ON THE PHRAGMÉN-LINDELÖF PRINCIPLE

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1. **Introduction.** This paper supplements the conclusions of the classical Phragmén-Lindelöf principle as formulated for a half-plane(1) in such a manner that a question raised by Ahlfors(2) concerning this principle is settled. The basic facts concerning the Phragmén-Lindelöf principle for the half-plane are these: Let $f(z)$ denote a function which is defined and analytic for $Rz>0$ and which possesses the property that

$$\limsup_{z\to i\eta} |f(z)| \leqq 1 \tag{1.1}$$

for all η which are finite and real and for $Rz>0$. The following notation is introduced:

$$M(r) = \underset{|\theta|<\pi/2}{\text{l.u.b.}} \left| f(re^{i\theta}) \right|,$$

where r and θ are the customary polar coordinates of a point z of $Rz>0$;

$$\alpha = \liminf_{r\to+\infty} \frac{\log M(r)}{r}; \qquad \beta = \limsup_{r\to+\infty} \frac{\log M(r)}{r}.$$

Positively and negatively infinite values will be admitted for α and β. Under the hypotheses imposed upon $f(z)$ it is concluded, and this is essentially the usual statement of the conclusions of the classical Phragmén-Lindelöf principle, that:

1. *If* $\alpha=-\infty$, *then* $\beta=-\infty$ *and* $f\equiv 0$;
2. *If* $-\infty<\alpha\leqq 0$, *then* $\beta\leqq 0$ *and* $|f|\leqq 1$ *for* $Rz>0$;
3. *If* $\beta=+\infty$, *then* $\alpha=+\infty$;
4. *If* $0<\alpha<+\infty$, *then* $\beta\leqq 4\alpha/\pi$ *and in fact* $[\log M(r)]/r\leqq 4\alpha/\pi$ *for all positive* r.

The question raised by Ahlfors (loc. cit.) is *whether or not* $\alpha=\beta$ *in all cases.* We shall see that *for all functions* $f(z)$ *admitted by the hypotheses of the theorem*

$$\lim_{r\to+\infty} \frac{\log M(r)}{r}$$

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(1) Cf. R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936.

(2) Lars Ahlfors, *On Phragmén-Lindelöf's principle*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 1–8. See in particular p. 6.

exists (as a finite or infinite limit). Further we shall see that α *can never be finite and negative.* Hence if $f(z)$ is of modulus not exceeding one for $Rz>0$ and is not identically zero, then

$$\lim_{r\to+\infty} \frac{\log M(r)}{r} = 0.$$

It is clear that our attention may be confined to the cases (one of which will turn out to be vacuous) where α is finite and either strictly positive or strictly negative. The proof is based upon the representation of $f(z)$

$$f(z) \equiv e^{4\alpha z/\pi}\phi(z), \tag{1.2}$$

where $\phi(z)(\not\equiv 0)$ is of modulus not exceeding one for $Rz>0$, this representation following from the proof of the classical Phragmén-Lindelöf principle as given for example by R. Nevanlinna (loc. cit.). The basic idea which is used throughout this paper is embodied in a lemma established in §2 concerning the measure of the set on $[|z|=\text{const.}, |\theta|<\pi/2]$ where $|\phi|<\exp\{-\epsilon|z|\}$, ϵ being a given positive constant.

In addition to establishing the above results a brief proof of Ahlfors' formulation of the Phragmén-Lindelöf-Nevanlinna principle will be given. This proof is based upon the principle of the harmonic majorant, the Poisson integral for the circle, and the symmetry properties of the Poisson kernel[3]. A comparison is then made with the aid of the lemma of §2 between $\lim_{r\to+\infty}[\log M(r)]/r$ and the analogous limiting value of the Phragmén-Lindelöf-Nevanlinna-Ahlfors principle.

It should be remarked that many of the results of the present paper could be obtained with the aid of the Poisson-Stieltjes integral. In fact, the present results are intimately connected with the behavior of the distribution function appearing in the Poisson-Stieltjes representation. The proofs given here are more "elementary" in character since they involve only the classical Poisson integral and the most primitive properties of harmonic functions.

The author plans to consider related questions in another paper.

