ON CONVEX TOPOLOGICAL LINEAR SPACES[1]

BY

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Introduction. In an earlier article [9] the author developed at some length the theory of certain mathematical objects which he called linear systems. It is the purpose of the present paper to apply this theory to the study of convex topological linear spaces. This application is based on the many-to-one correspondence between convex topological linear spaces and linear systems which may be set up by assigning to each such space X_t the linear system X_L where X is the abstract linear space underlying X_t and L is the set of all continuous linear functionals on X. First of all the nature of the family of all X_t's belonging to a given X_L is studied and it is shown that it has a weakest and a strongest member. The bulk of the paper is then devoted to correlating the properties of a convex topological linear space with those of its linear system and with the strength of its topology relative to that of the others with the same linear system. A general survey of the contents of this paper will be found in [10].

1. **Preliminary definitions and remarks.** In·this section we recall briefly some of the notions from the theory of topological linear spaces and from [9] which we shall need in the sequel. By a topological linear space[2] we mean a real linear space which is at the same time a T_1 space in the sense of Alexandroff and Hopf [1] and in which the topology is related to the algebra in such a manner that the operations of addition and multiplication by reals are continuous in both variables together. By a convex topological linear space[2] we mean a topological linear space in which every point has a complete system of convex neighborhoods. By a linear functional on a linear space we mean a function l defined from the space to the reals such that $l(\lambda x + \mu y) = \lambda l(x) + \mu l(y)$ for all x and y in the space and all real numbers λ and μ. If X is a linear space we denote by X^* the linear space of all linear functionals on X. By a linear system X_L we mean a linear space X together with a distinguished

subspace L of X^*. If L is total in the sense that $l(x) = 0$ for all l in L implies that $x = 0$ we say that X_L is regular. The linear system L_F where F is the subset of L^* consisting of those functionals defined by members of X is called the conjugate of X_L and is denoted by X_L^0 or by L_X. A subset A of X_L is said to be bounded if for each countable sequence of elements of A, $x_1, x_2, \cdots$, and each l in L the sequence of real numbers $\{l(x_n)\}$ is a bounded sequence. If l is a member of X^* such that $\{l(x_n)\}$ is a bounded sequence of real numbers whenever $\{x_n\}$ is a bounded sequence of members of X_L then l is said to be in the bounded closure of L. If each such l is in L then L and X_L are said be boundedly closed. If $\{l_n(x_m)\}$ is bounded for $n, m = 1, 2, \cdots$ whenever $\{l_n\}$ is a bounded subset of L_X and $\{x_m\}$ is a subset of $A \subseteq X$ we say that A is a uniformly bounded subset of X_L. If every bounded subset of X_L is uniformly bounded we say that X_L is uniform. If there exists a bounded subset of X_L whose linear span is X we say that L and X_L are relatively bounded. If X is the union of countably many bounded subsets of X_L we say that L and X_L are almost relatively bounded. Let $x_1, x_2, \cdots$ be a sequence of members of X. If there exists a member f of L^* and a sequence $\gamma_1, \gamma_2, \cdots$ of positive real numbers converging to zero such that $\{(l(x_n) - f(l))/\gamma_n\}$ is bounded for all l in L we say that x_n converges to f in the linear system X_L. If whenever x_n converges to f there exists $\bar{x}$ in X such that $f(l) = l(\bar{x})$ for all l in L we say that X_L is a complete linear system. A function N from a linear space X to the non-negative reals such that $N(x+y) \leq N(x) + N(y)$ and $N(\lambda x) = |\lambda| N(x)$ for all x and y in X and all real numbers λ is what we call a pseudo-norm. If $N(x) = 0$ implies that $x = 0$ we call N a norm. If N is a pseudo-norm, the pseudo-norm set of N is the set of all l in X^* such that, for some positive real number μ depending on l, the inequality $|l(x)| \leq \mu N(x)$ holds for all x in X. It is a trivial generalization of a theorem of Fichtenholz [5] that the pseudo-norm set of N_1 includes that of N_2 if and only if there exists a positive real number μ such that $N_2(x) \leq \mu N_1(x)$ for all x in X. A linear system X_L is simple if the bounded closure of L is a pseudo-norm set or, equivalently, if there exists a bounded subset of X_L whose linear span is the whole space. If A is a bounded subset of the conjugate of the linear system X_L then $\text{l.u.b.}_{l \in A} |l(x)|$ is a pseudo-norm as a function of x. The linear span L_r of the pseudo-norm sets of all such pseudo-norms is the reach of L and X_{L_r} is the reach of X_L.

2. **The convex topologies associated with a fixed linear system.** Our theory of the family of convex topologies associated with a fixed linear system is based upon von Neumann's [12] observation that a convex topology may be described by pseudo-norms. Let X be a convex topological linear space and let O' be any open set containing the origin. Then there exists a second open set O which in addition to containing the origin and being contained in O' is convex and such that $-O = O$. For each x in X let $N(x) = \text{l.u.b.} |\lambda|$ for $x/\lambda \notin O$. It is readily verified that N is a pseudo-norm and that O is the set of all x in X with $N(x) < 1$. Furthermore N is continuous. In fact N is obvi-

ously continuous at the origin and it follows from the inequality $\mid M(x) - M(y) \mid$ $\leq M(x-y)$ that any pseudo-norm M is continuous if and only if it is continuous at the origin. Thus the sets (all x with $N(x) < 1$) for the continuous pseudo-norms N form a complete system of neighborhoods of the origin. The truth of the following theorem is an immediate consequence.

THEOREM 1. *Let X be a linear space. Then two convex topologies in X are identical if and only if the same pseudo-norms are continuous with respect to each.*

According to Theorem 1 there is a natural one-to-one correspondence between the convex topologies in X and certain families of pseudo-norms. Theorem 2 tells us just what these certain families are.

