INTERPOLATION TO CERTAIN ANALYTIC FUNCTIONS BY RATIONAL FUNCTIONS

BY

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1. **Introduction.** The problem of interpolation by rational functions to an analytic function defined by a line integral has been treated in great detail by J. L. Walsh. The object of the present discussion is to obtain some results in the corresponding problem in connection with certain functions defined by surface integrals, namely, functions of class S_2. By a function of class S_2 we mean a function $f(z)$, which is analytic in the open interior K of the unit circle $|z|=1$, integrable together with its square on K, and (hence) capable of the integral representation

$$f(z) = \frac{1}{\pi}\iint_K \frac{f(t)}{(1 - z\bar{t})^2}\, dS, \qquad |z| < 1.$$

(This can be verified by expanding $(1-z\bar{t})^{-2}$ and integrating term by term[1].)

Let a_{ni}, $i=1, 2, \cdots, n$; $n=1, 2, \cdots$, be a set of points pre-assigned on K and subject to the conditions: (i) a_{ni} have no limit point on K; (ii) $a_{ni}\neq 0$ for all n and i; (iii) $a_{ni}\neq a_{nj}$ if $i\neq j$. None of these is essential to our results; (i) serves merely to exclude trivial cases, whereas (ii) and (iii) will be removed in the course of discussion.

The set a_{ni} will be called a normal set if, for each n, the n points a_{ni} all lie on a circular arc C_n which is orthogonal to the unit circle and passes through a fixed point P on K.

For a given function $f(z)$ of class S_2, let $f_n(z)$ denote the rational function of the form

$$\sum_{i=1}^{n} \frac{A_{ni}}{(1 - \bar{a}_{ni}z)^2}, \tag{1}$$

found by interpolation to $f(z)$ at the points a_{ni};

$$f_n(a_{ni}) = f(a_{ni}), \qquad i = 1, 2, \cdots, n. \tag{2}$$

Our results are as follows:

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(1) This class of functions has been discussed by J. L. Walsh in his book, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, 1935. The present writer begs to offer apology for not being able to give any quotation, because this book has been inaccessible to him for many years, although it is from this book that he got the impetus for the investigation.

THEOREM A. *If we have*

$$\lim_{n\to\infty} n \prod_{i=1}^{n} |a_{ni}|^2 = 0, \tag{A}$$

then, for every function $f(z)$ of class S_2, the corresponding sequence $f_n(z)$ converges to $f(z)$ on K, uniformly on any closed point set on K.

THEOREM B. *If the set a_{ni} is normal, then condition* (A) *in Theorem* A *can be replaced by the condition*

$$\lim_{n\to\infty} \prod_{i=1}^{n} |a_{ni}| = 0. \tag{B}$$

From Theorems A and B follows the theorem: *Let $f(z)$ be a function of class S_2 which vanishes on a set of points a_n, $0<|a_n|<1$. If these points satisfy condition* (A), *or if they form a normal set and satisfy condition* (B), *then $f(z)$ vanishes identically.*

For conciseness, we shall write a_i instead of a_{ni} when the consideration is confined to a fixed n; and shall omit to indicate the region of integration of surface integrals because that region is always K.

2. **The remainder.** Since the determinant

$$\Delta_n = \left| \frac{1}{(1 - a_i\bar{a}_j)^2} \right|$$

of the systems of linear equations (2) is precisely the Gramian determinant(2) for the n linearly independent functions $(1-\bar{a}_i z)^{-2}$, and is therefore positive, the function $f_n(z)$ is uniquely determined. Set

$$\Delta_{n+1}(z;\bar{t}) = \left| \begin{array}{ccc:c} & & & \frac{1}{(1-a_1\bar{t})^2} \\ & \Delta^n & & \vdots \\ & & & \frac{1}{(1-a_n\bar{t})^2} \\ \hdashline \frac{1}{(1-z\bar{a}_1)^2} & \cdots & \frac{1}{(1-z\bar{a}_n)^2} & \frac{1}{(1-z\bar{t})^2} \end{array} \right|,$$

and set

$$r_n(z;\bar{t}) = \frac{\Delta_{n+1}(z;\bar{t})}{\Delta_n}.$$

(2) See G. Kowalewski, *Einführung in die Determinantentheorie*, Berlin and Leipzig, 2d ed., 1925, p. 224.

