A STUDY OF THE PROJECTIVE DIFFERENTIAL GEOMETRY OF SURFACES BY MEANS OF A MODIFIED TENSOR ANALYSIS

BY

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1. Introduction. The present paper is a study of the projective differential geometry of surfaces in ordinary space by means of a system of linear homogeneous differential equations of the first order. The use of a tensor notation with intrinsic differentiation (a generalized covariant differentiation introduced in §2) enables us to express general results with great formal simplicity.

Simple forms of the analytic conditions for fixity of points, lines, planes and algebraic surfaces (obtained in §§2 and 3) make possible a remarkably compact formulation of the theory of envelopes. Applications of a particular aspect of this theory yield the characteristics of significant families of planes, thereby providing (i) basic elements for the geometric characterizations of the differential invariants introduced in §§6 and 10, and (ii) means of determining (in §12) dual systems of hypergeodesics.

The conditions that a given net of an arbitrarily selected surface $S$ be a conjugate net or the asymptotic net, respectively, are expressed in new ways (in §4) by means of simple relations among the local coordinates of a generic point of $S$ and their first and second intrinsic derivatives. Each of these conditions assumes its simplest form if the given net is the parametric net.

§5 is devoted to a study of the effects on the coefficients of the differential equations of the most general transformations of independent and dependent variables which leave the surface unchanged. These effects, expressed by the law of transformation of the connection, are first interpreted in terms of local point coordinates of a generic point of the surface and then in terms of local plane coordinates of a general tangent plane to the surface. These interpretations lead to the introduction and geometric characterization (in §6) of a new quadratic differential invariant $\Omega$ of a general pair of analytic surfaces $S, S'$. An invariant quadratic differential form which plays a basic role in a former paper by the author\(^1\) will be shown in the present paper to be a projective generalization of the Euclidean second fundamental form of $S$. We shall denote this form by $\phi_2$ and its discriminant by $d$. It is shown (in §7) that the form $\phi_1$ defined by the relation

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\(^{1}\) P. O. Bell [1, (2.6)]. (Numbers in brackets refer to the bibliography at the end of the paper.)
\[ \phi_1 = \Omega d/h, \]

in which we denote by \( h \) the discriminant of \( \Omega \), becomes the Euclidean first fundamental form when the surface \( S' \) is selected to be the locus of the center of mean curvature of \( S \). The form \( \phi_1 \) for the general pair of surfaces \( S, S' \) will therefore be called the projective first fundamental form of \( S \) relative to \( S' \). We define the projective normal curvature \( \kappa_n \) of \( S \) relative to \( S' \), for a given direction at a point \( x \), by the ratio of the projective fundamental forms

\[ \kappa_n = \phi_2 / \phi_1. \]

The sum and product of the extremal values of \( \kappa_n \) at \( x \) are projective invariants which we denote by \( K_m \) and \( K \) and which we call the projective mean curvature and the projective total curvature, respectively, of \( S \) relative to \( S' \). The class of \( R \)-associate surfaces of \( S \) is geometrically characterized and the invariants \( K_m \) and \( K \) of \( S \) relative to a general member of this class are then determined.

In §8 a geometric characterization for \( \phi_2 \) is obtained and a geometric relation among the forms \( \phi_2^{1/2}, \phi_2 \) and the form for the projective linear element is established. In §9 a system of projective geodesics of \( S \) is defined in association with a general \( R \)-conjugate congruence. A projective theorem concerning the cusp-axis of this system is proved, and a metric analogue of this theorem is established.

Differential invariants of a general tetrad of surfaces are defined and geometrically characterized (in §10), special cases of which are shown (in §11) to be the classical projective linear element of a surface, the elementary forms of Bompiani, and Fubini’s quadratic normal form. A principle of duality is outlined (in §12) and used to characterize dual systems of hypergeodesics of which union-curves of a congruence \( \Gamma' \), the \( \rho \)- and \( \sigma \)-tangeodesics, and the duals of these are special cases. Certain properties of the \( \rho \)- and \( \sigma \)-tangeodesics serve to characterize the first canonical pencil.

2. The fundamental differential equations. Consider in ordinary projective space four analytic surfaces \( S_i, i = 0, 1, 2, 3 \), whose corresponding generic points \( x_i \) are linearly independent. The projective homogeneous coordinates of \( x_i \) form a square matrix of rank and order four whose elements are analytic functions \( x_{i}^{(p)} \) of two independent variables \( u^1, u^2 \). The general coordinates of any point in space may consequently be expressed as a linear combination of the corresponding coordinates of the points \( x_i \). It follows that a set of functions \( \Gamma^{h}_{i \alpha} \) of \( u^1, u^2 \), which we call the connection of the surfaces \( S_i \), can be uniquely determined such that the functions \( x_{i}^{(p)} \) are solutions of the system of differential equations

\[ \frac{\partial x_i}{\partial u^\alpha} - \Gamma^{h}_{i \alpha} x_h = 0, \quad i = 0, 1, 2, 3; \alpha = 1, 2, \]

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in which \( h \) denotes a dummy index. Throughout the present paper, except when otherwise specified, Latin indices have the range 0, 1, 2, 3 whereas Greek indices have the range 1, 2; repeated indices in upper and lower positions of adjoining symbols indicate summing over their respective ranges. The most general sets of solutions of (2.1) are the sets of coordinates of the points \( \bar{x}_i \) which correspond to the points \( x_i \) by a general projective transformation

\[
\bar{x}_i^p = c_j^p x_i^j.
\]

Thus the equations (2.1) determine the projective differential properties common to all projective transforms of the tetrad of surfaces \( S_i \).

For the sake of typographical simplicity, where no confusion can arise we shall denote the partial derivative of a quantity with respect to \( u^\alpha \) by the symbol for the quantity with the subscript \( \alpha \) adjoined. Thus

\[
x_{ia} = \frac{\partial x_i}{\partial u^\alpha}, \quad \Gamma_{ia\beta} = \frac{\partial x_{ia}}{\partial u^\beta}, \quad x_{i\alpha\beta} = \frac{\partial^2 x_i}{\partial u^\alpha \partial u^\beta}.
\]

Plückerian line coordinates \( \omega_{ij}^{pq} \) of the line \( \omega_{ij} \) joining the points \( x_i, x_j \) are defined by the formula

\[
\omega_{ij}^{pq} = x_i^p x_j^q - x_j^p x_i^q.
\]

By making use of (2.1) in an obvious manner we find that the coordinates \( \omega_{ij}^{pq} \) are solutions of the system of equations

\[
(2.2) \quad \omega_{i\alpha} - \omega_{i\alpha} \Gamma_{h}^{\alpha} - \omega_{h}^{\alpha} \Gamma_{i\alpha}^{h} = 0.
\]

Let \( |x| \) denote the determinant whose elements are the functions \( x_i^p \) and let \( \xi^r_i \) denote the normalized cofactor of \( x_i^r \) in \( |x| \), defined by the relations

\[
(2.3) \quad \xi^r_i x_h^i = \delta^r_h,
\]

in which the right member represents the Kronecker deltas. The functions \( \xi^r_i \), \( i = 0, 1, 2, 3 \), form a set of general homogeneous plane coordinates of the plane determined by the three points \( x_h \) wherein \( h \neq r \). Performing partial differentiation with respect to \( u^\alpha \) of (2.3) and making use of (2.1) yields

\[
x_h^{i\alpha} \xi^r_i + \Gamma_{r\alpha} = 0.
\]

Forming the inner product of the left member of this equation with \( \xi^r_h \) we find that the plane coordinates \( \xi^r_i \) are solutions of the system of equations

\[
(2.4) \quad \xi^r_i + \xi^r_h \Gamma_{r\alpha} = 0.
\]

The left members of (2.1), (2.2), and (2.4) will be called the intrinsic derivatives of \( x_i, \omega_{ij}, \) and \( \xi^r \), respectively, and will be denoted simply by \( x_{i,\alpha}, \omega_{ij,\alpha}, \) and \( \xi^r_{\alpha} \). We define, similarly, the intrinsic derivatives of aggregates \( a_{ij}, a_{ij}, a_{i_1i_2 \ldots i_p}, a_{i_1i_2 \ldots i_p} \) by the respective equations
\[ a_{ij,\alpha} = a_{ij,\alpha} + a^{ik}\Gamma_{k\alpha} + a^{kj}\Gamma_{k\alpha}, \]
\[ a_{ij,\alpha} = a_{ij,\alpha} - a_{i\alpha}\Gamma_{j\alpha} - a_{j\alpha}\Gamma_{i\alpha}, \]
\[ (2.5) \]
\[ a_{i_1i_2\ldots i_p,\alpha} = a_{i_1i_2\ldots i_p,\alpha} - \sum_{r=1}^{p} a_{i_1i_2\ldots i_{r-1}i_{r+1}\ldots i_p,\Gamma_{i_r,\alpha}}, \]
\[ a_{i_1i_2\ldots i_p,\alpha} = a_{i_1i_2\ldots i_p,\alpha} + \sum_{r=1}^{p} a_{i_1i_2\ldots i_{r-1}i_{r+1}\ldots i_p,\Gamma_{i_r,\alpha}}. \]

It is evident from the forms of these equations that intrinsic differentiation of sums and of outer and inner products of aggregates of the types here considered obeys the same rules as ordinary differentiation.

If we put \( u^\alpha = u^\alpha(t), \alpha = 1, 2, \) a point \( x \) whose general coordinates are functions of \( u^1, u^2 \) describes a curve \( C \) as \( t \) varies. The aggregates
\[ \frac{du^\alpha}{dt}, \frac{a_{ij,\ldots}}{dt}, \frac{du^\alpha}{dt}, \]
will be called the intrinsic derivatives along the curve \( C \) of the corresponding aggregates \( a_{ij,\ldots}, a_{ij,\ldots}^l \). Clearly these aggregates reduce to the ordinary intrinsic derivatives
\[ a_{ij,\ldots}, a_{ij,\ldots}^l, \]
when the curve \( C \) is the parametric \( u^\alpha \) curve.