THEOREM 2. *A family $\mathfrak{N}$ of pseudo-norms N_α on a linear space X is the family of all pseudo-norms continuous with respect to some convex topology t in X if and only if the following conditions are satisfied.*
  (a) *If $N_1 \in \mathfrak{N}$ and N_2 is such that $N_2(x) \leq \mu N_1(x)$ for all x in X and some positive μ independent of x then N_2 is also in $\mathfrak{N}$.*
  (b) *If N_1 and N_2 are in $\mathfrak{N}$ then $N_1 + N_2$ is in $\mathfrak{N}$.*
  (c) *If $x \in X$ and $x \neq 0$ then there exists N in $\mathfrak{N}$ with $N(x) \neq 0$.*

Proof. To prove the "only if" part of the theorem we observe that if $\mathfrak{N}$ is the family of continuous pseudo-norms for a convex topology in X then condition (c) follows from the fact that the sets (all x with $N(x) < 1$) for the N in $\mathfrak{N}$ form a complete system of neighborhoods of the origin and that condition (b) is a consequence of the continuity of the sum of two continuous functions. (It is clear of course that the sum of two pseudo-norms is always a pseudo-norm.) Condition (a) follows from the fact that if N_1 is a continuous pseudo-norm and $N_2(x) \leq N_1(x)$ for all x in X then the set where $N_2(x) < 1$ contains the open set on which $N_1(x) < 1$. Turning to the converse suppose that $\mathfrak{N}$ is a family of pseudo-norm sets satisfying (a), (b), and (c). Define a set S in X to be open if and only if for each x in X there is an N in $\mathfrak{N}$ such that $N(x-y) < 1$ implies that y is in S. Then it may be verified without difficulty that with these open sets X becomes a convex topological linear space. Let N_1 be any pseudo-norm continuous with respect to this topology. Then the set where $N_1(x) < 1$ is open. Hence there exists N_2 in $\mathfrak{N}$ such that $N_2(x) < 1$ implies that $N_1(x) < 1$. It follows easily that $N_1(x) \leq N_2(x)$ for all x in X and hence by (a) that $N_1 \in \mathfrak{N}$. This completes the proof of the theorem.

Let X be a convex topological linear space and let P be a pseudo-norm set in X^*. It follows from one of the remarks in the latter part of §1 that if N_1 and N_2 are pseudo-norms having P as pseudo-norm set then there exist positive constants λ and μ such that $\lambda N_1(x) \leq N_2(x) \leq \mu N_1(x)$ for all x in X. Hence it is clear from the argument given in the proof of (a) of Theorem 2 that N_1 and N_2 are either both continuous or both discontinuous. Accordingly we make:

Definition 1. If X is a convex topological linear space and P is a pseudo-norm set in X^*, one, and hence all, of whose generating pseudo-norms are continuous, we shall call P a continuity pseudo-norm set.

Theorem 1 now tells us that there is a natural one-to-one correspondence between the convex topologies in a linear space X and certain families of pseudo-norm sets and it is not difficult to deduce from Theorem 2 just what these families are. In fact let $\mathfrak{F}$ be such a family of continuity pseudo-norm sets. It follows from (a) under Theorem 2 that: (a') If $P_1 \in \mathfrak{F}$ and P_2 is a pseudo-norm set with $P_2 \subseteq P_1$ then $P_2 \in \mathfrak{F}$. Furthermore it is proved in Theorem VII-5 of [9] that if P_1 and P_2 are pseudo-norm sets then $P_1 + P_2$ is a pseudo-norm set. It follows from this together with (b) and (a') that: (b') If P_1 and P_2 are in $\mathfrak{F}$ then $P_1 + P_2$ is in $\mathfrak{F}$. Finally it is an easy consequence of the Hahn-Banach extension theorem that the intersection of the null spaces of the members of a pseudo-norm set is precisely the common null space of its generating pseudo-norms. We thus see that a family of pseudo-norms has property (c) if and only if the linear unions of their pseudo-norm sets is total. Hence $\mathfrak{F}$ has the property: (c') The linear union of the members of $\mathfrak{F}$ is a total subspace of X^*. It follows from what we have said and the theorem of [9] to the effect that the intersection of two pseudo-norm sets is again a pseudo-norm set that the set of all pseudo-norm sets in the X^* of a linear space X forms a lattice under partial ordering by inclusion. A subset $\mathfrak{F}$ with properties (a') and (b') is what is called an ideal in lattice theory. Hence we make:

Definition 2. A family of pseudo-norm sets satisfying conditions (a'), (b'), and (c') above will be called a total ideal of pseudo-norm sets.   •

We may now state:

Theorem 3. *Let X be a linear space. Then for each convex topology in X the family of continuity pseudo-norm sets is a total ideal. Conversely, given a total ideal $\mathfrak{F}$ of pseudo-norm sets in X^*, there is a unique convex topology in X the continuity pseudo-norm sets of which are precisely the members of $\mathfrak{F}$.*

Proof. The first part of this theorem has just been proved. The converse is an easy consequence of Theorem 2 and now familiar arguments and will be left to the reader.

If l is a linear functional on a convex topological linear space X it is clear that $l(x)$ is a continuous function of x if and only if this is the case for $|l(x)|$. It is also clear that $|l(x)|$ as a function of x is a pseudo-norm. It follows at once from Theorem 3 that the continuous linear functionals on X are precisely the members of the linear union L of the continuity pseudo-norm sets in X^*. Since every one-dimensional subspace of L is a continuity pseudo-norm set (for the continuous pseudo-norm $|l(x)|$ where l is a nonzero member of the pseudo-norm set) it follows that every finite-dimensional subspace of L is a continuity pseudo-norm set. Combining these observations with Theorem 3 we conclude the truth of:

THEOREM 4. *Let X be a convex topological linear space and let L be the set of all continuous linear functionals defined on X. Then the linear system X_L is regular, every continuity pseudo-norm set is contained in L and every finite-dimensional subspace of L is a continuity pseudo-norm set. Conversely let X_L be any regular linear system and let $\mathfrak{F}$ be any ideal of pseudo-norm sets contained in L which includes every finite-dimensional subspace of L. Then there is a unique convex topology in X whose ideal of continuity pseudo-norm sets is $\mathfrak{F}$. The continuous linear functionals on the resulting convex topological linear space are precisely the members of L.*

DEFINITION 3. If X is a convex topological linear space and L is the set of continuous linear functionals defined on X we shall call X_L the linear system associated with or belonging to the convex topological linear space.

DEFINITION 4. If X_L is a linear system and $\mathfrak{F}$ is a total ideal of pseudo-norm sets such that (1) every finite-dimensional subspace of L is a member of $\mathfrak{F}$ and (2) every member of $\mathfrak{F}$ is a subspace of L we shall say that $\mathfrak{F}$ is an ideal associated with or belonging to the system.