Then the remainder $R_n(z) = f(z) - f_n(z)$ is given by

$$R_n(z) = \frac{1}{\pi} \iint f(t) r_n(z; \bar{t}) dS, \qquad |z| < 1. \tag{3}$$

On comparing this formula with a well known formula in the theory of orthogonal functions(3), we see that $f_n(z)$ is also the unique function of the form (1) which minimizes the integral

$$\iint \left| f(z) - \sum_{i=1}^{n} \frac{A_i}{(1 - \bar{a}_i z)^2} \right|^2 dS.$$

Thus our problem of integration is equivalent to one of approximation in the sense of least squares.

In particular, if we choose $f(z) = (1 - z\bar{\zeta})^{-2}$, where ζ, $|\zeta| < 1$, is a constant, the corresponding $R_n(z)$ is precisely $r_n(z; \bar{\zeta})$. Hence $r_n(z; \bar{\zeta})$ is the unique function of the form

$$\frac{1}{(1 - \bar{\zeta} z)^2} - \sum_{i=1}^{n} \frac{A_{ni}}{(1 - \bar{a}_i z)^2},$$

whose norm on K:

$$\frac{1}{\pi} \iint |r_n(z; \bar{\zeta})|^2 dS = r_n(\zeta; \bar{\zeta}) \tag{4}$$

is least. This minimum property of $r_n(z; \bar{t})$ will be useful in the sequel.

Applying Schwarz's inequality to (3) and using (4), we find

$$|R_n(z)|^2 \leq \frac{1}{\pi} \iint |f(t)|^2 dS \cdot r_n(z; \bar{z}), \qquad |z| < 1. \tag{5}$$

Thus the study of $R_n(z)$ is reduced to a study of $r_n(z; \bar{z})$.

3. **The form of** $r_n(z; \bar{t})$. Let D_n denote the n-rowed determinant $|(1 - a_i \bar{a}_j)^{-1}|$, and E_n denote the corresponding n-rowed permanent

$$\left| \frac{1}{1 - a_i \bar{a}_j} \right|^{+} = \sum \frac{1}{1 - a_1 \bar{a}_{r_1}} \frac{1}{1 - a_2 \bar{a}_{r_2}} \cdots \frac{1}{1 - a^n \bar{a}_{r_n}},$$

where the summation extends over all the $n!$ permutations of the n numbers $1, 2, \cdots, n$. Then, by an identity due to Borchardt(4), the following relations are valid:

(3) See G. Kowalewski, ibid. p. 229.

(4) C. W. Borchardt, *Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden Function*, J. Reine Angew. Math. vol. 53 (1857) pp. 193–198. The identity is also found in, for instance, R. F. Scott, *The theory of determinants and their applications*, Cambridge, 1904, pp. 159–161.

$$\Delta_n = D_n \cdot E_n; \qquad \Delta_{n+1}(z;\bar{t}) = D_{n+1}(z;\bar{t}) \cdot E_{n+1}(z;\bar{t})$$

where $D_{n+1}(z;\bar{t})$ $(E_{n+1}(z;\bar{t}))$ is the $(n+1)$-rowed determinant (permanent) obtained from D_n (E_n) in the same manner as $\Delta_{n+1}(z;\bar{t})$ is from Δ_n. Hence

$$r_n(z;\bar{t}) = \frac{D_{n+1}(z;\bar{t})}{D_n} \cdot \frac{E_{n+1}(z;\bar{t})}{E_n}.$$

