_iN)$, it follows that $y$ belongs to the cross cut ${}_{i+1}M$ of ${}_iM$ and ${}_{i+1}P(H, M)$, proving that $\eta$ induces an isomorphism of $P(H/{}_{i+1}M)$ into $P(K/{}_{i+1}N)$.

4. **Interdependence of invariants.** In the first part of this investigation we have derived from every representation $H/M$ of a group $G$ a vast system of invariants of the class of representations to which $H/M$ belongs. It is the object of this section to show that under certain circumstances some of these invariants determine the others.

THEOREM 1. *If* $M$ *and* $N$ *are normal subgroups of the groups* $H$ *and* $K$ *respectively, and if the homomorphism* $\eta$ *of* $H$ *into* $K$ *induces a homomorphism of* $H/M$ *upon* $K/N$, *a homomorphism of* $H/{}^1H$ *upon* $K/{}^1K$ *and, for some positive* $i$, *a homomorphism of* ${}_{i-1}M/{}_iM$ *upon* ${}_{i-1}N/{}_iN$, *then* $\eta$ *induces, for every* $j\geqq i$, *a homomorphism of* ${}_{j-1}M/{}_jM$ *upon* ${}_{j-1}N/{}_jN$.

**Proof.** From our hypotheses we infer $M^\eta\leqq N$, $K=H^\eta N$, $K={}^1KH^\eta$ and ${}_iM^\eta\leqq{}_iN$, ${}_{i-1}N={}_{i-1}M^\eta{}_iN$. Clearly ${}_{i+1}M^\eta\leqq{}_{i+1}N$ so that $\eta$ induces a homomorphism of ${}_iM/{}_{i+1}M$ into ${}_iN/{}_{i+1}N$. Consequently we may deduce from formulas (2) and (3) of I.1 and the inequality (2) of I.3 that

$$\begin{aligned} {}_iM^\eta{}_{i+1}N \leqq {}_iN &= (K, {}_{i-1}N) = (K, {}_{i-1}M^\eta{}_iN) = (K, {}_{i-1}M^\eta)(K, {}_iN) \\ &= (H^\eta N, {}_{i-1}M^\eta)_{i+1}N = (H^\eta, {}_{i-1}M^\eta)(N, {}_{i-1}M^\eta)_{i+1}N \\ &\leqq (H, {}_{i-1}M)^\eta({}^1KH^\eta, {}_{i-1}M^\eta)_{i+1}N \\ &\leqq {}_iM^\eta({}^1K, {}_{i-1}N)(H^\eta, {}_{i-1}M^\eta)_{i+1}N \leqq {}_iM^\eta{}_{i+1}N; \end{aligned}$$

and we have therefore ${}_iN = {}_iM^\eta{}_{i+1}N$, proving that $\eta$ induces a homomorphism of ${}_iM/{}_{i+1}M$ *upon* ${}_iN/{}_{i+1}N$. But from this fact our contention is easily deduced by complete induction.

COROLLARY 1. *If the homomorphism $\eta$ of the group $H$ into the group $K$ induces a homomorphism of $H/(H, H)$ upon $K/(K, K)$, then $\eta$ induces, for every $i$, a homomorphism of $H/{}^iH$ upon $K/{}^iK$ and of ${}^iH/{}^{i+1}H$ upon ${}^iK/{}^{i+1}K$.*

**Proof.** The second contention is an immediate consequence of Theorem 1 if we let $M=H$ and $N=K$. If the first contention has been verified for some $i$, then $K={}^iKH^\eta$. But we may infer from the second contention that ${}^iK = {}^{i+1}K{}^iH^\eta$. Hence $K={}^{i+1}K{}^iH^\eta H^\eta = {}^{i+1}KH^\eta$, and the validity of our contention may be verified by complete induction.

THEOREM 2. *If $M$ and $N$ are normal subgroups of the groups $H$ and $K$ respectively, and if the homomorphism $\eta$ of $H$ into $K$ induces a homomorphism of $H/M$ upon $K/N$, then $\eta$ induces, for every $i$, a homomorphism of ${}^iH/{}_iM$ upon ${}^iK/{}_iN$.*

**Proof.** Our assertion being true for $i=0$, we may make the induction hypothesis that $\eta$ induces a homomorphism of ${}^{i-1}H/{}_{i-1}M$ upon ${}^{i-1}K/{}_{i-1}N$. Consequently $\eta$ meets the following requirements:

$$M^\eta \leqq N, \; K = H^\eta N \text{ and } {}_{i-1}M^\eta \leqq {}_{i-1}N, \; {}^{i-1}K = {}^{i-1}H^\eta{}_{i-1}N.$$

Furthermore we find ${}_iM^\eta \leqq {}_iN$ and ${}^iH^\eta \leqq {}^iK$. Consequently we may deduce from formulas (2) and (3) of I.1 and from the inequality (2) of I.3 that

$$\begin{aligned} {}^iH^\eta{}_iN \leqq {}^iK &= (K, {}^{i-1}K) = (K, {}^{i-1}H^\eta{}_{i-1}N) = (K, {}^{i-1}H^\eta)(K, {}_{i-1}N) \\ &= (H^\eta N, {}^{i-1}H^\eta)_iN = (H^\eta, {}^{i-1}H^\eta)(N, {}^{i-1}H^\eta)_iN \\ &\leqq (H, {}^{i-1}H)^\eta(N, {}^{i-1}K)_iN \leqq {}^iH^\eta{}_iN, \end{aligned}$$

and we have therefore obtained the proof of ${}^iK = {}^iH^\eta{}_iN$, completing the proof of the theorem.

COROLLARY 2. *If $M$ and $N$ are normal subgroups of the groups $H$ and $K$ respectively, and if the homomorphism $\eta$ of $H$ into $K$ induces a homomorphism of $H/M$ upon $K/N$ and a homomorphism of $H/(H, H)$ upon $K/(K, K)$, then $\eta$ induces, for every $i$, a homomorphism of $H/{}_iM$ upon $K/{}_iN$.*

**Proof.** It is a consequence of Corollary 1 that $\eta$ induces a homomorphism of $H/{}^iH$ upon $K/{}^iK$, and we infer from Theorem 2 that $\eta$ induces a homomorphism of ${}^iH/{}_iM$ upon ${}^iK/{}_iN$. Consequently ${}_iM^\eta \leqq {}_iN$, ${}^iK = {}^iH^\eta{}_iN$, $K = H^\eta{}^iK = H^\eta{}^iH^\eta{}_iN = H^\eta{}_iN$, proving our contention.

*Remark* 1. If $H$ is the direct product of two abelian groups $A$ and $M$ neither of which is equal to 1, and if $\eta$ is the endomorphism of $H$ which maps $M$ upon 1 and which leaves invariant every element in $A$, then $\eta$ induces the identity automorphism in $H/M$, though it maps $M = M/(M, H)$ upon $1 < M$, and $H$ upon $A < M$. Thus it is impossible to omit in Corollary 2 the hypothesis that $\eta$ induces a homomorphism of $H/{}^1H$ upon $K/{}^1K$.

*Remark* 2. If $H$ is the additive group of all the rational numbers, if $M$ is the subgroup of the integers, and if the endomorphism $\eta$ of $H$ maps the rational number $x$ upon $2x$, then $\eta$ is a homomorphism of $H$ upon $H$, it induces a homomorphism of $H/M$ upon $H/M$, but it maps $M = M/(H, M)$ upon a proper part of itself. This shows that the last hypothesis in Theorem 1 cannot be omitted.