The integrability conditions of the system (2.1) may be obtained by demanding that the order of differentiation of \( x_i \) be immaterial, that is
\[ x_{i\alpha\beta} = x_{i\beta\alpha}. \]
Making use of (2.1) to express the members of this equation as linear combinations of \( x_k \) we obtain
\[ \Gamma^k_{i\alpha,\beta}x_k = \Gamma^k_{i\beta,\alpha}x_k. \]
Since the points \( x_k \) are linearly independent, we obtain the following form for the conditions of integrability
\[ \Gamma^k_{i\alpha,\beta} = \Gamma^k_{i\beta,\alpha}, \]
wherein intrinsic differentiation is with respect to the index \( k \). It may be easily demonstrated that these integrability conditions, which insure the existence of integral surfaces of a system of equations (2.1), may also be geometrically interpreted as follows. Let \( S_i^\alpha \) denote a general transversal surface of the congruence of \( u^\alpha \) tangents to \( S_i, \) generated by the point \( \rho^\alpha \) whose general homogeneous coordinates are given by
\[ \rho^\alpha = x_{i\alpha} + a^\alpha x_i, \]
wherein \( a^\alpha \) is a function of \( u^1, u^2 \). In correspondence with an arbitrarily se-
lected surface $S^\alpha$ there exist infinitely many surfaces $S^\beta$ with the property that the $u^\beta$-tangent to $S^\alpha$ at $\rho^\alpha$ intersects the $u^\alpha$-tangent to $S^\beta$ at $\rho^\beta$ if, and only if, the integrability conditions are satisfied. The points $\rho^\beta$ are defined by the above equation wherein $a^\beta$ satisfies the condition

$$a^\beta_a = a^\alpha_\beta, \quad \alpha \neq \beta.$$  

3. Conditions for fixity of points, lines and algebraic surfaces. The linearly independent points $x_i$, $i = 0, 1, 2, 3$, serve as vertices of a moving reference frame for local point coordinates. If we define the general coordinates of the unit point by the vector relation

$$U = x_0 + x_1 + x_2 + x_3,$$

a point $X$ whose general coordinates are given by

$$X = x^h x_h$$

has the functions $x^h$ as its local coordinates.

A point $X$ is fixed as $t$ varies if, and only if, the general coordinates of $X$ satisfy a relation of the form

$$\frac{dX}{dt} = \lambda X,$$

wherein $\lambda$ is a function of $t$. If we substitute in this relation the right member of (3.1) and make use of (2.1), we obtain, on equating coefficients of $x_k$, the relations

$$x^{k,a} \frac{du^a}{dt} = \lambda x^k,$$

where $\lambda$ is a function of $t$, which are necessary and sufficient that the point $X$ be fixed as $t$ varies.

Local line coordinates of the line joining two points $X$, $Y$ are defined by

$$\omega^{ij} = x^i y^j - x^i y^i,$$

in which $x^i$, $y^i$ are the local coordinates of $X$, $Y$. We prove the following theorem.

**Theorem 3.1.** Necessary and sufficient conditions that the line $\omega$ be fixed as $t$ varies are that the local line coordinates $\omega^{ij}$ satisfy a system of equations of the form

$$\omega^{ij,a} \frac{du^a}{dt} = \eta \omega^{ij}, \quad (3.1)$$

in which $\eta$ is an arbitrary function of $t$. 

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The line $\omega$ is fixed as $t$ varies if, and only if, the general coordinates $X$, $Y$ satisfy a system of equations of the form

$$\frac{dX}{dt} = a_1X + a_2Y, \quad \frac{dY}{dt} = b_1X + b_2Y,$$

in which the coefficients are functions of $t$. If we substitute for $X$ and $Y$ the respective expressions $x^h x_h$ and $y^h y_h$ in these equations and make use of (2.1) we find the systems of equations

$$(3.2) \quad x^h, a \frac{du^a}{dt} = a_1x^h + a_2y^h, \quad y^h, a \frac{du^a}{dt} = b_1x^h + b_2y^h,$$

which are necessary and sufficient that the line $\omega$ joining the points $X$, $Y$ be fixed as $t$ varies.

If we differentiate the functions $\omega^{ij}$ intrinsically we find

$$\omega^{ij}, a \frac{du^a}{dt} = (x^i y^j, a + y^i x^j, a - x^i y^j, a - y^i x^j, a) \frac{du^a}{dt}.$$ 

Making use of equations (3.2) in the right member of this equation we obtain (3.1) in which $\eta = a_1 + b_2$. Hence equations (3.1) are necessary conditions. To prove them sufficient let the $(i, j)$-line coordinate of the line joining a general pair of points $z, w$ be denoted by $(z, w)$. Thus

$$(z, w) = z^i w^j - w^i z^j.$$ 

Equations (3.1) may consequently be written in the determinantal form

$$\{(x, y, a) + (x, a, y) \} \frac{du^a}{dt} + \eta(y, x) = 0.$$ 

Combining the first and third determinants we have

$$\left( x, y, a \frac{du^a}{dt} - \eta y \right) = (y, x, a) \frac{du^a}{dt}.$$ 

Therefore $x, a \frac{du^a}{dt}$ and $y, a \frac{du^a}{dt}$ must be defined by equations of the form (3.1). This completes the proof.

Similar arguments may be applied to prove the following theorem.

**Theorem 3.2.** A point $X$ and a line $\omega$ are fixed as $u^1, u^2$ vary independently if and only if their local coordinates satisfy respective systems of equations

$$x^{i, a} = \lambda_{a} x^{i}, \quad \omega^{i, a} = \eta_{a} \omega^{i}, \quad i, j = 0, 1, 2, 3; \alpha = 1, 2,$$

in which $\lambda$ and $\eta$ are arbitrary solutions of the equations

$$\theta_{a} = \theta_{b}.$$
It may be easily verified that these equations may be reduced to normal forms

$$x^i, \alpha = 0, \quad \omega^{ij}, \alpha = 0$$

by making the substitutions

$$x^i = r x^i, \quad \omega^{ij} = s \omega^{ij}$$

where $r$, $s$ satisfy the respective relations

$$\log r = \lambda, \quad \log s = \eta.$$

If the equation of a system of surfaces is expressed in terms of local point coordinates, the usual theory of envelopes may be applied to the system, providing the derivatives of local point coordinates are calculated by means of the conditions of fixity of a point\(^{(2)}\). An equation of the form

$$f(x^0, x^1, x^2, x^3, u^1, u^2) = 0$$

which is homogeneous in the variables $x^i$ represents a two-parameter family of surfaces. The characteristic points of the family corresponding to the parameter values $u^1, u^2$ have local coordinates which are solutions of the three equations

$$f = 0, \quad f_\alpha - x^i \Gamma^i_{\alpha \beta} \frac{\partial f}{\partial x^\beta} = 0.$$

The locus of the characteristic points is the envelope of the family.

If we put $u^\alpha = u^\alpha(t), \alpha = 1, 2$, the equation $f = 0$ represents a one-parameter family of surfaces. The equations of the characteristics are found to be

$$f = 0, \quad \left( f_\alpha - x^i \Gamma^i_{\alpha \beta} \frac{\partial f}{\partial x^\beta} \right) \frac{du^\alpha}{dt} = 0.$$

The focal points of the characteristics are defined by these equations together with a third equation obtained by differentiating the second equation and making use of the fixed point conditions.

It follows that the edge of regression of a one-parameter family of planes $x^i = 0$, in which the local coordinates $x^i$ are functions of $u^\alpha(t)$, is the locus of focal points whose coordinates satisfy the equations

$$(3.3) \quad x^i x^j = 0, \quad x^i \xi_{i, \alpha} \frac{du^\alpha}{dt} = 0, \quad x^i \left( \xi_{i, \alpha} \frac{d^2 u^\alpha}{dt^2} + \xi_{i, \alpha, \beta} \frac{du^\alpha}{dt} \frac{dw^\beta}{dt} \right) = 0.$$

The first two of these equations define the characteristics.

Let $a_0, \ldots, a_p$ denote an aggregate of functions of $u^\alpha(t)$. The characteristics of the one-parameter family of algebraic surfaces

\(^{(2)}\) E. P. Lane [1, p. 207].
are readily found to be defined by the equations
\[ a_{i_1 \ldots i_p} x^{i_1} \cdots x^{i_p} = 0, \quad a_{i_1 \ldots i_p, \alpha} \frac{d u^\alpha}{d t} x^{i_1} \cdots x^{i_p} = 0. \]

In virtue of the form of the second of these equations we have

**Theorem 3.3.** The algebraic surface whose equation in local point coordinates is
\[ a_{i_1 \ldots i_p} x^{i_1} \cdots x^{i_p} = 0 \]
is fixed as \( t \) varies if, and only if,
\[ a_{i_1 \ldots i_p, \alpha} \frac{d u^\alpha}{d t} = \mu a_{i_1 \ldots i_p} \]
in which \( \mu \) is a function of \( t \).

**Corollary.** The surface \( a_{i_1 \ldots i_p} x^{i_1} \cdots x^{i_p} = 0 \) is fixed as \( u^1, u^2 \) vary independently if, and only if,
\[ a_{i_1 \ldots i_p, \alpha} = \mu a_{i_1 \ldots i_p}, \]
where \( \mu_\alpha \) are functions of \( u^1, u^2 \).

Necessary and sufficient conditions that a plane whose local coordinates are \( \xi_i \) be fixed as \( u^1, u^2 \) vary independently are, consequently,
\[ \xi_{i, \alpha} = \mu_\alpha \xi_i \]
where \( \mu_\alpha \) are functions of \( t \).

