CONCERNING SIMPLE PLANE WEBS

BY

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A compact continuum W is said [1]([1]) to be a simple web if there exists an upper semi-continuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that (1) G is a dendron with respect to its elements and so is H and (2) if g and h are elements of G and H, respectively, the common part of g and h exists and is totally disconnected. Hence, a simple web is a web([2]).

Examples of simple plane webs. Although a simple web is not necessarily a subset of the plane, this paper will deal only with those of this type. A square plus its interior is a simple web. We may consider the elements of G to be intervals parallel to one pair of sides of the square and the elements of H to be intervals parallel to the other pair of sides. We see from Theorem 1 of this paper that if $C_1, C_2, \cdots$ are circles no two of which intersect each other and such that C_1 incloses C_i $(i = 2, 3, \cdots)$, then C_1 plus its interior minus the sum of the interiors of $C_2, C_3, \cdots$ is a simple web.

A square plus its interior plus an interval intersecting both the interior and the exterior of the square is not a simple web. Hence, a plane web is not necessarily a simple web. By the use of Theorem 1 of this paper, we find that the continuous curve described on page 273 of [7] which is left connected but not locally connected on the omission of some countable subset is not a simple web.

This paper gives necessary and sufficient conditions that a compact plane continuum be a simple plane web. Professor R. L. Moore suggested the problem of finding such conditions. Much credit is due him for the development of this paper.

It has been shown [6] that every simple plane web is a continuous curve. In the present paper, the following theorem will be established.

THEOREM 1. *In order that a compact plane continuous curve be a simple plane web, it is necessary and sufficient that it remain connected and locally connected on the omission of any countable subset.*

Proof of necessity. The following argument is similar to that found in the proof of Theorem 1 of [1]. Suppose that G and H are collections satisfying with respect to the simple plane web W all([3]) of the conditions described in the preceding definition. Suppose that K is a countable subset of W. If W_Z $(Z = G, H)$ denotes the sum of all of the continua of the collection Z that do not intersect K, then any two points of $W_G + W_H$ lie on a continuum which is the sum of three continua of the collection $G + H$ that do not intersect K. As $W_G + W_H$ is dense in W, $W - K$ is connected.

If $W - K$ is not locally connected at P, there exist a domain([4]) d containing P and a sequence of points of $W - K$ converging to P such that each belongs to a different component of $(W - K) \cdot d$. Let C_1, C_2, C_3 denote three circles such that C_1 incloses P, C_2 incloses C_1, C_3 incloses C_2 but no element of G or H and C_3 plus its interior is a subset of d. Now C_1 incloses a sequence of points P_1, P_2, $\cdots$ of W_G no two of which belong to the same component of $(W - K) \cdot d$ such that if g_i is the component containing P_i of the common part of d and the element of G containing P_i, then g_1, g_2, $\cdots$ has a sequential limiting set F. There exists a sequence Q_1, Q_2, $\cdots$ in W_H converging to a point of C_2 such that Q_n is a point of a subcontinuum of g_m $(m \geqq n)$ irreducible from C_1 to C_3. If h_i denotes the component containing Q_i of the common part of d and the element of H containing Q_i, then some nondegenerate subcontinuum of F belongs (see [6, Theorem 2]) to the limiting set of h_1, h_2, $\cdots$. However, as G and H are upper semi-continuous collections, this nondegenerate continuum is both a subset of an element of G and a subset of an element of H. This is contrary to the hypothesis that the common part of an element of G and an element of H is totally disconnected.

This establishes the necessity of the condition of Theorem 1. A number of preliminary theorems will be proved before the establishment of its sufficiency. When the term "simple plane web" is used in the following theorems, it is to be understood that we refer to a compact plane continuum that remains connected and locally connected on the omission of any countable subset.

THEOREM 2. *Each complementary domain of a simple plane web is bounded by a simple closed curve but no two boundaries of its individual complementary domains have more than one point in common.*

Proof. The outer boundary of a bounded complementary domain of a continuous curve is a simple closed curve [4]. As a simple plane web has no cut points, the boundary of one of its complementary domains is also the outer boundary of this domain. If the boundaries of two such complementary

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([3]) However, in the argument no use is made of the condition that G and H are dendrons with respect to their elements.

([4]) A domain is an open set not necessarily connected. However, a complementary domain is connected [7, pp. 11 and 153].

domains intersected in a pair of points, this pair of points would separate the simple plane web.

THEOREM 3. *If the connected set x is the sum of a countable number of points and complementary domains of the simple plane web W, then each point of $W \cdot \bar{x}$ is a limit point of $W - W \cdot \bar{x}$.*

Proof. Suppose that d is a domain containing a point P of $W \cdot \bar{x}$. As $W - W \cdot x$ is a connected and locally connected inner limiting set, the common part of it and d contains [8] an arc AQB. There exists in W an arc ACB containing no point of the countable point set $W \cdot x + Q$. The point set $AQB + ACB$ contains a simple closed curve J which contains an interval $A'QB'$ of AQB. There exists in W an arc α from Q to $J - A'QB'$ which contains no point of $W \cdot x + A' + B'$. The arc α contains an arc $Q'OL$ having in common with J only its end points Q' and L which belong to $A'QB'$ and $J - A'QB'$ respectively. Let $Q'A'L$ and $Q'B'L$ be arcs lying on J with points as indicated. Then $Q'A'L + Q'B'L + Q'OL$ divides space into three mutually exclusive domains only one of which contains the connected set x. Now one of the segments[5] $\langle Q'A'L \rangle$, $\langle Q'B'L \rangle$ and $\langle Q'OL \rangle$ contains no point of $\bar{x}$ and each of them contains a point of d. Therefore d contains a point of $W - W \cdot \bar{x}$. Since any domain containing a point of $W \cdot \bar{x}$ contains a point of $W - W \cdot \bar{x}$, then each point of $W \cdot \bar{x}$ is a limit point of $W - W \cdot \bar{x}$.

THEOREM 4. *Under the hypotheses of Theorem 3, the closure of x does not separate W.*

Proof. If P is a point of the plane S, then $S - (x + P)$ is a connected and locally connected inner limiting set. Hence [8] it is arcwise connected. Furthermore [2, Theorem 12], $S - \bar{x}$ is connected.

