ON LINEAR EXPANSIONS. I

BY

LEOPOLDO NACHBIN

1. **Introduction.** The main purpose of the present note is to establish the equivalence between the Cantor-Lebesgue and the Lusin-Denjoy properties for linear expansions(1). The statement of the sense in which this equivalence shall be understood requires some definitions.

Let us consider a sequence of real (finite, single-valued) functions of the real variable x: $\phi_n(x)$ $(n=0, 1, 2, \cdots)$ simultaneously defined in a set Ω of real numbers. Such a sequence will be called a base and will be briefly denoted by $\Phi(\Omega)$. Any series of functions having the form $\sum_0^\infty \lambda_n\phi_n(x)$, where the λ_n $(n=0, 1, 2, \cdots)$ are real numbers, will be called a linear expansion associated with the base $\Phi(\Omega)$. A base $\phi(\Omega)$ is said to be measurable if all the functions of the base are measurable in the Lebesgue sense in the set Ω. All bases considered here are supposed to be measurable. The extension of our considerations to the general case of non-measurable bases requires certain new details in the definitions and in the proofs.

We shall say that two given properties P' and P'' of a measurable base are equivalent and write $P' \sim P''$ provided that any measurable base possessing the property P' also possesses the property P'', and conversely. This equivalence is reflexive, symmetric and transitive. If P is a property of a measurable base, we shall denote by $n(P)$ the negative of P, that is, the property of a measurable base expressed by the fact that the base does not possess the property P.

We shall say that a measurable base $\Phi(\Omega)$ possesses the Cantor-Lebesgue property if the following condition is satisfied:

CL. Any linear expansion associated with the base is almost everywhere nonconvergent in the set Ω if the condition $\lambda_n \to 0$ as $n \to \infty$ is not satisfied.

We shall also say that the measurable base possesses the Lusin-Denjoy property if the following condition is satisfied:

LD. Any linear expansion associated with the base is almost everywhere absolutely divergent in the set Ω if $\sum_0^\infty |\lambda_n| = +\infty$.

We are now in a position to state the announced equivalence and in fact we shall prove that for any measurable base we have CL$\sim$LD. This equiva-

Presented to the Society, April 27, 1946; received by the editors August 12, 1944, and, in revised form, November 8, 1945.

(1) For the statements of these properties in the trigonometric case see [5, pp. 267 and 131]. Numbers in brackets refer to the bibliography at the end of the paper.

lence was first pointed out by Stone [4, Theorems 2 and 4](2).

2. **The equivalence theorem.** Throughout this section we shall suppose that the base $\Phi(\Omega)$ is measurable and that Ω is a positive set (the case where $m(\Omega)=0$ is trivial)(3). Now we define the following properties of a measurable base:

P$_1$. A sub-sequence of the sequence $\{\phi_n(x)\}$ can be obtained which converges to zero in a positive set $\Delta\subset\Omega$.

P$_2$. A sub-sequence of the sequence $\{\phi_n(x)\}$ can be obtained which converges uniformly to zero in a positive set $\Delta\subset\Omega$.

P$_3$. A positive set $\Delta\subset\Omega$ can be obtained such that:

$$\liminf_{n\to\infty}\int_\Delta |\phi_n(x)|\,dx = 0.$$

The following theorem can now be proved [4, 3].

THEOREM. *For any measurable base we have* $n(\mathrm{P}_1)\sim n(\mathrm{P}_2)\sim n(\mathrm{P}_3)\sim\mathrm{CL}\sim\mathrm{LD}$.

Proof. We divide the proof into the following parts.

(a) P_1 *implies* P_2. This follows immediately from Egoroff's theorem [1, p. 144].

(b) P_2 *implies* P_3. This is obvious if we consider a positive set $\Delta\subset\Omega$ of finite measure in the condition stated by P_2.

(c) P_3 *implies* P_1. In fact, from P_3 it follows that a sub-sequence of the sequence $\{\phi_n(x)\}$ can be obtained which converges on the average to zero in a positive set $\Delta\subset\Omega$ [1, p. 245]; hence a further sub-sequence of the sequence $\{\phi_n(x)\}$ can be obtained which converges to zero almost everywhere in Δ.

(d) P_2 *implies* $n(\mathrm{LD})$. In fact, by virtue of P_2 there exist a positive set $\Delta\subset\Omega$ and an increasing sequence of positive integers $\{n_r\}$ such that $|\phi_{n_r}(x)|\leq 1/2^r$ for $x\in\Delta$ and $r=1, 2, \cdots$; we may then define a sequence λ_n $(n=0, 1, 2, \cdots)$ by taking $\lambda_n=1$ for $n=n_r$ $(r=1, 2, \cdots)$ and $\lambda_n=0$ otherwise. Then $\sum_0^\infty\lambda_n\phi_n(x)$ is absolutely convergent in Δ and $\sum_0^\infty|\lambda_n|=+\infty$; hence the base does not have the LD property.

(e) P_2 *implies* $n(\mathrm{CL})$. This follows also from the preceding argument.