Since in general there are many ideals belonging to a given regular linear system and always at least one (the ideal of all finite-dimensional subspaces of L), Theorem 4 tells us that there is a natural many-to-one correspondence between convex topological linear spaces and regular linear systems. Furthermore it tells us that there is a natural one-to-one correspondence between the ideals belonging to a regular linear system and the convex topological linear spaces to which the system belongs.

It is clear that if two convex topologies are introduced into the same linear space and if the ideal of continuity pseudo-norm sets of the first contains that of the second then every open set of the first is an open set of the second; that is, the first topology is stronger than the second and the second is weaker than the first. In particular it is clear that if X_L is a regular linear system then the convex topology associated with the ideal of all finite-dimensional subspaces of L is weaker than any other convex topology associated with X_L and that the ideal of all[3] pseudo-norm sets contained in L is associated with a convex topology that is stronger than any other. Thus we may state:

THEOREM 5. *If X_L is a regular linear system, then among the convex topologies associated with X_L there is both a weakest and a strongest.*

This suggests the following definition.

DEFINITION 5. A relatively weak (relatively strong) convex topological linear space is a convex topological linear space which has a weaker (stronger) topology than any other such space with the same linear system.

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[3] That this is indeed an ideal follows at once from the fact noted above that the linear union of two pseudo-norm sets is again a pseudo-norm set.

It is clear that a normed linear space in its norm topology is relatively strong and in its weak topology relatively weak. In case L contains no infinite-dimensional pseudo-norm sets, if for example L is $\aleph_0$-dimensional, then the relatively weak and strong topologies for X associated with X_L coincide. If however there exists at least one infinite-dimensional pseudo-norm set in L then there are infinitely many convex topologies associated with X_L which are neither relatively weak nor relatively strong. In fact we may prove the following theorem.

THEOREM 6. *If X is a non relatively weak convex topological linear space then there exists a second non relatively weak convex topology in X which is properly weaker than the first and has the same linear system as the first.*

Proof. Let P_1 be an infinite-dimensional continuity pseudo-norm set for the first topology. Let X_1 be the intersection of the null spaces of the members of P_1 and let X_2 be a complement of X_1 (that is, a not necessarily closed subspace of X such that $X_1 + X_2 = X$ and $X_1 \cap X_2 = 0$). Then P_1 as on X_2 is total and hence a norm set. Choose a subspace of X_2 closed with respect to the norms defining P_1 and having an infinite-dimensional quotient space. That this can always be done follows, for example, from the argument given on page 174 of [9]. Let P_2 be the set of all members of P_1 which vanish on this subspace. Then P_2 is an infinite-dimensional pseudo-norm set whose deficiency in P_1 is infinite. Finally consider the set $\mathfrak{J}$ of all pseudo-norm sets contained in pseudo-norm sets of the form $P_2 + F$ where F is a finite-dimensional subspace of L. It is clear that $\mathfrak{J}$ is an ideal associated with X_L and that it does not contain P_1 but does contain infinite-dimensional pseudo-norm sets. Thus it defines a convex topology with the desired properties.

3. **Boundedness and continuity of transformations.** Kolmogoroff [8] calls a subset A of a topological linear space bounded if whenever $x_1, x_2, \cdots$ is a sequence of members of A and $\gamma_1, \gamma_2, \cdots$ a sequence of real numbers converging to zero then $\gamma_n x_n$ converges to zero with respect to the topology. On the other hand von Neumann [12] calls such a subset A bounded if and only if for every neighborhood of the origin O in the space there exists a real number μ such that $A \subseteq \mu O$. Since Hyers [6] has proved the equivalence of these definitions we may speak without ambiguity of the bounded subsets of a topological linear space. We now prove a theorem making connections with the theory of linear systems.

THEOREM 7[4]. *Let X be a convex topological linear space and let X_L be its linear system. Then a subset A of X is bounded if and only if it is bounded as a subset of the linear system X_L; that is, if and only if, for each l in L, l.u.b.$_{x \in A} |l(x)| < \infty$.*

Proof. The "only if" half of the theorem is obvious. Suppose conversely

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[4] Cf. Šmulian [14, Hilfsatz 1].

that l.u.b.$_{x \in A}|l(x)| < \infty$ for all l in L. Let O be any neighborhood of the origin in X. Let N be a pseudo-norm such that if $N(x) < 1$ then $x \in O$. Now a fortiori l.u.b.$_{x \in A}|l(x)| < \infty$ for all l in the pseudo-norm set of N. If we observe that N is a norm on a complement of its null space it is easy to see that Théorème 6 on page 80 of [2] applies so that we may conclude the existence of a positive constant μ such that $N(x) < \mu$ for all x in A. It follows that $A \subseteq \mu O$. Thus A is bounded.

It is an immediate consequence of Theorem 7 that two convex topologies in X generate the same bounded sets if and only if their families of continuous linear functionals have identical bounded closures. Thus it is clear that not only are there in general many convex topologies with the same continuous linear functionals but also many convex topologies with different continuous linear functionals and the same bounded sets. Examining this situation a little more closely we can gain a certain amount of insight into a problem discussed by Wehausen [18]; namely, that of the relationship between continuous linear transformations and additive transformations which take bounded sets into bounded sets. We begin by proving a lemma showing that we need only deal with linear transformations.

LEMMA. *If T is an additive transformation from one topological linear space X_1 into another X_2 which takes bounded sets into bounded sets then T is linear.*

Proof. Let x be an arbitrary nonzero member of X. If r is a rational number it follows from the additivity of T that $T(rx) = rT(x)$. Let λ be any non-rational real number and let $r_1, r_2, \cdots$ be a sequence of rational numbers converging to λ. For each $n = 1, 2, \cdots$ let s_n be a rational number such that $|\lambda - r_n| \leq s_n \leq 2|\lambda - r_n|$. Then $T(\lambda x) - \lambda T(x) = (T(\lambda x) - T(r_n x)) + (r_n T(x) - \lambda T(x)) = s_n(T(((\lambda - r_n)/s_n)x)) + (r_n - \lambda)T(x)$. Now $\{((\lambda - r_n)/s_n)x\}$ is clearly a bounded sequence of members of X_1. By hypothesis then $\{T(((\lambda - r_n)/s_n)x)\}$ is a bounded sequence of members of X_L. Hence since s_n converges to zero $s_n T(((\lambda - r_n)/s_n)x)$ must converge to zero in the topology of X_2. Since finite sets are bounded $(r_n - \lambda)T(x)$ also converges to zero. Thus $T(\lambda x) = \lambda T(x)$ and the lemma is proved.