This can be further simplified by observing that

$$\frac{D_{n+1}(z;\bar{t})}{D_n} = \frac{B_n(z)\cdot\overline{B_n(t)}}{1 - z\bar{t}} \qquad \left(B_n(z) = \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z}\right),$$

and by writing

$$\frac{E_{n+1}(z;\bar{t})}{E_n} = \frac{1}{1 - z\bar{t}} + H_n(z;\bar{t}), \tag{6}$$

where

$$H_n(z;\bar{t}) = \sum_{i,j} \frac{A_{ij}}{(1 - a_i\bar{t})(1 - \bar{a}_j z)}, \qquad A_{ij} = \frac{E_{ij}}{E_n}, \tag{7}$$

E_{ij} being the $(n-1)$-rowed permanent obtained from E_n by striking out the ith row and the jth column. The result is

$$(1 - z\bar{t})^2 \cdot r_n(z;\bar{t}) = B_n(z)\cdot\overline{B_n(t)}\left\{1 + (1 - z\bar{t})H_n(z;\bar{t})\right\}. \tag{8}$$

It is easy to verify that equation (8) is invariant when z as well as $t, a_1, \cdots, a_n$ are subjected to transformations of the form $\zeta = (z-c)/(1-\bar{c}z)$, $|c| < 1$, that is, to transformations which carry $|z| \leqq 1$ into $|\zeta| \leqq 1$ so that $z = c$ corresponds to $\zeta = 0$.

Finally, we have the following identity:

$$\sum_{i,j=1}^{n} \frac{A_{ij}}{(1 - \bar{a}_i z)(z - a_j)} = \sum_{i=1}^{n} \frac{1 - |a_i|^2}{(1 - \bar{a}_i z)(z - a_i)}, \tag{9}$$

which can be verified by means of the relations:

$$\frac{1}{(1 - \bar{a}_i z)(z - a_j)} = \frac{\bar{a}_i}{1 - \bar{a}_i\bar{a}_j}\,\frac{1}{1 - \bar{a}_i z} + \frac{1}{1 - \bar{a}_i a_j}\,\frac{1}{z - a_j};$$

$$\sum_{i=1}^{n} \frac{A_{ij}}{1 - \bar{a}_i a_j} = \sum_{j=1}^{n} \frac{A_{ij}}{1 - \bar{a}_i a_j} = \frac{E_n}{E_n} = 1.$$

A particular case of (9) is

$$H_n(z;\bar{z}) = \sum_{i=1}^{n} \frac{1 - |a_i|^2}{|1 - \bar{a}_i z|^2}, \qquad |z| = 1. \tag{10}$$

4. **Removal of the restriction on a_i.** The results of §§2 and 3 do not depend on the restriction $a_i \neq 0$; they depend only on the restriction that the n points in question should be distinct. But those results remain valid in the general case if we adopt the usual convention that, in case the mth point a_n $(1 < m \leqq n)$ coincides with k $(k > 0)$ points in the set $a_1, a_2, \cdots, a_{m-1}$, then

(a) the function $(1-\bar{a}_m z)^{-2}$ in (1) is replaced by

$$\frac{\partial^k}{\partial a^k} \frac{1}{(1-az)^2}\bigg|_{a=\bar{a}_m};$$

and

(b) the condition in (2) which corresponds to a_m is replaced by $f_n^{(k)}(a_m) = f^{(k)}(a_m)$. In fact, formulas (3), (4), (5) still hold, except that the determinants Δ_n and $\Delta_{n+1}(z; \bar{t})$ assume slightly different forms. But, when these are simplified by the process set forth in §3, we get the same results (6)–(10).

5. **Lemmas.** We proceed to establish some lemmas.

LEMMA 1. *For any n fixed points $a_1, a_2, \cdots, a_n$ on K, we have*

$$H_n(t; \bar{t}) \leqq \sum_{i=1}^{n} \frac{1 - |a_i t|^2}{|1 - \bar{a}_i t|^2}, \qquad |t| \leqq 1, \tag{11}$$

the equality sign being valid only in the following two cases:

(a) $|t| = 1$ *(as has been shown by* (10));

(b) $a_1 = a_2 = \cdots = a_n = 0$.