THEOREM 3. *If $N$ is a normal subgroup of the group $G$, and if the endomorphism $\epsilon$ of $G$ induces the identity in $G/(G, G)$ and in ${}_iN/{}_{i+1}N$, then:*

(a) *$\epsilon$ induces the identity in ${}_{i+1}N/{}_{i+2}N$.*

(b) *$\epsilon$ induces the identity in ${}_{i-1}N/{}_iN$, provided ${}_{i-1}N^\epsilon \leqq {}_{i-1}N$ and*[19] *${}_iN = {}_{i+1}N \div G$.*

**Proof.** If $x$ is in $G$ and $y$ in ${}_iN$, then $x^\epsilon = xx'$ for $x'$ in $(G, G)$ and $y^\epsilon = yy'$ for $y'$ in ${}_{i+1}N$, since $\epsilon$ induces the identity in $G/(G, G)$ and in ${}_iN/{}_{i+1}N$. Since $(x', y)$ belongs to $({}^1G, {}_iN) \leqq {}_{i+2}N$ by (2) of I.3, since $(xx', y')$ belongs to $(G, {}_{i+1}N) = {}_{i+2}N$, we deduce from formulas (2) and (3) of I.1 that

$$(x, y)^\epsilon = (x^\epsilon, y^\epsilon) = (xx', yy') \equiv (x, y) \text{ modulo } {}_{i+2}N.$$

But ${}_{i+1}N$ is generated by the elements $(x, y)$ for $x$ in $G$ and $y$ in ${}_iN$, and thus we have shown that the identity is induced by $\epsilon$ in ${}_{i+1}N/{}_{i+2}N$.

Assume now that ${}_{i-1}N^\epsilon \leqq {}_{i-1}N$ and ${}_iN = {}_{i+1}N \div G$. If $x$ is an element in $G$ and $z$ an element in ${}_{i-1}N$, then $x^\epsilon = xx'$ for $x'$ in $(G, G)$, since $\epsilon$ induces the identity in $G/(G, G)$, and $z^\epsilon = zz'$ where the uniquely determined element $z'$ belongs to ${}_{i-1}N$, since $\epsilon$ maps the element $z$ in ${}_{i-1}N$ upon an element in ${}_{i-1}N^\epsilon \leqq {}_{i-1}N$. Since $(x, z)$ belongs to $(G, {}_{i-1}N) = {}_iN$, and since $\epsilon$ induces the identity in ${}_iN/{}_{i+1}N$, we have $(x, z) \equiv (x, z)^\epsilon$ modulo ${}_{i+1}N$. Since $(x', zz')$ belongs to $({}^1G, {}_{i-1}N) \leqq {}_{i+1}N$ by (2) of I.3, we may deduce from formulas (2) and (3) of I.1 that

$$(x, z) \equiv (x, z)^\epsilon \equiv (x^\epsilon, z^\epsilon) \equiv (xx', zz') \equiv (x, z)(x, z') \text{ modulo } {}_{i+1}N.$$

From this congruence we infer that $(x, z')$ is an element in ${}_{i+1}N$ for every $x$ in $G$ so that $(G, z') \leqq {}_{i+1}N$. Thus $z'$ has been shown to be an element in ${}_{i+1}N \div G = {}_iN$. Hence $z^\epsilon = zz' \equiv z$ modulo ${}_iN$ for every $z$ in ${}_{i-1}N$, proving that $\epsilon$ induces the identity in ${}_{i-1}N/{}_iN$.

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[19] The concept of commutator quotient has been defined in I.1.

COROLLARY 3. *If $M$ and $N$ are normal subgroups of the groups $H$ and $K$ respectively, if the homomorphism $\eta$ of $H$ into $K$ satisfies $M^\eta \leqq N$ and the homomorphism $\kappa$ of $K$ into $H$ satisfies $N^\kappa \leqq M$, and if $\eta$ and $\kappa$ induce reciprocal isomorphisms between $H/(H, H)$ and $K/(K, K)$ and between ${}_iM/{}_{i+1}M$ and ${}_iN/{}_{i+1}N$, then:*

(a) *$\eta$ and $\kappa$ induce reciprocal isomorphisms between ${}_{i+1}M/{}_{i+2}M$ and ${}_{i+1}N/{}_{i+2}N$.*

(b) *$\eta$ and $\kappa$ induce reciprocal isomorphisms between ${}_{i-1}M/{}_iM$ and ${}_{i-1}N/{}_iN$, provided ${}_iM = {}_{i+1}M \div H$ and ${}_iN = {}_{i+1}N \div K$.*

**Proof.** It is readily seen that $\eta$ induces, for every $j$, a homomorphism of ${}_{j-1}M/{}_jM$ into ${}_{j-1}N/{}_jN$, and that $\kappa$ induces a homomorphism of ${}_{j-1}N/{}_jN$ into ${}_{j-1}M/{}_jM$. Furthermore $\eta\kappa$ is an endomorphism of $H$ which induces the identity in $H/(H, H)$ and in ${}_iM/{}_{i+1}M$. Hence it follows from Theorem 3 that $\eta\kappa$ induces the identity in ${}_{i-1}M/{}_iM$ and in ${}_{i+1}M/{}_{i+2}M$, and likewise we see that $\kappa\eta$ is an endomorphism of $K$ which induces the identity in ${}_{j-1}N/{}_jN$ for $j = i, i+1, i+2$. Our contention is an immediate consequence of these facts.

If two true representations of a group are related, then they are similar. Two groups are therefore certainly similar if they are related and if each of them is self-similar. Of this last fact we need a generalization.

LEMMA 1. *If $H$ and $K$ are related groups, if $K$ is a self-similar group, then the representations $H/M$ and $K/N$ of the group $G$ are related by every homomorphism $\eta$ of $H$ into $G$ which induces an isomorphism of $H/M$ upon $K/N$.*

**Proof.** $H$ and $K$ are related groups, and $H/M \sim K/N$. Consequently there exists a homomorphism $\gamma$ of $H$ into $K$ and a homomorphism $\kappa$ of $K$ into $H$ which induce reciprocal isomorphisms between $H/M$ and $K/N$. Denote by $\alpha$ the isomorphism of $H/M$ upon $K/N$ which is induced by $\eta$, and denote by $\beta$ the isomorphism of $K/N$ upon $H/M$ which is induced by $\kappa$ so that $\beta^{-1}$ is induced by $\gamma$. Then $\beta\alpha$ and $(\beta\alpha)^{-1}$ are automorphisms of $K/N$. But $K$ is self-similar so that $K/N$ is, by Theorem 1 of II.4, a true representation of $G$. Consequently there exists an endomorphism $\epsilon$ of $K$ which induces the automorphism $(\beta\alpha)^{-1}$ of $K/N$. Clearly $\epsilon\kappa$ is a homomorphism of $K$ into $H$ which induces the isomorphism $(\beta\alpha)^{-1}\beta = \alpha^{-1}$ of $K/N$ upon $H/M$. Thus reciprocal isomorphisms between $H/M$ and $K/N$ are induced by the homomorphisms $\eta$ and $\epsilon\kappa$, and the representations $H/M$ and $K/N$ of $G$ are related by $\eta$, as we intended to show.

Throughout the remainder of this section we shall make use of the following notations and hypotheses without actually restating them.

(H) *$M$ and $N$ are normal subgroups of the related groups $H$ and $K$ respectively, $K$ is self-similar and $M \leqq (H, H)$, and the homomorphism $\eta$ of $H$ into $K$ induces a homomorphism of $H/M$ upon $K/N$ whose kernel is $W/M$.*

We note that $\eta$ induces an isomorphism of $H/W$ upon $K/N$, and thus it follows from Lemma 1 that $\eta$ relates the representations $H/W$ and $K/N$ (of the same group $G$).

