ON SOKOLOVSKY'S "MOMENTLESS SHELLS"

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Professor I. S. Sokolnikoff has kindly brought to my notice a memoir of V. V. Sokolovsky(1) which gives solutions of the differential equations of the membrane theory of shells of revolution for several types of surfaces. If the Cartesian equation of the middle surface meridian in some axial plane be

$$r = f(z), \tag{1}$$

the three families treated may be put in the forms

$$f = kz^{\mu}; \tag{2}$$

$$f = a \sin^c\phi, \qquad z = -ca\int \sin^c\phi\, d\phi; \tag{3}$$

$$f = a \sec^c\phi, \qquad z = -ca\int \sec^c\phi \tan^2\phi\, d\phi. \tag{4}$$

For (3) and (4) the parameter ϕ may be regarded as the colatitude angle.

The family (2) is included in a more general family of surfaces for which Nemenyi and I have more recently given the same solutions(2).

Sokolovsky's expression for the meridial stress resultant Fourier coefficient $N_{\phi n}$ for the family (3) in the case of no load is

$$\begin{aligned} N_{\phi n} = (\sin\phi)^{(\Delta-3c-1)/2}\Bigg[C_n F\left(\frac{\Delta+c+1}{4}, \frac{\Delta-c+1}{4}; 1+\frac{\Delta}{2}; \sin^2\phi\right) \\ + D_n \sin^{-\Delta}\phi\, F\left(\frac{-\Delta+c+1}{4}, \frac{-\Delta-c+1}{4}; 1-\frac{\Delta}{2}; \sin^2\phi\right)\Bigg], \end{aligned} \tag{5}$$

where

$$\Delta \equiv ((c-1)^2 + 4n^2c)^{1/2}, \tag{6}$$

and he gives a solution of similar type for the family (4). Our first observation in looking at the solution (5) is that the family of curves (3) must be unnecessarily special, since there are only two parameters n and c occurring in the hypergeometric functions. In fact (3) and (4) are both special cases of

$$f = a \sin^p\xi, \qquad z = -pb\int \sin^p\xi \tan^q\xi\, d\xi, \tag{7}$$

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(1) [1]. Numbers in brackets refer to the list of references at the end of this note.

(2) [3, equations (8)]; [4, equations (21.9), (21.10)].

by the choices

$$p = c, \qquad b = a, \qquad q = 0, \qquad \xi = \phi, \tag{8}$$

$$p = -c, \qquad b = -a, \qquad q = -2, \qquad \xi = \pi/2 - \phi, \tag{9}$$

respectively. This family of surfaces is not directly amenable to treatment by Sokolovsky's method because ξ is not in general the colatitude angle.

Nemenyi and I([3]) have shown that all stress resultant Fourier coefficients for a shell with meridian (1) can be expressed in terms of the complete stress functions $\mathfrak{U}_{nc}$, which satisfy the differential equation

$$\frac{d^2\mathfrak{U}_{nc}}{dz^2} + (n^2 - 1)\frac{f''(z)}{f(z)}\mathfrak{U}_{nc} = 0. \tag{10}$$

The family (2) is the most natural and easy one to study in view of this equation, but Sokolovsky's surfaces (3) and (4) are not directly amenable to treatment because they are not formally expressible in the form (1). They can, however, be put in the form

$$z = z(f). \tag{11}$$

For convenience in studying surfaces whose meridian is expressible in the form (11), by elementary calculus we may put the equation (10) into the corresponding form

$$\frac{z'(f)}{z''(f)}\frac{d^2\mathfrak{U}_{nc}}{df^2} - \frac{d\mathfrak{U}_{nc}}{df} + \frac{1-n^2}{f}\mathfrak{U}_{nc} = 0, \tag{12}$$

where primes denote differentiation with respect to f.

