ON THE ASSOCIATE AND CONJUGATE SPACE FOR THE DIRECT PRODUCT OF BANACH SPACES

BY

NELSON DUNFORD AND ROBERT SCHATTEN(1)

The direct product $E_1 \otimes_N E_2$ of two Banach spaces E_1, E_2 has been defined before [5](2) as the closure of the normed linear set $\mathfrak{A}_N(E_1, E_2)$ (that is, linear set $\mathfrak{A}(E_1, E_2)$ of expressions $\sum_{i=1}^{n} f_i \otimes \phi_i$, in which N is a norm) [5, p. 200, Definition 1.3] and [6, p. 499, b].

Let N denote a crossnorm whose associate N' is also a crossnorm [5, p. 208]. Then, the cross-space $E_1 \otimes_N E_2$ determines uniquely a "conjugate space" $(E_1 \otimes_N E_2)'$ and an "associate space" $E_1' \otimes_{N'} E_2'$. It is shown [5, p. 205] that $E_1' \otimes_{N'} E_2'$ is always included in $(E_1 \otimes_N E_2)'$. While there are many known examples of cross-spaces for which the associate space coincides with the conjugate space—for example, the cross-space generated by the self-associate crossnorm constructed for Hilbert spaces by F. J. Murray and John von Neumann [3, p. 128] and [5, pp. 212–214]—it is not without interest to construct a cross-space for which the associate space forms a proper subset of the conjugate space (§§1–2).

For reflexive Banach spaces E_1, E_2 (that is, such that $E_i'' = E_i$), and a reflexive crossnorm N [6, p. 500], the reflexivity of $E_1 \otimes_N E_2$ implies $(E_1 \otimes_N E_2)' = E_1' \otimes_{N'} E_2'$ [6, p. 505]. Thus, the finding of the exact conditions imposed upon reflexive Banach spaces and a reflexive crossnorm for which the resulting cross-space is reflexive is closely connected with the above-mentioned problem.

In §1, we show that for a "natural crossnorm" N, $L' \otimes_N L'$ is a proper subset of $(L \otimes_N L)'$. In §2 we prove that for a "natural crossnorm" N, $l' \otimes_{N'} l'$ is a proper subset of $(l \otimes_N l)'$. In §3 we show that for any $p > 1$, $l_p \otimes_N l_q$ is not reflexive, provided $1/p + 1/q = 1$ and N denotes the least crossnorm whose associate is also a crossnorm [5, p. 208]. The last one is reflexive [6, p. 501].

1. Let $L_{(1)}$ and $L_{(2)}$ denote the Banach spaces of all functions integrable in the sense of Lebesgue on the interval $0 \leqq s \leqq 1$, and on the square $0 \leqq s$, $t \leqq 1$ respectively. Similarly, let $M_{(1)}$ and $M_{(2)}$ denote the Banach spaces of all functions Lebesgue measurable and essentially bounded on the interval $0 \leqq s \leqq 1$ and the square $0 \leqq s$, $t \leqq 1$ respectively [1, pp. 10, 12]. We recall that

Presented to the Society, September 17, 1945; received by the editors September 24, 1945.

(1) National Research Fellow.

(2) Numerals in square brackets refer to bibliography at the end of the paper. We shall use the notation of [6].

$$\text{for} \quad f(s) \in L_{(1)}, \quad \|f(s)\| = \int_0^1 |f(s)|\,ds;$$

$$\text{for} \quad f(s, t) \in L_{(2)}, \quad \|f(s, t)\| = \int_0^1 \int_0^1 |f(s, t)|\,dsdt;$$

$$\text{for} \quad F(s) \in M_{(1)}, \quad \|F(s)\| = \underset{0 \leqq s \leqq 1}{\text{ess. l.u.b.}} |F(s)|;$$

$$\text{for} \quad F(s, t) \in M_{(2)}, \quad \|F(s, t)\| = \underset{0 \leqq s,\, t \leqq 1}{\text{ess. l.u.b.}} |F(s, t)|$$

(ess. l.u.b. stands for "essential least upper bound").