LEMMA 2. *$W \leqq (H, H)$ if, and only if, $\eta$ induces an isomorphism of $H/(H, H)$ upon $K/N(K, K)$.*

**Proof.** The sufficiency of the condition is a consequence of $W^\eta \leqq N$. If conversely $W \leqq (H, H)$, and if $x$ is an element in $H$ such that $x^\eta$ belongs to $N(K, K)$, then there exists an element $y$ in $(H, H)$ such that $x^\eta \equiv y^\eta$ modulo $N$, since a homomorphism of $H/M$ upon $K/N$ maps the commutator subgroup of $H/M$ upon the commutator subgroup of $K/N$. Hence $xy^{-1}$ is an element in $W \leqq (H, H)$ so that $x$ itself belongs to $(H, H)$, proving that $\eta$ induces an isomorphism of $H/(H, H)$ into, and therefore upon, $K/N(K, K)$.

Elsewhere[20] we have proved the following two facts which we shall need during the proof of Theorem 4 and which we restate here for convenient reference.

(A) If $J$ and $U$ are subgroups of the group $B$, and if $L$ and $V$ are normal subgroups of $J$ and $U$ respectively, then the following properties imply each other.

(i) *$U = JV$ and $L \leqq J \cap V$.*

(ii) *The identity automorphism of $B$ induces a homomorphism of $J/L$ upon $U/V$.*

(iii) *$JV \leqq U$, $L \leqq J \cap V$, and there exists a homomorphism $\eta$ of $B$ into a group $D$ and a normal subgroup $S$ of a subgroup $R$ of $D$ such that $\eta$ induces an isomorphism of $U/V$ upon $R/S$ and a homomorphism of $J/L$ upon $R/S$.*

(B) If $J$ and $U$ are subgroups of the group $B$, and if $L$ and $V$ are normal subgroups of $J$ and $U$ respectively, then the following properties imply each other.

(i) *$U = JV$ and $L = J \cap V$.*

(ii) *The identity automorphism of $B$ induces an isomorphism of $J/L$ upon $U/V$.*

(iii) *$JV \leqq U$, $L \leqq J \cap V$, and there exists a homomorphism $\eta$ of $B$ into a group $D$ and a normal subgroup $S$ of a subgroup $R$ of $D$ such that $\eta$ induces an isomorphism of $U/V$ upon $R/S$ and an isomorphism of $J/L$ upon $R/S$.*

THEOREM 4. *Suppose that $\eta$ induces an isomorphism of $H/(H, H)$ upon $K/N(K, K)$. Then:*

(a) *$\eta$ induces, for every positive $i$, an isomorphism of ${}_{i-1}W/{}_iW$ upon $({}_{i-1}N \cap {}^iK)/{}_iN$.*

(b) *$\eta$ induces, for every $i$, a homomorphism of $H/{}_iM$ into $K/{}_iN$ and a homomorphism of ${}^iH/{}_iM$ upon ${}^iK/{}_iN$, and the kernel of both these induced homomorphisms is ${}_iW/{}_iM$.*

(c) *$\eta$ induces a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N \cap {}^iK)/{}_iN$ if, and*

[20] Baer [3].

*only if*, ${}_{i-1}W={}_{i-1}M_iW$.

(d) *If* $\eta$ *induces a homomorphism of* ${}_{i-1}M/{}_iM$ *upon* $({}_{i-1}N\cap {}^iK)/{}_iN$, *then* ${}_{i-1}W\leqq {}_{i-1}P(H, M)$, *and* $\eta$ *induces a homomorphism of* ${}_iM/{}_{i+1}M$ *upon* $({}_iN\cap {}^{i+1}K)/{}_{i+1}N$.

(e) $\eta$ *induces an isomorphism of* ${}_{i-1}M/{}_iM$ *upon* $({}_{i-1}N\cap {}^iK)/{}_iN$ *if, and only if, an isomorphism of* ${}_iW/{}_iM$ *upon* ${}_{i-1}W/{}_{i-1}M$ *is induced by the identity automorphism of* $H$.

**Proof.** It is a consequence of Lemma 2 that $W\leqq (H, H)$. Hence it follows from Theorem 3 of I.3 that the lower central series of $H$ relative to $M$ and the lower central series of $H$ relative to $W$ are both interlocking.

It is a consequence of Lemma 1 that $\eta$ relates the representations $H/W$ and $K/N$ of the same group $G$. Thus it follows from Theorem 2 of I.4 and (3) of I.4 that $\eta$ induces, for every $i$, an isomorphism of ${}^iH/{}_iW$ upon ${}^iK/{}_iN$ and an isomorphism of ${}_iW/{}_{i+1}W$ upon $({}_iN\cap {}^{i+1}K)/{}_{i+1}N$. This shows the validity of (a).

Clearly ${}_iM\leqq {}_iW$ and ${}_iM^\eta\leqq {}_iW^\eta\leqq {}_iN$. Thus $\eta$ induces both a homomorphism of $H/{}_iM$ into $K/{}_iN$ and a homomorphism of ${}^iH/{}_iM$ into ${}^iK/{}_iN$. It is a consequence of a result obtained in the preceding paragraph that $\eta$ induces a homomorphism of ${}^iH/{}_iM$ *upon* ${}^iK/{}_iN$. Clearly the kernels of these two homomorphisms which are induced by $\eta$ contain ${}_iW/{}_iM$.

It is a consequence of Lemma 1 that $\eta$ relates the representations $H/(H, H)$ and $K/N(K, K)$ of the same abelian group $G/(G, G)$. Hence it follows from Theorem 2 of I.4 that $\eta$ induces an isomorphism of ${}^iH/{}^{i+1}H$ upon ${}^iK/{}_i[N(K, K)]={}^iK/[{}_iN{}^{i+1}K]$. It is furthermore obvious that $\eta$ induces a homomorphism of ${}^iH/{}^{i+1}H$ into ${}^iK/{}^{i+1}K$, and that this homomorphism is an isomorphism is immediately deduced. But now it follows by complete induction that $\eta$ induces an isomorphism of $H/{}^iH$ into $K/{}^iK$. It has been shown in the second paragraph of this proof that $\eta$ induces an isomorphism of ${}^iH/{}_iW$ upon ${}^iK/{}_iN$. Thus we have shown that $\eta$ induces an isomorphism of $H/{}_iW$ into $K/{}_iN$. Combining this result with those obtained in the preceding paragraph of this proof, we see that ${}_iW/{}_iM$ is the kernel of the homomorphisms of $H/{}_iM$ into $K/{}_iN$ and of ${}^iH/{}_iM$ upon ${}^iK/{}_iN$ which are induced by $\eta$, completing the proof of (b).

If $\eta$ induces a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N\cap {}^iK)/{}_iN$, then we deduce from (a) and Theorem (A) that ${}_{i-1}W={}_{i-1}M_iW$. If conversely ${}_{i-1}W={}_{i-1}M_iW$, then it follows from Theorem (A) that a homomorphism of ${}_{i-1}M/{}_iM$ upon ${}_{i-1}W/{}_iW$ is induced by the identity automorphism of $H$; and we infer from (a) that $\eta=1\eta$ induces a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N\cap {}^iK)/{}_iN$, proving (c).