Suppose that $W - W \cdot \bar{x}$ is the sum of two mutually separated sets S_1 and S_2. There exists in $S - \bar{x}$ an arc AB from a point A of S_1 to a point B of S_2. Let P denote the last point of the closed point set $S_1 \cdot AB$ in the order from A to B on AB. There is a point Q following P on AB such that the interval PQ of AB contains no point of S_2. Let P' denote the first point in the order from Q to B that the interval QB of AB has in common with β, the boundary of d, the complementary domain of W that contains Q. Since the closure of $x + d$ does not separate S, one arc of β from P to P' contains no point of $\bar{x}$. But P belongs to S_1 and P' belongs to S_2. Thus, the supposition that $W - W \cdot \bar{x}$ is not connected leads to a contradiction.

THEOREM 5. *If R is a connected point set consisting of a plane domain plus a totally disconnected point set and the complement of R is a locally connected inner limiting set, then $\bar{R}$ is a continuous curve.*

Proof. If $\bar{R}$ is not locally connected, there exist a domain d, a sequence

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[5] By $\langle \alpha \rangle$ is meant the arc α minus its end points.

of components $c_1, c_2, \cdots$ of $\bar{d} \cdot \bar{R}$, and a sequence of arcs $\alpha_1, \alpha_2, \cdots$ such that (1) for each positive integer j, α_j is a subset of $\bar{d} - \bar{d} \cdot \bar{R}$; (2) if h, i, j are positive integers such that $h < i < j$, then α_i separates c_h from c_j in d and c_i separates α_h from α_j in d; (3) d intersects c, the sequential limiting set of $c_1, c_2, \cdots$. The sequence $\alpha_1, \alpha_2, \cdots$ converges to c.

As no point of c is a point of a domain lying in R and some nondegenerate subcontinuum c' of c is a subset of d, the complement of R contains two points P_1 and P_2 of c'. There exist domains d_1 and d_2 containing P_1 and P_2 respectively such that d_1 and d_2 are mutually exclusive subsets of d. As the complement of R is locally arcwise connected at P_1 and P_2, there exists a positive integer j such that each of the point sets $d_1 - d_1 \cdot R$ and $d_2 - d_2 \cdot R$ contains an arc intersecting both α_j and α_{j+2}. Therefore, there is a simple closed curve J lying in $d - d \cdot R$ and intersecting α_j and α_{j+2}. As R is connected and contains no point of J, it does not intersect both the interior and the exterior of J. Therefore, no component of $\bar{d} \cdot \bar{R}$ separates α_j from α_{j+2} in d and $\bar{R}$ is locally connected.

THEOREM 6. *Under the hypotheses of Theorem 3, $\bar{x}$ is a continuous curve each of whose degenerate cyclic elements is a point of W in x and each of whose nondegenerate cyclic elements is the closure of a complementary domain of W in x.*

Proof. As the complement of x is a locally connected inner limiting set, it follows from Theorem 5 that $\bar{x}$ is a continuous curve. If P is a cut point of $\bar{x}$, it is a cut point of x and therefore must be a point of W in x.

If J is a simple closed curve in $\bar{x}$, we shall show that it is a subset of the closure of a complementary domain of W in x. Since $\bar{x}$ does not separate space (see Theorem 4), one of the complementary domains of J is a subset of $\bar{x}$. By Theorem 3, we have that J separates no two points of $\bar{x} - x$ from each other. Therefore, one of the complementary domains of J is a subset of x and is therefore a subset of a complementary domain of W in x.

THEOREM 7. *Suppose W is a simple plane web, β is its outer boundary and X is the collection of all nondegenerate continua x such that x is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and bounded complementary domains of W. If A and B are points of W, there is an arc AB in W such that AB minus its end points contains no point of either β or any element of X.*

Proof. Suppose that A belongs to x_A of X and that B belongs to neither β nor any element of X. As $W - W \cdot x_A + A$ is connected, A is accessible [10, Theorem 1] from $W - W \cdot x_A$. Since the continua of the collection X are mutually exclusive and no one of them separates the plane, then x_A plus the complement of the sum of the elements of X is a connected and locally connected inner limiting set which if regarded as space satisfies Axioms 1–5 of [7]. In this space, there exists an arc AB having no point except A in common with

$x_A + \beta$. The arc AB is a subset of W.

THEOREM 8. *Under the hypotheses of Theorem 7, if P is a point belonging to both β and an element x_P of X, Q is another point of β and c is a component of $x_P - P$, then there exists in W a simple closed curve J inclosing c but no point of $x_P - c$ and containing P and Q but no other point either of β or of an element of X.*

Proof. Let X' denote the set of all continua of the collection X except x_P and x_Q where x_Q denotes the element of X containing Q if there is one but denotes Q if there is not. Let Z denote the collection whose elements are the continua of X' and the points of the plane that belong to no element of X'. The collection Z is upper semi-continuous and no one of its elements separates the plane. It follows [5] that the space Σ_1 whose "points" are elements of Z is topologically equivalent to the plane. If Σ_2 denotes the space whose "points" are the "points" of Σ_1 other than the elements of X' not intersecting β, then it is topologically equivalent to a plane minus a countable number of points. If β' denotes the "point" set whose points are the points of β that belong to no element of X' and the continua of X' that intersect β, then in the space Σ_2, β' is a simple closed curve and x_P and x_Q are continuous curves each lying except for one point wholly in the interior of β'. In this space, there exists an arc EQ from a point E of c to Q such that $x_P + x_Q + \beta'$ intersects EQ only at $E + Q$. Furthermore [7, Theorem 12, p. 217], there exists a simple closed curve intersecting $x_P + x_Q + \beta'$ only at $P + Q$ and inclosing $c + EQ - Q$ but no point of $x_P - c$.

Description of dendrons G and F. In this section, W will designate a compact plane continuous curve which is left connected and locally connected on the omission of any countable point set. Also, X will denote the set of all nondegenerate continua x such that x is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and bounded complementary domains of W.