For $f_i(s) \in L_{(1)}$, $\phi_i(t) \in L_{(1)}$ let the expression $\sum_{i=1}^n f_i(s) \otimes \phi_i(t)$ denote the function $\sum_{i=1}^n f_i(s)\phi_i(t)$. The last function naturally belongs to $L_{(2)}$. For an expression $\sum_{i=1}^n f_i(s) \otimes \phi_i(t)$ in $\mathfrak{A}(L_{(1)}, L_{(1)})$ we define

$$N\left(\sum_{i=1}^n f_i(s) \otimes \phi_i(t) \right) = \int_0^1 \int_0^1 \left| \sum_{i=1}^n f_i(s)\phi_i(t) \right| dsdt.$$

LEMMA 1.1. *N is a crossnorm in $\mathfrak{A}(L_{(1)}, L_{(1)})$* [5, p. 205].

Proof. The proof is elementary. In particular, the invariance of the norm under equivalence can be readily verified, since the equivalence of two expressions $\sum_{i=1}^n f_i(s) \otimes \phi_i(t)$, $\sum_{j=1}^m g_j(s) \otimes \psi_j(t)$ implies $\sum_{i=1}^n f_i(s)\phi_i(t) = \sum_{j=1}^m g_j(s)\psi_j(t)$ for almost every s and almost every t.

LEMMA 1.2. $L_{(1)} \otimes_N L_{(1)} = L_{(2)}$.

Proof. Obviously, $\mathfrak{A}_N(L_{(1)}, L_{(1)}) \subset L_{(2)}$. Since $L_{(2)}$ is complete, the closure of $\mathfrak{A}_N(L_{(1)}, L_{(1)})$, that is, $L_{(1)} \otimes_N L_{(1)} \subset L_{(2)}$. On the other hand, it is well known that functions in $L_{(2)}$ can always be approximated in norm by a sequence of expressions $\{\sum_{k=1}^{p_n} f_k^{(n)}(s)\phi_k^{(n)}(t)\}$, where $f_k^{(n)}(s) \in L_{(1)}$, $\phi_k^{(n)}(t) \in L_{(1)}$. This completes the proof.

LEMMA 1.3. $L_{(i)}' = M_{(i)}$ for $i = 1, 2$.

Proof. The proof may be found in [1, p. 65].

LEMMA 1.4. $(L_{(1)} \otimes_N L_{(1)})' = M_{(2)}$.

Proof. This is a consequence of Lemmas 1.2 and 1.3.

THEOREM 1. *$L_{(1)}' \otimes_{N'} L_{(1)}'$ is a proper subset of $(L_{(1)} \otimes_N L_{(1)})'$.*

Proof. Clearly, $L_{(1)}' \otimes_{N'} L_{(1)}' \subset (L_{(1)} \otimes_N L_{(1)})'$ [5, p. 205]. Due to Lemmas 1.3 and 1.4, the last statement may be expressed as $M_{(1)} \otimes_{N'} M_{(1)} \subset M_{(2)}$. We shall prove our theorem by showing that not every function in $M_{(2)}$ can be approximated in norm by a sequence of functions $\{\sum_i^{p_n} F_i^{(n)}(s)\Phi_i^{(n)}(t)\}$ where $F_i^{(n)}(s)$, $\Phi_i^{(n)}(t)$ belong to $M_{(1)}$. We shall show in particular that the function $K(s, t)$ defined for $0 \leqq s, t \leqq 1$ as follows: $K(s, t)=1$ if $s \leqq t$, otherwise $K(s, t)=0$, cannot be approximated in norm by such a sequence of expressions. Suppose to the contrary, that

$$\lim_{n\to\infty} \operatorname{ess.\,l.u.b.}_{0 \leqq s, t \leqq 1} \left| \sum_{i=1}^{p_n} F_i^{(n)}(s)\Phi_i^{(n)}(t) - K(s, t) \right| = 0.$$