Assume now that $\eta$ induces a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N\cap {}^iK)/{}_iN$. Then we infer ${}_{i-1}W={}_{i-1}M_iW$ from (c). Hence ${}_{i-1}W/{}_{i-1}M\leqq P(H/{}_{i-1}M)$. But $P(H/{}_{i-1}M)={}_{i-1}P(H, M)/{}_{i-1}M$ by Theorem 1 of §3, proving ${}_{i-1}W \leqq {}_{i-1}P(H, M)$. From ${}_{i-1}W = {}_{i-1}M_iW$ and the formulas (2) and (3) of I.1 we deduce ${}_iW = {}_iM_{i+1}W$, and it follows from (c) that $\eta$ induces a homomorphism of ${}_iM/{}_{i+1}M$ upon $({}_iN \cap {}^{i+1}K)/{}_{i+1}N$, completing the proof of (d).

It is an immediate consequence of Theorem (B) that an isomorphism of ${}_iW/{}_iM$ upon ${}_{i-1}W/{}_{i-1}M$ is induced by the identity automorphism of $H$ if, and only if, an isomorphism of ${}_{i-1}M/{}_iM$ upon ${}_{i-1}W/{}_iW$ is induced by the identity automorphism of $H$, since all the subgroups under consideration are normal subgroups of $H$. If $\eta$ induces an isomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N \cap {}^iK)/{}_iN$, then we deduce from (a) and Theorem (A) that an isomorphism of ${}_{i-1}M/{}_iM$ upon ${}_{i-1}W/{}_iW$ is induced by the identity automorphism of $H$. If conversely the identity automorphism of $H$ induces an isomorphism of ${}_{i-1}M/{}_iM$ upon ${}_{i-1}W/{}_iW$, then (a) implies that $\eta = 1\eta$ induces an isomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N \cap {}^iK)/{}_iN$, completing the proof of (e).

*Remark* 3. The second assertion of (d) could have been obtained as an application of Theorem 1 and assertion (a). But Theorem 1 cannot be proven as a special case of Theorem 4.

The following theorem shows that the first assertion of (d) is a "best" result.

COROLLARY 4. *If $\eta$ induces an isomorphism of $H/(H, H)$ upon $K/N(K, K)$, and if ${}_{i-1}W = {}_{i-1}P(H, M)$ for some positive $i$, then $\eta$ induces a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N \cap {}^iK)/{}_iN$.*

**Proof.** It is a consequence of Corollary 1 of §3 that

$$ {}_{i-1}W = {}_{i-1}P(H, M) = {}_iP(H, M)_{i-1}M = {}_iW\,{}_{i-1}M, $$

and our contention is therefore a consequence of (c) of Theorem 4.

COROLLARY 5. $W/M = (H/M, W/M)$ *if, and only if, $\eta$ meets the following requirements.*

(i) *$\eta$ induces an isomorphism of $H/(H, H)$ upon $K/N(K, K)$.*

(ii) *$\eta$ induces a homomorphism of $M/{}_1M$ upon $(N \cap {}^1K)/{}_1N$.*

**Proof.** It is a consequence of (c) of Theorem 4 that conditions (i) and (ii) imply $W = M\,{}_1W = M(H, W)$, proving the sufficiency of our conditions. If conversely $W/M = (H/M, W/M)$, then $W = M(H, W) \leqq (H, H)$. Thus Lemma 2 implies that $\eta$ induces an isomorphism of $H/(H, H)$ upon $K/N(K, K)$, and we may deduce from $W = M\,{}_1W$ and (c) of Theorem 4 that $\eta$ induces a homomorphism of $M/{}_1M$ upon $(N \cap {}^1K)/{}_1N$, proving the necessity of our conditions.

*Remark* 4. Using (d) of Theorem 4 one proves by complete induction that the conditions (i) and (ii) imply the following property of $\eta$:

(ii') *$\eta$ induces, for every positive $i$, a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N \cap {}^iK)/{}_iN$.*

*Remark* 5. If in particular $H = K$, $N = P(H, M)$ and $\eta$ induces the natural homomorphism of $H/M$ upon $H/P(H, M)$, then $W=P(H, M)$, and it follows from (1) of §3 that $W/M=(H/M, W/M)$. Hence it follows from Corollary 5 and Remark 4 that $\eta$ induces a homomorphism of ${}_{i-1}M/{}_iM$ upon ${}_{i-1}P(H, M)/{}_iP(H, M)$.

COROLLARY 6. *If $\eta$ induces an isomorphism of $H/(H, H)$ upon $K/N(K, K)$, and if $\eta$ induces, for some $i$, an isomorphism of $H/{}_iM$ into $K/{}_iN$, then:*

(a) *$\eta$ induces, for every $j>i$, an isomorphism of $H/{}_jM$ into $K/{}_jN$ and an isomorphism of ${}_{j-1}M/{}_jM$ upon $({}_{j-1}N\cap{}^jK)/{}_jN$.*

(b) *The following properties of $\eta$ (and $i\neq 0$) imply each other:*

(b′) *$\eta$ induces an isomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N\cap{}^iK)/{}_iN$.*

(b″) *$\eta$ induces a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N\cap{}^iK)/{}_iN$.*

(b‴) *$\eta$ induces an isomorphism of $H/{}_{i-1}M$ into $K/{}_{i-1}N$.*

**Proof.** It is a consequence of (b) of Theorem 4 that $\eta$ induces an isomorphism of $H/{}_jM$ into $K/{}_jN$ if, and only if, ${}_jW={}_jM$. But $\eta$ induces an isomorphism of $H/{}_iM$ into $K/{}_iN$. Hence ${}_iW={}_iM$ and therefore, for $i\leqq j$, ${}_jW={}_jM$. Now (a) is an immediate consequence of (a) and (b) of Theorem 4.

It is obvious that (b′) implies (b″). If (b″) is valid, then we deduce ${}_{i-1}W={}_{i-1}M_iW={}_{i-1}M_iM={}_{i-1}M$ from (c) of Theorem 4 and (b‴) is a consequence of facts already deduced ((b) of Theorem 4!). If finally (b‴) is satisfied by $\eta$, then the validity of (b′) is an immediate consequence of (a).

THEOREM 5. *$\eta$ induces an isomorphism of $H/M$ upon $K/N$ if, and only if:*

(i) *$\eta$ induces an isomorphism of $H/(H, H)$ upon $K/N(K, K)$.*

(ii) *$\eta$ induces a homomorphism of $M/{}_1M$ upon $(N\cap{}^1K)/{}_1N$.*

(iii) *There exists an integer $n\geqq 0$ such that $\eta$ induces an isomorphism of $P(H/{}_nM)$ into $K/{}_nM$.*

**Proof.** The necessity of (i) is obvious. If (i) is satisfied, then we may deduce the necessity of (ii) from (a) and (b) of Theorem 4. The necessity of (iii) is obvious too; select $n=0$.

Assume conversely the validity of conditions (i) to (iii). Then we deduce from (i), (ii) and (d) of Theorem 4 by complete induction that $\eta$ induces, for every positive $i$, a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N\cap{}^iK)/{}_iN$, and we infer therefore ${}_{i-1}W\leqq{}_{i-1}P(H, M)$ from (d) of Theorem 4. It is a consequence of (b) of Theorem 4 that ${}_{i-1}W/{}_{i-1}M$ is the kernel of the homomorphism of $H/{}_{i-1}M$ into $K/{}_{i-1}N$ which is induced by $\eta$. Thus condition (iii) implies $1={}_nW/{}_nM\cap P(H/{}_nM)$. Since $P(H/{}_nM)={}_nP(H, M)/{}_nM$ by Theorem 1 of §3, we find now that ${}_nM={}_nW\cap{}_nP(H, M)={}_rW$. It is therefore a consequence of (b) of Theorem 4 that $\eta$ induces an isomorphism of $H/{}_nM$ into $K/{}_nN$. There exists consequently a smallest integer $w$ such that $\eta$ induces an isomorphism of $H/{}_wM$ into $K/{}_wN$. If $w$ were positive, then we would deduce from Corollary 6 and the fact that $\eta$ induces a homomorphism of ${}_{w-1}M/{}_wM$ upon $({}_{w-1}N\cap{}^wK)/{}_wN$ that $\eta$ induces an isomorphism of $H/{}_{w-1}M$

into $K/_{w-1}N$, a contradiction which proves $w=0$. Thus $\eta$ has been shown to induce an isomorphism of $H/M$ into, and therefore upon, $K/N$, as we desired to prove.

