MEROMORPHIC FUNCTIONS WITH SIMULTANEOUS MULTIPLICATION AND ADDITION THEOREMS

BY

IRVING GERST

Introduction. Let \( f(z) \) be a nonconstant meromorphic function. Further, let there exist two numbers \( m \neq 0, 1 \) and \( h \neq 0 \), and two rational functions \( R \) and \( S \), such that

\[
(1) \quad f(mz) = R[f(z)],
\]

and

\[
(2) \quad f(z + h) = S[f(z)].
\]

Then the function \( f(z) \) has both a rational multiplication and a rational addition theorem. We propose to determine all such functions.

Our problem is a generalization of one treated in a paper by Ritt\(^1\), in which all periodic meromorphic functions having a multiplication theorem were determined. This is equivalent to taking \( S(z) = z \) in (2). For this case, Ritt proved that \( |m| \geq 1 \); and if \( |m| > 1 \), then \( f(z) \) is a linear function of one of the functions \( e^{az}, \cos(az + \beta), \varphi(z + \beta) \); when \( g_2 = 0, \varphi'(z + \beta) \); when \( g_2 = 0, \varphi''(z + \beta) \) and \( \varphi^3(z + \beta) \). Here \( \alpha \) is arbitrary while \( \beta \) is restricted to certain values. If \( |m| = 1 \), then it was shown that \( m \) is either -1 or a third, fourth or sixth root of unity. The forms for the function \( f(z) \) were also explicitly given in this case. Use will be made of the results quoted and also of the methods of Ritt’s paper.

We distinguish the following three cases:

(A) \( |m| > 1 \);

(B) \( |m| \leq 1 \), with \( m \) not a rational root of unity;

(C) \( m \) a primitive \( n \)th root of unity.

For Case (A), which is the one usually considered in multiplication theorems, we can restate the problem in a way which shows its connection with the theory of a class of functions first systematically studied by Poincaré\(^2\). He provided an existence theorem for meromorphic functions satisfying (1) assuming that \( R(z) \) has a fixed point\(^3\) \( a \) for which \( R'(a) \) has a modulus greater than unity. Thus our problem is equivalent to that of finding all Poincaré functions having a rational addition theorem.

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\(^1\) Ritt, Periodic functions with a multiplication theorem, Trans. Amer. Math. Soc. vol. 23 (1922) pp. 16–25.

\(^2\) H. Poincaré, Sur une classe nouvelle de transcendantes uniformes, Journal de Mathématiques (3) vol. 55 (1890).

\(^3\) That is, a point for which \( R(a) = a \).
We remark that, if \( f(z) \) is a linear function of \( z \), integral or fractional, then \( f(z) \) will evidently have both a multiplication and addition theorem for every value of \( m \) and \( h \). This solution is therefore common to all three cases and, as it turns out, is the only rational solution of our problem.

On the other hand, if \( f(z) \) is transcendental, we may state the chief result of this paper for Case (A) as follows:

*If a Poincaré function has a rational addition theorem, it must be a periodic function.*

Although \( f(z) \) is periodic in this case, the value of \( h \) in the addition theorem is not necessarily a period, and therefore the corresponding \( S(z) \) need not equal \( z \). The possibilities for \( h \) and \( S(z) \) are given in detail in §4. It is found that except for the case in which \( f(z) \) is a linear function of \( e^{az} \), \( h \), when not a period, must be a suitable half or third of a period. It is noteworthy that for each of these admissible values of \( h \), \( S(z) \) is a linear function of \( z \).

In Case (B), \( f(z) \) must be a linear function of \( z \).

In Case (C) there appear nonperiodic solutions in addition to the types already mentioned.

It will be noted that, essentially, only the conditions that \( f(mz) \) and \( f(z+h) \) are uniform functions of \( f(z) \) will enter into consideration.

1. **Preliminary transformations.** The case of \( f(z) \) rational will be handled in §5. From this point on through §4, we assume that \( f(z) \) is transcendental.

Supposing that \( |m| > 1 \), we wish to replace \( f(z) \) by another meromorphic function \( g(z) \), related to it in a simple way, but having the following special properties at the origin:

- \((\alpha)\) \( g(0) = 0 \);
- \((\beta)\) \( g'(0) \neq 0 \);
- \((\gamma)\) 0 is not an exceptional point of \( g(z) \); that is, \( g(z) \) has an infinite number of zeros.