As a step toward proving the sufficiency of Theorem 1, we shall describe an upper semi-continuous collection G of mutually exclusive continua filling up W such that G is a dendron with respect to its elements. At the same time, we shall describe a dendron F in W such that each element of G intersects F in only one point.

As the description of G is somewhat complicated, we shall describe it by steps. First, we consider a transformation T throwing a square plus a subset of its interior into W. Next, we describe an upper semi-continuous collection of mutually exclusive continua which is an arc with respect to its elements and which covers a part of the inverse of W under transformation T. We cover more of this inverse by another upper semi-continuous collection of mutually exclusive continua. The process is continued until we have an upper semi-continuous collection of mutually exclusive continua filling up the inverse

of W under T such that this collection is a dendron with respect to its elements. The image of the elements of this collection with respect to T gives the elements of G.

Transformation T. Let $P_1P_2P_3$ and $P_1P_4P_3$ be two arcs whose sum is the boundary β of the unbounded complementary domain of W and whose ends are not points of X^*[6]. There exist in W arcs $P_1P_5P_3$ and $P_1P_6P_3$ which except for their end points are mutually exclusive and contain no point of $X^*+\beta$. Let R be the interior of a square with diagonals V_1V_3 and V_2V_4 and let V_5, V_6 be the midpoints of V_1V_4, V_2V_3 respectively.

There exists a continuous transformation T of $\overline{R}$ into $W+X^*$ such that $T(V_1V_5) = T(V_4V_5) = P_1P_5$, $T(V_2V_6) = T(V_3V_6) = P_3P_6$, $T(V_5V_2) = P_5P_3$, $T(V_4V_6)=P_1P_6$ and each point of $W+X^*-P_1P_5-P_6P_3$ is the image of only one point of $\overline{R}- V_1V_5V_4- V_2V_6V_3$ under the transformation T.

Let M denote the set of all points P of $\overline{R}$ such that the image of P under T is a point of W. We note that M is a compact continuous curve which is left connected and locally connected on the omission of any countable subset. Let Y be the collection of all continua y such that y is the inverse under T of an element of X. We note that y is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and bounded complementary domains of M.

The elements of Y are ordered y_1, y_2, $\cdots$. By Theorem 7 and the fact that M is separable, there exists a countable point set z_1, z_2, $\cdots$ dense in M and such that for each positive integer i, z_i belongs to M but not to Y^* or the boundary of R. The bounded complementary domains of M are ordered d_1, d_2, $\cdots$.

We shall describe an upper semi-continuous collection G' of mutually exclusive continua filling up M such that (1) G' is a dendron with respect to its elements, (2) no element of G' contains more than two points of Y^*, (3) each element of G' intersects V_5V_2 and V_4V_6, and (4) each element of G' is a continuum of condensation of M. Also, we shall describe a dendron F' in M such that each element of G' intersects F' in only one point.

First step. If some element of Y has a point P in common with V_1V_2 or V_4V_3 and P is nearer the midpoint than either end point of the one of V_1V_2, V_4V_3 on which it lies, let i be the least positive integer n such that y_n is such an element of Y and let P be the point that y_i has in common with $V_1V_2+V_4V_3$. If there is no such i, let P be the midpoint of V_1V_2. Let P_1 and P_2 be the projections of P on V_1V_2 and V_4V_3 respectively. Let k be the least positive integer n, if there is one, such that y_n has no point in common with the boundary of R. There are two arcs P_1AP_2 and P_1DP_2 lying in M, intersecting Y^* plus the boundary of R only at P_1 and P_2 and forming a simple closed curve J such that (1) J incloses z_1 and y_k, (2) the interior of J together with P_1 and P_2 contains each element of Y (if any) that contains

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[6] By X^* is meant the set of all points P such that P is a point of an element of X.

P_1 or P_2, and (3) P_1AP_2 separates $\langle P_1DP_2 \rangle$ from V_1V_4 in M.

We shall consider the case in which P_1 and P_2 belong to elements $y(P_1)$ and $y(P_2)$ of Y respectively. Let $\langle P_1BP_2 \rangle$ and $\langle P_1CP_2 \rangle$ be mutually exclusive open arcs in M which are inclosed by J, which contain no point of Y^*, and which are such that P_1BP_2 separates $\langle P_1CP_2 \rangle$ from V_1V_4 in M, the simple closed curve formed by $P_1AP_2+P_1BP_2$ incloses $y(P_1)-P_1$, the simple closed curve formed by $P_1BP_2+P_1CP_2$ incloses $y(P_2)-P_2$, and P_2C of P_1CP_2 intersects V_5V_2.

If $y(P_1)-P_1$ is not connected, let $c_1, c_2, \cdots$ denote its components. Let i, j be the least positive integers m, n respectively such that z_m, y_n are inclosed by $P_1AP_2+P_1BP_2$. There is a simple closed curve J_1 in M containing P_1 and P_2 and inclosing c_1, z_i, y_j but no point of $y(P_1)-c_1$ which is such that $J_1-P_1-P_2$ contains no point of Y^* and is inclosed by $P_1AP_2+P_1BP_2$. Let O represent the simple closed curve which is a subset of $P_1AP_2+P_1BP_2+J_1$ and which incloses c_2 but no point of J_1. Let i, j be the least positive integers m, n such that z_m, y_n are inclosed by O. There is a simple closed curve J_2 in M containing P_1 and P_2 and inclosing c_2, z_i, y_j but no point of $Y(P_1)-c_2$ which is such that $J_2-P_1-P_2$ contains no point of Y^* and is inclosed by O. This process is continued until for each c_i there is a unique J_i inclosing it.

The interior of $P_1AP_2+P_1BP_2$ is topologically equivalent to the bounded complementary domain of $P_1BP_2+P_2C+P_1DP_2$. Treating $P_2BP_1+P_1DP_2$ as P_1AP_2 and P_2C+CP_2 as P_1BP_2, we consider curves $J_1', J_2', \cdots$ inclosing the components of $y(P_2)-P_2$ in a manner similar to that in which we considered simple closed curves inclosing the components of $y(P_1)-P_1$.