Let X be a linear space and let $\Im$ be any ideal of pseudo-norm sets in X^* total or not. Let M be the intersection of the null spaces of these pseudo-norm sets. It is clear that the natural isomorphism between $(X/M)^*$ and the linear subspace M' of members of X^* vanishing on M carries each pseudo-norm set whose members vanish throughout M into a pseudo-norm set in $(X/M)^*$ and carries $\Im$ into a total ideal in $(X/M)^*$. Hence it follows from Theorem 4 that there is a natural one-to-one correspondence between the (not necessarily total) ideals of pseudo-norm sets in X^* and the convex topological linear spaces defined on the quotient spaces X/M.

Now let T be any linear transformation from X onto a convex topological linear space Y. If M is the null space of T then T sets up an isomorphism

between X/M and Y as linear spaces. Hence in an obvious and trivial manner a convex topology may be introduced into X/M in such a way that T sets up an isomorphism between Y and X/M as topological linear spaces. Thus if we define two linear transformations to be equivalent whenever there is an isomorphism (as topological linear spaces) U between their range spaces such that $U(T_1(x)) = T_2(x)$ for all x in X, then each linear transformation T from X onto another convex topological linear space is equivalent to the natural transformation of X onto one of its quotient spaces where this quotient space has been endowed with a suitable convex topology.

Combining the observations of the last two paragraphs we see that there is a natural one-to-one correspondence between the classes of equivalent transformations of X onto topological linear spaces and the ideals of pseudo-norm sets in X^*. In other words, we can survey all possible types of linear transformations from one convex topological linear space into another by considering the ordered pairs of ideals of pseudo-norm sets, the first of which is total, in the X^*'s of the various dimensional X's. Because of the order preserving nature of the correspondence between ideals and convex topologies it is clear that such a transformation is continuous if and only if the second ideal is a subset of the first and, because of Theorem 7, that such a transformation is bounded if and only if the bounded closure of the linear union of the members of the second ideal is a subset of that of the first.

It is at once clear that any continuous linear transformation carries bounded sets into bounded sets and exactly when and why the converse is not true. In fact from our point of view the converse can fail to be true in two different ways. Let X be a convex topological linear space, let X_L be the associated linear system, and let $\mathfrak{F}$ be the ideal of continuity pseudo-norm sets. Let $\mathfrak{F}_1$ be any ideal of pseudo-norm sets in X^* and let L_1 be the linear union of its members. If L is not boundedly closed then $\mathfrak{F}_1$ may be chosen so that while the bounded closure of L_1 is contained in that of L it is not true that L_1 is contained in L and hence not true that $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. In other words, the corresponding linear transformations will carry bounded sets into bounded sets but be discontinuous. Even if L is boundedly closed, if $\mathfrak{F}$ fails to contain some pseudo-norm set contained in L then by choosing $\mathfrak{F}_1$ as the set of all pseudo-norm sets contained in L we again get a pair of ideals corresponding to a class of discontinuous transformations carrying bounded sets into bounded sets. On the other hand these are the only things that can happen. Suppose that X is relatively strong and that X_L is boundedly closed. If $\mathfrak{F}_1$ corresponds to a class of transformations which carry bounded sets into bounded sets so that the bounded closure of L_1 is contained in that of L then, since L is boundedly closed, $L_1 \subseteq L$. Hence every member of $\mathfrak{F}_1$ is a subspace of L and, since X is relatively strong, is in $\mathfrak{F}$. Thus $\mathfrak{F}_1$ corresponds to a class of continuous transformations. This paragraph may be summarized by the following theorem.

THEOREM 8. *Let X be a convex topological linear space. Then the following two statements about X are equivalent.*

(a) *Every additive (linear) transformation from X into another convex topological linear space which takes bounded sets into bounded sets is continuous.*

(b) *X is relatively strong and has a boundedly closed linear system.*

Another obvious consequence of the preceding considerations generalizes Wehausen's Theorem 18.

THEOREM 9. *If X_1 and X_2 are relatively weak convex topological linear spaces with boundedly closed linear systems then any additive (linear) transformation from X_1 into X_2 which takes bounded sets into bounded sets is continuous.*

4. Metrizability. In this section we study conditions on the linear system and relative strength of a convex topological linear space under which it is metrizable as a topological space. We begin by observing that the proof of Theorem VII-6 on page 199 of [9] also yields a proof of the following lemma.

LEMMA. *Let X be a linear space and let P_1, P_2, $\cdots$ be an arbitrary sequence of pseudo-norm sets in X^*. Then if P is a pseudo-norm set contained in $P_1 + P_2 + \cdots$ there exists $n = 1, 2, \cdots$ such that $P \subseteq P_1 + P_2 + \cdots + P_n$.*

As an easy consequence of this we may prove:

THEOREM 10. *Let X be a convex topological linear space and let X_L be its linear system. Then X is metrizable if and only if it is relatively strong and L is the union of an ascending countable sequence of pseudo-norm sets.*

Proof. Suppose that X is metrizable. Then there exists a countable complete system of neighborhoods of the origin and hence a countable sequence N_1, N_2, $\cdots$ of continuous pseudo-norms such that every other continuous pseudo-norm is bounded by one of them. Clearly we may choose these so that their pseudo-norm sets form an ascending sequence, $P_1 \subseteq P_2 \subseteq \cdots$. Since every continuity pseudo-norm set is contained in one of the P_i it follows that $L = P_1 + P_2 + \cdots$. Our lemma now implies that every pseudo-norm set contained in L is contained in some P_i and hence is a continuity pseudo-norm set. In other words, X is relatively strong. Conversely let X be relatively strong and let L be the union of an ascending countable sequence of pseudo-norm sets, $P_1 \subseteq P_2 \subseteq \cdots$. For each P_i let N_i be an associated pseudo-norm. If O is any open set containing the origin and N is a continuous pseudo-norm such that $N(x) < 1$ implies that x is in O it follows upon applying the lemma to the pseudo-norm set of N that, for some $n = 1, 2, \cdots$ and some $\mu > 0$, $N(x) \leq \mu N_n(x)$ for all x in X so that the set where $N_n(x) < 1/\mu$ is included in O. Thus the open sets defined by the inequalities $N_i(x) < 1/j$ for $i, j = 1, 2, \cdots$ form a complete system of neighborhoods of the origin. It follows at once from the Birkhoff-Kakutani group metrizability criterion that X is metriza-

ble. As a matter of fact, as the reader may show directly without difficulty, $\rho(x, y) = \sum_{n=1}^{\infty} N_n(x-y)/2^n(1+N_n(x-y))$ is a metric.