COROLLARY 7. *$\eta$ induces an isomorphism of $H/M$ upon $K/N$ if, and only if*:

(a) $W/M=(H/M, W/M)$.

(b) *There exists an integer $n\geqq 0$ such that $\eta$ induces an isomorphism of $P(H/_nM)$ into $K/_nN$.*

This is an immediate consequence of Theorem 5 and Corollary 5.

*Remark* 6. The indispensability of condition (ii) of Theorem 5 will be discussed in §5. As a matter of fact, §5 will be devoted mainly to a study of this condition.

*Remark* 7. The impossibility of omitting condition (i) of Theorem 5 may be seen from the following example. Let $H=K$ be a free abelian group of infinite rank (a direct product of an infinity of infinite cyclic groups), and let $M=N=1$. Then there exists a homomorphism of $H$ upon itself which is not an isomorphism. The hypotheses (H) are clearly satisfied, and so are conditions (ii) and (iii). But (i) is not satisfied and the homomorphism is no isomorphism.

*Remark* 8. The impossibility of omitting condition (iii) of Theorem 5 may be seen from the following example. Let $H=K$ be a nonabelian simple group, $M=1$, $N=K$ and $\eta=1$. Clearly $H$ is self-similar, since 1 and $H$ are its only normal subgroups. Furthermore $M=1\leqq(H, H)=H$. Finally $_{i-1}M/_iM=1/1=1$ and $_{i-1}N/_iN=K/K=1$. Thus the hypotheses (H) are satisfied as well as conditions (i), (ii) of Theorem 5. But condition (iii) is not satisfied, and $\eta$ does not induce an isomorphism of $H/M$ into $K/N$.

*Remark* 9. Condition (iii) of Theorem 5 is clearly satisfied whenever $P(H/_nM)=1$. It is, however, a consequence of Corollary 2 of §3 that this latter condition is satisfied if, and only if, the potence $P(H/M)$ of $H/M$ is equal to 1.

5. **Multiplicator preserving homomorphisms.** If $G$ is any group, then we denote by $C(G)$ the abelian group generated by elements $f(x, y)$ for $x$ and $y$ in $G$, subject to the relations:

(C) $f(1, y)=f(x, 1)=1$ *and* $f(x, y)f(xy, z)=f(x, yz)f(y, z)$ *for* $x, y, z$ *in* $G$.

In IV.1 we have termed $f(x, y)$ *the free central factor set of* $G$, and $C(G)$ is the abelian group generated by the free central factor set of $G$. There exists one and essentially only one central extension $E(G)$ of $C(G)$ by $G$ which realizes this free central factor set $f(x, y)$, and we have termed $E(G)$ in IV.3 *the free central extension* (*of* $C(G)$) *by* $G$. In Remark 1 of IV.3 we have pointed out that $M(G)={}^1E(G)\cap C(G)$ *is the multiplicator of* $G$.

If $\gamma$ is a homomorphism of the group $G$ upon the group $H$, then a homomorphism $\gamma'$ of $C(G)$ upon $C(H)$ is defined by the rule:

$$f(x, y)^{\gamma'} = f(x^{\gamma}, y^{\gamma}) \qquad \text{for } x, y \text{ in } G. \tag{1}$$

Since the homomorphism $\gamma'$ is clearly uniquely determined by the homomorphism $\gamma$, we may say that $\gamma$ *induces the homomorphism* $\gamma'$ *of* $C(G)$ *upon* $C(H)$.

THEOREM 1. *If the homomorphism* $\gamma$ *of* $G$ *upon* $H$ *induces the homomorphism* $\gamma'$ *of* $C(G)$ *upon* $C(H)$, *then* $M(G)^{\gamma'} \leqq M(H)$.

**Proof.** There exists a homomorphism $\eta$ of $E(G)$ upon $E(H)$ which induces $\gamma'$ in the subgroup $C(G)$ of $E(G)$ and which induces the homomorphism $\gamma$ of $G=E(G)/C(G)$ upon $H=E(H)/C(H)$. Clearly ${}^1E(G)^\eta={}^1E(H)$ and $C(G)^\eta = C(H)$. Thus

$$M(G)^{\gamma'} = ({}^1E(G) \cap C(G))^\eta \leqq {}^1E(G)^\eta \cap C(G)^\eta = M(H).$$

We shall show later the impossibility of substituting equality for the inequality in Theorem 1. Thus we are led to the following *definition*:

The homomorphism $\gamma$ of $G$ *upon* $H$ is *multiplicator preserving*[21] if

$$M(G)^{\gamma'} = M(H) \tag{2}$$

where $\gamma'$ is the homomorphism of $C(G)$ upon $C(H)$ which is induced by $\gamma$.

It is a consequence of Theorem 1 that every homomorphism of $G$ upon $H$ induces a homomorphism of the multiplicator $M(G)$ of $G$ *into* the multiplicator $M(H)$ of $H$. If this latter homomorphism happens to be a homomorphism *upon*, then (and only then) the homomorphism is multiplicator preserving.

THEOREM 2. *The homomorphism* $\gamma$ *of the group* $G$ *upon the group* $H$ *is multiplicator preserving if it induces an isomorphism in* $(G, G)$.

**Proof.** If $\gamma'$ is the homomorphism of $C(G)$ upon $C(H)$ which is induced by $\gamma$, then there exists clearly a homomorphism $\eta$ of $E(G)$ upon $E(H)$ which coincides with $\gamma'$ in $C(G)$ and which induces the homomorphism $\gamma$ of $G=E(G)/C(G)$ upon $H=E(H)/C(H)$. From $E(G)^\eta=E(H)$ we deduce ${}^1E(G)^\eta={}^1E(H)$. Hence there exists to every element $x$ in $M(H)\leqq {}^1E(H)$ an element $y$ in ${}^1E(G)$ such that $x=y^\eta$. But $x$ belongs to $M(H)\leqq C(H)$ so that $C(H)=C(H)x=C(H)[C(G)y]^\eta=[C(G)y]^\gamma$, since $\eta$ induces $\gamma$ in $E(G)/C(G)$. The coset $C(G)y$ is an element in $[C(G){}^1E(G)]/C(G)=(G, G)$ and $\gamma$ induces an isomorphism in $(G, G)$. Hence $C(G)y=C(G)$ so that $y$ belongs to $M(G) = C(G)\cap {}^1E(G)$. Thus it follows from Theorem 1 that

$$M(G)^{\gamma'} \leqq M(H) \leqq M(G)^\eta = M(G)^{\gamma'},$$

and this completes the proof.

This theorem has two important simple consequences.

(a) Every isomorphism of $G$ upon $H$ is multiplicator preserving.

(b) Every homomorphism of the abelian group $G$ upon $H$ is multiplicator preserving.

Let us note finally that every homomorphism of $G$ upon $H$ is multiplicator

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[21] See the Appendix below.

preserving (by Theorem 1) if $M(H)=1$; this condition is satisfied, for example, whenever $H$ is a free group.