2. **A lemma.** As in §1, we shall use $\phi(z)$ to denote a function which is analytic, not identically zero, and of modulus not exceeding unity for $Rz>0$. Let $E(r, \epsilon)$ denote the set of points, θ, of the interval $|\theta|<\pi/2$ for which

$$\log|\phi(re^{i\theta})| < -\epsilon r, \tag{2.1}$$

for r (>0) fixed and ϵ a given positive number. The lemma is:

(3) Cf. A. Dinghas, *Ueber das Phragmén-Lindelöfsche Prinzip und den Julia-Carathéodoryschen Satz*, Preuss. Akad. Wiss. Sitzungsber. (1938) pp. 32–48. The proof of Dinghas makes use of the Poisson integral for a semicircle. By using the Poisson integral for the circle and the symmetry properties of the associated kernel practically all computational details are eliminated.

LEMMA 2.1(4). *If* $\lim\sup_{r\to+\infty}[\text{meas. } E(r, \epsilon)]>0$, *then there exists a positive constant* κ *such that*

$$\log |\phi(z)| \leqq -\kappa\cdot Rz \tag{2.2}$$

for $Rz>0$.

Proof. Let $X(\theta; r, \epsilon)$ denote the characteristic function of the set $E(r, \epsilon)$ for $|\theta|<\pi/2$ and let X be defined for other values of θ by

$$\begin{aligned} X\left(\frac{\pi}{2}; r, \epsilon\right) &= X\left(-\frac{\pi}{2}; r, \epsilon\right) = 0, \\ X(\theta; r, \epsilon) &= -X(\pi-\theta; r, \epsilon) \qquad \text{for } \pi/2<\theta<3\pi/2, \end{aligned} \tag{2.3}$$

and by the requirement that X shall be periodic in θ with period 2π. Further let $K(r, \theta; \rho, \psi)$ denote the Poisson kernel

$$K(r, \theta; \rho, \psi) \equiv \frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\psi-\theta)} \qquad (r<\rho). \tag{2.4}$$

It follows from the principle of the harmonic majorant and the nature of X that

$$\log |\phi(re^{i\theta})| \leqq -\frac{\epsilon\rho}{2\pi}\int_{-\pi}^{\pi} X(\psi; \rho, \epsilon)K(r, \theta; \rho, \psi)d\psi \tag{2.5}$$

for $|\theta|<\pi/2$ if $r<\rho$. Use will be made of the following local representation of K for r/ρ small:

$$K \equiv 1+\frac{2r}{\rho}\cos(\psi-\theta)+O\left[\left(\frac{r}{\rho}\right)^2\right]. \tag{2.6}$$

Note that

$$\int_{-\pi}^{\pi} X(\psi; \rho, \epsilon)d\psi = \int_{-\pi}^{\pi} X(\psi; \rho, \epsilon)\sin\psi d\psi = 0. \tag{2.7}$$

In order to obtain an upper bound for $\log|\phi(re^{i\theta})|$ in terms of meas. $E(\rho, \epsilon)$ it will be convenient to obtain a lower bound for

$$\int_{-\pi}^{\pi} X(\psi; \rho, \epsilon)\cos\psi d\psi. \tag{2.8}$$

To this end note that (2.8) is twice

(4) It may be shown that $\lim_{r\to+\infty}$ meas $E(r, \epsilon)$ exists (the measure of a null-set being zero by convention). Also if this limit is positive and if κ_0 denotes the *largest* positive value of κ for which (2.2) is true throughout $Rz>0$, then $\lim_{r\to+\infty}$ meas $E(r, \epsilon)=2\arccos(\epsilon/\kappa_0)$ for $\epsilon\leqq\kappa_0$. These results will not be needed in this paper.

$$\int_{-\pi/2}^{\pi/2} X(\psi;\rho,\epsilon)\cos\psi d\psi \tag{2.9}$$

and that (2.9) in turn is not less than

$$\frac{\lambda(\rho,\epsilon)}{4}\equiv 2\left[1-\cos\left\{\frac{\text{meas. }E(\rho,\epsilon)}{2}\right\}\right][5]. \tag{2.10}$$

This appraisal is readily deduced by comparing (2.9) with the integral obtained by replacing $X(\psi;\rho,\epsilon)$ in (2.9) by the characteristic function of the set

$$\left[\frac{\pi}{2}-\frac{\text{meas. }E(\rho,\epsilon)}{2}\leqq|\psi|<\frac{\pi}{2}\right].$$

It follows from (2.5), (2.6), (2.7), and (2.10) that

$$\log|\phi(re^{i\theta})|\leqq-\frac{\epsilon\rho}{2\pi}\left\{\frac{r\cos\theta}{\rho}\lambda(\rho,\epsilon)+O\left[\left(\frac{r}{\rho}\right)^2\right]\right\}. \tag{2.11}$$

The lemma is an immediate consequence of (2.11), since $\limsup_{\rho\to+\infty}\{\text{meas. } E(\rho,\epsilon)\}>0$ implies that $\limsup_{\rho\to+\infty}\lambda(\rho,\epsilon)>0$.