We further require that there exist relations of the form

\[
(3) \quad g(Mz) = U[g(z)],
\]

and

\[
(4) \quad g(z + h) = V[g(z)].
\]

Here \( U \) and \( V \) are rational and \( M \) is a number for which \( |M| > 1 \).

To find such a function \( g(z) \), we consider \( g(z) = f(z + \xi) - f(\xi) \), where \( f(z) \) is analytic at \( \xi \). We shall show that \( \xi \) can be chosen so that the resulting function \( g(z) \) fulfills all the conditions placed upon it.

Evidently for any value of \( \xi \) at which \( f(z) \) is analytic, property \((\alpha)\) holds, and also a relation of type \((4)\) exists with \( V(z) = S[z + f(\xi)] - f(\xi) \).

Write \( \sigma(z) = mz \), \( \tau(z) = z + h \). Let a positive integral subscript appended to a function denote the corresponding iterate of that function; for example \( R_r(z) \) denotes the \( r \)th iterate of \( R(z) \). Then if \( \xi \) is a finite fixed point of one of
the transformations $\sigma$ or $\tau\sigma_r$ ($r = 1, 2, \ldots$) and if $f(z)$ is analytic at $\xi$, a relation of type (3) will exist.

For if $\xi = 0$, the finite fixed point of $\sigma$, we may take $M = m$ and $U(z) = R[z + f(0)] - f(0)$.

And if for a positive integral $r$, $\tau\sigma_r(\xi) = \xi$, we may take $M = m^r$ and $U(z) = S[R[z + f(\xi)] - f(\xi)]$. This follows since

$$g(Mz) = g[\sigma_r(z)] = f[\sigma_r(z) + \xi] - f(\xi) = f[\tau\sigma_r(z + \xi)] - f(\xi);$$

and using (1) and (2) we have

$$f[\tau\sigma_r(z + \xi)] = S_f[\sigma_r(z + \xi)] = S[R[z + f(\xi)].$$

In each of these cases $|M| > 1$.

It remains to show the existence of a fixed point $\xi$ at which $f(z)$ is analytic and for which the corresponding function $g(z)$ has properties ($\beta$) and ($\gamma$). To this end consider the sequence

$$0, \frac{h}{1 - m}, \frac{h}{1 - m^2}, \ldots, \frac{h}{1 - m^r}, \ldots,$$

consisting of the finite fixed points of $\sigma$ and $\tau\sigma_r$ ($r = 1, 2, \ldots$).

These points converge to the origin since $|m| > 1$. Hence there exists a point $\xi$ in the above sequence at which $f(z)$ is analytic, at which $f'(z)$ is not zero, and which is not an exceptional point of $f(z)$ in the sense of Picard. With this value of $\xi$, the corresponding function $g(z)$ will have the required properties ($\beta$) and ($\gamma$).

2. The automorphism of $g(z)$. Adapting a method of Ritt(4), we now establish a relation for $g(z)$ which will enable us to show that $g(z)$ is periodic.

As $g(z)$ has an infinite number of zeros, suppose that $z_1 \neq 0$ is one of them. Then, since $g(0) = 0$ and $g'(0) \neq 0$, there exists a neighborhood of the origin, $\Gamma_0$, in which $g(z)$ assumes no value more than once.

Since $g(z_1) = 0$, there exists a neighborhood $\Gamma_1$ of $z_1$, in which $g(z)$ takes no value not assumed in $\Gamma_0$.

Define $\phi(z)$, for $z$ in $\Gamma_1$, as that unique point $z_2$ of $\Gamma_0$ for which $g(z_2) = g(z)$. Then

(5) $g[\phi(z)] = g(z)$

for $z$ in $\Gamma_1$.

The function $\phi(z)$ is analytic in $\Gamma_1$, not identically 0, and has a Taylor development at $z_1$. Since $\phi(z_1) = 0$, we may write that expansion as

(6) $\phi(z) = \alpha_1(z - z_1) + \alpha_2(z - z_1)^2 + \cdots + \alpha_n(z - z_1)^n + \cdots$.