Let g_1' denote the closure of $J+P_1BP_2+P_2C+J_1+J_2+\cdots+J_1'+J_2'+\cdots$. The continuum g_1' is a subset of an element of G'. It is to be noted that g_1' is a continuum of condensation of M intersecting both $V_5'V_2$ and V_4V_6 and if C is the common part of M and a bounded complementary domain of g_1', then (1) C intersects the interiors of triangles $V_1V_2V_5$ and $V_4V_3V_6$, (2) $\overline{C}$ is a simple plane web, and (3) no two elements of Y intersect both $\dot{C}$ and the boundary of C with respect to M.

Second step. Let i be the least positive integer n such that y_n intersects one of the intervals V_1P_1 and V_4P_2 in a point P nearer the midpoint than either end point of that interval. If there is no such i, let P be the midpoint of V_1P_1. Let P_3, P_4 be the projections of P on V_1P_1 and V_4P_2 respectively. By treating $V_1P_1+P_1AP_2+P_2V_4+V_4V_1$ as the boundary of R and following a procedure similar to that used in finding g_1', we obtain g_2'. By treating $P_1V_2+V_2V_3+V_3P_2+P_2DP_1$ as the boundary of R, we get g_3'. This process is continued until every point of M belongs to either the closure of the sum of all the g_n''s or to a bounded complementary domain of one of them.

If U is a component of M minus the common part of M and the sum of the g_n''s together with their bounded complementary domains and U is closed, U will have a point in common with V_1V_2 and a point in common with V_4V_3

and is to be considered as a g'. If U is not closed, it will not have a point in common with either V_1V_2 or V_4V_3 and, as every two g_n''s are separated in M by a third g_n', there is only one g_n' which contains a limit point of U. If there are such U's, g_n' plus all U's which have a point of g_n' as a limit point is a g'. Each g' described is a subset of an element of G'.

If G_1' is the set of maximal g''s thus described, G_1' is an arc with respect to its elements, each point of $G_1'^*$ is a limit point of $M' - G_1'^*$, no element of G_1' has two points in common with an element of Y and each element of G_1' has both a point in common with V_1V_2 and a point in common with V_4V_3. There exists a dendron F_1' in M such that $F_1' \cdot G_1'^*$ is V_4V_3, $F_1' - V_4V_3$ intersects neither V_4V_6 nor any element of Y and if C is a component of $M - G_1'^*$, then $F_1' \cdot C$ is an arc minus an end point of diameter less than the diameter of V_1V_2 and if K is a connected subset of C that intersects both this arc and an element of Y that intersects the boundary of C with respect to M, then K intersects both V_5V_2 and V_4V_6. Let D_1 be a domain containing $F_1' - V_4V_3$ such that each point of D_1 is nearer F_1' than $G_1'^* + V_4V_6$. Then $\overline{D}_1$ intersects $G_1'^*$ only in V_4V_3.

Third step. If C is a component of $M - G_1'^*$, C is the common part of M and a bounded complementary domain of some element of G_1'. We now consider the breaking up of $\overline{C}$ into elements of G'. We note that C intersects the interiors of the triangles $V_1V_2V_5$ and $V_4V_3V_6$. If β denotes the boundary of C with respect to M, then $C + \beta$ is a simple plane web and no more than one element of Y intersects both C and β.

Let y_a be such an element of Y intersecting β at P_1 and let c denote the closure of the common part of y_a and C. Let Q_1Q_2 be an arc such that $Q_1Q_2 - Q_1$ is $F_1' \cdot C$. Let i be the least positive integer n such that d_n is a subset of c. There is a point P_2 of the boundary of d_i such that $c + \beta - P_2$ is connected. There is a subcontinuum w of c such that (1) w contains P_1 and P_2, (2) if a point of a complementary domain of M belongs to w, the complementary domain is a subset of w, and (3) no subcontinuum of w has properties (1) and (2). There is an upper semi-continuous collection E of mutually exclusive continua filling up w such that E is an arc from P_1 to P_2 with respect to its elements and such that every element of E is either a point or an arc which has only its end points in common with the boundary of c. Designate the arc of elements of E by $\xi(P_1P_2)$. Each connected subset of C that intersects $\xi(P_1P_2)$ and Q_1Q_2 also intersects V_5V_2 and V_4V_6. To adjust the argument to the case where no element of Y has a point in common with both C and β, let $\xi(P_1P_2)$ be an arc in $C + P_1$ from a point P_1 of β to a point P_2 of C such that $\xi(P_1P_2)$ contains no point of Y^* and any connected subset of C intersecting both $\xi(P_1P_2)$ and Q_1Q_2 also intersects both V_5V_2 and V_4V_6.

Let j, k be the least positive integers m, n respectively such that z_m, y_n are subsets of C. There is a simple closed curve J lying in C, inclosing y_k, intersecting $\xi(P_1P_2) + Q_1Q_2$ only at $P_2 + Q_2$ and such that $Q_1Q_2 + J$ contains z_j.

We note that J plus the common part of M and the interior of J is a simple plane web intersecting both $V_5 V_2$ and $V_4 V_6$.

A breaking up of the portion of $\overline{C}$ not in the interior of J into elements of G' will now be considered. Considering the elements of $\xi(P_1 P_2)$ as points, $\xi(P_1 P_2)$ as $V_1 V_2$, β as $V_1 V_4$, $\xi(P_1 P_2)$ as $V_4 V_3$, J as $V_2 V_3$, and $Q_1 Q_2$ as $V_5 V_6$, we divide the portion of $\overline{C}$ not in the interior of J in a manner somewhat analogous to that described in the second step. We divide the portion of $\overline{C}$ corresponding to $V_1 V_2 V_6 V_5$ in the manner in which $V_1 V_2 V_3 V_4$ was divided in the second step and we divide the portion of $\overline{C}$ corresponding to $V_5 V_6 V_3 V_4$ in the same fashion, care being taken that each pair of g''s thus defined that intersect on $Q_1 Q_2$ also intersect the same elements of $\xi(P_1 P_2)$.

The closures of all components of $M - G_1'^*$ are broken up just as was $\overline{C}$. If a g' as defined in this step intersects another g' defined in either this or the second step, the sum of the two is a g'. Let G_2' denote the set of maximal g''s defined. We note that the boundary of d_i of y_a is a subset of $G_2'^*$. Also, each element of G_2' intersects $V_5 V_2$ and $V_4 V_6$ and so does each component of $M - G_2'^*$.