(We note that the right-hand member of (11):

$$\sum_{i=1}^{n} \frac{1 - |a_i t|^2}{|1 - \bar{a}_i t|^2} = \frac{1}{2}\left\{\sum_{i=1}^{n} \frac{1 + \bar{a}_i t}{1 - \bar{a}_i t} + \sum_{i=1}^{n} \frac{1 + a_i \bar{t}}{1 - a_i \bar{t}}\right\}$$

is harmonic in (x, y), $x + iy = t$, for $|t| \leqq 1$, and is therefore a harmonic majorant of $H_n(t; \bar{t})$ on $|t| \leqq 1$.)

Assume that the $n+1$ points $a_1, a_2, \cdots, a_n, t$ (on K) are distinct, and sent

$$s_n(z; \bar{t}) = \frac{z}{1 - z\bar{t}} B_n(z) \cdot \overline{B_n(t)},$$

which, when resolved into partial fractions, is

$$s_n(z; \bar{t}) = \frac{z}{1 - z\bar{t}} - \sum_{i=1}^{n} \frac{\bar{A}_i z}{1 - \bar{a}_i z},$$

where

$$A_i = \frac{1 - |a_i|^2}{t - a_i} \frac{B_n(t)}{B_{ni}(a_i)}, \qquad B_{ni}(t) = B_n(t) \frac{1 - \bar{a}_i t}{t - a_i}. \tag{12}$$

By differentiation with respect to z, we have

$$(13)\quad s_n'(z;\bar{t}) = \overline{B_n(t)}\left\{\frac{B_n(z)}{(1-z\bar{t})^2}+\frac{zB_n'(z)}{1-z\bar{t}}\right\} = \frac{1}{(1-z\bar{t})^2} - \sum_{i=1}^{n}\frac{\bar{A}_i}{(1-\bar{a}_iz)^2}.$$

Thus $s_n'(z;\bar{t})$ is of the same form as $r_n(z;\bar{t})$. It follows from the minimum property of $r_n(z;\bar{t})$ that

$$(14)\quad \frac{1}{\pi}\iint |r_n(z;\bar{t})|^2dS \leqq \frac{1}{\pi}\iint |s_n'(z;\bar{t})|^2dS.$$

The value I of the first integral in (14) is, by (4) and (8),

$$(15)\quad I = r_n(t;\bar{t}) = \frac{|B_n(t)|^2}{1-|t|^2}\left\{\frac{1}{1-|t|^2}+H_n(t;\bar{t})\right\}.$$

To compute the value J of the second integral in (14), we multiply the middle member of (13) by the conjugate of the last member of (13) and integrate. The result can be easily written down:

$$J = \frac{|B_n(t)|^2}{(1-z\bar{t})^2}+\frac{\overline{B_n(t)}\,B_n'(t)\cdot t}{1-|t|^2} - \overline{B_n(t)}\sum_{i=1}^{n}\frac{A_ia_iB_n'(a_i)}{1-\bar{a}_it}.$$

When the last two terms are simplified by means of (12), we have

$$(16)\quad J = \frac{|B_n(t)|^2}{1-|t|^2}\left\{\frac{1}{1-|t|^2}+\sum_{i=1}^{n}\frac{1-|a_it|^2}{|1-\bar{a}_it|^2}\right\}.$$

Since $B_n(t)\neq 0$, inequality (11) follows from (15) and (16).

This result is sufficient to ensure the validity of (11) in general. But, in order to single out the case (b), we add the following remarks.

(i) In the case where the n points $a_1, a_2, \cdots, a_n$ are distinct from t but not distinct among themselves, the above argument is obviously valid; and, without integration, we see that the norms I and J are still given by (15) and (16) respectively. Hence (11) still holds.