It will now be shown that \( \phi(z) \) is linear. Then evidently \( \alpha \neq 0 \), since \( \phi(z) \neq 0 \) and our argument will show that \( \alpha_n = 0 \) for \( n > 1 \). Take \( s \), an integer, so great that \( z_i + h/M^s \) is in \( \Gamma_1 \). Then, from (5),

\[
(7) \quad g\left[ \phi\left( z_1 + \frac{h}{M^s} \right) \right] = g\left( z_1 + \frac{h}{M^s} \right).
\]

Now by (3) and (7) with \( U_s \) denoting the \( s \)th iterate of \( U \), we have

\[
g\left[ M^s \phi\left( z_1 + \frac{h}{M^s} \right) \right] = U_s \left\{ g\left[ \phi\left( z_1 + \frac{h}{M^s} \right) \right] \right\} = U_s \left[ g\left( z_1 + \frac{h}{M^s} \right) \right],
\]

or

\[
(8) \quad g\left[ M^s \phi\left( z_1 + \frac{h}{M^s} \right) \right] = g\left[ M^s \left( z_1 + \frac{h}{M^s} \right) \right] = g\left[ M^s z_1 + h \right].
\]

Using (4) and then (3), we find

\[
g\left[ M^s z_1 + h \right] = V\left[ g(M^s z_1) \right] = VU_s\left[ g(z_1) \right].
\]

But, since \( g(z_1) = 0 \), \( VU_s\left[ g(z_1) \right] = VU_s(0) = V(0) \). Thus (8) becomes

\[
(9) \quad g\left[ M^s \phi\left( z_1 + \frac{h}{M^s} \right) \right] = V(0).
\]

We have by (6)

\[
M^s \phi\left( z_1 + \frac{h}{M^s} \right) = \alpha_1 h + \alpha_2 \frac{h^2}{M^s} + \cdots.
\]

Then by (9)

\[
(10) \quad g\left( \alpha_1 h + \alpha_2 \frac{h^2}{M^s} + \cdots \right) = V(0)
\]

for sufficiently large \( s \).

It follows from (10) that \( \alpha_n = 0 \) for \( n > 1 \), otherwise allowing \( s \) to pass through all sufficiently large values we would have an infinite number of distinct points accumulating at the point \( \alpha_1 h \) for each of which \( g(z) \) assumes the value \( V(0) \), and \( \alpha_1 h \) would be an essential singularity of \( g(z) \).

Thus (5) becomes

\[
(11) \quad g\left[ \alpha_1 (z - z_1) \right] = g(z).
\]

This equation, proved for \( z \) in \( \Gamma_1 \), must hold for the whole plane since the functions in it are analytic.

This is the desired relation.

3. Proof of periodicity. It now follows that \( g(z) \) is periodic. For in (11),
if \( \alpha_1 = 1, \) \(-z_1\) is a period of \( g(z) \). If \( \alpha_1 \neq 1 \), then by using (4) and (11) we find
\[
g(z + h) = V[g[\alpha_1(z - z_1)] = g[\alpha_1(z - z_1) + h].
\]
But from (11) with \( z \) replaced by \( z + h \), we get
\[
g(z + h) = g[\alpha_1(z - z_1) + \alpha_1 h].
\]
Then
\[
g[\alpha_1(z - z_1) + h] = g[\alpha_1(z - z_1) + \alpha_1 h],
\]
and \( h(\alpha_1 - 1) \neq 0 \) is a period of \( g(z) \).

The function \( f(z) \) which is \( g(z) + f(0) \) or \( g[z - h/(1 - mr)] + f[h/(1 - mr)] \) is, of course, also periodic. This is the result stated in the introduction.

4. The values of \( h \). Having established the periodicity of our functions, our next step is to determine the values of \( h \) for which addition theorems are possible if a multiplication theorem also holds. Evidently \( h \) may be a period and then \( S(z) = z \). But there are other values besides periods and it is these in which we are particularly interested.

Use will be made of the fact that together with \( f(z) \), \( F(z) = \frac{4/(z) + 73}{C/(z) + 17} \) will also have a multiplication and addition theorem. In what follows, we will use \( F(z) \) to denote such a linearly transformed \( f(z) \).

As stated in the introduction, the periodic functions with multiplication theorems are divided into six categories and each of these will be considered in turn.

Case (a) : \( f(z) \) a linear function of \( e^{az} \). Here \( \alpha \) is arbitrary and the multiplier \( m \) must be an integer.