4. **Conjugate nets of a general surface.** The general coordinates of a generic point \( X \) of a general analytic surface \( S \) are given by a vector relation of the form
\[ X = x^i x_i. \]
The analytic condition that two directions \( d u^2 / d u^1, \delta u^2 / \delta u^1 \) be conjugate with respect to \( S \) at \( X \) is that they satisfy the determinantal equation
\[ \left( X, \frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2}, X_{\alpha \delta} d u^\alpha \delta u^\delta \right) = 0. \]

In view of the vector relations
\[ dX = x_i x^i, d u^\alpha, \quad \delta X = x_i x^i, \delta u^\delta, \quad d \delta X = x_i x^i, \delta u^\delta d u^\alpha, \]
which follow from (2.1), this determinantal equation may be expressed in terms of local coordinates of \( X \) as follows

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Since an asymptotic direction of $S$ at $X$ is a self-conjugate direction, the differential equation of the net of asymptotic curves of $S$ is

$$(x^i, x^i_1, x^i_2, x^i_{\beta, \delta} d\delta du^\alpha) = 0.$$  

It follows that the parametric net of $S$ is (a) a conjugate net if, and only if,

$$(x^i, x^i_1, x^i_2, x^i_{1,1}) = 0,$$

(b) the asymptotic net of $S$ if, and only if

$$(x^i, x^i_1, x^i_2, x^i_{\alpha, \alpha}) = 0, \quad \alpha = 1, 2.$$  

Consequent to these relations we have the following theorems.

**Theorem 4.1.** Necessary and sufficient conditions that the parametric net of $S$ be a conjugate net are that the local coordinates $x^i$ of $X$ be solutions of a system of differential equations of the form

$$x^i_{1,2} + bx^i + a^\alpha x^i_\alpha = 0,$$

in which $b, a^\alpha$ are functions of $u^1, u^2$.

**Theorem 4.2.** Necessary and sufficient conditions that the parametric net of $S$ be the asymptotic net of $S$ are that the coordinates $x^i$ satisfy a system of equations of the form

$$x^i_{\beta, \beta} + c_{\beta} x^i + d_\beta x^i_\alpha = 0.$$  

If $S$ is the surface $S_0$ of the reference tetrad, the curvilinear differential equations defining a conjugate net of $S_0$ are found to be

$$(\Gamma^i_{01}, \Gamma^i_{02}, \Gamma^i_{0\alpha, \delta} d\delta du^\alpha) = 0, \quad du^2/du^1 = \lambda(u^1, u^2), \quad i = 1, 2, 3.$$  

The differential equation of the asymptotic net of $S_0$ is, therefore,

$$(\Gamma^i_{01}, \Gamma^i_{02}, \Gamma^i_{0\alpha, \delta} d\delta du^\alpha) = 0.$$  

If we assume that the tangent plane to $S_0$ at $x_0$ is the plane $x^3 = 0$, we have

$$x_{0\alpha} = \Gamma^h_{0\alpha} x_h, \quad \Gamma^3_{0\alpha} = 0, \quad \alpha = 1, 2.$$  

It follows that, in this case, a conjugate net of $S_0$ is defined by a pair of equations of the form

$$\Gamma^i_{0\alpha} \Gamma^3_{\delta} d\delta du^\alpha = 0, \quad du^2/du^1 = \lambda(u^1, u^2),$$

and the asymptotic net of $S_0$ is defined by the equation

$$\Gamma^i_{0\alpha} \Gamma^3_{\delta} d\delta du^\alpha = 0.$$  

Furthermore, if $\Gamma^i_{01} \Gamma^3_{1} = \Gamma^i_{01} \Gamma^3_{2} = 0$, the parametric curves of $S_0$ form a con-
jugate net. Similarly, if $\Gamma'_{\alpha\beta} = \Gamma'_{\alpha\gamma} = 0$, the parametric curves of $S_0$ form the asymptotic net of $S_0$.

The condition that the directions $du^2/du^1$, $\delta u^2/\delta u^1$ be conjugate to $S_0$ at $x_0$ assumes its simplest form

$$
\Gamma_{a\beta} du^a \delta u^\beta = 0
$$

if the vertices $x_\alpha$, $\alpha = 1, 2$, lie on the corresponding tangents to the $u^a$ curves of $S_0$ at $x_0$, for in this case we have $\Gamma_{0\alpha} = 0$, $\alpha \neq \beta$, $\alpha, \beta = 1, 2$. If we define differential forms $\phi^i$ by the relations

$$
\phi^i = \Gamma_{i\beta} \delta u^\beta,
$$

the direction $du^2/du^1$ conjugate to $\delta u^2/\delta u^1$ is given by the relation

$$
du^2/du^1 = - \phi^3/\phi^2.
$$

5. Transformation of the connection. A general analytic surface $S'$ is undisturbed by the general transformations: (i) of independent variables

(5.1) $u^a = u^a(\check{u}^1, \check{u}^2)$, $J(u, \check{u}) \neq 0$;

(ii) of dependent variables of the form

(5.2) $x^i = \check{x}^i x_h$,

in which $\check{x}^i$ are functions of $\check{u}^1$, $\check{u}^2$; (iii) of proportionality factor

(5.3) $X = \check{X}$

in which $X$ and $\check{X}$ denote general homogeneous projective coordinates of a generic point of $S'$ defined by the vector forms

$$
X = x^i x_i, \quad \check{X} = \check{x}^i \check{x}_i,
$$

and $\check{\lambda}$ is a function of $\check{u}^1$, $\check{u}^2$.

In view of the linear independency of the points $\check{x}_h$ the set of functions $\Gamma_{i\alpha}^h$ exists for which

$$
\frac{\partial \check{x}_i}{\partial \check{u}^\alpha} = \Gamma_{i\alpha}^h \check{x}_h.
$$

Equating the right member of this equation to the derivative of the right member of (5.2) and making use of (2.1) yields

(5.4) $\Gamma_{i\alpha}^h \check{\lambda}_h x_j = \check{\lambda}_h \Gamma_{i\beta}^j x_j \frac{\partial u^\beta}{\partial \check{u}^\alpha} + \frac{\partial \check{\lambda}_i^j}{\partial \check{u}^\alpha} x_j$.

From (5.2) and (5.3) it follows that

(5.5) $x^h = \check{x}^i \check{\lambda}_i^h$,

and therefore
On equating corresponding coefficients of $x_i$ in (5.4), multiplying by $\lambda$ and making use of (5.5), we obtain the law of transformation of the connection $\Gamma^i_{\alpha\beta}$ under the most general transformation which leaves the surface $S'$ undisturbed:

$$\frac{\partial x^h}{\partial \bar{x}^i} = \lambda \delta^h_i. \tag{5.6}$$

Writing (5.5) in the form

$$x^i = \bar{x}^i \frac{\partial x^i}{\partial \bar{x}^i}, \tag{5.7}$$

we find, on differentiating with respect to $\bar{u}^\alpha$,

$$\bar{x}^i \frac{\partial^2 x^i}{\partial \bar{u}^\alpha \partial \bar{x}^i} = \frac{\partial x^i}{\partial \bar{u}^\beta} \frac{\partial x^\beta}{\partial \bar{x}^i} - \frac{\partial \bar{x}^i}{\partial \bar{u}^\alpha} \frac{\partial x^i}{\partial \bar{x}^i}. \tag{5.8}$$

On substituting the right member of this relation in the equation obtained by forming the inner products of the members of (5.7) with $\bar{x}^i$, we obtain the law of transformation of intrinsic derivatives of local point coordinates

$$x^i_{\beta} = \bar{x}^i_{\alpha} \frac{\partial x^i}{\partial \bar{u}^\alpha} + x^i \frac{\partial (\log \lambda)}{\partial \bar{u}^\beta}, \tag{5.9}$$

in which $x^i_{\beta}$ and $\bar{x}^i_{\alpha}$ denote the intrinsic derivatives of $x^i$ and $\bar{x}^i$ with respect to $\bar{u}^\beta$ and $\bar{u}^\alpha$, respectively.

Let $\xi^i$ and $\bar{\xi}^i$ denote the faces of the tetrahedra $(x_0, x_1, x_2, x_3)$ and $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$, respectively, which are opposite the corresponding vertices $x_i$ and $\bar{x}_i$. General homogeneous projective plane coordinates of these faces may be defined by the functions $\xi^i_{h}$ and $\bar{\xi}^i_{h}$, $h = 0, 1, 2, 3$, which are solutions of the systems of equations

$$(5.9) \quad \xi^i_{h} x^j_{h} = \bar{\xi}^i_{h} \bar{x}^j_{h} = \delta^i_j. \tag{5.10}$$

According as the old or new local reference tetrahedron is employed, the general homogeneous projective plane coordinates of a generic tangent plane of a surface $S$ are given by the vector forms

$$(5.10) \quad \pi = \xi^i \xi^i, \quad \bar{\pi} = \bar{\xi}^i \bar{\xi}^i$$

in which the local coordinates $\xi_i$ and $\bar{\xi}_i$ are functions of $u^1, u^2$ and $\bar{u}^1, \bar{u}^2$, respectively.

(\textsuperscript{(*)} For $i, j = 1, 2$ this becomes the law of transformation of an affine connection if we impose the conditions that $\frac{\partial x^h}{\partial \bar{x}^i} = 0$, $h = 0, 3$, $\frac{\partial x^i}{\partial \bar{x}^i} = \lambda \frac{\partial u^1}{\partial \bar{u}^i}$, in which $\lambda$ is a constant.)
spectively. In order that the forms of (5.10) represent a common plane they must be related by a transformation of proportionality factor

\[ \pi = \mu \pi. \]

In view of equations (5.9) the general transformation (5.2) may be expressed in terms of general homogeneous projective plane coordinates as

\[ \xi^i = \mu^i \xi^h, \]

in which the coefficients \( \mu^i \) are the functions of \( \tilde{u}^1, \tilde{u}^2 \) defined by the equations

\[ \mu^i \delta^j = \delta^i_j. \]

On substituting the right member of (5.12) for \( \xi^i \) in (5.11) and equating corresponding coefficients of \( \xi_h \), we obtain the following relations connecting local plane coordinates

\[ \xi_h = \mu \mu^i \xi^i. \]

In virtue of these relations and (5.5), we find that

\[ \frac{\partial \xi_h}{\partial \xi^i} = \frac{\partial \bar{\xi}^h}{\partial \bar{x}^i} = \mu \delta^i_j. \]

If now we differentiate the members of (5.11) with respect to \( u^a \) and equate corresponding coefficients of \( \xi^i \), we readily obtain the law of transformation of intrinsic derivatives of local plane coordinates

\[ \xi_{j,a} = \xi_{i,\beta} \frac{\partial \bar{u}^\beta}{\partial u^a} \frac{\partial \xi_j}{\partial \xi^i} + \xi_j \frac{\partial (\log \mu)}{\partial u^a}. \]

The law of transformation of the connection \( \Gamma^j_{ia} \) may be written in the form

\[ \Gamma^j_{ia} \xi_h = \Gamma^j_{i\beta} \xi^h \frac{\partial \bar{u}^\beta}{\partial u^a} \frac{\partial \bar{\xi}_i}{\partial \bar{u}^a} - \frac{\partial (\log \bar{\lambda})}{\partial \bar{u}^a} \bar{\lambda}_i. \]

Compounding the members of this equation with \( \mu^i_k \) yields

\[ \Gamma^k_{ia} = \left( \frac{\partial \bar{\lambda}_i^j}{\partial \bar{u}^a} + \bar{\lambda}_i^j \Gamma^j_{i\beta} \right) \mu^k_j \frac{\partial \bar{u}^\beta}{\partial u^a}. \]

If we denote by \( \lambda_i^j \) the functions of \( u^1, u^2 \) defined by the identities

\[ \lambda_i^j(u^1, u^2) \equiv \bar{\lambda}_i^j(\bar{u}^1, \bar{u}^2) \]

we have for the determination of the functions \( \Gamma^k_{ia} \) the forms

\[ \Gamma^k_{ia} = \lambda_{i,\beta} \mu^k_j \frac{\partial \bar{u}^\beta}{\partial u^a}, \]
in which intrinsic differentiation is with respect to the upper index.