Put

$$K_n(s, t) = \sum_{i=1}^{p_n} F_i^{(n)}(s)\Phi_i^{(n)}(t) - K(s, t).$$

Thus, there exists a set E_0 of points (s, t) in the square $0 \leqq s, t \leqq 1$, and a sequence $\{\epsilon_n\}$ of positive numbers such that:

(a) $mE_0=0$,

(b) $\epsilon_n \to 0$,

(c) $|K_n(s, t)| \leqq \epsilon_n$ for $(s, t) \notin E_0$.

Let $H(s, t)$ denote the characteristic function of E_0. Its Lebesgue integral over the square $0 \leqq s, t \leqq 1$ is 0. Fubini's theorem [7, p. 77] gives

$$\int_0^1 \left(\int_0^1 H(s, t)dt \right) ds = 0.$$

Therefore, there exists a linear set S of measure 1 in the interval $0 \leqq s \leqq 1$ such that, for every $s_0 \in S$,

$$\int_0^1 H(s_0, t)dt = 0.$$

The last statement implies for each $s \in S$ the existence of a linear set T_s of measure 1 in the interval $0 \leqq t \leqq 1$ such that $s \in S$ and $t \in T_s$ implies $H(s, t)=0$, consequently $(s, t) \notin E_0$, and therefore $|K_n(s, t)| \leqq \epsilon_n$ for $n=1, 2, \cdots$. This proves the existence of a linear set S in $0 \leqq s \leqq 1$, of measure 1, such that, for every $s_0 \in S$,

$$\lim_{n\to\infty} \operatorname{ess.\,l.u.b.}_{0 \leqq t \leqq 1} \left| \sum_{i=1}^{p_n} F_i^{(n)}(s_0)\Phi_i^{(n)}(t) - K(s_0, t) \right| = 0. \tag{1}$$

Let $\mathfrak{M}$ denote the closed linear manifold determined by all $\Phi_i^{(n)}(t)$; $n=1, 2, 3, \cdots; i=1, 2, \cdots, p_n$. Clearly, $\mathfrak{M}$ is separable, and a subset of $M_{(1)}$. For a fixed point $s_0 \in S$, $\sum_{i=1}^{p_n} F_i^{(n)}(s_0)\Phi_i^{(n)}(t)$ is a function of one variable t, $0 \leqq t \leqq 1$, and obviously belongs to $\mathfrak{M}$. Since $\mathfrak{M}$ is closed, $K(s_0, t) \in \mathfrak{M}$ by virtue of (1). Furthermore, for $s_0 \in S$, $s_1 \in S$, and $s_0 \neq s_1$,

$$\operatorname{ess.\,l.u.b.}_{0 \leqq t \leqq 1} | K(s_0, t) - K(s_1, t) | = 1.$$

Thus, $\mathfrak{M}$ contains a "continuum number" of elements $K(s_0, t)$ whose "distance" from each other is 1. The last implication contradicts the separability of $\mathfrak{M}$ [2, p. 126]. This completes the proof.

2. Let l denote the space of all sequences of real numbers $\{x_i\}$ for which $\sum_{i=1}^{\infty}|x_i| < \infty$, and m the space of all bounded sequences of real numbers [1, pp. 11–12]. Let $\mathfrak{a}$ denote the Banach space of all infinite matrices $(a_{i,j})$ for which $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{i,j}| < \infty$, and $\mathfrak{b}$ the Banach space of all bounded matrices $(b_{i,j})$. We recall that

$$\text{for} \quad (x_1, x_2, \cdots) \in l, \qquad \|(x_1, x_2, \cdots)\| = \sum_{i=1}^{\infty} |x_i|;$$

$$\text{for} \quad (\alpha_1, \alpha_2, \cdots) \in m, \qquad \|(\alpha_1, \alpha_2, \cdots)\| = \sup_{1 \leq i < \infty} |\alpha_i|;$$

$$\text{for} \quad (a_{i,j}) \in \mathfrak{a}, \qquad \|(a_{i,j})\| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{i,j}|;$$

$$\text{for} \quad (b_{i,j}) \in \mathfrak{b}, \qquad \|(b_{i,j})\| = \sup_{1 \leq i,j < \infty} |b_{i,j}|.$$