There exists a dendron F_2' in M such that $F_2' \cdot G_2'^*$ is F_1', $F_2' - F_1'$ is a subset of D_1, $F_2' - F_1'$ intersects no element of Y, and if C is a component of $M - G_2'^*$, then $F_2' \cdot C$ is an arc minus one end point of diameter less than one-half the diameter of $V_1 V_2$. Let D_2 be a domain containing $F_2' - F_1'$ such that each point of D_2 is nearer F_2' than $G_2'^* \overline{D}_2$ and is a subset of $D_1 + F_1'$.

Fourth step. We shall now describe the breaking up of a component C of $M - G_2'^*$. The boundary of C with respect to M is a simple closed curve J. If an element of Y intersects both C and J, break up $\overline{C}$ as described in the third step. Otherwise, let j be the least integer equal to n such that y_n is inclosed by J. Let P be a point of y_j such that $y_j - P$ is connected. Let $Q_1 Q_2 - Q_1$ be the common part of F_2' and the interior of J. There is an arc $Q_1 P$ in M which except for its end points intersects neither $J + Q_1 Q_2$ nor any element of Y and which is such that any connected subset of the interior of J intersecting $Q_1 Q_2$ and y_j but not $Q_1 P$ intersects both $V_5 V_2$ and $V_4 V_6$. Treating $Q_1 P + J$, $Q_1 Q_2$, and y_j as β, $Q_1 Q_2$, and c respectively were treated in the third step, we break up the common part of M and the interior of J in a manner analogous to that in which C was broken up as considered in the third step.

Let G_3' be the set of all g''s considered after all components of $M - G_2'^*$ are broken up as described above. We define a dendron F_3' in D_2 containing F_2' and satisfying conditions analogous to those satisfied by F_2'. In like manner, we consider G_4', G_5', $\cdots$ and F_4', F_5', $\cdots$. It is to be noted that (1) for each positive integer j, there is an n such that z_j and the boundary of d_i belongs to $G_n'^*$, (2) for no n do three points of Y^* belong to the same element of G_n', (3) each element of G_n' intersects $V_5 V_2$ and $V_4 V_6$, and (4) each component of $M - G_n'^*$ intersects $V_5 V_2$ and $V_4 V_6$.

Definition of dendron G. Let G' be the set of all continua such that g'

is an element of G' only if either (1) for some positive integer n, g' belongs to G_n' and G_{n+1}' or (2) g' is a component of $M-(G_1'^{*}+G_2'^{*}+\cdots)$. With respect to its elements, G' is a dendron. Each element of G' is a continuum intersecting both $V_5 V_2$ and $V_4 V_6$. Each element of G' is a continuum of condensation of M. No element of G' intersects an element of Y in more than two points.

We define G to be the collection of all point sets g such that g is the image of an element of G' under T. We note that G is an upper semi-continuous collection of mutually exclusive continua filling up W such that G is a dendron with respect to its elements.

Definition of dendron F. We define F' to be the limiting set of F_1', F_2', $\cdots$. We note that it is a dendron that intersects each element of G' in only one point. We define F to be the image of F' under transformation T. We note that F is a dendron that intersects each element of G in only one point and the outer boundary of W in only an arc.

Description of arc H. We shall describe an upper semi-continuous collection H of mutually exclusive continua filling up W such that H is an arc with respect to its elements and such that if g and h are elements of G and H respectively, then $g \cdot h$ is totally disconnected. First, we shall establish four additional theorems.

THEOREM 9. *If R is the interior of a square and E is a collection of point sets covering $\overline{R}$ such that for each point P and each domain D there is a point P' of D which belongs to no element of E that contains P, then for any positive ϵ and any two points A and B of $\overline{R}$ there is in $R+A+B$ an arc ACB such that (1) for any two points Q_1 and Q_2 of ACB[7], $\cos(Q_1Q_2, AB) > 1-\epsilon$ and (2) no element of E contains a subarc of ACB.*

Proof. There is a point P_1 such that[8] $d(A, P_1)=d(P_1, B)$, the intervals AP_1 and P_1B are subsets of $R+A+B$, and $\cos(AP_1, AB) > 1-\epsilon$. There exists a quadrilateral Q lying in $R+A+B$ and inclosing $AP_1+P_1B-(A+B)$. Within Q there exist (1) a point P_2 not belonging to any element of E that contains A and such that $\max[d(A, P_2); d(P_2, P_1)] < (2/3)d(A, P_1)$, $\cos(AP_2, AB) > 1-\epsilon$ and $\cos(P_2P_1, AB) > 1-\epsilon$ and (2) a point P_3 not belonging to any element of E that contains P_1 and such that $\max[d(P_1, P_3); d(P_3, B)] < (2/3)d(P_1, B)$, $\cos(P_1P_3, AB) > 1-\epsilon$ and $\cos(P_3B, AB) > 1-\epsilon$. Likewise we get within Q a point P_4 not belonging to any element of E containing A and such that $\max[d(A, P_4); d(P_4, P_2)] < (2/3)d(A, P_2)$ and $\cos(AP_4, AB) > 1-\epsilon$ and $\cos(P_4P_2, AB) > 1-\epsilon$. This process is continued.

Let L denote the closure of the sum of all such P_n's. Since L is closed and compact and as its projection on the line AB is the interval AB and each line

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[7] By $\cos(AB, CD)$ is meant the cosine of the smaller angle between the lines AB and CD. If AB and CD are coincident or parallel, $\cos(AB, CD)=1$.

[8] By $d(A, B)$ is meant the distance between A and B.

perpendicular to this interval contains only one point of L, then L is an arc from A to B.

As each of the numbers $\cos(P_2P_1, AB)$ and $\cos(P_1P_3, AB)$ is greater than $1-\epsilon$ and as the projection of P_1 on AB is between the projections of P_2 and P_3, it follows that $\cos(P_2P_3, AB)>1-\epsilon$. By induction, we get for different integers j and k that $\cos(P_jP_k, AB)>1-\epsilon$. Hence if Q_1 and Q_2 are points of the continuum L, $\cos(Q_1Q_2, AB)>1-\epsilon$.