6. **A differential invariant of a general pair of surfaces.** If the point $X$ of a surface $S'$ does not lie in the plane $\pi$ of a surface $S$, a relation free from $\mu$ and $\lambda$ may be obtained by making use of relations (5.5), (5.8), (5.13) and (5.15). Such a relation is the tensor equation

$$\bar{a}_{\gamma \delta} = a_{\alpha \beta} \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial u^\beta}{\partial \bar{u}^\delta}$$

in which the quantities $a_{\alpha \beta}$ are the functions of $u^1, u^2$ defined by

$$a_{\alpha \beta} = \left( \frac{x^i \xi_\alpha \eta_\beta}{x^i} - x^i x^j \alpha x^j, \beta \right)/(x^i)^2,$$

and the quantities $\bar{a}_{\gamma \delta}$ are the corresponding barred functions of $\bar{u}^1, \bar{u}^2$. The quadratic differential form $\Omega$ defined by

$$\Omega = a_{\alpha \beta} du^\alpha du^\beta$$

is, therefore, an absolute invariant of the pair of surfaces $S, S'$.

Let $X_1$ and $\pi_1$ denote the point of $S'$ and the plane of $S$, respectively, which correspond to the parameter values $u^1 + du^1, u^2 + du^2$. A geometric characterization of the invariant (6.1) will now be derived.

The general point coordinates of $X_1$ are given by the development

$$X(u^1 + du^1, u^2 + du^2) = X + \frac{\partial X}{\partial u^\alpha} du^\alpha + (\omega)$$

in which $(\omega)$ denotes terms of order at least two. In view of (2.1) we may write

$$\frac{\partial X}{\partial u^\alpha} = x^i, \alpha x^i.$$

Hence, to terms of order one, the local point coordinates of $X_1$ are given by

$$x^i + x^i, \alpha du^\alpha.$$

Similarly, we find that the local plane coordinates of $\pi_1$ are, to terms of order one, given by

$$\xi^i + \xi^i, \alpha du^\alpha.$$

Let $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ denote the forms which are defined by the relations

$$\theta_{00} = x^i \xi_\alpha, \theta_{01} = x^i \xi_\alpha du^\alpha, \theta_{10} = \xi, \alpha x^i du^\alpha, \theta_{11} = \xi, \alpha x^i, \beta du^\alpha du^\beta.$$

Let $Y, Y_1$ denote the respective points of intersection of the planes $\pi, \pi_1$ with the line joining $X, X_1$. The local coordinates of any point collinear with $X, X_1$ are expressible in the form

$$z^i = \mu x^i + x^i, \alpha du^\alpha.$$
The values of \( \mu \) with which the points \( X, X_1 \) are associated are \( \infty, 1 \), respectively. The point \( z \) lies in the plane \( \pi \) or in the plane \( \pi_1 \) according as \( \mu \) is the root of the first or of the second of the equations

\[
(\mu_1 x^i + x^i, adu^a)\xi_i = 0, \quad (\mu_2 x^i + x^i, adu^a)(\xi_i + \xi_i, adu^a) = 0
\]

The values of \( \mu \) with which \( Y, Y_1 \) are associated are, therefore,

\[
\mu_1 = -\theta_{10}/\theta_{00}, \quad \mu_2 = - (\theta_{10} + \theta_{11})/(\theta_{00} + \theta_{01}),
\]

respectively. The cross-ratio of the four points \( X, Y, X_1, Y_1 \) is the cross-ratio of the four corresponding values of \( \mu \)

\[
(\infty, \mu_1, 1, \mu_2) = (\theta_{00}\theta_{10} - \theta_{00}\theta_{11})/(\theta_{00} + \theta_{01})(\theta_{00} + \theta_{10}).
\]

Hence, we find that the quadratic differential invariant (6.1) is the principal part of the cross-ratio

\[
(X, Y, X_1, Y_1).
\]

Since the form (6.1) is symmetric in \( \xi_i, x^i \), the above characterization may be dualized as follows: A plane \( \pi \) of \( S \) and a neighboring plane \( \pi_1 \), corresponding to the increments \( du^1, du^2 \) (except for infinitesimals of order at least two), intersect in the characteristic of \( \pi \) for the direction \( du^2/du^1 \). Let \( \pi_x, \pi_{x_1} \) denote the planes which pass through the respective points \( X, X_1 \) and contain the characteristic of \( \pi \). The principal part of the cross-ratio

\[
(\pi_x, \pi, \pi_{x_1}, \pi_1)
\]

is the differential invariant (6.1).

The dual characterizations just described apply to pairs of surfaces \( S', S \) generated by point \( X \) and plane \( \pi \), respectively, such that \( X \) does not lie in \( \pi \). Let us consider, briefly, the case of surfaces \( S', S \) for which the generic point \( X \) of \( S' \) lies in the plane \( \pi \) of \( S \). For this case we have

\[
(6.2) \quad \xi_i x^i = 0, \quad (x^i\xi_i, a + x^i, a\xi)du^a = 0,
\]

in which the second equation is satisfied identically in \( du^a \), that is, for every direction. If we assume that the point \( X \) lies on the characteristic of the plane \( \pi \) for a direction \( du^2/du^1 \), the point \( X \) is fixed, except for variations involving infinitesimals of order at least two, as \( u^1, u^2 \) vary in this direction. The condition for this is that the local coordinates of \( X \) satisfy the relations

\[
x^i, adu^a = \lambda x^idt, \quad (u^a = u^a(t)).
\]

Forming the inner products of the members of this equation with \( \xi_i \), we find, in view of equations (6.2),

\[
(6.3) \quad \theta_{00} = \theta_{10} = \theta_{01} = 0.
\]
Since the sets of equations (6.2) and (6.3) are symmetric in $\xi_i$ and $x^i$, we may state our results in the following theorem.

**Theorem 6.1.** If at a point $X$ of $S'$ the tangent line to $S'$ in a direction $du^2/du^1$ lies in the plane $\pi$ of $S$, the characteristic of $\pi$ for this direction passes through the point $X$. This condition is fulfilled if and only if the local coordinates $x^i$ and $\xi_i$ of $X$ and $\pi$ satisfy the equations

$$\theta_{00} = \theta_{10} = \theta_{01} = 0.$$

It follows that the equations (6.3) are satisfied independently of direction if the plane $\pi$ of $S$ is tangent to $S'$ at $X$. If the surface $S$ enveloped by $\pi$ is identical with $S'$ and the generic point $X$ is the contact point of $\pi$ with $S$, these equations are identities in $u^1, u^2$.

7. Projective fundamental forms of a surface and associated curvatures. Let the surface $S_0$ be referred to asymptotic parameters with Grove's normal coordinates$^{(4)}$ for the point $x_0$. This choice allows us to select as an edge of the local tetrahedron, with equal formal simplicity, the line through $x_0$ of any $R$-conjugate congruence. Let the point $x_3$ lie on the given $R$-conjugate line passing through $x_0$, and let the points $x_1, x_2$ be defined by the relations

$$x_a = x_{0a}, \quad u^1 = u, \quad u^2 = v,$$

so that the coordinates of the points $x_i$ satisfy the system of linear equations

$$(7.1) \quad x_{ia} = \Gamma^h_{ia} x_h$$

whose coefficients are related to those of Grove's canonical system by the equations

$$(7.2) \quad \Gamma^h_{0a} = \delta^h_a, \quad \Gamma^0_{11} = p, \quad \Gamma^0_{aa} = \theta_a, \quad \Gamma^3_{11} = \beta,$$

$$\Gamma^a_{12} = 0, \quad \Gamma^3_{22} = q, \quad \theta = \log R, \quad \Gamma^1_{22} = \gamma,$$

$$\Gamma^a_{12} = (\theta_{12} + \beta \gamma - \Gamma^0_{12})/\Gamma^3_{12}, \quad \Gamma^3_{2a} = (\Gamma^0_{aa} + (\Gamma^3_{aa}) \beta + \Gamma^0_{aa} \Gamma^3_{aa})/\Gamma^3_{12},$$

in which a repeated index does not indicate summation, and $a \neq b$.

The multiplier of $x_3$ may be so selected that the functions $\Gamma^3_{3a}$ vanish. This selection, which is defined by the condition $\Gamma^3_{12} = R$, does not restrict the location of the point $x_3$. We readily find that the form $\Omega$ for $S_0, S_3$ in which $S_0, S_3$ play the roles of $S, S'$ respectively is given by the equation

$$(7.3) \quad \Omega = \pi (du^1)^2 + 2(\kappa - \Gamma^0_{12}) du^1 du^2 + \chi (du^2)^2,$$

in which $\pi = p + \beta_2 + \beta \theta_2, \chi = q + \gamma_1 + \gamma \theta_1, \kappa = \beta \gamma + \theta_1$.