Since the linear union of countably many pseudo-norm sets is always boundedly closed [9, Theorem V-13] we may state the following corollary.

COROLLARY. *The linear system of a metrizable convex topological linear space is always boundedly closed.*

Theorem 10 tells us that there is a natural one-to-one correspondence between the metrizable convex topologies in a linear space X and the total subspaces of X^* which are the unions of ascending sequences of pseudo-norm sets. It follows from the lemma at the beginning of this section that such a union is a norm set if and only if all of the terms past a certain one are identical. Thus the metrizable but non-normable convex topologies correspond to the strictly increasing sequences. It follows from the general theory of pseudo-norm sets as developed in [9] that such sequences exist in great variety whenever X is infinite-dimensional.

An interesting special case occurs when all the pseudo-norm sets are finite-dimensional. In this case L is not more than $\aleph_0$-dimensional and the space itself is relatively weak as well as relatively strong. Conversely if X_L is an arbitrary regular linear system in which L is not more than $\aleph_0$-dimensional, it follows from the fact that an infinite-dimensional pseudo-norm set must be at least C-dimensional [9, Theorem I-1] that the only pseudo-norm sets in L are finite-dimensional and that L is the union of a countable ascending sequence of them. Thus there is a unique convex topology associated with X_L and it is metrizable. The most familiar example of such a space is Banach's [2, p. 10] space (s) consisting of all sequences of real numbers where

$$\rho(x_1, x_2, \cdots ; y_1, y_2, \cdots) = \sum_{n=1}^{\infty} |x_n - y_n|/2^n(1 + |x_n - y_n|).$$

It is readily verified that the topology defined by this metric is the unique convex topology whose continuous linear functionals are the finite linear combinations of the functionals f_i where, for each $i = 1, 2, \cdots, f_i(x_1, x_2, \cdots) = x_i$. We show now that the space (s) is a universal space of this character.

THEOREM 11. *A relatively weak convex topological linear space is metrizable if and only if it is isomorphic (as a topological linear space) to a subspace of the space (s) of Banach. It is normable if and only if it is finite-dimensional.*

Proof. The "if" parts of both statements are obvious. The "only if" part of the second statement is an immediate consequence of the fact that a strictly ascending sequence of pseudo-norm sets never has a norm set for its sum. To prove the "only if" part of the first statement let X_L be the linear system of a metrizable relatively weak convex topological linear space. Then L is $\aleph_0$-dimensional and admits a Hamel basis $l_1, l_2, \cdots$. For each x in X

let $F(x) = l_1(x), l_2(x), \cdots$. Then F sets up a one-to-one linear map of X into part of (s) in such a way that $l_i(x) = f_i(F(x))$ for all x in X. Thus since the topology of X is defined by the l_i and that of (s) by the f_i it is easily seen that F is a homeomorphism. This completes the proof of the theorem.

We close this section with an example of a metrizable non-normable topological linear space of interest in function theory. Let R be a bounded open set in the complex z-plane. Let $\mathfrak{F}$ be the linear space of all single-valued functions defined and analytic throughout R. We introduce a topology in $\mathfrak{F}$ as follows. For each $n = 1, 2, \cdots$ let C_n be the set of all points in R whose distance from the boundary of R is at least $1/n$. Then C_n is a closed subset of R. Let $N_n(f) = \text{l.u.b.}_{z \in C_n} |f(z)|$ for each f in $\mathfrak{F}$ and each $n = 1, 2, \cdots$. Clearly N_n is not only a pseudo-norm but, for n large enough so that C_n is not empty, also a norm. Let P_n be the norm set associated with N_n. Let $L = P_1 + P_2 + \cdots$ and introduce in $\mathfrak{F}$ the relatively strong convex topology associated with $\mathfrak{F}_L$. It follows easily from well known theorems in function theory that uniform boundedness on C_n does not imply uniform boundedness on C_{n+1}. Accordingly it is clear that the sequence of pseudo-norm sets $P_1, P_2, \cdots$ is strictly ascending so that the topology we obtain is metrizable but not normable. It is readily verified that convergence in this topology is equivalent to uniform convergence on every closed subset of R.

5. Completeness. A topological linear space, being a topological group, has a natural uniform structure. (See Weil [19] for a discussion of this notion.) Hence one may speak of its totally bounded subsets, its Cauchy directed systems of elements, and of whether or not it is complete. Completeness with respect to general Cauchy directed systems turns out to be rather rare and various authors have introduced several weaker notions which we shall now formulate.

DEFINITION 6. A topological linear space is C_4 complete if it is complete as a uniform structure with respect to the convergence of directed systems. It is C_3 complete if every closed and bounded subset is C_4 complete. It is C_2 complete if every closed and totally bounded subset is C_4 complete. It is C_1 complete if it is complete as a uniform structure with respect to the convergence of sequences. It is T_2 complete if every closed and totally bounded subset is bicompact. It is T_1 complete if every closed and totally bounded subset is sequentially compact.

Completeness in topological linear spaces was apparently first discussed by von Neumann [12] who introduced what we have called C_1 and T_1 completeness. C_4 completeness was introduced by G. Birkhoff [4], C_3 completeness by Taylor [15, 16], and C_2 completeness by Šmulian [14]. It is readily verified that for $i = 2, 3$, or 4, C_i completeness implies C_{i-1} completeness and that T_2 completeness implies C_2 completeness. That C_2 completeness implies T_2 completeness follows from the generalization to uniform structures of the well known theorem that a complete totally bounded metric space is bicom-

pact [19, Théorème 7] or from a trivial modification of G. Birkhoff's [4] proof that C_4 completeness implies T_2 completeness. Finally von Neumann [12] has proved that T_1 completeness implies C_1 completeness. The relationship between the various notions of completeness may thus be represented schematically as follows:

$$C_4 \to C_3 \to C_2 = T_2 \to T_1 \to C_1.$$

In discussing completeness in convex spaces our principal tool will be a sort of partial completion, the elements of which depend only upon the linear system of the space in question. Let X_L be an arbitrary regular linear system and in the usual manner regard X as a subspace of L^*. Consider the family $\mathfrak{A}$ of all subsets A of L which are uniformly bounded as members of the conjugate system L_X. Let $X^- \supseteq X$ be the set of all f in L^* for which l.u.b.$_{l \in A} |f(l)|$ $< \infty$ for all A in $\mathfrak{A}$. Next let N be any pseudo-norm defined in X whose pseudo-norm set is contained in L. Following the procedure familiar for norms we define $N(l) = $ l.u.b.$_{N(x)=1} |l(x)|$ for each l in L. Finally we define $N^-(f)$ $= $ l.u.b.$_{N(l)=1} |f(l)|$ for all f in X^-.