If $\gamma$ is a homomorphism of the group $G$ upon the group $H$, and if $\gamma'$ is the homomorphism of $C(G)$ upon $C(H)$ which is induced by $\gamma$, then it follows from Theorem 1 that $\gamma'$ induces a homomorphism of $C(G)/M(G)$ upon $C(H)/M(H)$. If this latter homomorphism happens to be an isomorphism, then $\gamma$ is certainly multiplicator preserving. For every element in $M(H)$ is the image, under $\gamma'$, of an element in $C(G)$, and no element outside $M(G)$ is mapped by $\gamma'$ upon an element in $M(H)$. The criterion just verified may be considered a counterpart to Theorem 2, since $C(X)/M(X)$ is essentially the same as the subgroup $[{}^1E(X)C(X)]/{}^1E(X)$ of $E(X)/{}^1E(X)$—for $X=G, H$—and since therefore our condition would certainly be satisfied whenever there exists a homomorphism $\eta$ of $E(G)$ upon $E(H)$ which induces the homomorphism $\gamma$ of $E(G)/C(G)$ upon $E(H)/C(H)$, the homomorphism $\gamma'$ of $C(G)$ upon $C(H)$ and an isomorphism of $E(G)/{}^1E(G)$ upon $E(H)/{}^1E(H)$.

THEOREM 3. *If $M$ and $N$ are normal subgroups of the similarly reduced free groups*[22] *$R$ and $S$ respectively, and if $\gamma$ is a multiplicator preserving homomorphism of the group $G=R/M$ upon the group $H=S/N$, then there exists a homomorphism $\phi$ of $R$ into $S$ which induces $\gamma$ in $R/M$ and which induces a homomorphism of $({}^1R\cap M)/{}_1M$ upon $({}^1S\cap N)/{}_1N$.*

**Proof.** Since $R$ and $S$ are similarly reduced free groups, there exist fully invariant subgroups $J$ and $L$ of free groups $F$ and $V$ respectively such that $R=F/J$, $S=V/L$ and[23] $J^{(F\to V)}\leqq L$, $L^{(V\to F)}\leqq J$. It is a consequence of Theorem 3 of III.3 that these representations of the groups $R$ and $S$ are essentially uniquely determined. We denote by $M^*$ and $N^*$ the uniquely determined normal subgroups of $F$ and $V$ respectively which satisfy $J\leqq M^*$, $M^*/J=M$ and $L\leqq N^*$, $N^*/L=N$.

In III.1 we have introduced the regular representation $D/T$ of the group $G$. This representation $D/T$ of $G$ is characterized by the existence of a set of representatives of the cosets, not $T$, of $D/T$ which is at the same time a free set of generators of the free group $D$. It will be convenient to denote by $\delta$ the isomorphism of $G$ upon $D/T$ which maps the element $x$ in $G$ upon the corresponding coset of $D/T$. It is a consequence of Theorem 2 of IV.1 that $T/{}_1T$ is essentially the same as $C(G)$, that $D/{}_1T$ is essentially the same as $E(G)$, and that therefore $({}^1D\cap T)/{}_1T$ and $M(G)$ are essentially the same. Likewise we consider the regular representation $B/Q$ of the group $H$ and we denote by $\beta$ the isomorphism of $H$ upon $B/Q$ which maps the element $y$ in $H$ upon the corresponding coset of $B/Q$. As before $Q/{}_1Q$ and $C(H)$, $B/{}_1Q$ and $E(H)$, $({}^1B\cap Q)/{}_1Q$ and $M(H)$ are essentially the same.

Since the homomorphism $\gamma$ of $G$ upon $H$ and the homomorphism $\kappa=\delta^{-1}\gamma\beta$

(22) This concept has been introduced in I.4, Theorem 1.

(23) This signifies that every homomorphism of $F$ into $V$ maps $J$ into part of $L$; see II.3.

of $D/T$ upon $B/Q$ are essentially the same, and since $\gamma$ is multiplicator preserving, it follows that $\kappa$ too is multiplicator preserving. Denote by $d(x)$ for $x$ in $G$ an element in the coset of $D/T$ corresponding to $x$ such that $d(1)=1$ and such that the elements $d(x)$ for $x\neq 1$ form a free set of generators of $D$, and likewise select elements $b(y)$ for $y$ in $H$ in the coset of $B/Q$ corresponding to $y$ such that $b(1)=1$ and such that the elements $b(y)$ for $y\neq 1$ form a free set of generators of $B$. There exists one and only one homomorphism $\nu$ of $D$ upon $B$ such that $d(x)^{\nu}=b(x^{\gamma})$ for every $x$ in $G$. Clearly $\nu$ induces the homomorphism $\kappa$ of $D/T$ upon $B/Q$ and $\nu$ induces the homomorphism $\kappa'$ of $T/{}_1T$ upon $Q/{}_1Q$ where we denote—as usual—by $\kappa'$ the homomorphism induced by $\kappa$ in $C(D/T)$. But $\kappa$ is multiplicator preserving. Hence $({}^1D\cap T)/{}_1T$ is mapped by $\kappa'$ and therefore by $\nu$ upon $({}^1B\cap Q)/{}_1Q$.

There exists by Lemma 1 of III.1 a homomorphism $\delta_0$ of the free group $F$ into the free group $D$ which induces the isomorphism $\delta$ of $F/M^*=G$ upon $D/T$, and we deduce from Corollary 1 of III.1 and (3) of I.4 that $\delta_0$ induces an isomorphism of $({}^1F\cap M^*)/{}_1M^*$ upon $({}^1D\cap T)/{}_1T$. Likewise we verify the existence of a homomorphism $\beta_0$ of the free group $B$ into the free group $V$ which induces the isomorphism $\beta^{-1}$ of $B/Q$ upon $V/N^*=H$ and which induces therefore an isomorphism of $({}^1B\cap Q)/{}_1Q$ upon $({}^1V\cap N^*)/{}_1N^*$. Thus $\omega=\delta_0\nu\beta_0$ is a homomorphism of $F$ into $V$ which induces the homomorphism $\delta\kappa\beta^{-1}=\gamma$ of $F/M^*=G$ upon $V/N^*=H$ and which induces a homomorphism of $({}^1F\cap M^*)/{}_1M^*$ upon $({}^1V\cap N^*)/{}_1N^*$.

From $J^{(F\to V)}\leqq L$ we deduce that $\omega$ effects a homomorphism $\phi$ of $R=F/J$ into $S=V/L$. From $J\leqq M^*$ and $L\leqq N^*$ we infer that $\phi$ induces the homomorphism $\gamma$ of $G=R/M=F/M^*$ upon $H=S/N=V/N^*$ and that $\phi$ induces a homomorphism of $({}^1R\cap M)/{}_1M=({}^1FJ\cap M^*)/{}_1M^*J=({}^1F\cap M^*)J/{}_1M^*J$ upon $({}^1V\cap N^*)L/{}_1N^*L=({}^1VL\cap N^*)/{}_1N^*L=({}^1S\cap N)/{}_1N$, and thus we have shown the existence of the desired homomorphism $\phi$ of $F$ into $V$.

If $\Xi$ is a system of similarly reduced free groups, then we shall denote by $\Xi_c$ the system of all the groups which may be represented in the form $R/N$ where $R$ is a group in $\Xi$ and where the normal subgroup $N$ of $R$ is part of the commutator subgroup $(R, R)$ of the reduced free group $R$. This system $\Xi_c$ is a proper part of the system $\Xi_h$ of all the homomorphic images of groups in $\Xi$ which we investigated in III.5.