Each subarc of L contains a P_j and a P_k such that P_j does not belong to any element of E containing P_k. Hence, no subarc of L is a subset of an element of E.

THEOREM 10. *Let R be the interior of a square with diagonals V_1V_3 and V_2V_4; let E be a collection of mutually exclusive point sets filling up $\overline{R}$ such that (1) no element of E contains a nondegenerate connected subset of $V_1V_4+V_2V_3$ and (2) no element of E is dense in a domain; then there exists an upper semicontinuous collection H of mutually exclusive arcs filling up $\overline{R}$ such that H is an arc with respect to its elements, each element of H has one end point on V_1V_2 and one on V_4V_3, each point of $V_1V_2+V_4V_3$ is an end point of an element of H; and no element of E contains a subarc of an element of H.*

Proof. With the help of the preceding theorem, it will be shown that there exists a set of mutually exclusive arcs $\alpha_1, \alpha_2, \cdots$ such that (1) each of them has one end point on V_1V_2 and the other on V_4V_3 and has no arc in common with an element of E and (2) the components of $\overline{R}-(\alpha_1+\alpha_2+\cdots)$ are arcs having one end point on V_1V_2 and one on V_4V_3 and, furthermore, no such component has an arc in common with an element of E.

We shall assume that $\overline{R}-R$ is a unit square. Consider points Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 in the orders $V_1Q_1Q_2Q_3V_4$ and $V_2Q_4Q_5Q_6V_3$ and such that Q_1, Q_2, Q_3 and Q_4, Q_5, Q_6 divide V_1V_4 and V_2V_3 respectively into four equal parts. Choose a positive number ϵ less than 1. Let P_0 be the midpoint of V_1V_2 and P_1, P_2 be points of V_1V_2 at a distance from P_0 equal to $\epsilon/6$ and in the order $V_1P_1P_0P_2V_2$. Let P_3 and P_4 denote points on V_1V_2 at a distance from P_0 equal to $2\epsilon/6$ and in the order $V_1P_3P_0P_4V_2$. Similarly, consider points P_5, P_6, $\cdots$, P_{2j-1}, P_{2j} where P_{2j-1} and P_{2j} are at a distance from P_0 equal to $j\epsilon/6$ and from V_1 and V_2 respectively of not more than $\epsilon/6$ and are in the order $V_1P_{2j-1}P_0P_{2j}V_2$. Let P_{2j+1} be the midpoint of V_1P_{2j-1} and in general let $P_{2j+2k+1}$ be the midpoint of $V_1P_{2j+2k-1}$ for all positive integers k. Also $P_{2j+2k+2}$ is the midpoint of $P_{2j+2k}V_2$. Let $P_{i,1}$, $P_{i,2}$, $P_{i,3}$, $P_{i,4}$ be the projections of P_i on Q_1Q_4, Q_2Q_5, Q_3Q_6, V_4V_3 respectively.

It is assumed that each of the following arcs has the properties that (1) no subarc of it is a subset of E and (2) if A and B are two points of it, $\cos(AB, V_1V_4)>1/(1+\epsilon^2)^{1/2}$. There exist in P_0+P_2+R mutually exclusive arcs $P_0P_{0,1}$ and $P_2P_{2,1}$ satisfying the above two conditions and such that no arc from P_0P_2 to $P_{0,1}P_{2,1}$ in $P_0P_2+P_{0,1}P_{2,1}$ plus the interior of $P_0P_2+P_2P_{2,1}$

$+P_{2,1}P_{0,1}+P_{0,1}P_0$ is a subset of an element of E. In the case where the elements of E contain arcs, this may be seen by constructing $P_0P_{0,1}$ and $P_2P_{2,1}$ so that two elements of E contain arcs from $P_0P_{0,1}$ to $P_2P_{2,1}$ in $P_0P_{0,1}+P_2P_{2,1}$ plus the interior of $P_0P_2+P_2P_{2,1}+P_{2,1}P_{0,1}+P_{0,1}P_0$. Also, there exist in R mutually exclusive arcs $P_{0,2}P_{0,3}$ and $P_{2,2}P_{2,3}$ which satisfy the above two conditions and such that no arc from $P_{0,2}P_{2,2}$ to $P_{0,3}P_{2,3}$ in $P_{0,2}P_{2,2}+P_{0,3}P_{2,3}$ plus the interior of $P_{0,2}P_{2,2}+P_{2,2}P_{2,3}+P_{2,3}P_{0,3}+P_{0,3}P_{0,2}$ is a subset of an element of E. Furthermore, there are mutually exclusive arcs $P_{0,1}P_{0,2}$ and $P_{1,1}P_{1,2}$ in R satisfying the above two conditions and such that no arc from $P_{1,1}P_{0,1}$ to $P_{1,2}P_{0,2}$ in $P_{1,1}P_{0,1}+P_{1,2}P_{0,2}$ plus the interior of $P_{1,1}P_{0,1}+P_{0,1}P_{0,2}$ $+P_{0,2}P_{1,2}+P_{1,2}P_{1,1}$ is a subset of an element of E. Arcs satisfying corresponding conditions join $P_{1,3}$, $P_{0,3}$ to $P_{1,4}$, $P_{0,4}$ respectively. Furthermore, arcs which intersect no arc previously described and which satisfy conditions analogous to those given join P_3, P_1 to $P_{3,1}$, $P_{1,1}$; $P_{3,2}$, $P_{1,2}$ to $P_{3,3}$, $P_{1,3}$; $P_{2,1}$, $P_{4,1}$ to $P_{2,2}$, $P_{4,2}$; $P_{2,3}$, $P_{4,3}$ to $P_{2,4}$, $P_{4,4}$; $P_{5,1}$, $P_{3,1}$ to $P_{5,2}$, $P_{3,2}$; $\cdots$ in such a manner that if α_i designates $P_iP_{i,1}+P_{i,1}P_{i,2}+P_{i,2}P_{i,3}+P_{i,3}P_{i,4}$, then V_1V_4, V_2V_3 are the sequential limiting sets of α_1, α_3, $\cdots$ and α_0, α_2, $\cdots$ respectively.