3. **The existence of** $\lim_{r\to+\infty}[\log M(r)]/r$.

Case I. $0<\alpha<+\infty$. In this case it is immediate from (1.2) that $\beta\leqq 4\alpha/\pi$. To proceed let κ equal the largest *non-negative* number for which

$$\log|\phi(z)|\leqq-\kappa\cdot Rz \tag{3.1}$$

holds *throughout* $Rz>0$. Such a number κ clearly exists since $\phi\not\equiv 0$. Set

$$\phi(z)\equiv e^{-\kappa z}\Phi(z) \tag{3.2}$$

defining $\Phi(z)$ thereby. Clearly Φ is analytic, is of modulus not exceeding unity for $Rz>0$, and is not subject to a domination of the type (3.1) for a *strictly positive* κ. From the definition of κ and (1.2), we have

$$\beta\leqq 4\alpha/\pi-\kappa. \tag{3.3}$$

If $\alpha<\beta$, then $4\alpha/\pi-\kappa>\alpha$. From the definition of α it follows that there would exist a positive number ϵ such that

$$\alpha+\epsilon<4\alpha/\pi-\kappa,$$

and a monotone strictly increasing sequence of positive numbers $\{r_k\}$ with $\lim_{k\to\infty}r_k=+\infty$, such that for all k $(k=1,2,\cdots)$

$$\log M(r_k)<(\alpha+\epsilon)r_k.$$

Hence if $\alpha<\beta$, there would exist a positive number δ less than $\pi/2$ and a positive number γ ($-\gamma$ may be taken as $(\alpha+\epsilon)-(4\alpha/\pi-\kappa)\cos\delta$ if δ is chosen

[5] Defining $\lambda(\rho,\epsilon)$.

sufficiently small) such that for all k and for $|\theta| \leqq \delta$,

$$\log |\Phi(r_k e^{i\theta})| \leqq -\gamma r_k. \tag{3.4}$$

The contradiction follows from Lemma 2.1. Hence $\alpha = \beta$. Further $\alpha = 4\alpha/\pi - \kappa$. Hence

$$f(z) \equiv e^{\alpha z}\Phi(z). \tag{3.5}$$

To sum up, we have:

THEOREM 3.1. *If* $0 < \alpha < +\infty$, *then* $\alpha = \beta$. *Further*

$$\frac{\log M(r)}{r} \leqq \alpha \tag{3.6}$$

for all positive r. *If equality is attained in* (3.6) *for any finite positive value of* r, *then equality prevails for all positive* r *and* $f(z) \equiv ce^{\alpha z}$, *where* c *is a constant of modulus one.*

Case II. $-\infty < \alpha < 0$. In this case we may assume that $f \not\equiv 0$ and we obtain a contradiction as follows. Let κ denote the largest positive number such that

$$\log |f(z)| \leqq -\kappa \cdot Rz \tag{3.7}$$

holds throughout $Rz > 0$. That such a number κ exists follows from (1.2). Hence $f(z)$ admits the representation

$$f(z) \equiv e^{-\kappa z}F(z) \tag{3.8}$$

where $F(z)$ is analytic and of modulus not exceeding unity for $Rz > 0$, and in addition is not subject to a domination of the type (3.7) for a strictly positive κ. The remainder of the argument is similar to that of Case I save that here the θ-interval, $|\theta| \leqq \delta$, is to be replaced by the intervals $\pi/2 - \delta^* \leqq |\theta| < \pi/2$ for some appropriate positive δ^*. The lemma is then immediately applicable and we infer the following theorem.