Replacing \( f(z) \) by a suitable linear function of itself, we may suppose that
\[
F(z) = e^{az}.
\]
Then since
\[
F(z + h) = e^{ah} F(z),
\]
\( F(z) \) and therefore \( f(z) \) has an addition theorem for every value of \( h \).

Case (b) : \( f(z) \) a linear function of \( \cos(az + \beta) \). Here \( \beta = k\pi/(m - 1) \), \( k \) an integer, while \( \alpha \) is arbitrary. The multiplier \( m \) must be an integer.

We may suppose
\[
F(z) = \cos(az + \beta),
\]
and have to determine for which values of \( h \), \( \cos(az + ah + \beta) \) is a rational function of \( \cos(az + \beta) \). Replacing \( az + \beta \) by \( z \) and \( ah \) by \( \gamma \), we must have
\[
\cos(z + \gamma) = S[\cos z],
\]
with \( S \) rational.

Let \( z_1 \) be any value of \( z \) and determine another value of \( z \), \( z_2 \) so as to satisfy the congruence
\( z_1 + z_2 \equiv 0 \pmod{2\pi} \).

Then, since \( \cos z_1 = \cos z_2 \),

\[ \cos (z_1 + \gamma) = \cos (z_2 + \gamma). \]

We have the following possibilities; either

\( z_1 + \gamma \equiv z_2 + \gamma \pmod{2\pi}, \)

or

\( z_1 + \gamma + z_2 + \gamma \equiv 0 \pmod{2\pi}, \)

or perhaps both of these congruences hold.

In any event, if (13) holds, then adding (12) to it we get

\[ 2z_1 \equiv 0 \pmod{2\pi}; \]

that is, \( z_1 = k\pi, \) \( k \) an integer. If \( z_1 \), which was arbitrary, is given any value not of the form \( k\pi \), only (13') can hold. We may suppose this done. Then the subtraction of (12) from (13') gives

\[ 2\gamma \equiv 0 \pmod{2\pi}; \]

that is, \( \gamma = k\pi, \) \( k \) an integer. Then \( h = k\pi/\alpha \).

Since \( 2\pi/\alpha \) is a period of \( \cos(az + \beta) \), the values of \( h \), other than the periods, must be the half-periods.

That this necessary condition for \( h \) is also sufficient is obvious. Thus, for \( h \) the half-period \( \pi/\alpha \), to which all other half-periods are congruent modulo \( 2\pi/\alpha \), we have

\[ \cos \left[ a(z + \pi/\alpha) + \beta \right] = - \cos (az + \beta). \]

Case (c): \( f(z) \) a linear function of \( \varphi(z + \beta) \). The multiplier \( m \) must satisfy the congruences

\[ 2m_1 \equiv 0, \quad 2m_3 \equiv 0 \pmod{2\omega_1, 2\omega_3}, \]

and the constant \( \beta \) is given by the equation

\[ \beta = \frac{l\omega_1 + k\omega_3}{m - 1}, \]

where \( l \) and \( k \) are integers, and \( 2\omega_1, 2\omega_3 \), a pair of primitive periods of \( \varphi(z) \).

By repeating the procedure of the preceding case, except that all congruences are taken modulis \( 2\omega_1, 2\omega_3 \), we find that

\[ h = l\omega_1 + k\omega_3, \]

where \( l \) and \( k \) are any integers.

Again we get periods and half-periods for \( h \).
To set up the addition theorem for the half-periods, it suffices to consider \( \omega_1, \omega_2 \) and \( \omega_3 = -\omega_1 - \omega_2 \), for any other half-period will be congruent to one of these modulis \( 2\omega_1, 2\omega_3 \).

The following formula is well known in the theory of elliptic functions (§)

\[
\varphi(z + \omega_a) = e_a + \frac{(e_b - e_a)(e_c - e_a)}{\varphi(z) - e_a},
\]

where \( \varphi(\omega_a) = e_a \) and \( a, b, c \) is any permutation of 1, 2, 3.

This is the required addition theorem if \( z \) is replaced by \( z + \beta \).

Case (d): \( f(z) \) a linear function of \( \varphi^2(z + \beta) \). Here \( \varphi(z) \) is lemniscatic, \( m = p + qi \), with \( p \) and \( q \) integral, and

\[
\beta = \frac{2l\omega_1 + 2k\omega_3}{(m - 1)(1 - i)},
\]

where \( l \) and \( k \) are any integers.