(ii) In the case where t coincides with k $(1<k\leqq n)$ of the points $a_1, a_2, \cdots, a_n$, our argument still remains valid if $r_n(z;\bar{t})$ and $s_n(z;\bar{t})$ are interpreted as standing for

$$\left.\frac{\partial^k}{\partial\zeta^k}r_n(z;\zeta)\right|_{\zeta=\bar{t}} \quad\text{and}\quad \left.\frac{\partial^k}{\partial\zeta^k}s_n(z;\zeta)\right|_{\zeta=\bar{t}}$$

respectively. The corresponding norms I and J can be obtained from (15) and (16) by differentiation. The results again lead to (11).

It is easy now to complete the proof. If $a_1=a_2=\cdots=a_n=0$, (11) obviously reduces to an equality. Conversely, if (11) reduces to an equality, then, in virtue of the uniqueness of $r_n(z;t)$, we have the identity in z:

$r_n(z;\bar{t}) \equiv s_n'(z;\bar{t})$. Since, in each case, $r_n(z;\bar{t})$ has exactly $n+1$ roots, namely, $z=0, a_1, \cdots, a_n$, and since $r_n(z;\bar{t})$ vanishes at $a_1, a_2, \cdots, a_n$, the identity implies that each a_i is a multiple root of $s_n(z;\bar{t})$. For such a case to happen, it is necessary and sufficient that $s_n(z;\bar{t})$ has $n+1$ equal roots: $a_1=a_2=\cdots=a_n=0$. The proof is complete.

LEMMA 2. *If the n points a_i are such that $0<r_1\leqq|a_i|<1$, $i=1, 2, \cdots, n$, then, on every circle $|z|=r$, $0<r<r_1$, there exists a point z_n such that*

$$H_n(z_n;\bar{z}_n)\cdot|B_n(z_n)|^2 < H_n(0;0)\cdot\prod_{i=1}^{n}|a_i|^2 < n\cdot\prod_{i=1}^{n}|a_i|^2.$$

Let

$$u(x,y) = \sum_{i=1}^{n}\frac{1-|a_iz|^2}{|1-\bar{a}_iz|^2}, \qquad v(x,y) = |B_n(z)|^2 \qquad (z = x+iy).$$

Then $\log u$ is superharmonic in $|z|<1$, and $\log v$ is harmonic in $|z|<r_1$. It follows that $\log uv = \log u + \log v \neq \text{const.}$ is superharmonic in $|z|<r_1$, and therefore its minimum in $|z|\leqq r$ occurs at a point z_n on $|z|=r$. This fact, together with Lemma 1, proves Lemma 2.

LEMMA 3. *Let the points a_{ni} be uniformly bounded from zero: $0<r_1<|a_{ni}|<1$, and be such that, for each n, the n points a_{ni} all lie on a radius of the unit circle which makes an angle $\theta=\theta_n$ with the axis of reals; then, for every r, $0<r<1$, the sequence of points $z=-re^{i\theta_n}$ satisfies the condition*

$$\lim_{n\to\infty} H_n(z_n;\bar{z}_n)\cdot|B_n(z_n)| = 0$$

provided that the set a_{ni} satisfies condition (B).

First of all, we have

$$|B_n(z_n)| = \prod_{i=1}^{n}\frac{r+|a_{ni}|}{1+r|a_{ni}|}.$$

This last product, which we shall denote by $1/p_n$, is known to approach zero with $\prod_{i=1}^{n}|a_{ni}|$. Next, by setting $z=t=z_n$ in (7) and $z=t=0$ in (9), we find

$$H_n(z_n;\bar{z}_n) < H_n(0;0) < \sum_{i,j}\frac{A_{ij}}{|a_{ni}|} = \sum_{i=1}^{n}\frac{1-|a_{ni}|^2}{|a_{ni}|} < \frac{2}{r_1}\sum_{i=1}^{n}(1-|a_{ni}|).$$

Since we have also

$$\sum_{i=1}^{n}(1-|a_{ni}|) < \frac{1+r}{1-r}\sum_{i=1}^{n}\left(1-\frac{r+|a_{ni}|}{1+r|a_{ni}|}\right) < \frac{1+r}{1-r}\log p_n,$$

it follows from the above that

$$H_n(z_n; \bar{z}_n) \cdot | B_n(z_n) | < \frac{2}{r_1} \frac{1 + r}{1 - r} \frac{\log p_n}{p_n}.$$

The last member approaches zero with $1/p_n$. Lemma 3 is thus established.