Grove$^{(6)}$ has defined analytically the projective curvature tensor of the

---

$^{(4)}$ We signify by “Grove's normal coordinates” a set of solutions of Grove's canonical system of differential equations. V. G. Grove [1, p. 582].

$^{(5)}$ V. G. Grove [2, pp. 121–122].
surface $S$ relative to the $R$-conjugate line $xy$ and the point $y$, and he has
given the analytic basis for the introduction of an associated metric on the
surface. It is a remarkable fact, which can be easily verified, that the form $\Omega$
for $S_0$, $S_3$ is characterized by the relation

$$\Omega = Kds^2$$

in which $K$ is the projective curvature of $S_0$ relative to the $R$-conjugate line and
the point $x_3$, and $ds^2$ is the associated metric of $S_0$. The curves defined by $\Omega = 0$
will be called the projective minimal curves of $S_0$ relative to $S_3$. We shall hereafter
refer to $K$ as the projective total curvature of $S_0$ relative to $S_3$.

Let a quadratic differential invariant of the form

$$\phi_2 = 2Rdu^1du^2$$

serve as the projective second fundamental form of $S_0$ relative to the $R$-con-
jugate congruence, and let $h$, $g$ and $d$ denote the discriminants of $\Omega$, $ds^2$ and $\phi_2$,respectively. The projective total curvature $K$ may be defined by the formula

$$K = d/g.$$

Since $h=K^2g$, it follows that

$$K = h/d = [(\kappa - \Gamma_0^2)^2 - \pi \chi]/R^2.$$

Introducing $\phi_1$ by the relation $\phi_1 = \Omega/K$, we now define by the relation

$$\kappa_n = \phi_2/\phi_1$$

the projective normal curvature $\kappa_n$ of $S_0$ relative to $S_3$ for the direction $du^2/du^1$
at $x_0$. The maximum and minimum values of $\kappa_n$ at a point $x_0$, which are the
roots of the equation

$$R\kappa_n^2 - 2(\kappa - \Gamma_0^2)\kappa_n + KR = 0,$$

will be called the projective principal normal curvatures of $S_0$ at $x_0$ relative to $S_3$.
The directions of $S_0$ at $x_0$ which correspond to these curvatures will be called the
projective principal directions of $S_0$ at $x_0$ relative to $S_3$. The curves of $S_0$
having these directions at each point $x_0$ will be called the projective lines of
curvature of $S_0$ relative to $S_3$. Their curvilinear differential equation may be
readily found to be

$$\pi(du^1)^2 - \chi(du^2)^2 = 0$$

in which $\pi = \rho + \beta_2 + \beta_\theta$, $\chi = \sigma + \gamma_1 + \gamma_\theta$, $\theta = \log R$. The sum of the projective principal normal curvatures of $S_0$ at $x_0$ serves to define the projective mean
curvature $K_m$ of $S_0$ at $x_0$ relative to $S_3$. Hence,

$$K_m = 2(\kappa - \Gamma_0^2)/R,$$

in which $\kappa = \beta_\gamma + \theta_1$, $\theta = \log R$. 

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Homogeneous cartesian coordinates $x^i$, $i=0, 1, 2, 3$, of the point $x_0$ of $S_0$ may be obtained by adjoining to ordinary rectangular coordinates of $x_0$ a fourth coordinate $x^3=1$. The point $z$ at infinity on the normal to $S_0$ at $x_0$ has homogeneous cartesian coordinates $z_0, z_1, z_2, 0$, the first three of which may be taken as the direction cosines of the normal to $S_0$ at $x_0$. If the generic point $x_3$ of $S_3$ is defined by the vector relation

$$x_3 = z + K_m x_0$$

in which $K_m$ denotes the mean curvature of $S_0$, and if $\phi_2$ is the second fundamental form of $S_0$, the associated entities $ds^2, K, \kappa, K_m$ defined, as above, relative to $S_3$ are the Euclidean first fundamental form, Gaussian curvature, normal curvature for a given direction, and mean curvature, respectively, of $S_0(6)$. Let us now specialize in projective manners the results of this section. The harmonic invariant of the differential forms $\Omega$ and $\phi_2$ vanishes if, and only if, $\Gamma^{0}_{12} = \kappa$. This is a necessary and sufficient condition that the projective minimal curves of $S_0$ relative to $S_3$ form a conjugate net. The surface $S_3$ thus selected is, moreover, such that the projective mean curvature of $S_0$ relative to $S_3$ vanishes. This surface $S_3$ will be called the $R$-associate of $S_0$. The following additional geometric characterization of the $R$-associate of $S_0$ may be readily verified. The harmonic conjugate of the point $x_0$ with respect to the points in which the $R$-conjugate line through $x_0$ intersects the quadric of Lie at $x_0$ is the point $x_3$ which generates the $R$-associate of $S_0$ as $x_0$ varies over $S_0$. The conjugate net characterized above as the projective minimal net of $S_0$ relative to the $R$-associate of $S_0$ is the mean conjugate net of $S_0$ when the $R$-conjugate congruence is the congruence of metric normals of $S_0(7)$. Let $P_\omega, P_0, P_1, x_3$ denote the points on the Fubini-Green projective normal to $S_0$ at $x_0$ whose general homogeneous coordinates are given by the forms

$$x_0 = x, \quad z = [x_{12} - 2^{-1}(\theta_{12} + 2\beta\gamma)x]/\beta\gamma, \quad z + x, \quad z + kx,$$

respectively. The point $P_1$ is the intersection of the projective normal to $S_0$ at $x_0$ with the quadric of Wilczynski at $x_0$, and the point $P_0$ is the harmonic conjugate of $P_1$ with respect to the points in which the projective normal intersects the quadric of Lie. The point $x_3$ is, therefore, geometrically characterized by the cross-ratio equation

$$(P_\omega, P_0, P_1, x_3) = k.$$ 

A convenient way to complete the geometric characterization of $x_3$ is to let the function $k$ be the projective mean curvature $K_m$ of $S_0$ relative to $S_3$. The imposition of this condition results in the characterization of $K_m$ by the relation

$$K_m = -\theta_{12}/\beta\gamma$$

See P. O. Bell [2, pp. 567–569].

P. O. Bell [2, Theorem 5].
and of the point $x_3$ by the vector form
\[ x_3 = \left[ x_{12} - 2^{-1}(3\theta_{12} + 2\beta\gamma)x \right]/\beta\gamma. \]

The form of the coordinates of $x_3$ reveals that $P_0$ and $x_3$ are harmonic conjugates with respect to $x_0$ and the point of the $R$-associate of $S_0$, where $R = \beta\gamma$, which lies on the Fubini-Green projective normal of $S_0$ at $x_0$. We shall call this point $x_3$ the center of mean projective curvature of $S_0$ at $x_0$ relative to the points $P_0, P_1$ on the projective normal to $S_0$ at $x_0$. The associated projective total curvature of $S_0$ and the metric of $S_0$ are defined by the equations
\[ K = (\theta_{12} - 4\pi\chi)/4\beta^2\gamma^2, \quad Kds^2 = \pi(du^1)^2 - \theta_{12}du^1du^2 - \chi(du^2)^2, \]
respectively.

8. Projective analogues of the second fundamental form and the projective linear element of a surface. Let us again suppose that the surface $S_0$ is referred to asymptotic parameters. Again, let $\Gamma^3_{12} = \Gamma^3_{21} = R$, and let the points $x_\alpha, \alpha = 1, 2$, be the points defined by $x_\alpha = x_0$. We need not restrict $x_3$ to lie on the $R$-conjugate line which passes through $x_0$, but we shall assume that the form $\phi_2$ defined by
\[ \phi_2 = 2Rdu^1du^2 \]
is a projective invariant of $S_0$ and that the vertices of the local reference tetrahedron and the point $x_3 + x_0$ are covariantly determined points. We shall show that the form $\phi_2$ is a projective analog of the second fundamental form of $S_0$.

Let $\pi$ denote the tangent plane to $S_0$ at $x_0$ and let $\eta$ denote the plane determined by the points $x_1, x_2, x_3 + x_0$. The local equations for $\pi$ and $\eta$ are $x^3 = 0$ and $x^3 - x^0 = 0$, respectively.

Let $dp$ denote the differential defined by the relation
\[ dp = \phi_2^{1/2}, \]
and let $C_\lambda$ denote a curve of $S_0$ whose direction at each of its points is defined by
\[ du^2/du^1 = \lambda(u^1, u^2). \]
Along this curve we have, therefore,
\[ dp/du^1 = (2R\lambda)^{1/2}. \]
Let us regard $p$ as independent variable\(^{(9)}\) for points of $C_\lambda$, where $p$ is defined by the integral
\[ p = \int_{u_0}^{u_1} (2R\lambda)^{1/2}du^1. \]

\(^{(9)}\) For the geometric interpretation of the intrinsic parameter $p$ see P. O. Bell [1, pp. 532–534].
The general coordinates \( x_0(dp) \) of a point \( X_0 \) of \( C_\lambda \) "near" to \( x_0 \) are given by the expansion

\[
x_0(dp) = x_0 + x'dp + x''dp^2/2 + x'''dp^3/3! + \cdots,
\]
in which accents indicate differentiation with respect to \( p \). Let \( P \) and \( Q \) denote the intersections of the line joining the points \( X_0, x_3 \) with the planes \( \pi \) and \( \eta \), respectively. The general coordinates of \( P, Q \) may be found to be given by the forms

\[
P = x_0(dp) + \mu_1 x_3, \quad Q = x_0(dp) + \mu_2 x_3,
\]
in which \( \mu_1, \mu_2 \) are defined by

\[
\mu_1 = -(3\phi_2 + \phi_3)/3! + \cdots, \quad \mu_2 = 1 + (2),
\]
in which we denote by \( \phi_3 \) the cubic differential form \( R(\gamma(du^2)^3 + \beta(du^1)^3) \) and by \((2)\) terms of order \((dp)^2\) at least. The cross-ratio equations

\[
(P, x_3, X_0, Q) = (\mu_1, \infty, 0, \mu_2) = (3\phi_2 + \phi_3)/3! + (4),
\]
in which \((4)\) denotes terms of order \((dp)^4\) at least, may now be readily obtained. Let us call the cross-ratio \((P, x_3, X_0, Q)\) the projective distance \( D_{\pi X_0} \) from \( \pi \) to \( X_0 \) with respect to \( x_3 \). We observe that the difference between this projective distance and its principal part is equal to \( \phi_3/3! \) plus terms of order \((dp)^2\) at least. Since the curves of Darboux are integral curves of the differential equation \( \phi_3 = 0 \), we may state our results in the following theorem.