We may replace \( z + \beta \) by \( z \) and we shall suppose this done here and in the following cases. The values of \( h \) do not depend on those of \( \beta \).

Taking \( F(z) = \varphi^2(z) \), we must have \( \varphi^2(z + h) = S[\varphi^2(z)] \).

Since \( \varphi^2(iu) = \varphi^2(u) \) in this case, the fourth order function \( \varphi^2(u) \) will take on the same value at points congruent to \( i^s u \) (\( s = 0, 1, 2, 3 \)) modulis \( 2\omega_1, 2\omega_3 \), and only at these points if the \( i^s u \) are all incongruent.

If any two of \( i^s u \) are congruent, it follows that they are all congruent, and \( \varphi^2(u) \) will have a point of the fourth order at any one of them; so that again \( \varphi^2(u) \) will take the same value only at the points \( i^s u \).

Choose an arbitrary \( z_1 \), and determine \( z_2 \) so as to satisfy the congruence

(15) \( z_2 \equiv iz_1 \pmod{2\omega_1, 2\omega_3} \).

Then since \( \varphi^2(z_2) = \varphi^2(z_1) \), it follows that

\[
\varphi^2(z_2 + h) = \varphi^2(z_1 + h),
\]

and

\[
z_2 + h \equiv i^s(z_1 + h) \pmod{2\omega_1, 2\omega_3},
\]

where \( s \) may be one of 0, 1, 2, 3, or perhaps any of them.

Subtracting (15) from this congruence gives

\[
h(1 - i^s) \equiv (i^s - i)z_1 \pmod{2\omega_1, 2\omega_3};
\]

and unless \( s = 1 \), this will determine \( z_1 \) as belonging to a particular residue class modulis \( 2\omega_1/(i^s - i), 2\omega_3/(i^s - i) \), whereas \( z_1 \) was chosen without restriction.

If \( s = 1 \), we find that

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where \( l \) and \( k \) are any integers.

Simplifying this expression, we get \( h = (l - k) \omega_1 + (l + k) \omega_2 \). Thus the values of \( h \) must be periods or suitable half-periods; namely, since \( l - k \) and \( l + k \) are both of the same parity, those half-periods which are congruent to \( \omega_2 \) modulis \( 2 \omega_1, 2 \omega_2 \).

If we note that
\[ i \omega_2 \equiv i(- \omega_1 - \omega_2) \equiv - \omega_3 + \omega_1 \equiv \omega_2 \ (\text{mod } 2 \omega_1, 2 \omega_2), \]
and that \( \varphi(iu) = - \varphi(u) \) for this case, then
\[ e_2 = \varphi(\omega_2) = \varphi(i \omega_2) = - \varphi(\omega_2) = - e_2, \]
and \( e_2 = 0 \).

Substituting \( a = 2 \) in (14) and squaring we get the addition formula
\[ \varphi^2(z + \omega_2) = \frac{e_1 e_3}{\varphi^2(z)}. \]

Case (e): \( f(z) \) a linear function of \( \varphi'(z + \beta) \). Here \( \varphi'(z) \) is equianharmonic, \( m = p + q e^{2 \pi i/3} \), with \( p \) and \( q \) integral and
\[ (m - 1)(1 - e^{2 \pi i/3}) \beta \equiv 0 \ (\text{mod } 2 \omega_1, 2 \omega_2). \]

This case and the next one may be treated in the same way as case (d) and we simply state the results.

Since \( \varphi'(e^{2 \pi i/3} u) = \varphi'(u) \) in this case, \( e^{2 \pi i/3} \) takes the place of \( i \) in the preceding case and thus
\[ h = \frac{2 \omega_1 + 2 \omega_3}{1 - e^{2 \pi i/3}}, \]
where \( l \) and \( k \) are any integers.

Rationalizing the denominator of this fraction, we get
\[ h = \frac{2 \omega_1(2l - k) + 2 \omega_3(l + k)}{3}. \]

Thus, apart from a period, \( h \) must be a suitable third of a period.