LEMMA 1. $N^-(f)$ *is finite for all f in X^- and is a pseudo-norm as a function of f. Furthermore $N(x) = N^-(x)$ for all x in X. Finally if $N_1(x) \leq N_2(x)$ for all x in X then $N_1^-(f) \leq N_2^-(f)$ for all f in X^-.*

Proof. Let B be any bounded subset of X_L. Then in particular $l(B)$ is a bounded set of real numbers for each l in the pseudo-norm set P_N of N. Hence $N(B)$ is a bounded set of real numbers[5]. But $|l(x)| \leq N(l)N(x)$ for all l in P_N and all x in X. Thus l.u.b.$_{x \in B, \, N(l)=1,} |l(x)| < \infty$ so that the set of all l with $N(l) = 1$ is uniformly bounded. Hence N^- is finite on X^-. That N^- is a pseudo-norm follows from the easily proved general theorem that the supremum of any pointwise bounded family of pseudo-norms is again a pseudo-norm. That $N(x) = N^-(x)$ for x in X follows readily from its well known truth in the case where N is a norm. Finally the last statement is an obvious consequence of the definitions concerned.

Now let t be any convex topology associated with X_L and let $\mathfrak{M}$ be the class of all pseudo-norms M defined on X^- such that, for some continuous pseudo-norm N defined on X, $M(f) \leq N^-(f)$ for all f in X^-.

LEMMA 2. *There is a unique convex topology t^- in X^- whose continuous pseudo-norms are precisely the members of $\mathfrak{M}$. This topology coincides on X with t. If t_1 and t_2 are two convex topologies associated with X_L and t_1 is stronger than t_2 then t_1^- is stronger than t_2^-.*

Proof. If N_1 and N_2 are pseudo-norms on X continuous with respect to t then $N_1 + N_2$ is again such and by the preceding lemma $(N_1 + N_2)^-(f)$ $\geq \max(N_1^-(f), N_2^-(f))$ for all f in X. If $f \neq 0$ choose l in L with $f(l) \neq 0$ and a con-

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[5] Cf. proof of Theorem 7.

tinuous pseudo-norm N containing l in its pseudo-norm set. Then $N^-(f) \neq 0$. It follows from these remarks and the mode of definition of $\mathfrak{M}$ that $\mathfrak{M}$ satisfies the conditions (a), (b), and (c) of Theorem 2. The truth of the first statement of this lemma now follows from Theorem 2. The truth of the second statement follows at once from Theorem 1 and the truth of the third follows from the fact (obvious from the proof of Theorem 1) that one convex topology is stronger than another if and only if it has more continuous pseudo-norms.

LEMMA 3. *With respect to the topology t^- every bounded Cauchy directed system $\{x_\alpha\}$ in X has a limit in X^-.*

Proof. Let l be an arbitrary member of L. Since l is continuous and linear it is readily verified that $\{l(x_\alpha)\}$ is a Cauchy directed system of real numbers and hence has a limit $f(l)$. The number $f(l)$ as a function of l is clearly a member of L^*. Let A be any uniformly bounded subset of L_X. Then

$$\underset{\alpha}{\text{l.u.b.}}_{l \in A} \; |l(x_\alpha)| = c < \infty.$$

Hence $|f(l)| \leq c$ for all l in A. Thus $f \in X^-$. To show that $x_\alpha \to f$ it is manifestly sufficient to show that for each continuous pseudo-norm N on X we have $N^-(x_\alpha - f)$ tending to zero. Given $\epsilon > 0$, choose γ so that if $\alpha > \gamma$ and $\beta > \gamma$ then $N^-(x_\alpha - x_\beta) < \epsilon/2$. Choose any $\alpha > \gamma$. Since for each l in L we have $|l(x_\alpha) - f(l)|$ $= \lim_\beta |l(x_\alpha) - l(x_\beta)|$ and for each β we have $|l(x_\alpha) - l(x_\beta)| \leq N(l)N(x_\alpha - x_\beta)$ it follows that $|l(x_\alpha) - f(l)| \leq N(l)(\epsilon/2) \leq \epsilon/2$ if $N(l) \leq 1$. Thus since $N^-(x_\alpha - f)$ $= \text{l.u.b.}_{N(l)=1} |l(x_\alpha) - f(l)|$ it follows that $N^-(x_\alpha - f) \leq \epsilon/2 < \epsilon$. Thus $N^-(x_\alpha - f)$ $\to 0$ and the lemma is proved.

As a first consequence of these considerations we prove the following theorem.

THEOREM 12. *If i is 1, 2, or 3 and X is a C_i complete convex topological linear space then under any "relatively stronger" convex topology X is again C_i complete.*

Proof. We shall consider only the case $i = 2$. The other cases are similar and slightly easier. Let t_1 be the weaker and t_2 the stronger of the two topologies. Let X_L be their common linear system and let X^- be the subspace of L^* introduced above. Extend t_1 and t_2 to topologies t_1^- and t_2^- defined in X^- as described in Lemma 2. Let $\{x_\alpha\}$ be a directed system of elements of X, Cauchy and totally bounded with respect to t_2. By Lemma 3 the system $\{x_\alpha\}$ has a limit f in X^- with respect to t_2^-. Since, by Lemma 2, t_2^- is stronger than t_1^- it follows that f is also a limit of $\{x_\alpha\}$ with respect to t_1^-. Hence $\{x_\alpha\}$ is Cauchy with respect to t_1. Furthermore since t_1 is weaker than t_2, $\{x_\alpha\}$ is totally bounded with respect to t_1. But X is C_2 complete in the t_1 topology. Since limits of directed systems are unique it follows that f is in X. Thus X is C_2 complete in the t_2 topology and the theorem is proved.