**Theorem 8.1.** The invariant form \( \phi_2 \) is equal to twice the principal part of the projective distance from \( \pi \) to \( X_0 \) with respect to \( x_3 \). The difference between this projective distance and its principal part is equal to \( \phi_3/3! \) plus terms of order at least \((dp)^4\). This difference consists of terms of order at least \((dp)^4\) if the points \( x_0, X_0 \) lie on a curve defined by \( R = 0 \) or on a curve of Darboux. The ratio \( \phi_3/\phi_2 \) is independent of \( R \) and is equal to the projective linear element of \( S_0 \).

Let \( x^0, x^1, x^2, 1 \) again represent homogeneous cartesian coordinates of the point \( x_0 \) of \( S_0 \) and let the coordinates of the point \( x_3 \) at infinity on the normal to \( S_0 \) at \( x_0 \) be represented by the three direction cosines \( z^0, z^1, z^2 \) of the normal and \( z^3 = 0 \). Let a local reference tetrahedron \((x_0, x_1, x_2, x_3)\) with equal units on the three axes through \( x_0 \) be established with \( x_1 \) and \( x_2 \) the intersections of the ideal line \( \text{line at infinity} \) in the plane \( \pi \) with the asymptotic \( u^1, u^2 \) tangents to \( S_0 \) at \( x_0 \). The form \( \phi_2 \) and the corresponding projective distance \( D_{\pi X_0} \) associated with the plane \( \eta \) determined by the ideal line \( x_2 x_3 \) and the unit point \( x_0 + x_3 \) are the second fundamental form of \( S_0 \) and associated metric distance from \( \pi \) to \( X_0 \), respectively.

**9. A theorem on projective geodesics and an application to metric geometry.** The projective first fundamental form of \( S_0 \) relative to the \( R \)-associ-
ate of $S_0$ and the associated projective total curvature of $S_0$ (characterized in §7) are defined analytically by the equations

$$K ds^2 = \pi(du^1)^2 + x(du^2)^2, \quad K = -\pi x/R^2.$$  

We may, therefore, write

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2$$

in which $g_{11} = -R^2/x$, $g_{22} = -R^2/\pi$. We shall call the extremals of the integral invariant

$$\int_{u_0}^{u} (g_{11} + g_{22} \lambda^2)^{1/2} du^1,$$

in which $\lambda = du^2/du^1$, the projective geodesics of $S_0$ relative to the $R$-associate of $S_0$. Since the above integral invariant is of the form

$$\int_{u_0}^{u} \phi(u, v, v') du, \quad v' = dv/du,$$

and Euler's equation for the extremals of this integral is known to be

$$4v''v''v' = \phi_v - \phi_{vv'} - \phi_{vv'}v',$$

the equation for these projective geodesics may be found to be

$$2d^2 u^2/(du^1)^2 = \left( g_{11}/g_{22} + (\log g_{11}/(g_{22})^2) du^2/du^1 \right)^2$$

$$+ \left( \log (g_{11})^2/g_{22} du^2/du^1 \right)^2 - \left( (g_{22})_1/g_{11} \right)(du^2/du^1)^3.$$  

The cusp-axis at $x_0$ of a system of hypergeodesics defined by an equation of the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3,$$

where $u^1 = u$, $u^2 = v$, is known (9) to be a line $l'$ which passes through the points $x_0, x_1 - ax_1 - bx_2$ where $a = (\theta_2 + C)/2, b = (\theta_1 - B)/2$. The cusp-axis at $x_0$ of the projective geodesics (9.2) is therefore the line $l'$ for which

$$a = (\log R^4\pi/\chi^3)_{12}/4, \quad b = (\log R^4\pi/\pi^3)_{12}/4.$$  

Since the condition $b_2 = a_1$ may be expressed in the present case as

$$(\log \pi/\chi)_{12} = 0$$

we have the following theorem.

**Theorem 9.1.** If the $\Gamma'$ curves of the cusp-axis congruence of the projective geodesics of $S_0$ relative to the $R$-associate of $S_0$ form a conjugate net, or coincide with an asymptotic family, or are indeterminate, the associated projective lines of curvature form an isothermally conjugate net, and conversely.

(9) Cf. E. P. Lane [1, p. 193].
Since a distinct form (9.1) is associated uniquely with each $R$-conjugate congruence, the above theorem describes a representative result which is characteristic of every member of the class of conjugate congruences.

An analogous Euclidean geometric theorem may be derived as follows. Let us consider a surface $S_0$ having the property that its asymptotic net is orthogonal. Let $x^0, x^1, x^2, 1$ and $z^0, z^1, z^2, 0$ denote homogeneous rectangular coordinates of a generic point $x_0$ of $S_0$ and the point $x_3$ at infinity on the normal to $S_0$ at $x_0$, respectively, in which $z^0, z^1, z^2$ denote direction cosines of the normal at $x_0$. Let points $x_\alpha$, $\alpha = 1, 2$, be defined by the relations $x_\alpha = x_{0\alpha}$. It follows that the homogeneous coordinates of the points $x_{ja}, j = 0, 1, 2, 3$, satisfy a system of equations of the form $x_{ja} = \Gamma^h_{ja} x_h$, in which $\Gamma^h_{jk}, j, \alpha, h = 1, 2$, are the Christoffel symbols of the second kind for the first fundamental form of $S_0$. According to the hypotheses we have

$$g_{13} = 0, \quad \Gamma^3_{11} = \Gamma^3_{22} = 0, \quad \Gamma^3_{12} \neq 0.$$  

The geodesics are the extremals of the arc length integral

$$\int_{u_0}^{u_1} (g_{11} + g_{22}\lambda^2)^{1/2} du^1$$

whose cusp-axis at $x_0$ is the normal to $S_0$. The $\Gamma'$ curves of the congruence of normals (known as the lines of curvature) form a conjugate net if and only if

$$(9.3)\quad g_{11} \neq 0, \quad g_{22} \neq 0, \quad (\log g_{11}/g_{22})_{12} = 0,$$

and conversely. The equation for the lines of curvature then assumes the form

$$g_{11}(du^1)^2 - g_{22}(du^2)^2 = 0.$$  

Fulfillment of conditions (9.3) is assured by the two Codazzi equations. Equations (9.3) are necessary and sufficient conditions that the lines of curvature form an isothermally conjugate net; moreover (9.3) with the added condition $g_{12}=0$ are the conditions that the parametric net be isothermally orthogonal. Hence we have the following theorem.

**Theorem 9.2.** If the asymptotic curves of $S_0$ form an orthogonal net, it is isothermally orthogonal and the lines of curvature of $S_0$ form an isothermally conjugate net.

10. **Intrinsic geometry of a tetrad of surfaces.** The surfaces $S_i$ of a tetrad are undisturbed by the general transformation (5.1) of independent variables. The generic points of $S_i$ are maintained as the vertices $x_i$ of the local moving reference tetrahedron by a transformation of dependent variables of the form

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\(^{(10)}\) For the geometric significance of isothermal conjugacy see E. J. Wilczynski [1, pp. 211–221].

\(^{(11)}\) See, for example, E. P. Lane [2, p. 236].
(10.1) \[ x_h = \lambda_h^i x_j, \]
in which

\[ \lambda_h^i = 0, \quad h \neq j, \]
\[ \lambda_h^i \neq 0, \quad h = j. \]

Corresponding to this transformation of dependent variables we have that the normalized cofactors \( \mu_j^h \) of \( \lambda_h^i \) in \( \lambda_h^i \) are defined by the relations

\[ \mu_j^h = 0, \quad h \neq j, \]
\[ \mu_j^h = 1/\lambda_h^h, \quad h = j, \]

where \( \lambda_h \) denotes the function \( \lambda_h^i \) for \( h = j \).

The law of transformation of the connection \( \Gamma_{h \alpha}^i \) under the general transformation combining (5.1), (5.2) and (5.3) is given by (5.16) with \( \lambda_h^i, \mu_j^h \) satisfying the above relations. Incorporating these relations in the law (5.16) we find

\[ \Gamma_{h \alpha}^i = \delta \lambda_h \partial (\log \lambda_h)/\partial \bar{u}^\alpha + \lambda_h \Gamma_{h \alpha}^i (\partial w^\beta/\partial \bar{u}^\alpha)/\lambda_j \]
in which \( h \) and \( j \) are not summed. It follows that the linear forms \( \phi_j^h \) defined by

\[ (10.2) \quad \phi_j^h = \Gamma_{h \alpha}^i du^\alpha \]
are relative invariants of the general transformation combining (5.1) and (10.1). Hence the directions defined by

\[ \phi_j^h = 0, \quad h \neq j, \]
possess intrinsic geometric significance with respect to the tetrad of surfaces \( S_i \). On comparing the right member of (10.2) with the forms for the relative invariants \( \theta_{01}, \theta_{10} \) of the vertex \( x_h \) and face \( x_k = 0 \), we find that the direction defined by \( \phi_j^h = 0 \) is that for which these forms vanish. The geometric significance of the vanishing of these forms has already been given in theorem (6.1).