To get our addition theorem in the latter instance, we note that taken modulis \( 2 \omega_1, 2 \omega_3 \) all these thirds of periods are congruent to \( \gamma = (2 \omega_1 - 2 \omega_3)/3 \) or \(- \gamma \). It suffices to consider \( \gamma \). Since \( \varphi(e^{2 \pi i/3} u) = e^{2 \pi i/3} \varphi(u) \) and
\[ e^{2 \pi i/3} \gamma \equiv \gamma \ (\text{mod } 2 \omega_1, 2 \omega_3), \]
we have
\[ \psi(\gamma) = \psi(e^{2\pi i/3}\gamma) = e^{2\pi i/3}\psi(\gamma) \]

and \( \psi(\gamma) = 0. \)

If we let \( u = \gamma \) in the addition formula for \( \psi(z+u) \)\(^{(s)} \) and differentiate the resulting expression with respect to \( z \), we find

\[ \psi'(z + \gamma) = \frac{\psi'(z)\psi'(\gamma) - 3[\psi'(\gamma)]^2}{\psi'(z) + \psi'(\gamma)} \]

Case (f): \( \psi(z) \) a linear function of \( \psi(z+\beta) \). Here \( \psi(z) \) is equianharmonic, the multiplier \( m = p + q e^{2\pi i/3} \), with \( p \) and \( q \) integral, and

\[ (m - 1)(1 - e^{\pi i/3})\beta \equiv 0 \pmod{2\omega_1, 2\omega_3}. \]

Since \( \psi^2(e^{\pi i/3}u) = \psi^2(u) \), we may use \( e^{\pi i/3} \) in the same manner as \( e^{2\pi i/3} \) and \( i \) in the preceding two cases. The result is that

\[ h = \frac{2\omega_1 + 2k\omega_3}{1 - e^{\pi i/3}}, \]

with \( l \) and \( k \) any integers.

This simplifies to

\[ h = (1 + e^{2\pi i/3})(2l\omega_1 + 2k\omega_3). \]

Then the only values of \( h \) in this case are periods of \( f(z) \).

5. The case of \( |m| \leq 1, m \) not a rational root of unity. We first show that, in this case, \( f(z) \) must be rational. For otherwise, suppose that \( f(z) \) is transcendental.

Then again considering the sequence

\[ 0, \frac{h}{1 - m}, \frac{h}{1 - m^2}, \ldots, \frac{h}{1 - m^r}, \ldots, \]

whose points are all distinct, we can find a point in it which is not an exceptional point of \( f(z) \). If \( f(z) \) is analytic at this point, the transformations of \$1$ will then give us a \( g(z) \) having properties (a) and (c) and satisfying relations of type (3) and (4).

If \( f(z) \) has a pole at the non-exceptional point, we take \( g(z) = 1/f(z) \) or \( g(z) = 1/f[z+h/(1-m^r)] \) according as our point is 0 or \( h/(1-m^r) \), and again get a \( g(z) \) fulfilling these same four conditions. In either case, \( |M| \leq 1, M \) not a rational root of unity.

If \( z_1 \neq 0 \) is one of the zeros of \( g(z) \), then the relation

\[ g(Mz) = U[g(z)] \]

shows that \( Mz_1 \) is a zero of \( g(z) \) for every positive integral \( s \). If \( |M| < 1 \), these

\(^{(s)}\) See, for example, Tannery and Molk, loc. cit. p. 172.
zeros will converge to the origin; if \(|M| = 1\), \(M\) not a rational root of unity, they will be everywhere dense on the circle \(|z| = |z_1|\). Both of these results are impossible for a nonconstant meromorphic function.

Therefore, let \(f(z)\) be rational\(^7\). Then \(R\) and \(S\) are both linear. Replacing \(f(z)\) by a suitable linear function of itself, which will also be rational, we may obtain the addition theorem in one of the forms:

\[
F(z + h) = aF(z), \quad a \neq 1
\]
or

\[
F(z + h) = F(z) + b.
\]

The first of these is impossible for a nonconstant rational \(F(z)\), since the existence of a zero or a pole at \(z_1\) would imply the existence of an infinite number of zeros or poles at the points \(z_1 + nh\), \(n\) any integer.

The second gives

\[
F(z) = \frac{bz}{h} + \Pi(z),
\]

where \(\Pi(z)\) is a meromorphic periodic function of period \(h\).

Here \(\Pi(z)\) must be rational and therefore constant. Then \(F(z)\) and also \(f(z)\) is a linear function of \(z\).