If in the equation (12) we substitute for z from the family (7), we obtain the differential equation

$$\frac{pf}{q+1}\left[1 - \left(\frac{f}{a}\right)^{2/p}\right]\frac{d^2\mathfrak{U}_{nc}}{df^2} - \frac{d\mathfrak{U}_{nc}}{df} + \frac{1-n^2}{f}\mathfrak{U}_{nc} = 0, \tag{13}$$

which by the substitution

$$x = (f/a)^{2/p} \tag{14}$$

is easily shown to have the integral

$$\begin{aligned}\mathfrak{U}_{nc} = x^{(p+q+\Delta+1)/4}\Bigg[&C_nF\left(\frac{\Delta+p+q+1}{4}, \frac{\Delta-p+q+1}{4}; 1+\frac{\Delta}{2}; x\right)\\ &+ D_nx^{-\Delta/2}F\left(\frac{-\Delta+p-q+1}{4}, \frac{-\Delta-p-q+1}{4}; 1-\frac{\Delta}{2}; x\right)\Bigg],\end{aligned} \tag{15}$$

([3]) [3, equations (4), (5)]; [4, equations (11.8), (11.13)].

where

$$\Delta \equiv ((p - q + 1)^2 + 4pn^2)^{1/2}. \tag{16}$$

The substitutions (8) reduce this result to the equivalent of Sokolovsky's solution (5); the substitutions (9) reduce it to a form which can with the aid of familiar transformations of the hypergeometric function be shown equivalent to Sokolovsky's solutions for the family (4).

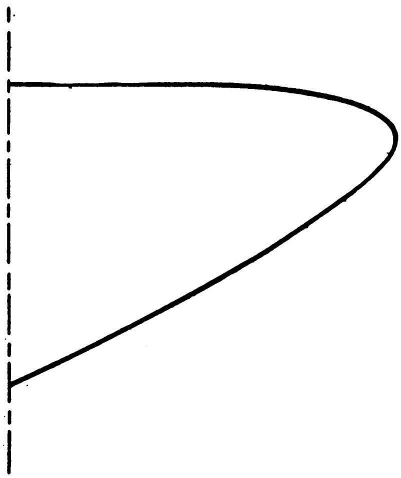

FIG. 1.
Meridian $f=\sin 2\phi$, $z=-\phi/2+\sin 2\phi/4$

In the family (7) we still are restricting ourselves unnecessarily, for I have shown previously[4] that the complete stress functions for the surface (1) are also complete stress functions for the family of surfaces

$$r = f\left[A + B\int \frac{dz}{f^2}\right]. \tag{17}$$

Hence the solution (15) is valid also for the family

$$\begin{aligned} f &= a \sin^p \xi\left[A + B\int \csc^p\xi \tan^q\xi \, d\xi\right], \\ z &= - pb\int \sin^p\xi \tan^q\xi \, d\xi. \end{aligned} \tag{18}$$

Since our generalization depends upon the invariance of the differential equation (10) when the surface (1) is replaced by the family (17), we must take care that for a given value of the parameter ξ we compute f for use in the

(4) [4, equation (19.4)].

solution (15) from the *original* formula (7), not from the generalized formula (18).

The family (18) contains some interesting special cases. The case when $p=1$, $q=0$ I have treated elsewhere(5). When $A=0$, $p=2$, $q=0$ we have the curves

$$f = B \sin 2\xi, \qquad z = b\left[-\frac{\xi}{2} + \frac{\sin 2\xi}{4}\right], \tag{19}$$

one of which is sketched in Fig. 1.

The equation (12) suggests to us other surfaces for which we can write down complete stress functions in terms of tabulated functions. For example, suppose

$$z = aErf(kf); \tag{20}$$

we easily see that equation (12) now becomes simply

$$\frac{d^2\mathfrak{U}_{nc}}{df^2} + 2k^2f\frac{d\mathfrak{U}_{nc}}{df} + 2k^2(n^2-1)\mathfrak{U}_{nc} = 0; \tag{21}$$

hence

$$\mathfrak{U}_{nc} = \exp\,(-\,k^2f^2/2)\left[C_nD_{n^2-2}(2^{1/2}kf) + D_nD_{1-n^2}(2^{1/2}ikf)\right], \tag{22}$$

where $D_n(x)$ is Weber's parabolic cylinder function. If in the formula (22) we still obtain f in terms of z from the relation (20), the complete stress functions (22) are valid for the more general family

$$z = a\mathrm{Erf}\left(\frac{kf}{A + B\int f^{-2}\exp\,(-\,k^2f^2)df}\right). \tag{23}$$

A meridian of the form (20) is sketched in Fig. 2.