6. **Proof of Theorem** A. We are now in a position to prove Theorem A. Since $f_n(z)$ gives the best approximation to $f(z)$ in the sense of least squares,

$$\frac{1}{\pi} \iint | R_n(t) |^2 dS = \frac{1}{\pi} \iint | f(t) - f_n(t) |^2 dS \leqq \frac{1}{\pi} \iint | f(t) |^2 dS = M.$$

Hence, for $|z| \leqq \rho < 1$, we have

$$| R_n(z) | = \left| \frac{1}{\pi} \iint \frac{R_n(t)}{(1 - z\bar{t})^2} dS \right| \leqq \frac{M^{1/2}}{1 - \rho^2},$$

by Schwarz's inequality. It follows that the functions $R_n(z)$ form a normal family on K. From every sub-sequence of $R_n(z)$ can be extracted a sub-sequence $R_{n_k}(z)$ which converges to an analytic function $R(z)$ on K, uniformly on any closed point set on K. To prove Theorem A, it is sufficient to prove that, under condition (A), any such limit function is identically zero.

Since the points a_{ni} are different from zero (a fact implied in (A)) and have no limit point on K, there exists r_1 such that $0 < r_1 < |a_{ni}| < 1$. Now suppose that the limit function $R(z)$ of the sub-sequence $R_{n_k}(z)$ does not vanish identically. For the sake of simplicity, we shall take $R_n(z)$ for $R_{n_k}(z)$. Then there exists r, $0 < r < r_1$, such that $R(z)$ has a positive minimum m on the circle $|z| = r$. Hence, for n sufficiently large, we have $|R_n(z)| > m/2$ on $|z| = r$.

But, by (5), we have

$$| R_n(z) |^2 \leqq M \cdot r_n(z; \bar{z}), \qquad | z | = r.$$

Let z_n be the point on $|z| = r$ given by Lemma 2 corresponding to the n points a_{ni}. Then, by (8) and Lemma 2, we have

$$| R_n(z_n) |^2 < \frac{M}{(1 - r^2)^2} \left\{ | B_n(z_n) |^2 + (1 - r^2)\, n \prod_{i=1}^{n} | a_{ni} |^2 \right\}.$$

The second term in the braces approaches zero by hypothesis, and the first term therein is again dominated by $1/p_n$ (where p_n has the same meaning as in the proof of Lemma 3), and hence approaches zero with $\prod_{i=1}^{n} | a_{ni} |$. Hence, for n sufficiently large, we have $|R_n(z_n)| < m/2$. The contradiction proves Theorem A.

Remark. If the condition $a_{ni} \neq 0$ is dropped, condition (A) should be interpreted as implying

$$\lim_{n \to \infty} n \prod_{i=1}^{n} | a_{ni}^{*} |^2 = 0,$$

where the set a_{ni} is obtained from the original set by omitting all those numbers which are zeros. Then Theorem A is true for the sequence $f_n^*(z)$ corresponding to the set a_{ni}^*. It follows from the minimum property of $f_n(z)$ that Theorem A is also true for the sequence $f_n(z)$. The restriction $a_{ni}\neq 0$ is thus removed.

7. **Proof of Theorem** B. Consider first the particular case where the normal set a_{ni} satisfies the hypothesis of Lemma 3. In this case the proof follows the same line as that of Theorem A; Lemma 3 now plays the rôle of Lemma 2.

Next, consider the less particular case where the normal set a_{ni} is such that, after a transformation of the form $\zeta=(z-c)/(1-\bar{c}z)$, $|c|<1$, it goes into a set b_{ni} satisfying the hypothesis of Lemma 3. In this case, the remainder $R_n(z)$, which was originally bounded by

$$| R_n(z) |^2 \leqq \frac{M}{(1-|z|^2)^2}\{1+(1-|z|^2)H_n(z;\bar{z})\}\,| B_n(z) |^2,$$

is bounded by

$$| R_n(z) |^2 \leqq \frac{M}{(1-|z|^2)^2}\{1+(1-|\zeta|^2)H_n(\zeta;\bar{\zeta})\}\,| B_n(\zeta) |^2$$

after the transformation. Hence our argument is still applicable, and the truth of the theorem follows.