6. **The case** \(m\) **a primitive** \(n\)th **root of unity.** If \(m\) is a primitive \(n\)th root of unity,

\[
f(m^nz) = f(z) = R_n[f(z)],
\]

and the rational function \(R(z)\) must be a periodic linear function of \(z\) with period \(n\) or a divisor of \(n\).

For every positive integral \(p\) and \(r\), we have

\[
f(m^rz + ph) = S_p[f(m^rz)] = S_pR_r[f(z)].
\]

Then, if \(r\) is not a multiple of \(n\), the function \(g(z) = f(z + ph/(1-m^r))\) obeys the relation

\[
g(m^rz) = S_pR_r[g(z)].
\]

This shows that \(S_pR_r(z)\) is a linear periodic function of \(z\) with period \(n\) or a divisor of \(n\) for every positive integral \(p\) and \(r\), \(r\) not a multiple of \(n\).

The function \(S(z)\) itself then must be linear.

Those cases with \(R(z)\) of period \(d\) < \(n\) are easily handled. For with \(r = d\) and \(p = 1\), \(S_pR_r(z)\) becomes \(S(z)\) which must be periodic. Then \(f(z)\) is periodic\(^8\).

\(^7\) The following argument holds for any value of \(m\). We thus complete our discussion of Case (A) for \(f(z)\) rational.

\(^8\) We refer to the paper of Ritt, loc. cit., for the enumeration of the functions in this case.
In a similar manner, it may be shown that \( f(z) \) is periodic if the period of \( S_pR(z) \) is less than \( n \).

These cases disposed of, we may assume both \( R(z) \) and \( S_pR(z) \) to be of period \( n \) \((p=1, 2, \cdots )\).

Then, replacing \( f(z) \) by a suitable linear function of itself, we have two sub-cases according as

\[
F(z + h) = aF(z), \quad a \neq 1
\]
or

\[
F(z + h) = F(z) + b.
\]

Subcase 1: \( S(z) = az, \ a \neq 1 \). Suppose first that \( m = -1 \), that is, \( n = 2 \). Then \( R(z) \), being of period 2, is of the form \((Az+B)/(Cz-A)\). The function \( SR(z) = a(Az+B)/(Cz-A) \) must also be of period 2.

If \( A \neq 0 \), \( a \) must be \(-1\) and \( F(z) \) is periodic.

If \( A = 0 \), \( a \) is arbitrary. Solving the addition relation for \( F(z) \) by elementary means, we get

\[
F(z) = e^{a\pi\Pi(z)},
\]

where \( \alpha = \log a/h \) and \( \Pi(z) \) is a meromorphic function with period \( h \).

In order for this function, \( F(z) \), to satisfy the multiplication theorem, we find that \( \Pi(z) \), a periodic function, must also have a multiplication theorem, namely

\[
\Pi(-z) = \frac{B}{C\Pi(z)}.
\]

Thus \( \Pi(z) \) and also \( F(z) \) are characterized.

Let us now consider the values of \( m \) for which \( n > 2 \). Then with \( R(z) = (Az+B)/(Cz+D) \), we have

\[
S_pR(z) = a^p \cdot \frac{Az+B}{Cz+D};
\]

and this function must be of period \( n \) for every positive integral \( p \).

The multiplier, \( K \), of the periodic linear transformation \( S_pR(z) \) is a primitive \( n \)th root of unity which we denote by \( e_p \), the subscript indicating its dependence on \( p \). Using the known relation between the multiplier and the coefficients of the transformation\(^*\), we find that \( K \) is algebraic in \( a^p \), and we may write

\[
e_p = W(a^p) \quad (p = 1, 2, \cdots)
\]

where \( W(z) \) is an algebraic function of \( z \).

\(^*\) See, for example, Ford, *Automorphic functions*, p. 16.
There being only a finite number of primitive \( n \)th roots of unity, at least one of them will occur in \((16)\) as \( \epsilon_p \) an infinite number of times as \( p \) grows large. This shows that \( a \) must be a rational root of unity since otherwise the algebraic function \( W(z) \) would take the same value, \( \epsilon_p \), at an infinite number of distinct points \( z = a^p \). Thus \( S(z) \) is periodic. We conclude that if \( n > 2, f(z) \) must be periodic.