LEMMA 1. *If $\Xi$ is a system of similarly reduced free groups, if $G$ is a group in $\Xi_c$ and $\gamma$ a homomorphism of $G$ upon a group $J$, and if $\gamma$ induces an isomorphism of $G/(G, G)$ upon $J/(J, J)$, then $J$ belongs to $\Xi_c$ too.*

**Proof.** If $K$ is the kernel of the homomorphism $\gamma$, then $K\leqq(G, G)$, since $\gamma$ induces an isomorphism of $G/(G, G)$ upon $J/(J, J)$. Since $G$ belongs to $\Xi_c$, there exists a representation $G=R/N$ for $R$ in $\Xi$ and $N\leqq(R, R)$. Denote by $M$ the uniquely determined normal subgroup of $R$ which satisfies $N\leqq M$ and

$M/N=K$. From $(G, G)=(R, R)/N$ we infer $M\leqq(R, R)$. Hence $J$ belongs to $\Xi_c$, since $J\sim R/M$.

COROLLARY 1. *If $\Xi$ is a system of similarly reduced free groups, if $\gamma$ is a multiplicator preserving homomorphism of the group $G$ in $\Xi_c$ upon the group $J$ in $\Xi_c$, and if $G=R/M$ and $J=S/N$ for $R$ and $S$ in $\Xi$, then there exists a homomorphism $\eta$ of $R$ into $S$ which induces the homomorphism $\gamma$ of $R/M$ upon $S/N$ and which induces, for every positive $i$, a homomorphism of $({}_{i-1}M\cap {}^iR)/{}_iM$ upon $({}_{i-1}N\cap {}^iS)/{}_iN$.*

**Proof.** There exist normal subgroups $D$ and $T$ of groups $H$ and $K$ in $\Xi$ such that $D\leqq(H, H)$, $H/D\sim G$ and $T\leqq(K, K)$, $K/T\sim J$, since $G$ and $J$ belong to $\Xi_c$. Since $R, S, H, K$ are all pairs of similarly reduced free groups, we deduce from Corollary 2 of III.4 and Theorem 1 of II.4 that $R/M$ and $H/D$ are similar representations of $G$ and that $S/N$ and $K/T$ are similar representations of $J$. We deduce from Theorem 3 of I.3 that the lower central series of $H$ relative to $D$ and of $K$ relative to $T$ are interlocking. Hence we infer from Theorem 2 of I.4 and (3) of I.4 the existence of a homomorphism $\upsilon$ of $R$ into $H$ which induces an isomorphism $\alpha$ of $R/M$ upon $H/D$ and, for every positive $i$, an isomorphism of $({}_{i-1}M\cap {}^iR)/{}_iM$ upon ${}_{i-1}D/{}_iD$, and the existence of a homomorphism $\sigma$ of $K$ into $S$ which induces an isomorphism $\beta$ of $K/T$ upon $S/N$ and, for every positive $i$, an isomorphism of ${}_{i-1}T/{}_iT$ upon $({}_{i-1}N\cap {}^iS)/{}_iN$. Clearly $\alpha^{-1}\gamma\beta^{-1}$ is a multiplicator preserving homomorphism of $H/D$ upon $K/T$. We infer from Theorem 3 the existence of a homomorphism $\phi$ of $H$ into $K$ which induces the homomorphism $\alpha^{-1}\gamma\beta^{-1}$ of $H/D$ upon $K/T$ and which induces a homomorphism of $D/{}_1D$ upon $T/{}_1T$. Since $D\leqq {}^1H$ and $T\leqq {}^1K$, it is clear that $\phi$ induces a homomorphism of $H/{}^1H$ upon $K/{}^1K$. Hence it follows from Theorem 1 of §4 that $\phi$ induces, for every positive $i$, a homomorphism of ${}_{i-1}D/{}_iD$ upon ${}_{i-1}T/{}_iT$. Clearly $\eta=\upsilon\phi\sigma$ is a homomorphism of $R$ into $S$ which induces the homomorphism $\gamma$ of $R/M$ upon $S/N$ and which induces, for every positive $i$, a homomorphism of $({}_{i-1}M\cap {}^iR)/{}_iM$ upon $({}_{i-1}N\cap {}^iS)/{}_iN$, as we desired to show.

COROLLARY 2. *If $M$ and $N$ are normal subgroups of the similarly reduced free groups $R$ and $S$ respectively, if $M\leqq(R, R)$, and if the multiplicator preserving homomorphism $\gamma$ of $R/M$ upon $S/N$ induces an isomorphism of $R/(R, R)$ upon $S/N(S, S)$, then there exists a homomorphism $\eta$ of $R$ into $S$ which induces the homomorphism $\gamma$ of $R/M$ upon $S/N$ and which induces, for every positive $i$, a homomorphism of ${}_{i-1}M/{}_iM$ upon $({}_{i-1}N\cap {}^iS)/{}_iN$.*

This is an almost immediate consequence of Corollary 1, Lemma 1 and Theorem 3 of I.3.

*Remark* 1. Let $R$ be a reduced free group such that ${}^2R<{}^1R$. Then there exists a normal subgroup $N$ of $R$ satisfying ${}_1N<N<(R, R)$. Let $M=1$ and

$R=S$, and let $\gamma$ be the natural homomorphism of $R/M$ upon $S/N$. It is obvious that all the conditions of Corollary 2 are satisfied with the exception of the requirement that $\gamma$ be multiplicator preserving. But ${}_{i-1}M/{}_iM=1$ for positive $i$, though $N/{}_1N\neq 1$. Hence it is impossible that there exists an endomorphism of $R$ which induces $\gamma$ and which maps $M/{}_1M$ upon $N/{}_1N$. This shows the impossibility of omitting in Corollaries 1 and 2 the requirement that $\gamma$ be multiplicator preserving, and it proves, incidentally, the existence of a homomorphism of some group upon another group which is *not multiplicator preserving.*

THEOREM 4. *Assume that the group $G$ meets the following requirements:*

(a) *$G/(G, G)$ is the direct product of cyclic groups of equal order*(24) *$n$.*

(b) $G^n=1$.

(c) *There exists a set of representatives of a basis of $G/(G, G)$ which generates $G$.*

*Then the homomorphism $\gamma$ of $G$ upon $H$ is an isomorphism if, and only if:*

(i) *$\gamma$ induces an isomorphism of $G/(G, G)$ upon $H/(H, H)$.*

(ii) *$\gamma$ induces an isomorphism in $P(G)$.*

(iii) *$\gamma$ is multiplicator preserving.*

**Proof.** The necessity of conditions (i) and (ii) is practically obvious, and the necessity of condition (iii) may be deduced from Theorem 2.

Assume conversely the validity of the conditions (i) to (iii). We deduce from conditions (a) to (c) and the corollary of III.6 that $G$ may be represented in the form $R/M$ where the normal subgroup $M$ of $R$ is part of $(R, R)$ and where $R=F/F^n$ is a reduced free group (a free group reduced modulo $n$). We denote by $N$ the uniquely determined normal subgroup of $R$ such that $M\leqq N$ and such that $N/M$ is the kernel of $\gamma$. We deduce from (i) that $N\leqq(R, R)$, since $(G, G)=(R, R)/M$ because of $M\leqq(R, R)$, and we infer from (ii) that $P(R, M)\cap N=M$. Since $N/M$ is the kernel of $\gamma$, and since isomorphisms are, by Theorem 2, multiplicator preserving, it follows that the natural homomorphism of $R/M$ upon $R/N$ is multiplicator preserving. There exists therefore by Theorem 3 an endomorphism $\epsilon$ of $R$ which induces the natural homomorphism of $R/M$ upon $R/N$ and which induces a homomorphism of $M/{}_1M$ upon $N/{}_1N$. Since $M\leqq N\leqq(R, R)$, and since $\epsilon$ induces the natural homomorphism of $R/M$ upon $R/N$, it follows that $\epsilon$ induces the identity automorphism in $R/(R, R)$. Hence it follows from (d) of Theorem 4 of §4 that $N\leqq P(R, M)$ and that therefore $N=N\cap P(R, M)=M$. Thus $N/M=1$, proving that $\gamma$ is an isomorphism, as we desired to show.