Obviously l is equivalent to $\mathfrak{a}$ [1, p. 180]. Similarly, $\mathfrak{b}$ is equivalent to m. For $(x_1^{(k)}, x_2^{(k)}, \cdots) \in l$, $(y_1^{(k)}, y_2^{(k)}, \cdots) \in l$, the expression $\sum_{k=1}^{n}(x_1^{(k)}, x_2^{(k)}, \cdots) \otimes (y_1^{(k)}, y_2^{(k)}, \cdots)$ will mean the infinite matrix $(a_{i,j})$ of rank not greater than n, where $a_{i,j} = \sum_{k=1}^{n} x_i^{(k)} y_j^{(k)}$. Clearly, two expressions $\sum_{k=1}^{n}(x_1^{(k)}, x_2^{(k)}, \cdots) \otimes (y_1^{(k)}, y_2^{(k)}, \cdots)$, $\sum_{k=1}^{m}(\bar{x}_1^{(k)}, \bar{x}_2^{(k)}, \cdots) \otimes (\bar{y}_1^{(k)}, \bar{y}_2^{(k)}, \cdots)$ are equivalent [5, p. 196] if and only if $\sum_{k=1}^{n} x_i^{(k)} y_j^{(k)} = \sum_{k=1}^{m} \bar{x}_i^{(k)} \bar{y}_j^{(k)}$, for $i, j = 1, 2, \cdots$ [5, p. 202, Theorem 2.1]. Let

$$N\left(\sum_{k=1}^{n} (x_1^{(k)}, x_2^{(k)}, \cdots) \otimes (y_1^{(k)}, y_2^{(k)}, \cdots) \right) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left| \sum_{k=1}^{n} x_i^{(k)} y_j^{(k)} \right| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{i,j}|.$$

LEMMA 2.1. *N is a crossnorm in $\mathfrak{A}(l, l)$* [5, p. 205].

Proof. The proof is elementary. In particular, the previous remark shows that the norm is invariant under equivalence.

LEMMA 2.2. $l \otimes_N l = \mathfrak{a}$.

Proof. Clearly, every matrix with a finite number of rows in $\mathfrak{a}$, and every infinite matrix $(a_{i,j}) \in \mathfrak{a}$ is the limit of the sequence of finite rowed square matrices $(a_{i,j})_{j=1,2,\cdots,n}^{i=1,2,\cdots,n}$; $n = 1, 2, \cdots$. Thus, $\mathfrak{a} \subset l \otimes_N l$. On the other hand, every element of $\mathfrak{A}(l, l)$ belongs to $\mathfrak{a}$. Since $\mathfrak{a}$ is complete, $l \otimes_N l \subset \mathfrak{a}$. This completes the proof.

LEMMA 2.3. $(l \otimes_N l)' = \mathfrak{a}' = \mathfrak{b}$.

Proof. The proof is analogous to the case $l' = m$ [1, p. 97].

For $(\alpha_1^{(k)}, \alpha_2^{(k)}, \cdots)\in m$, $(\beta_1^{(k)}, \beta_2^{(k)}, \cdots)\in m$, let the expression $\sum_{k=1}^{n}(\alpha_1^{(k)}, \alpha_2^{(k)}, \cdots)\otimes(\beta_1^{(k)}, \beta_2^{(k)}, \cdots)$ in $\mathfrak{A}(m, m)$ denote the infinite bounded matrix $(b_{i,j})$ of rank not greater than n where $b_{i,j}=\sum_{k=1}^{n}\alpha_i^{(k)}\beta_j^{(k)}$.