(f) $n(\mathrm{CL})$ *implies* P_1. In fact, by virtue of $n(\mathrm{CL})$ there exists a certain

(2) This paper, previously overlooked by me, was kindly called to my attention by the referee. Stone proves the equivalences $\mathrm{CL}\sim\mathrm{LD}\sim n(\mathrm{P}_3)$ (property P_3 is defined below in §2) under the assumptions that Ω is bounded and the functions $\{\phi_n(x)\}$ are uniformly bounded in Ω. These assumptions are superfluous for Theorems 2 and 4 of Stone, but this is not the case for his Theorems 1 and 3.

(3) A positive set is a measurable set whose measure is positive. We observe that the measure of a measurable set and the integral of a non-negative measurable function may be finite or infinite.

sequence $\lambda_n (n=0, 1, 2, \cdots)$ such that (1) lim sup $|\lambda_n|>0$ as $n\rightarrow\infty$, and such that (2) $\sum_0^\infty \lambda_n\phi_n(x)$ is convergent in a certain positive subset $\Delta\subset\Omega$. From (2) we obtain (3) $\lambda_n\phi_n(x)\rightarrow 0$ as $n\rightarrow\infty$ for $x\in\Delta$. Next P_1 follows from (1) and (3).

(g) n(LD) *implies* P_3. In fact, by virtue of n(LD) there exists a sequence λ_n $(n=0, 1, 2, \cdots)$ such that (1) $\sum_0^\infty|\lambda_n|=+\infty$, and such that (2) $S(x)=\sum_0^\infty|\lambda_n\phi_n(x)|<+\infty$ holds in a positive subset of Ω. By (2) there exist a positive set $\Delta\subset\Omega$ of finite measure and a number $K\geqq 0$ such that (3) $S(x)\leqq K$ for $x\in\Delta$. Integrating (3) over Δ we obtain

$$\sum_0^\infty |\lambda_n| \int_\Delta |\phi_n(x)|\, dx \leqq Km(\Delta). \tag{4}$$

From (1) and (4) we may infer P_3.

Having proved these implications, the theorem follows readily. In fact, (a), (b) and (c) show that $P_1\sim P_2\sim P_3$; next from (e) and (f) and from (d) and (g) we obtain the remainder of the theorem.

3. **An example.** Now we consider as an example of the preceding argument the following base which contains as a particular case the trigonometric series and is of interest in the theory of almost periodic functions. Let $f(x)$ be a real function of the real variable x, defined for $-\infty<x<+\infty$, periodic with period $a>0$, and essentially distinct from the identically zero function. Let ω_n and θ_n $(n=0, 1, 2, \cdots)$ be two sequences of real numbers; also we suppose that $\omega_n\rightarrow\infty$ as $n\rightarrow\infty$. The base constituted by the following functions:

$$\phi_n(x) = f(\omega_n x + \theta_n) \qquad (n = 0, 1, 2, \cdots)$$

possesses the Cantor-Lebesgue and the Lusin-Denjoy properties. In fact, according to a theorem recently proved by Mazur and Orlicz [2] which generalizes Steinhaus' theorem [5, p. 269], we can assert that the relation:

$$\limsup_{r\rightarrow\infty} |\phi_{n_r}(x)| = \operatorname*{ess.\,l.u.b.}_{-\infty<x<+\infty} |f(x)|$$

holds almost everywhere for any increasing sequence of positive integers n_r $(r=1, 2, \cdots)$. This relation enables us to conclude that the base possesses the property $n(P_1)\sim$CL$\sim$LD. If the function $f(x)$ is summable on any finite interval, we can make use of the following elementary equation:

$$\lim_{n\rightarrow\infty} \int_\Delta |\phi_n(x)|\, dx = \frac{m(\Delta)}{a}\int_0^a |f(x)|\, dx$$

where Δ denotes a measurable set of real numbers. From this equation we may infer that the base possesses the property $n(P_3)\sim$CL$\sim$LD.

I wish to express my warmest thanks to Professor Lelio I. Gama, of the National Observatory, Rio de Janeiro.

Bibliography

1. Hobson, E. W., *The theory of functions of a real variable and the theory of Fourier's series*, vol. 2, Cambridge, 1926.

2. Mazur, S. and Orlicz, W., *Sur quelques propriétés des fonctions périodiques et presque-périodiques*, Studia Mathematica vol. 9 (1940) pp. 1–16.

3. Nachbin, L., *On the series of functions almost everywhere absolutely divergent* (in Portuguese), Revista de Matemáticas y Fisica Teórica de la Universidad Nacional de Tucumán vol. 3 (1942) pp. 311–315.

4. Stone, M. H., *A note on the theory of infinite series*, Ann. of Math. vol. 32 (1931) pp. 233–238.

5. Zygmund, A., *Trigonometrical series*, Monografje Matematyczne, vol. 5, Warsaw-Lwów, 1935.

National School of Engineering,
University of Brazil, Rio De Janeiro.