Similar considerations give us the following theorem.

Theorem 13. *If X is a convex topological linear space which is C_1 complete (and hence if X is complete in any sense) then the linear system X_L associated with X is a complete linear system.*

Proof. Let x_1, x_2, $\cdots$ be any sequence of elements of x which converges as a sequence of elements of the linear system X_L to a member f of L^*. Then by the definition of convergence there exists a sequence γ_1, γ_2, $\cdots$ of real numbers, converging to zero, such that $\{(l(x_n)-f(l))/\gamma_n\}$ is bounded for each l in L. It follows in particular that $l(x_n) \to f(l)$ for each l and that $\{l(x_n-x_m)/(\gamma_n+\gamma_m)\}$ is bounded in m and n for each l. Let l_1, l_2, $\cdots$ be any uniformly bounded sequence of elements of L_X. Then $|l_k(x_n)| \leq \mu$ for all $k = 1, 2, \cdots$ and $n = 1, 2, \cdots$ so that $|f(l_k)| \leq \mu$ for all $k = 1, 2, \cdots$ where μ is some suitable positive real number. Hence f is a member of X^-. Let N be any continuous pseudo-norm defined on X. Then $N^-(x_n - f) = \text{l.u.b.}_{N(l)=1}|l(x_n) - f(l)|$ $=\text{l.u.b.}_{N(l)=1}\lim_{m\to\infty}|l(x_n-x_m)| = \text{l.u.b.}_{N(l)=1}\lim_{m\to\infty}(\gamma_n+\gamma_m)l((x_n-x_m)/(\gamma_n+\gamma_m))$. But since the set of all l with $N(l) = 1$ is uniformly bounded and the set of all $\{(x_n-x_m)/(\gamma_n+\gamma_m)\}$ for n, $m = 1, 2, \cdots$ is bounded it follows that there exists a constant $\mu > 0$ with $|l((x_n-x_m)/(\gamma_n+\gamma_m))| \leq \mu$. Thus $N^-(x_n-f)$ $\leq \text{l.u.b.}_{N(l)=1}\lim \sup_{m\to\infty}(\gamma_n+\gamma_m)\mu \leq \gamma_n\mu$. Hence $N^-(x_n-f) \to 0$. It follows then from the connection between the N^-'s and the topology of X^- that $x_n \to f$. But since X is C_1 complete, f is in X. Thus X_L is a complete linear system.

We do not know whether or not it is true that every complete linear system is associated with a, in some sense complete, convex topological linear space. We do have, however, the following theorem.

Theorem 14. *If X_L is a complete linear system such that X is a boundedly closed subspace of L^* then every convex topology associated with X_L is C_1, C_2, C_3, T_1, and T_2 complete.*

Proof. By Theorem VI-4 of [9], X_L is uniform. Hence since X is boundedly closed as a subspace of L^* it follows that $X^- = X$. The truth of the theorem now follows from Lemma 3 and the fact that C_3 completeness implies all the other kinds under consideration.

That C_4 completeness cannot be included may be seen from the following theorem which is a trivial generalization of related theorems of G. Birkhoff [4, page 49] and Taylor [16, Theorem 4.9].

Theorem 15. *If X is a relatively weak C_4 complete convex topological linear space then the linear system X_L associated with X is of the form Y_Y^*, that is, $X = L^*$.*

Proof. Let f be any member of L^*. For each finite-dimensional subspace M of L let x_M be a member of X such that x as a member of L^* coincides on M with f. This is possible since X as on M, being total, is the whole of M^*. If we partially order the finite subspaces of L by inclusion $\{x_M\}$ becomes a directed system. Using the fact that the topology of X is relatively weak it is

easy to show that $\{x_M\}$ converges to f. Since X is C_4 complete, f must be in X and the theorem follows.

The question arises as to whether or not Theorem 14 has a converse; that is, as to whether or not the linear system of a relatively weak convex topological linear space which is C_3 complete necessarily has a boundedly closed conjugate. We do not know the answer to this question. We note, however, in view of an obvious converse of Theorem 15 that an affirmative answer would imply that a linear space X is always a boundedly closed subspace of X^{**} and hence, by virtue of Theorem IV-12 of [9], answer a measure theoretic question raised by Ulam [17]. Furthermore we do have the following theorem.

THEOREM 16. *If X_L is the linear system of a T_2 complete relatively weak convex topological linear space and if X_L is relatively bounded then X_L^0 is boundedly closed.*

Proof. By Theorem 13, X_L is a complete linear system. Hence, since X_L is relatively bounded, it follows from the corollary to Theorem VI-4 of [9] that X_L is simple. Thus by Theorem V-17 of [9] the bounded closure $\bar{L}$ of L is a norm set. Let N_1 be a generating norm. For each x in X let $N(x)$ $=$l.u.b.$_{N_1(l)=1}|l(x)|$. Then $N(x) \leqq N_1(x)$ so that the norm set of N is contained in L. On the other hand, this norm set contains L. Hence $\bar{L}$ is also the norm set of N. Let K be the set of all x in X for which $N(x) \leqq 1$. An element x of X is in K if and only if $|l(x)| \leqq 1$ for all l in L such that $N(l) = 1$. Hence K is an intersection of sets closed in the topology of X and is consequently itself closed in this topology. Obviously K is bounded. But it has been shown by Šmulian [14, Hilfsatz 2] and essentially by Taylor [16, Theorem 4.3] that in a relatively weak convex topological linear space boundedness and total boundedness are equivalent. Thus K is totally bounded. Since K is closed and totally bounded and X is T_2 complete it follows that K is bicompact. Thus by Theorem VII-4 of [9] $K+$ is a norm set. Since $K+ = X$ and norm sets are boundedly closed the theorem is proved.

Theorems 14 and 16 tell us that the relatively weak convex topological linear space associated with a regular relatively bounded linear system X_L is C_3 (T_2, C_2) complete if and only if X_L is complete and has a boundedly closed conjugate. We characterize such linear systems now in terms of the notion of an almost complete normed linear space [9, p. 196].