In the remainder of this section we denote a chosen permutation of \( (0, 1, 2, 3) \) by \( (i, j, k, l) \). An index to be summed through \( i, j, k \) will be represented by \( s \) and an index to be summed through \( j, k \) will be denoted by \( r \). The form \( F^k_i \) defined by the formula

\[ F^k_i = - \phi_r \phi^{k}_r / \phi^{k}_i \]
is a linear differential invariant. To obtain a geometric characterization of this form consider the point \( X_i \) whose general homogeneous coordinates \( X_i^p \) are given by the vector form

\[ X_i = x_i (u^1 + du^1, u^2 + du^2), \]
and the point $P_i$ determined by the projection of $X_i$ upon the plane $x^k = 0$ from the point $x_k$. The general homogeneous coordinates of $P_i$ may be represented by the vector form

$$P_i = x_i + \Gamma^i_{ia}x_a du^a.$$  

The line joining the points $x_i$ and $P_i$ intersects the line joining the points $x_i$ and $x_k$ in the point $Q$ whose general coordinates $Q^p$ are given by the form

$$Q = \Gamma^p_{ia}x_a du^a.$$  

Since the characteristic of the plane $x^k = 0$ corresponding to the direction $du^2/du^1$ is defined by the equations

$$x^k = 0, \quad x^r \Gamma^k_{ra} du^a = 0,$$

the intersection of the line joining the points $x_i$ and $P_i$ with this characteristic is the point $R$ whose general coordinates are defined by

$$R = x_i - \phi^k \phi_i^r x_r / \phi^r \phi^k_r.$$  

Let $P_{ki}$ denote the intersection of the characteristic of the plane $x^k = 0$, for the direction $du^2/du^1 = \lambda$, with the plane $x^i = 0$. The local point coordinates of $P_{ki}$ satisfy the equations

$$(10.3) \quad x^k = 0, \quad x^i = 0, \quad x^r \phi^k_r = 0.$$  

The proof of the following theorem may now be easily supplied by the reader.

**Theorem 10.1.** The form $F_{ki}$ is the principal part of the invariant cross-ratio $(x_i, Q, P_i, R)$. The point $P_{ki}$ lies in the plane determined by the line joining the points $x_i$ and $x_k$ and the tangent to $S_i$ at $x_i$ in the direction $\lambda$ if, and only if, $\lambda$ is a root of the differential equation $F_{ki} = 0$.

The linear differential invariants which we shall call the projective linear elements $E^{li}_i$ we define by the relations

$$E^{li}_i = F^{li}_i + P^{li}_i.$$  

These elements may be geometrically characterized in the following simple manner. Let $O$ denote the intersection of the line joining $x_i$ and $P_i$ with the line joining $P_{jk}$ and $P_{k}$, the validity of the following characterization may be verified by simple calculations.

**Theorem 10.2.** The principal part of the cross-ratio $(x_i, Q, P_i, O)$ is the projective linear element $E^{li}_i$.

The points $P_{jk}$ and $P_{ki}$ for a given direction $\lambda$ are clearly collinear with the points $x_j$ and $x_i$. The curves which correspond to the developables of the congruence of lines joining corresponding points of $S_i$ and $S_i$ are the curves whose directions are characterized by the property that the points $x_i, x_i$
$\phi^r x_r, \phi^r x_r$ are coplanar. This condition may clearly be expressed by the differential equation $\phi^i \phi^k = \phi^i \phi^k$. This condition is also seen to be the condition that a direction be such that the points $P_{jk}$ and $P_{ki}$ coincide. Hence we have the following theorem.

**Theorem 10.3.** The directions for which the relative invariants

$$\phi^i \phi^k$$

are symmetric in the indices $j, k$ are the directions which correspond to the developables of the congruence of lines joining corresponding points of $S_i$ and $S_i$. Moreover, for only these directions do the points $P_{jk}$ and $P_{ki}$ coincide.

For a given invariant direction the cross-ratio $(x_i, x_i, P_{jk}, P_{ki})$ is an absolute invariant which we denote by $R_{iik}$. By a simple calculation we find that

$$R_{iik} = \frac{\phi^i \phi^k}{\phi^i \phi^k}.$$

Corresponding to a given invariant $R_{iik}$ there are two invariant directions defined by this equation. In particular if $R_{iik} = -1$, the points $P_{jk}, P_{ki}$ are harmonic conjugates with respect to the points $x_i, x_i$, and in view of the form of (10.3) we have the following theorem.

**Theorem 10.4.** The directions for which $P_{jk}, P_{ki}$ are harmonic conjugates with respect to the points $x_i, x_i$ are the two directions for which the form

$$\phi^i \phi^k$$

is skew-symmetric in $j, k$.

11. The projective linear element and Fubini's element of projective arc length. Let $(x_0, x_1, x_2, x_3)$ denote a moving reference tetrahedron intrinsically connected with the surface $S_0$ at $x_0$ in such a manner that the vertices $x_1, x_2$ are located on the tangents at $x_0$ to the $u^1, u^2$ curves, respectively, of $S_0$. The general homogeneous coordinates of $x_a$ are therefore defined by the forms

$$x_{oa} = x_a + \Gamma^0_{oa} x_0.$$ 

The special cases $F^1_0, F^2_0$ of the corresponding invariant forms $F^i_0$ are, consequently, given by

$$F^1_0 = -(\Gamma^1 du^1 du^2 + \Gamma^1 du^2)^2/du^1, \quad F^2_0 = -(\Gamma^2 du^1 du^2 + \Gamma^2 du^2)^2/du^2,$$

and the special projective linear element $E^{12}_0$ is given by

$$E^{12}_0 = -(\Gamma^2 du^2)^3 + \Gamma^2 du^2)^2 du^1 + \Gamma^2 du^2)^3 du^1 + \Gamma^2 (du^1)^2 du^1 du^2.$$ 

The classical projective linear element of a surface $S_0$ may now be identified as the special case of $E^{12}_0$ for which the points $x_1, x_2, x_3$ are the points $x_u, x_v, x_{uv}$ whose general coordinates are expressed in Fubini's normal coordinates. However, if these points be replaced by the similarly defined points whose general coordinates are normal with respect to an arbitrarily selected canonical form of Grove, the associated invariant form $E^{21}_0$ is again found to be the classical pro-
jective linear element of the surface \( S_0 \), and the invariants \( F^1_0, F^2_0 \) are the elementary linear differential forms of Bompiani\(^{(12)}\). Under the present restrictions the forms \( F^1_0, F^2_0, E^{12}_0 \) are given by the simple expressions

\[
F^1_0 = - \Gamma^{12}_{12}(du^2)^2/du^1, \quad F^2_0 = - \Gamma^{21}_{11}(du^1)^2/du^2, \quad E^{12}_0 = F^1_0 + F^2_0. 
\]

The product

\[
F^1_0F^2_0 = \Gamma^{12}_{12}\Gamma^{21}_{11}du^1du^2 
\]

is Fubini’s normal form for the square of his element of projective arc length\(^{(12)}\).

12. Dual characteristics. Systems of hypergeodesics and the first canonical pencil. Let us establish a one-to-one correspondence between the points of a surface \( S' \) and the planes of a surface \( S \) by defining the local coordinates \( x^i \) and \( \xi_i \) of a generic point \( X \) of \( S' \) and the corresponding tangent plane \( \pi \) of \( S \), respectively, to be single-valued functions of \( u^1, u^2 \). The following geometric elements of \( S' \) and \( S \) are placed in one-to-one correspondence: (i) the points of a curve \( C_\lambda \) of \( S' \) and the tangent planes of a developable surface \( D_\lambda \) of \( S \), (ii) the tangent to \( C_\lambda \) at a generic point \( X \) and the characteristic line of \( D_\lambda \) in the plane \( \pi \) which corresponds to \( X \), and (iii) the osculating plane of \( C_\lambda \) at \( X \) and the ray-point of \( D_\lambda \) in \( \pi \). These elements of \( S' \) and \( S \) thus attached to \( C_\lambda \) and \( D_\lambda \) will be called the characteristics of \( C_\lambda \) and \( D_\lambda \) respectively. We shall refer to these characteristics as dual characteristics with respect to the correspondence between the points of \( S' \) and the planes of \( S \).

To define a developable surface \( D_\lambda \) of \( S \) and a corresponding curve \( C_\lambda \) of \( S' \) we let \( u^\alpha \) be functions of an independent parameter \( t \) so that local coordinates \( x^i, \xi_i \) of a generic plane \( \pi \) of \( S \) and the corresponding point \( X \) of \( S' \) become functions of \( t \). The characteristics of \( D_\lambda \) described in (i), (ii), and (iii) above are defined analytically by

\[
\xi_i x^i = 0, \quad (12.1) \quad \xi_i x^i = 0, \quad x^i \xi_i du^\alpha/dt = 0, \quad x^i = 0, \quad \xi_i x^i du^\alpha/dt = 0, \quad x^i(\xi_i, x^i du^\alpha/dt^2 + \xi_i, x^i du^\alpha du^\beta/dt^2) = 0, 
\]

respectively, in which \( \xi_i \) are functions of \( t \). The equations in local plane coordinates \( \xi_i \) for the corresponding characteristics of \( C_\lambda \) at \( X \) may be written by simply interchanging the roles of \( \xi_i \) and \( x^i \) in the above equations. They are, therefore,

\[
\xi_i x^i = 0, \quad (12.2) \quad x^i \xi_i = 0, \quad \xi_i x^i, x^i du^\alpha/dt = 0, \quad x^i(\xi_i, x^i du^\alpha/dt^2 + \xi_i, x^i du^\alpha du^\beta/dt^2) = 0 
\]

in which \( x^i \) are functions of \( t \).

\(^{(12)}\) E. Bompiani [1, pp. 167–173].