Let H_1 be the set consisting of V_1V_4, V_2V_3, α_0, α_1, $\cdots$. If h_1 and h_2 are elements of H_1 which are not separated from each other in $\bar{R}$ by any element of H_1 and C is set of all points of $\bar{R}$ between h_1 and h_2, no arc from Q_2Q_5 to $V_1V_2+V_4V_3$ in $Q_2Q_5+V_1V_2+V_4V_3$ plus the bounded complementary domain of $V_1V_2+V_4V_3+h_1+h_2$ is a subset of an element of E.

Designate $V_1V_2\cdot\bar{C}$ and $V_4V_3\cdot\bar{C}$ by U_1U_2 and U_4U_3 respectively, where the points U_1, U_2, U_3, U_4 are in the orders $V_1U_1U_2V_2$ and $V_4U_4U_3V_3$. Treating U_1U_2 and U_4U_3 as V_1V_2 and V_4V_3 were treated, we break up $\bar{C}$ in a manner somewhat analogous to that in which $\bar{R}$ was divided except that instead of three lines Q_1Q_4, Q_2Q_5, Q_3Q_6, we use seven lines parallel to U_1U_2 that divide a segment from U_1U_2 to U_4U_3 into eight equal parts. The division of all such C's gives us H_2. This division of C is possible, for no use was made above of the fact that V_1V_4 and V_2V_3 were straight lines. In fact, it can be shown that if (1) P_1P_2 and $P_1'P_2'$ are parallel intervals, (2) P_1P_1' and P_2P_2' are mutually exclusive arcs such that if A and B are either two points of P_1P_1' or two points of P_2P_2', then sin $(AB, P_1P_2)>1/(1+\epsilon^2)^{1/2}$, (3) Q_1, Q_2, Q_1', Q_2' are points in the orders $P_1Q_1Q_2P_2$ and $P_1'Q_1'Q_2'P_2'$ and min [sin (Q_1Q_2', P_1P_2); sin $(Q_2Q_1', P_1P_2)]>1/(1+\epsilon^2)^{1/2}$, and (4) E is a collection of mutually exclusive point sets filling up $P_1P_2+P_2P_2'+P_2'P_1'+P_1'P_1$ plus its interior and such that no element of E is dense in a domain, then there exist in its interior plus the points Q_1, Q_2, Q_1' and Q_2' two mutually exclusive arcs Q_1Q_1' and Q_2Q_2' such that no arc from Q_1Q_2 to $Q_1'Q_2'$ in $Q_1Q_2+Q_2Q_2'+Q_2'Q_1'+Q_1'Q_1$ plus its interior is a subset of an element of E and such that if C and D are two points of Q_1Q_1' or two points of Q_2Q_2', then sin $(CD, P_1P_2)>1/(1+\epsilon^2)^{1/2}$.

No element of E contains an arc in $\bar{R}-(H_1^*+H_2^*)$ whose projection of V_1V_4 is of length 3/8. With the use of fifteen lines parallel to V_1V_2, H_3 is de-

fined in such a way that no element of E contains an arc in $\overline{R} - (H_1^* + H_2^* + H_3^*)$ whose projection on $V_1 V_4$ is of length 3/16. If a suitable pattern is used in defining H_4, H_5, $\cdots$ and H is the set such that h is an element of H only if it is an element of some H_n or a component of $\overline{R} - (H_1^* + H_2^* + \cdots)$, then H satisfies the conditions of our theorem.

Our method actually gives the following result.

THEOREM 11. *Let R be the interior of a square with diagonals $V_1 V_3$ and $V_4 V_2$; let E be a collection of point sets covering $\overline{R}$ such that no element of E contains a nondegenerate connected subset of $V_1 V_4 + V_2 V_3$ and if D_1 and D_2 are two domains, there exist nondegenerate connected subsets f_1 and f_2 of D_1 and D_2 respectively such that no element of E intersects both f_1 and f_2. Then there exists an upper semi-continuous collection H of mutually exclusive arcs filling up $\overline{R}$ such that H is an arc with respect to its elements, each element of H has one end point on $V_1 V_2$ and one on $V_4 V_3$, each point of $V_1 V_2 + V_4 V_3$ is an end point of an element of H and no element of E contains a subarc of an element of H.*

THEOREM 12. *Let R be the interior of a square with diagonals $V_1 V_3$ and $V_2 V_4$; let F be a dendron in $\overline{R}$ intersecting the boundary of R only in $V_4 V_3$; let E be a collection of point sets covering $\overline{R}$ such that no element of E contains a nondegenerate connected subset of $V_1 V_4 + V_2 V_3$ and if D_1 and D_2 are two domains, there exist nondegenerate connected subsets f_1 and f_2 in D_1 and D_2 respectively such that no element of E intersects both f_1 and f_2. There exists an upper semi-continuous collection H of mutually exclusive continua filling up $\overline{R}$ such that H is a dendron with respect to its elements, $V_1 V_4$ is a subset of an element of H and so is $V_2 V_3$, no element of E contains a nondegenerate subcontinuum of an element of H and each element of H intersects F in only one point and $V_1 V_2$ in only a totally disconnected point set.*

Indication of proof. We construct dendron H so that each element of H intersects F in only one point in a manner analogous to that in which we constructed G' so that it intersected F' in only one point. We use the methods of Theorem 10 to insure that no element of H intersects an element of E in a nondegenerate continuum.

Definition of arc H. We shall now define an upper semi-continuous collection H of mutually exclusive continua filling up W such that H is an arc with respect to its elements, each element of H intersects each element of G, and the intersection of an element of G and an element of H is a totally disconnected point set.