THEOREM 17. *Let X_L be a regular linear system. Then X_L is relatively bounded, complete, and such that X_L^0 is boundedly closed if and only if X_L^0 is the linear system of an almost complete normed linear space.*

Proof. If X_L is relatively bounded and complete it follows from Theorems VI-4 and VI-2 of [9] that X_L and X_L^0 are simple and uniform. Since X_L^0 is boundedly closed and simple it is in the linear system of a normed

linear space [9, Theorem V-17]. Since it is uniform the normed linear space is almost complete. Conversely, if X_L^0 is the linear system of an almost complete normed linear space then it follows immediately that X_L^0 is boundedly closed and relatively bounded. Hence X_L is relatively bounded. It remains to show that X_L is complete. But since X_L^0 is uniform and boundedly closed $X^- = X$. Hence, by Lemma 3, every convex topological linear space associated with X_L is C_3 complete. Thus, by Theorem 13, X_L is a complete linear system.

As is pointed out on page 195 of [9], if X_L is the linear system of a normed linear space then the uniformly bounded subsets of X_L are precisely the norm bounded subsets of L. Thus if the normed linear space is almost complete (that is, if X_L is uniform) then it is reflexive if and only if X_L^0 is boundedly closed. Hence since a reflexive space is always complete it follows that a normed linear space is reflexive if and only if its linear system is complete and has a boundedly closed conjugate. Thus we obtain from the foregoing another proof of the well known theorem:

THEOREM 18. *For a normed linear space X the following are equivalent*: (a) *X is reflexive*, (b) *X is C_3 complete in its weak topology*, (c) *X is T_2 complete in its weak topology.*

6. **Category.** Wehausen [18, Theorems 15 and 16] has shown that a complete topological linear space need not be of the second category by proving that an infinite-dimensional normed linear space in its weak topology is necessarily of the first category. In this section we give two necessary conditions that a convex topological linear space be of the second category, each of which implies Wehausen's result. The first of these is essentially due (in another formulation) to Sirvint [13].

THEOREM 19. *If X is a convex topological linear space of the second category then it is relatively strong and its linear system X_L is uniform.*

Proof. Let $\{l_\alpha\}$ be any bounded subset of L_X. For each x in X let $N(x) = $ l.u.b. $|l_\alpha(x)|$. Then N is a pseudo-norm and the set K of all x with $N(x) \le 1$ is an intersection of closed sets and hence closed. Since $K \cup 2K \cup 3K \cup \cdots = X$ and X is of the second category it follows that some nK has an interior point. Hence K has an interior point $\bar{x}$. Thus there exists a continuous pseudo-norm N_1 such that if $N_1(y) < 1$ then $\bar{x} + y$ is in K; that is, $N(\bar{x} + y) \le 1$. It follows easily that there exists a positive real number μ such that $N(x) \le \mu N_1(x)$ for all x in X. Hence, by Theorem 2, N is continuous. Since any pseudo-norm whose pseudo-norm set is in L can be generated in this way it follows that X is relatively strong. To prove the uniformity of X_L let $x_1, x_2, \cdots$ and $l_1, l_2, \cdots$ be bounded subsets of X_L and L_X respectively. By the argument just presented l.u.b.$_{i=1,2,\ldots}|l_i(x)|$ is a continuous pseudo-norm. Hence l.u.b.$_{i=1,2,\ldots}|l_i(x_j)|$ is bounded with respect to j; that is, l.u.b.$_{i,j=1,2,\ldots}|l_i(x_j)| < \infty$. Thus X_L is uniform and the theorem is proved.

If L is not "too large" a subspace of X^* we can conclude considerably more.

THEOREM 20. *If X is a convex topological linear space of the second category and its linear system X_L is almost relatively bounded then X is normable as an almost complete normed linear space.*

Proof. Since X_L is almost relatively bounded we have $X = B_1 \cup B_2 \cup \cdots$ where each B_i is bounded. By a theorem of von Neumann [12, Theorem 11] the closure of each B_i is bounded. Since X is of the second category at least one B_i has an interior point. Since X is convex B_i contains a convex open set. Thus X contains a bounded convex open set and hence by a well known theorem of Kolmogoroff [8] is normable. That the resulting normed linear space is almost complete follows from the uniformity part of the conclusion of Theorem 19.

7. The lattice of closed subspaces. It is an easy consequence of the Hahn-Banach extension theorem that the closed subspaces of the convex topological linear space are the intersections of null spaces of continuous linear functionals. Thus the closed subspaces of X are precisely the closed subspaces of the linear system X_L [9, chap. III]. It follows immediately that all of the convex topological linear spaces associated with X_L have mutually isomorphic lattices of closed subspaces. Thus the theorem of [11] to the effect that a normed linear space is determined to within isomorphism by its lattice of closed subspaces does not extend to general convex topological linear spaces. On the other hand, since it is true that a linear system is determined to within isomorphism by its lattice of closed subspaces [9, Theorem III-3], we may state the following theorem.

THEOREM 21. *If X and Y are relatively strong (relatively weak) convex topological linear spaces then they are isomorphic (that is, homeomorphic and algebraically isomorphic under the same mapping) if and only if their lattices of closed subspaces are isomorphic as abstract lattices.*

8. The ring of continuous linear transformations. Let X be a convex topological linear space and let T be a linear transformation of X into itself. It is easy to see that T is continuous if and only if for each continuous pseudo-norm N on X the pseudo-norm $N(T(x))$ is also continuous. Furthermore as may be verified without difficulty the pseudo-norm set of $N(T(x))$ is the set of all functionals f of the form $f(x) = l(T(x))$ where l is in the pseudo-norm set of N. In other words, T is continuous if and only if whenever M is a continuity pseudo-norm set then $T^*(M)$ is also. It follows that every continuous T is a homomorphism [9, page 160] of the linear system X_L into itself and that the converse is true if and only if the continuity pseudo-norm sets are collectively invariant under the conjugates of homomorphisms of X_L into itself. This last condition will certainly hold if the continuity pseudo-norm

sets are all pseudo-norm sets in L or all finite-dimensional pseudo-norm sets in L but, as one may show by examples, will not hold in general. One may show moreover that whenever the ring of continuous linear transformations is not equal to the ring of homomorphisms of the linear system it is not isomorphic to it. The situation for rings is then a little different than that for lattices. Some but not all of the different convex topological linear spaces associated with a given linear system will be mutually isomorphic. However, as in the lattice case, we may conclude that while the isomorphism theorem for normed linear spaces does not extend to general convex topological linear spaces we do have the following theorem.

THEOREM 22. *If X and Y are relatively strong (relatively weak) convex topological linear spaces then they are isomorphic if and only if their rings of continuous linear transformations are isomorphic as abstract rings.*

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