\(^{(13)}\) G. Fubini and E. Čech [1, pp. 64–69].
Let $p$ and $Y$ denote the tangent plane to $S'$ at $X$ and the point of contact of the plane $\pi$ with $S$, respectively, and let the local plane coordinates of $p$ and the local point coordinates of $Y$ be denoted by $\eta_i$ and $y^i$ respectively. As $X$ varies over $S$ the line joining $XY$ generates a congruence which we denote by $\Gamma_{XY}$. A curve $C_\lambda$ of $S'$ having the property that its osculating plane at a general one of its points $X$ contains the line $XY$ is called a union-curve(14) of the congruence $\Gamma_{XY}$. The differential equation of the union-curves of a general congruence $\Gamma_{XY}$ may be obtained by imposing the condition that the plane determined by the third line of equations (12.2) contain the point $Y$. This condition is clearly given by the determinantal relation

\[(12.3) \quad (y^i, x^i, x^i, a du^\alpha / dt, x^i, a d^2 u^\alpha / dt^2 + x^i, a, a du^\alpha du^\beta / dt^2) = 0.\]

The dual of the line $XY$ is the line $\pi p$ determined by the planes $\pi$ and $p$. The dual of a union-curve of a congruence $\Gamma_{XY}$ is a developable of $S$ having the property that the ray-point of the generic plane $\pi$ lies in the plane $p$. Such a developable will be called a union-developable of the congruence $\Gamma_{xy}$. The curve of $S$ generated by the contact point of $\pi$ with $S$ as $\pi$ varies over a union-developable of the congruence $\Gamma_{xy}$ will be called an adjoint union-curve(15) of the congruence $\Gamma_{xy}$. The differential equation of the adjoint union-curves of the congruence $\Gamma_{xy}$ is found, by replacing $y^i$ by $\eta_i$ and $x^i$ by $\xi_i$, to be

\[(12.4) \quad (\eta_i, \xi_i, \xi_i, a du^\alpha / dt, \xi_i, a d^2 u^\alpha / dt^2 + \xi_i, a, a du^\alpha du^\beta / dt^2) = 0.\]

As $X$ varies along a union-curve of the congruence $\Gamma_{XY}$ the point $Y$ describes a curve of $S$ which we call an $S'$-tangeodesic of $S$, and as $\pi$ varies over a union-developable of the congruence $\Gamma_{xy}$ the point $X$ describes a curve of $S'$ which we call an adjoint $S'$-tangeodesic of $S'$.

The author has defined in a different manner(16) the systems of $p$- and $s$-tangeodesics of $S$. We shall show that the above definition of the $S'$-tangeodesics of $S$ is equivalent to a generalization of the definition of either of these systems of tangeodesics. Let $R_\lambda$ denote the ruled surface generated by the line $XY$ as $X$, $Y$ describe corresponding curves $C_\lambda$ of $S'$ and $S$, respectively. If the curve $C_\lambda$ of $S'$ is a union-curve of the congruence $\Gamma_{XY}$, the osculating plane of $C_\lambda$ at $X$ coincides with the tangent plane to the ruled surface $R_\lambda$; the curve $C_\lambda$ is, therefore, an asymptotic curve of $R_\lambda$. Conversely, if the curve $C_\lambda$ of $S'$ is an asymptotic curve of $R_\lambda$, it is a union-curve of the congruence $\Gamma_{XY}$. It follows that the following generalization of the definition of the $p$-tangeodesics of $S$ serves to characterize the $S'$-tangeodesics of $S$: An $S'$-tangeodesics of $S$ is a curve $C_\lambda$ of $S$ whose associated ruled surface $R_\lambda$ (of the congruence, $\Gamma_{XY}$) intersects the surface $S'$ in an asymptotic curve of $R_\lambda$.

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(14) Union-curves were introduced by P. Sperry [1, p. 214].

(15) This is a generalization of the definition of adjoint-union curves (or dual-union curves) due to G. M. Green [1, p. 140] and P. Sperry [1, p. 222].

(16) P. O. Bell [2, p. 575].

If we let $S'$ and $S$ be surfaces $S_A$ and $S_B$ of the fundamental tetrad, respectively, the local coordinates of $X$ and $Y$ become the Kronecker deltas $\delta^i_A$ and $\delta^i_B$ respectively. Substituting these deltas for $x^i$ and $y^i$, respectively, in (12.3) and expanding the determinant yields the following curvilinear differential equation which defines on $S_A$ the $S_A$-tangent curves and on $S_B$ the union-curves of the congruence generated by the line $x_h x_k$:

\[
\frac{d^2u^\beta}{dt^3} = \left( \Gamma^{\gamma}_{\alpha \beta} \Gamma^{\alpha}_{\beta \gamma} - \Gamma^{\alpha}_{\alpha \beta} \Gamma^{\beta}_{\gamma \alpha} \right) \frac{du^\alpha du^\beta du^\gamma}{dt^3},
\]

(12.5)

in which intrinsic differentiation is with respect to upper indices and $(hmnk)$ represents a permutation of $(0123)$.

The equation of the adjoint union-curves of the congruence $\Gamma_r p$ generated by the line of intersection of the planes $\pi$ and $p$ defined by $x^h = 0$ and $x^k = 0$, respectively, may be written by interchanging the upper and lower left index in each of the symbols involving $\Gamma^i_{\alpha \beta}$, $\Gamma^j_{\alpha \beta}$, $j = m, n$, in (12.5). The equation is

\[
\frac{d^2u^\alpha}{dt^3} = \left( \Gamma^{\gamma}_{\alpha \beta} \Gamma^{\beta}_{\gamma \alpha} \right) \frac{du^\alpha du^\beta du^\gamma}{dt^3},
\]

(12.6)

in which each intrinsic differentiation is with respect to the lower left index.

Let us determine the equation of the $S_3$-tangent of $S_0$ for the special case in which the surface $S_0$ is the $R$-associate of $S_0$. This surface $S_0$, which has been geometrically characterized in $\S 7$, is generated by the point $x_3$ whose general homogeneous coordinates are given by the form

\[
R x_3 = x_{12} - (\beta \gamma + \theta_{12}) x
\]

in which $x$ denotes Grove's normal coordinates for $x_0$ and $u^1, u^2$ are asymptotic parameters. We find now that equation (12.5) for $h = 3, k = 0$ can be written in the form

\[
d\lambda/du^1 = A + B\lambda + C\lambda^2 + D\lambda^3, \quad \text{where} \quad \lambda = du^2/du^1,
\]

\[
A = - \Gamma^{13}_{11,1}/\Gamma^{13}_{32}, \quad B = \Gamma^{23}_{31,1}/\Gamma^{23}_{31} - (\Gamma^{13}_{32,1} + \Gamma^{13}_{13,2})/\Gamma^{13}_{32}, \quad C = (\Gamma^{23}_{13,1} + \Gamma^{23}_{31,1})/\Gamma^{23}_{31} - \Gamma^{13}_{32,1}/\Gamma^{13}_{32}, \quad D = \Gamma^{23}_{32,2}/\Gamma^{23}_{31},
\]

in which intrinsic differentiation is with respect to the upper index. In virtue of the relations (7.2) and the condition $\Gamma^{32}_3 = R$ we find, on evaluating the coefficients $B, C$ in terms of the coefficients of Grove's canonical form,

\[
B = (\log \pi/R x)_1 - \gamma \pi/\chi, \quad C = (\log \pi R/\chi)_2 + \beta \chi/\pi.
\]
The cusp-axis of these tangeodesies at $x_0$ is the line $l'$ which passes through the point $x_0$ and the point $z$ whose normal coordinates are given by $z = x_1 - bx_2$ in which $a$ and $b$ are defined by

$$a = ((\log \frac{\pi R^2}{\chi})_1 + \beta \pi / \chi)/2, \quad b = ((\log \frac{\chi R^2}{\pi})_1 + \gamma \pi / \chi)/2.$$ 

The proof of the following theorem may now be readily supplied by the reader.

**Theorem 12.1.** If the $\Gamma'$ curves of the congruence generated by this cusp-axis form a conjugate net, or coincide with an asymptotic family of $S_0$, or are indeterminate, then

$$2(\log \frac{\pi}{\chi})_{12} + (\gamma \pi / \chi)_2 - (\beta \pi / \chi)_1 = 0,$$

and conversely. If the axis curves of the net of projective lines of curvature of $S_0$ relative to $S_3$ form a conjugate net, or coincide with an asymptotic family of $S_0$, or are indeterminate, then

$$(\log \frac{\pi}{\chi})_{12} + (\beta \pi / \chi)_1 - (\gamma \pi / \chi)_2 = 0,$$

and conversely. The projective lines of curvature of $S_0$ relative to $S_3$ form an isothermally conjugate net if, and only if,

$$(\log \frac{\pi}{\chi})_{12} = 0.$$

The net of $R_\lambda$-derived curves of $S_0$, where $\lambda$ is a projective principal direction of $S_0$ relative to $S_3$, belongs to class $C$ if, and only if,

$$(\beta \pi / \chi)_1 - (\gamma \pi / \chi)_2 = 0(17).$$

If any two of these four conditions are fulfilled the other two are also fulfilled.

In conclusion we consider briefly the systems of $\rho$- and $\sigma$-tangeodesics of $S_0$ for which $\rho$ and $\sigma$ are points on the asymptotic $u^1$- and $u^2$-tangehts to $S_0$ at $x_0$ whose general coordinates are given by

$$\rho = x_1 - bx_0, \quad \sigma = x_2 - ax_0, \quad \text{where } x_0 = x_\alpha.$$

The joint-edge at $x_0$ of the systems of $\rho$- and $\sigma$-tangeodesics of $S_0$ was found(18) to be the line which passes through the points $x_0$ and $z$ whose general coordinates in Fubini's normal coordinates satisfy the relation $z = x_1 - \bar{a}x_1 - \bar{b}x_2$ in which $\bar{a} = -a + \psi / 2, \bar{b} = -b + \phi / 2$. Let $l$ denote the reciprocal with respect to $S_0$ of the joint-edge of the $\rho$- and $\sigma$-tangeodesics of $S_0$ at $x_0$, let $l$ denote the line joining $\rho \sigma$, and let $t$ denote the first canonical tangent to $S_0$ at $x_0$. Since $\bar{a}, \bar{b}$ are given by the formulas

$$\bar{a} = k_1 \psi, \quad \bar{b} = k \phi$$

where $k_1 = (1 - 2k)/2$, if we put

(17) P. O. Bell [3, p. 398].

(18) P. O. Bell [2, p. 576].

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we have, as a consequence, the following theorem.

**Theorem 12.2.** If the line $l$ joining $p\sigma$ is a canonical line $l_k$, the reciprocal $\overline{l}$ of the joint-edge of the $p$- and $\sigma$-tangeodesics of $S_0$ at $x_0$ is the canonical line $l_{k_1}$ for which $k_1 = (1 - 2k)/2$.

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