The surfaces (20) or (23) are special members of a more general family for which it is equally easy to write down the complete stress functions. Suppose

$$\begin{aligned} z &= aE_p(kf), \\ z &= ap!\int_0^{kf} \exp\,(-\,t^p)dt. \end{aligned} \tag{24}$$

Then the equation (12) becomes

$$-\,k^{-p}p^{-1}f^{1-p}\frac{d^2\mathfrak{U}_{nc}}{df^2} - \frac{d\mathfrak{U}_{nc}}{df} + \frac{1-n^2}{f}\mathfrak{U}_{nc} = 0. \tag{25}$$

The substitution

(5) [4, equations (19.5), (19.6)].

$$x = f^p$$

reduces this equation to the form

$$(26)\qquad px\frac{d^2\mathfrak{U}_{nc}}{dx^2} + \left[k^p + \left(1 - \frac{1}{p}\right)\frac{1}{x}\right]\frac{d\mathfrak{U}_{nc}}{dx} + \frac{k^2(n^2-1)}{px}\mathfrak{U}_{nc} = 0,$$

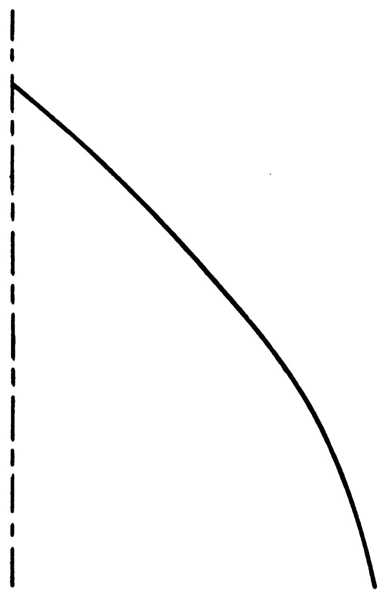

FIG. 2.
Meridian $z = \mathrm{Erf}(f)$

which without difficulty is transformed into Whittaker's equation. We find that

$$(27)\qquad \mathfrak{U}_{nc} = f^{(1-p)/2}\exp\left(-k^p f^p/2\right)\left[C_n W_{\alpha,\beta}(k^p f^p) + D_n W_{-\alpha,\beta}(-k^p f^p)\right],$$

where

$$\alpha \equiv -\frac{1}{2} + \frac{3 - 2n^2}{p}, \qquad \beta \equiv \frac{1}{2p},$$

the function $W_{\alpha,\beta}(y)$ being Whittaker's function. The family of surfaces (24) may of course be included in the more general family

$$(28)\qquad z = aE_p\left(\frac{kf}{A + B\int f^{-2}\exp\left(-k^p f^p\right)df}\right),$$

for which the complete stress functions (27) are still valid provided f in terms of z is calculated not from equation (28) but from equation (24).

Sokolovsky has given also expressions for the displacements for the families (3) and (4) for the case when all stress resultants vanish. While it is not difficult to modify my treatment(6) of the displacement equations so as conveniently to include and generalize his results, as I have mentioned elsewhere a knowledge of the complete displacement functions does not lead one to practical results without excessive computation, so we shall not take space to go into the matter.

Further investigations of Sokolovsky [2] have produced some very pretty solutions in terms of arbitrary functions for the equilibrium equations of the membrane theory for shells which are not shells of revolution, including quadric surfaces, surfaces of constant slope, and noncircular cones.

References

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(6) [4, equations (18.8), (18.12)].