LEMMA 2.4. *$(l\otimes_N l)'\supset l'\otimes_{N'} l' = m\otimes_{N'} m$.*

Proof. This is a consequence of [1, p. 97] and [5, p. 205].

THEOREM 2. *$l'\otimes_{N'} l'$ is a proper subset of $(l\otimes_N l)'$.*

Proof. It is sufficient to show that $m\otimes_{N'} m$ is a proper subset of $\mathfrak{b}$ (Lemmas 2.3, 2.4), or that not every bounded matrix can be approximated by a sequence of matrices of finite rank. We shall show that the bounded infinite matrix $(\delta_{i,j})$, where $(\delta_{i,j})=1$ if, and only if, $i=j$, otherwise $\delta_{i,j}=0$, cannot be approximated by a sequence of bounded matrices of finite rank, that is, does not belong to $m\otimes_{N'} m$. To see that, we notice first that every bounded matrix $(b_{i,j})$ represents a linear transformation T from l into m. Let $(l_1, l_2, \cdots)\in l$. Put $T(l_1, l_2, \cdots)=(m_1, m_2, \cdots)$ where $m_i=\sum_{j=1}^{\infty} b_{i,j}l_j$. We prove

$$|||T||| = \sup_{i,j} |b_{i,j}| \qquad (|||T||| \text{ denotes the bound of } T).$$

By definition

$$|||T||| = \sup_{\|(l_1, l_2, \cdots)\|=1} \|T(l_1, l_2, \cdots)\|.$$

The last number may be written as

$$\sup_{\|(l_1,l_2,\cdots)\|=1} \sup_i \left|\sum_{j=1}^{\infty} b_{i,j}l_j\right| = \sup_i \sup_{\|(l_1,l_2,\cdots)\|=1} \left|\sum_{j=1}^{\infty} b_{i,j}l_j\right|.$$

For a fixed i, $\sum_{j=1}^{\infty} b_{i,j}l_j$ denotes a linear functional on l [1, p. 97]; the bound of this linear functional is

$$\sup_{\|(l_1,l_2,\ldots)\|=1} \left|\sum_{j=1}^{\infty} b_{i,j}l_j\right| = \sup_j |b_{i,j}|.$$

Substituting the last number in the previous equation, we get

$$|||T||| = \sup_i \sup_j |b_{i,j}| = \sup_{i,j} |b_{i,j}|,$$

or the norm of the bounded matrix $(b_{i,j})$ is equal to the bound of the linear transformation T it represents.

It is easy to see that if the matrix is of finite rank, the corresponding linear transformation T is finite-dimensional. The linear transformation T_δ corresponding to the matrix $(\delta_{i,j})$ is obviously not completely continuous, therefore it can not be considered a limit of linear transformations whose ranges are finite-dimensional [1, p. 96]. Therefore, $(\delta_{i,j})$ is not a limit of bounded matrices of finite rank. This completes the proof.

3. THEOREM 3. *Let N denote the least crossnorm whose associate is also a crossnorm* [5, p. 208]. *If $p>1$, $1/p+1/q=1$ and l_p denotes the Banach space of all sequences of real numbers $\{x_i\}$ for which $\sum_{i=1}^{\infty}|x_i|^p<\infty$* [1, p. 12], *then $l_p\otimes_N l_q$ is not reflexive.*

Proof. Let $\Phi_1, \Phi_2, \Phi_3, \cdots$ and $\phi_1, \phi_2, \phi_3, \cdots$ denote the sequence of elements $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$, $(0, 0, 1, \cdots)$ in l_p and l_q respectively. Clearly, $\Phi_i(\phi_j)=0$ if $i\neq j$, and $\Phi_i(\phi_i)=1$; $i, j=1, 2, \cdots$. With a fixed sequence of real numbers $\{\lambda_i\}$ converging towards 0, consider the sequence of expressions

$$\lambda_1\Phi_1\otimes\phi_1, \qquad \sum_{i=1}^{2}\lambda_i\Phi_i\otimes\phi_i, \qquad \sum_{i=1}^{3}\lambda_i\Phi_i\otimes\phi_i, \cdots .$$