*Remark* 2. If $G$ is a direct product of cyclic groups of equal order, and if the homomorphism $\gamma$ of $G$ upon a group $H$ is not an isomorphism, then it is obvious that conditions (a) to (c) are satisfied by $G$. Clearly $P(G)=1$ so that

(24) Infinite cyclic groups and the elements generating them are said to be of order 0.

(ii) is satisfied too, and that (iii) is satisfied is a consequence of Theorem 2. But $\gamma$ is not an isomorphism, proving the impossibility of omitting (i).

*Remark* 3. Denote by $B$ a free abelian group of rank 2 and by $b'$, $b''$ a basis of $B$. There exists one and only one automorphism $\beta$ of $B$ satisfying $b'^{\beta}=b'^{2}b''$ and $b''^{\beta}=b'b''$, and it is readily seen that $B=B^{1-\beta}$. Denote by $A$ the group obtained by adjoining to $B$ an element $v$, subject to the relations: $v^{-1}xv=x^{\beta}$ for $x$ in $B$.

Clearly $v$ and $vb'$ induce in $B$ the same automorphism $\beta$, and $A$ may be obtained both by adjoining $v$ and $vb'$ to $B$. Hence there exists one and only one automorphism $\alpha$ of $A$ which maps $v$ upon $vb'$ and which leaves invariant every element in $B$. Denote by $G$ the group obtained by adjoining to $A$ an element $u$ subject to the relations $u^{-1}yu=y^{\alpha}$ for $y$ in $A$.

We note first that $(v, u)=b'$, and that $G$ may be obtained by adjoining $u$ and $v$ to $B$. Since $u$ permutes with every element in $B$, and since $v$ induces in $B$ the automorphism $\beta$, it follows now readily that

$$B = (G, G) = (G, (G, G)).$$

The group generated by $u$ and $v$ contains certainly $(v, u)=b'$ and $v^{-1}b'v=b'^{2}b''$. Thus the group generated by $u$ and $v$ contains $b''$ too. Hence $G$ is generated by $u$ and $v$.

If $u^{i}v^{j}\equiv 1$ modulo $B$, then $v^{j}$ permutes with every element in $B$, since $u$ permutes with every element in $B$. But $v$ induces in $B$ the automorphism $\beta$ which is readily seen to be of order 0, implying $j=0$. Hence $u^{i}$ is in $B$, and from this fact we deduce $i=0$. Thus we have shown that $G/B$ is a free abelian group of rank 2.

This group $G$ meets the requirements (a) to (c) of Theorem 4, and has the further property $P(G)=(G, G)\neq 1$. We denote by $\gamma$ the natural homomorphism of $G$ upon $G/(G, G)$. It is clear that $\gamma$ satisfies condition (i) of Theorem 4, and we may deduce from Corollary 4 of §4 by a slight argument involving the regular representations that $\gamma$ is multiplicator preserving. But $\gamma$ is certainly no isomorphism, proving the indispensability of condition (ii).

*Remark* 4. If $G$ is a nonabelian group, satisfying conditions (a) to (c) of Theorem 4, and satisfying furthermore $P(G)=1$, then the natural homomorphism $\gamma$ of $G$ upon $G/(G, G)$ clearly meets requirements (i) and (ii), though it is certainly no isomorphism, showing the indispensability of condition (iii).

COROLLARY 3. *Assume that the group $G$ meets the following requirements:*

(a) *$G/(G, G)$ is the direct product of a finite number of cyclic groups of equal order $n$.*

(b) $G^{n}=1$.

(c) *There exists a set of representatives of some basis of $G/(G, G)$ which generates $G$.*

(d) $P(G)=1$.

*Then $G$ is a $Q$-group if, and only if, every endomorphism of $G$ upon $G$ is multiplicator preserving.*

**Proof.** If $G$ is a $Q$-group, then every endomorphism of $G$ upon $G$ is a proper automorphism of $G$. But proper automorphisms are multiplicator preserving by Theorem 2.

Assume conversely that every endomorphism of $G$ upon $G$ is multiplicator preserving, and that $\epsilon$ is an endomorphism of $G$ upon $G$. Then $\epsilon$ induces an endomorphism of $G/(G, G)$ upon $G/(G, G)$ which group is a $Q$-group by (a). Hence $\epsilon$ induces a proper automorphism of $G/(G, G)$. It is a consequence of (d) that $\epsilon$ induces an isomorphism in $P(G)$. Consequently we may infer from Theorem 4 that $\epsilon$ is a proper automorphism of $G$, showing that $G$ is a $Q$-group.

*Remark* 5. If $G$ is the direct product of an infinity of cyclic groups of equal order $n \neq 1$, then conditions (b), (c), (d) are satisfied, and it is a consequence of Theorem 2 that every endomorphism of $G$ upon $G$ is multiplicator preserving. But $G$ is clearly not a $Q$-group, showing that the word "finite" in condition (a) cannot be omitted[25].

**Appendix: Generalization of multiplicator and multiplicator-preservation.** If $J$ is a fully invariant subgroup of the free group $F$ of countably infinite rank, and if $G$ is some group, then we put $J(G) = J^{(F \to G)}$. This is clearly a fully invariant subgroup of $G$. The fully invariant subgroups of $G$ which may be obtained in this fashion form a special class and have been termed by B. Neumann[26] "word subgroups" of $G$.

If $J$ is a fully invariant subgroup of the free group $F$ of countably infinite rank, if $\gamma$ is a homomorphism of the group $G$ upon the group $H$, and if $J(E(G)) \leqq C(G)$ and $J(E(H)) \leqq C(H)$, then one verifies easily that $J(E(G))^{\gamma'} \leqq J(E(H))$ where as usual $\gamma'$ is the homomorphism of $C(G)$ upon $C(H)$ which is induced by $\gamma$. Thus a homomorphism of $C(G)/J(E(G))$ upon $C(H)/J(E(H))$ is induced by $\gamma'$, and it is an immediate inference from Theorem 1 of §5 that this homomorphism maps the $J$-multiplicator $[M(G)J(E(G))]/J(E(G))$ of $G$ into part of the $J$-multiplicator $[M(H)J(E(H))]/J(E(H))$ of $H$. If the $J$-multiplicator of $G$ is mapped upon the full $J$-multiplicator of $H$, then we shall term $\gamma$ *$J$-multiplicator preserving*. Obviously $\gamma$ is $J$-multiplicator preserving if, and only if,

$$M(H) \leqq M(G)^{\gamma'} J(E(H)).$$

If $J$ and $L$ are fully invariant subgroups of $F$ such that $L \leqq J$ and $J(E(X)) \leqq N(X)$ for $X = G, H$, then one deduces readily from the above inequality that $\gamma$ is $J$-multiplicator preserving whenever $\gamma$ is $L$-multiplicator preserving.

[25] If $n \neq 0$, and if Burnside's conjecture is true for this particular $n$, then hypotheses (a) to (c) imply the finiteness of the group $G$ in which case this corollary turns trivial, since finite groups are clearly $Q$-groups.

[26] See Appendix to III.

An extension of the theory of multiplicator preserving homomorphisms which has been developed in §5 to $J$-multiplicator preserving homomorphisms may be effected without much difficulty.

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