THEOREM 3.2. *The case* $-\infty < \alpha < 0$ *never occurs.*

4. The Ahlfors formulation of the Phragmén-Lindelöf-Nevanlinna principle. In the form given to the Phragmén-Lindelöf principle by F. and R. Nevanlinna the same class of function $f(z)$ admitted in §1 is considered, but instead of treating $\log M(r)$, the integral

$$m(r) \equiv \int_{-\pi/2}^{\pi/2} \log^+ |f(re^{i\theta})| \cos\theta d\theta \tag{4.1}$$

is studied. Here

$$\log^+ |f(re^{i\theta})| = \max \{\log |f(re^{i\theta})|, 0\}.$$

The theorem of Ahlfors states: *The function $m(r)/r$ is a non-decreasing function of r.* Ahlfors' theorem appears as a corollary of a more general result which he obtained with the aid of a certain differential inequality. The proof given below depends upon the Poisson integral for the circle. The procedure followed here is related to that used in §2. We define the function $L(\rho, \psi)$ for positive ρ and all real ψ by the requirements

(i) $$L(\rho, \psi) \equiv \log^+|f(\rho e^{i\psi})| \qquad \text{for } |\psi| < \pi/2,$$

(ii) $$L(\rho, \pi/2) = L(\rho, -\pi/2) = 0,$$

(iii) $$L(\rho, \psi) \equiv -L(\rho, \pi - \psi) \qquad \text{for } \pi/2 < \psi < 3\pi/2.$$

For other values of ψ, L is defined by the requirement that L shall be periodic in ψ with the period 2π. It is to be observed that the condition (1.1) implies that $L(\rho, \psi)$ is continuous in ψ for all ψ. As in §2, we see that for $r < \rho$ and $|\theta| < \pi/2$

(4.2)
$$\begin{aligned}\log^+ |f(re^{i\theta})| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} L(\rho, \psi)K(r, \theta; \rho, \psi)d\psi \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \log^+ |f(\rho e^{i\psi})| \, [K(r, \theta; \rho, \psi) - K(r, \pi - \theta; \rho, \psi)]d\psi.\end{aligned}$$

Note that

(4.3)
$$\begin{aligned}&\int_{-\pi/2}^{\pi/2} \cos\theta[K(r, \theta; \rho, \psi) - K(r, \pi - \theta; \rho, \psi)]d\theta \\ &= \int_{-\pi}^{\pi} \cos\theta K(r, \theta; \rho, \psi)d\theta = \int_{-\pi}^{\pi} \cos\theta K(r, \psi; \rho, \theta)d\theta \\ &= 2\pi \frac{r}{\rho} \cos\psi\end{aligned}$$

by virtue of the symmetry properties of K. Hence

(4.4) $$m(r) \leq \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \log^+ |f(\rho e^{i\psi})| \cdot 2\pi \frac{r}{\rho} \cos\psi d\psi$$

or

$$\frac{m(r)}{r} \leq \frac{m(\rho)}{\rho}.$$

The theorem of Ahlfors follows.

5. $\lim_{r\to+\infty} (\log M(r)/r)$ vs. $\lim_{r\to+\infty} (m(r)/r)$. In this section we shall be concerned with determining the relation between the two indicated limits. The case where either of the two limits is zero is readily dismissed. It follows

from §3 and the definition of $m(r)$ that, if one of the limits is zero, then so is the other(6). On the other hand the relation

$$\log M(r) \geqq m(r)/2 \tag{5.1}$$

which prevails when $\log M(r) > 0$, implies that, if

$$\lim_{r\to+\infty} \frac{m(r)}{r} = +\infty, \tag{5.2}$$

then

$$\lim_{r\to+\infty} \frac{\log M(r)}{r} = +\infty. \tag{5.3}$$

Conversely, it may be shown that if (5.3) holds, then so does (5.2). The details follow from the proof of the Phragmén-Lindelöf-Nevanlinna principle(7).

There remains the case where both limits are finite and positive. Recall the representation (3.5) that prevails in this case. Here

$$\lim_{r\to+\infty} \frac{\log M(r)}{r} = \alpha.$$

From (3.5) it follows that

$$\alpha r \cos\theta + \log |\Phi(re^{i\theta})| \leqq \log^+ |f(re^{i\theta})| \leqq \alpha r \cos\theta. \tag{5.4}$$

Given $\epsilon > 0$, it follows from the nature of Φ and Lemma 2.1 that for r sufficiently large

$$\log |\Phi(re^{i\theta})| \geqq -\epsilon r$$

except for a set $E(r, \epsilon)$ of arbitrarily small measure. Hence

$$\int_{CE(r,\epsilon)} (\alpha r \cos\theta - \epsilon r)\cos\theta d\theta \leqq m(r) \leqq \alpha r \cdot \frac{\pi}{2}$$

and by virtue of the arbitrariness of ϵ, we have

$$\lim_{r\to+\infty} \frac{m(r)}{r} = \frac{\pi}{2}\alpha.$$

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(6) Except if $f \equiv 0$. This trivial case will be omitted from the discussion of this section.
(7) Cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, Leipzig, 1931, pp. 132–134.