Finally, the case of an arbitrary normal set can be disposed of with ease. The proof of the theorem is thus complete.

8. **The case of simple sequence** a_n. If we have a simple sequence a_n, the function $f_n(z)$ of the form (1) found by interpolation to a function $f(z)$ of class S_2 is the sum of the first n terms of the series

$$c_1\phi_1(z) + c_2\phi_2(z) + \cdots,$$

where the set of functions $\phi_n(z)$ is obtained from the set $(1-\bar{a}_iz)^{-2}$ by orthogonalization on the area K, and where the coefficients c_n are determined either by interpolation or by integration:

$$c_n = \frac{1}{\pi}\iint f(z)\,\overline{\phi_n(z)}\,dS, \qquad n = 1, 2, \cdots.$$

The following theorem, together with its corollary, is an obvious extension of a theorem due to Walsh in connection with interpolation to functions defined by line integrals.

For an arbitrary set a_n pre-assigned on K, and for a given function $f(z)$ of class S_2, the sequence $f_n(z)$ converges on K, uniformly on any closed point set on K, to an analytic function $g(z)$, which is characterized by the fact that, among all functions of class S_2 which coincide with $f(z)$ at the points a_{ni}, $g(z)$ is the unique one whose norm on K is the least.

As a corollary, we have:

Among all functions of class S_2 which vanish at $a_1, a_2, \cdots, a_n$ and take on the value unity at $z=t$, the function

$$\psi(z) = \frac{(1-|t|^2)B_n(z)G_n(z;\bar{t})}{(1-z\bar{t})B_n(t)G_n(t;\bar{t})} \qquad \left(G_n(z;\bar{t}) = \frac{1}{1-z\bar{t}} + H_n(z;\bar{t})\right)$$

is the unique one whose norm on K is least.

In this corollary, it is tacitly assumed that t is distinct from $a_1, a_2, \cdots, a_n$. If t coincides with k of these points, the factor $B_n(t)$ in the definition of $\psi(z)$ should be replaced by $B_n^{(k)}(t)$, the kth derivative of $B_n(z)$ at the point $z=t$. With this understanding, we prove the following theorem.

THEOREM. *For any sequence of points a_n pre-assigned on K, we have*

$$H_{n+1}(t;\bar{t}) > H_n(t;\bar{t}), \qquad |t| \leqq 1, n = 1, 2, \cdots.$$

In view of inequality (10), we need to prove the theorem merely for $|t|<1$. For this purpose, let $\psi(z)$ be defined as above, and let

$$\psi_1(z) = \frac{(1-|t|^2)B_n(z)G_{n-1}(z;\bar{t})}{(1-z\bar{t})B_n(t)G_{n-1}(t;\bar{t})} \qquad (n > 1).$$

Then $\psi(z)$ and $\psi_1(z)$ both vanish at $a_1, a_2, \cdots, a_n$, and both take on the value unity at $z=t$. Further, we have

$$\frac{1}{\pi}\iint |\psi(z)|^2 dS = \frac{1-|t|^2}{|B_n(t)|^2 G_n(t;\bar{t})},$$

$$\frac{1}{\pi}\iint |\psi_1(z)|^2 dS < \frac{1}{\pi}\iint \left|\frac{(1-|t|^2)B_{n-1}(z)G_{n-1}(z;\bar{t})}{(1-z\bar{t})B_n(t)G_{n-1}(t;\bar{t})}\right|^2 dS$$

$$= \frac{1-|t|^2}{|B_n(t)|^2 G_{n-1}(t;\bar{t})}.$$

The theorem then follows from the minimum property of $\psi(z)$.

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