As the points of $M - M \cdot Y^*$ and the elements of Y form an upper semicontinuous collection of mutually exclusive continua filling up $\overline{R}$, it follows [5] that there is a continuous transformation T' carrying $M + Y^*$ into $\overline{R}$ such that (1) T' carries each point of $V_5 V_2 + V_4 V_6$ plus the portion of the boundary of R not belonging to Y^* into itself, (2) T' carries each point of $M - M \cdot Y^*$

into a point, (3) T' carries each element of Y into a point, and (4) each point of $\overline{R}$ is either the image of only one point of $M - M \cdot Y^*$ or the image of only one element of Y. If G'' is the set of all $T'(g')$'s where g' is an element of G', the elements of G'' intersect $V_5 V_2$ and $V_4 V_6$. By the methods of Theorem 11, it can be shown that there is an upper semi-continuous collection H'' of mutually exclusive continua filling up $\overline{R}$ such that (1) H'' is with respect to its elements an arc with ends at V_1 and V_3, (2) every element of H'' other than V_1 and V_2 is an arc having only one point on $V_1 V_2 + V_2 V_3$ and only one point on $V_1 V_4 + V_4 V_3$, these points being its end points, (3) if g'' is an element of G'' intersecting an element h'' of H'', $g'' \cdot h''$ is totally disconnected, and (4) one element of H'' contains $V_5 + V_2$ and another contains $V_4 + V_6$ and each of these elements lies except for its end points on the interior of the parallelogram $V_5 V_2 V_6 V_4$.

If H' is a set such that h' is an element of H' only if it is a subcontinuum of M which is maximal with respect to $T'(h')$ being an element of H'', then H' is a collection of mutually exclusive continua filling up M. Furthermore, it is an arc with respect to its elements. As no element of G' has more than two points in common with the boundary of an element of Y, if g' is an element of G' having a point in common with an element h' of H', $g' \cdot h'$ is totally disconnected.

Let H be a set such that h is an element of H only if it is a continuum which is maximal with respect to being the image under T of the sum of two or three elements of H'. Then H is the required collection of mutually exclusive continua filling up W. The sufficiency of Theorem 1 is demonstrated.

Theorem 1 may also be stated as follows:

THEOREM 1. *A necessary and sufficient condition that a compact plane continuum W remain connected and locally connected on the omission of any countable subset is that there exist two continuous monotone transformations T_1 and T_2 such that both $T_1(W)$ and $T_2(W)$ are nondegenerate dendrons and if a and b are elements of $T_1(W)$ and $T_2(W)$ respectively, then $T_1^{-1}(a) \cdot T_2^{-1}(b)$ exists and is totally disconnected*[9].

Alternate definition of a simple plane web. Actually, we have shown that the following is an equivalent definition to that previously given.

Definition. A compact plane continuum W is a simple plane web provided there exist an upper semi-continuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that (1) G is a dendron with respect to its elements and H is an arc with respect to its elements and (2) if g and h are elements of G and H respectively, then $g \cdot h$ exists and is totally disconnected. It has been shown [1] that condition (1) is unnecessary.

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[9] G. T. Whyburn and some of his students have worked with monotone mappings onto dendrons. See [9] for references.

However, we cannot substitute for (1) in the above definition the condition (1′) that G is an arc with respect to its elements and so is H. Consider the plane continuum W obtained by subtracting from a circle C_0 plus its interior the interiors of five mutually exclusive circles C_1, C_2, C_3, C_4, C_5, each of which is tangent to C_0 internally. Now C_i is not both a subset of an element of G and also a subset of an element of H. Assume that C_j ($j = 1, 2, 3$) is not a subset of an element of G. Then there are uncountably many elements of G that intersect C_j. It may be seen that some element g_j of G separates a point P_j of C_j from $C_0 + C_j + \cdots + C_5 - C_j$ in W and intersects C_i ($i = 0, 1, \cdots, 5$) only if i is equal to j. No one of the elements g_1, g_2, g_3 separates the other two from each other in W. Hence, G is not an arc with respect to its elements.

THEOREM 13. *If W is a simple plane web, there exist a dendron F in W, an arc α in F, an upper semi-continuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that (1) G is a dendron with respect to its elements and H is an arc with respect to its elements, (2) if g and h are elements of G and H respectively, $g \cdot h$ exists and is totally disconnected, (3) each element of G intersects F in only one point, and (4) each element of H intersects α in only one point.*

Proof. Consider the dendron F and the upper semi-continuous collections G and H already defined. If α is the common part of F and the outer boundary of W, then F, α, G, and H satisfy the conditions of our theorem.

THEOREM 14. *If W is a simple plane web, there exist a dendron F in W, an upper semi-continuous collection G of mutually exclusive continua filling up W and another collection H also filling up W such that (1) G is a dendron with respect to its elements and so is H, (2) if g and h are elements of G and H respectively, $g \cdot h$ exists and is totally disconnected, (3) each element of G intersects F in only one point and so does each element of H.*

Proof. The dendron F and the upper semi-continuous collection G are the same as those previously described.

We shall use the same symbols as those used in our definition of the arc H. Let F'' be the image of F' under T'. If h is the element of H'' containing $V_4 + V_6$, let H_1'' denote the set of all continua h'' such that h'' is either V_1, h, or an element of H'' between V_1 and h. Using the methods of Theorem 12, it can be shown that there exists an upper semi-continuous collection H_2'' of mutually exclusive continua filling up $h + (\overline{R} - H_1''^*)$ such that (1) H_2'' is a dendron with respect to its elements, (2) h is a subset of an element of H_2'', (3) each element of H_2'' intersects F'' in only one point and $V_6 V_3$ in a totally disconnected set, and (4) if g'' is an element of G'' intersecting an element h'' of H_2'', $g'' \cdot h''$ is totally disconnected. We define H from $H_1'' + H_2''$ in the same manner in which the arc H was defined from H''.

Theorem 14 may also be stated as follows.

THEOREM 14. *Suppose that W is a compact plane continuum and that T_1, T_2 are continuous monotone transformations of W such that if a and b are elements of $T_1(W)$ and $T_2(W)$ respectively, then $T_1^{-1}(a)$ and $T_1^{-2}(b)$ are nondegenerate sets whose common part exists and is totally disconnected. Then there exist monotone retractions R_1 and R_2 of W such that $R_1(W)$ is a dendron, $R_2(W)$ is the same dendron and if a and b are elements of this dendron, $R_1^{-1}(a) \cdot R_2^{-1}(b)$ exists and is totally disconnected.*

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