Subcase 2: \( S(z) = z + b \). If \( b = 0, f(z) \) is periodic. Suppose \( b \neq 0 \). Taking \( R(z) = (Az + B)/(Cz + D) \) with \( AD - BC = 1 \), the linear transformation \( S_p R(z) = [(A + pbC)z + (B + pbD)]/(Cz + D), \ p = 1, 2, \ldots \), is periodic and hence elliptic. Using the classic condition that a linear transformation of determinant unity be elliptic\(^{10}\), we would have \(|A + pbC + D| < 2\) for every positive integral \( p \) and this implies that \( C = 0 \). Then \( A = \epsilon D \) where \( \epsilon \) is a primitive \( n \)th root of unity. By taking a suitable linear function of \( F(z) \) we may suppose our addition and multiplication theorem in the form

\[
F(mz) = \epsilon F(z), \tag{17}
\]

\[
F(z + h) = F(z) + b. \tag{18}
\]

These show that \( F'(z) \) is a periodic function with a multiplication theorem. If \( F'(z) \) is constant, \( F(z) \) is a linear function of \( z \). If \( F'(z) \) is not constant; then, as stated in the introduction, \( m \) is either \(-1\) or a primitive third, fourth, or sixth root of unity.

Taking the solution of \((18)\) as

\[
F(z) = bz/h + \Pi(z),
\]

where \( \Pi(z) \) is a meromorphic function of period \( h \), we must determine \( \Pi(z) \) so that \((17)\) is also satisfied.

If \( m = -1, \Pi(z) \) must be an odd function. Then, if \( \Pi(z) \) is simply periodic, it is the product of \( \sin(2\pi nz/h) \) by a meromorphic function of \( \cos(2\pi nz/h) \), \( n \) any integer. If \( \Pi(z) \) is elliptic, it is the product of \( \phi'(z) \) by a rational function of \( \phi(z) \).

If \( m \) is a primitive fourth root of unity, we may suppose that \( m = i \). Then if \( \epsilon = i \),

\[
\Pi(iz) = i\Pi(z),
\]

and \( \Pi(z) \) is the product of \( \phi'(z) \) by a rational function of \( \phi^2(z) \) with \( \phi(z) \) lemniscatic.

If \( \epsilon = -i \), we proceed in a different manner. Let \( \xi(z) \) be the Weierstrass zeta-function corresponding to a lemniscatic \( \phi(z) \) having \( h \) as a period.

The following relations hold for \( \xi(z) \).

\[
\xi(iz) = -i\xi(z), \quad \xi(z + h) = \xi(z) + \kappa_1, \quad \xi(z + ih) = \xi(z) + \kappa_2.
\]

Here \( \kappa_1 \) and \( \kappa_2 \) are constants each different from zero. To show this, we have

\footnote{\(^{10}\) Ford, loc. cit. p. 23.}
\[ \xi(iz) + \kappa_2 = \xi(iz + ih) = -i\xi(z + h) = -i\xi(z) - i\kappa_1 = \xi(iz) - i\kappa_1. \]

Thus \( \kappa_2 = -i\kappa_1 \), and the vanishing of either would imply the same for the other. But then \( \xi(z) \) would be elliptic, a contradiction.

If we let
\[
F(z) = \frac{b\xi(z)}{\kappa_1} + \Pi(z),
\]
then (17) and (18) will be satisfied only if \( \Pi(z) \) is of period \( h \) and obeys the relation
\[
\Pi(iz) = -i\Pi(z).
\]

Then \( \Pi(z) \) is a product of \( \varphi'''(z) \) by a rational function of \( \varphi^2(z) \) with \( \varphi(z) \) lemniscatic.

For the remaining cases the results are similar.

If \( m \) is a primitive cube root of unity,
\[
F(z) = b\xi/z + \varphi(z)T[\varphi'(z)],
\]
or
\[
F(z) = b\xi(z)/\kappa_1 + \varphi''(z)T[\varphi'(z)],
\]
where \( \varphi(z) \) is equianharmonic and \( T(z) \) is a rational function. If \( m \) is a primitive sixth root of unity, we replace \( \varphi(z), \varphi''(z) \) and \( \varphi'(z) \) respectively by \( \varphi''''(z), \varphi^5(z) \), and \( \varphi^3(z) \). In each case it is understood that any linear function of \( F(z) \) also has both a multiplication and addition theorem.

Columbia University,
New York, N. Y.