First we prove, if $n>m$, $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)=\max_{m\leqq i\leqq n}|\lambda_i|$. By definition [5, p. 208], $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)=\sup|\sum_{i=m}^{n}\lambda_i\Phi_i(\phi)\Phi(\phi_i)|$ where sup, that is, the least upper bound, is taken for all $\Phi\in l_p$, $\phi\in l_q$, such that $\|\Phi\|=\|\phi\|=1$. Substituting in the last equation Φ_i for Φ and ϕ_i for ϕ, we obtain $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)\geqq|\lambda_i|$. Thus,

$$N\left(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i\right)\geqq\max_{m\leqq i\leqq n}|\lambda_i|.$$

On the other hand, if $\Phi\in l_p$ and $\Phi=x_1\Phi_1+x_2\Phi_2+\cdots$, then $\|\Phi\|=1$ if, and only if, $\sum_{i=1}^{\infty}|x_i|^p=1$. Similarly, if $\phi\in l_q$ and $\phi=y_1\phi_1+y_2\phi_2+\cdots$, then $\|\phi\|=1$ if, and only if, $\sum_{i=1}^{\infty}|y_i|^q=1$. Furthermore, $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)=\sup|\sum_{i=m}^{n}\lambda_i x_i y_i|$ where sup is taken over the set of all sequences of real numbers $\{x_i\}$, $\{y_i\}$, for which $\sum_{i=1}^{\infty}|x_i|^p=1$ and $\sum_{i=1}^{\infty}|y_i|^q=1$. Hölder's inequality gives:

$$N\left(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i\right)\leqq\sup\left\{\left(\max_{m\leqq i\leqq n}|\lambda_i|\right)\left(\sum_{i=m}^{n}|x_i|^p\right)^{1/p}\left(\sum_{i=m}^{n}|y_i|^q\right)^{1/q}\right\}$$

where sup is as stated in the previous equation. Since

$$\sum_{i=m}^{n}|x_i|^p\leqq\sum_{i=1}^{\infty}|x_i|^p=1, \qquad \sum_{i=m}^{n}|y_i|^q\leqq\sum_{i=1}^{\infty}|y_i|^q=1,$$

$$N\left(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i\right)\leqq\max_{m\leqq i\leqq n}|\lambda_i|.$$

Since $\lambda_i\rightarrow 0$, the sequence of expressions $\lambda_1\Phi_1\otimes\phi_1, \sum_{i=1}^{2}\lambda_i\Phi_i\otimes\phi_i, \sum_{i=1}^{3}\lambda_i\Phi_i\otimes\phi_i, \cdots$ is fundamental. Therefore, it may be considered as an element of $l_p\otimes_N l_q$. Its norm is obviously [5, p. 205]

$$\sup_{1 \leq i < \infty} |\lambda_i|.$$

Thus, the well known non-reflexive space c_0 [1, p. 181] ($c_0'' = l' = m$ [1, pp. 66–67]) of all converging towards 0 sequences of real numbers may be considered a subspace of $l_p \otimes_N l_q$. Since a subspace of a reflexive space is also reflexive [4, p. 423], $l_p \otimes_N l_q$ is not reflexive. This completes the proof.

References

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.
2. F. Hausdorff, *Mengenlehre*, Göschen, Berlin, 1935.
3. F. J. Murray and John von Neumann, *On rings of operators*, Ann. of Math. vol. 37 (1936) pp. 116–229.
4. B. J. Pettis, *A note on regular Banach spaces*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 420–428.
5. R. Schatten, *On the direct product of Banach spaces*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 195–217.
6. ———, *On reflexive norms for the direct product of Banach spaces*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 498–506.
7. S. Saks, *Theory of the integral*, Warsaw, New York, 1937.

Yale University,
New Haven, Conn.