

TÖPLERIAN (L^2)-BASES

BY

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The following considerations deal with periodic functions $\phi(t)$ having the property that either the sequence $\phi(t), \phi(2t), \dots, \phi(nt), \dots$ or the sequence which results if the constant 1 is adjoined to these functions forms a basis of the (L^2)-space over a t -interval. By this is meant that $\phi(t)$ itself be of class (L^2) and that, corresponding to every $\epsilon > 0$ and to every function $f(t)$ of class (L^2) on the interval, the (L^2)-distance from $f(t)$ to

$$\sum_{n=1}^K C_n \phi(nt) \quad \left(\text{or } C_0 + \sum_{n=1}^K C_n \phi(nt) \right)$$

be less than ϵ , if K and K constants, C_n , are suitably chosen.

Since every periodic $\phi(t)$ can be written as the sum of two periodic functions one of which is odd and the other even, there will be no loss of generality in assuming that $\phi(t)$ is either odd or even. In addition, it will be clear from the proofs that the treatment of the case of an even $\phi(t)$ does not differ from that of an odd $\phi(t)$. For this reason, only the latter case will be considered. Then the adjunction of the function 1 becomes unnecessary.

The problem goes back to the formal considerations of Töpler [6]⁽¹⁾, a physicist who loc. cit. was concerned with generalized forms of wave analysis.

(Added in proof (23 January 1948). In the paper *A class of sequences of functions*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 475–518, some problems dealt with below are considered by D. G. Bourgin along the lines used in [9].)

1. If ϕ_1, ϕ_2, \dots is a sequence of numbers the first of which is not 0, then a unique sequence $^*\phi_1, ^*\phi_2, \dots$ is defined by the conditions

$$(1) \quad ^*\phi_1 \phi_1 = 1, \quad \sum_{j,k=n} ^*\phi_j \phi_k = 0, \quad \text{where } n = 2, 3, \dots$$

In fact, if $n=1$ is included, the first n of the equations (1) represent for $^*\phi_1, \dots, ^*\phi_n$ a recursive linear system, the determinant of which is n th power of $\phi_1 \neq 0$. Clearly, (1) is formally equivalent to the generating identity

$$(2) \quad ^*\Phi(s) \Phi(s) = 1,$$

where

$$(3) \quad \Phi(s) = \sum_{n=1}^{\infty} \phi_n / n^s$$

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(1) Numbers in brackets refer to the references cited at the end of the paper.

and

$$(4) \quad {}^* \Phi(s) = \sum_{n=1}^{\infty} {}^* \phi_n / n^s$$

(these Dirichlet series need not have half-planes of convergence).

Some of the facts concerning the (L^2) -bases in question can be collected as follows:

Let $\phi(t)$ be an odd function of period 2π and of class (L^2) , that is, let

$$(5) \quad \sum_{m=1}^{\infty} |\phi_m|^2 < \infty,$$

where

$$(6) \quad \phi(t) \sim \sum_{m=1}^{\infty} \phi_m \sin mt.$$

Then, in order that the sequence

$$(7) \quad \phi(t), \phi(2t), \dots, \phi(nt), \dots$$

be an (L^2) -basis of the (L^2) -class on the t -interval $(0, \pi)$,

(i) it is necessary and sufficient that the homogeneous equations

$$(8) \quad \sum_{m=1}^{\infty} \phi_m x_{nm} = 0 \quad (n = 1, 2, \dots)$$

do not possess within Hilbert's limitation

$$(9) \quad \sum_{m=1}^{\infty} |x_m|^2 < \infty$$

any solution distinct from

$$(10) \quad x_1 = x_2 = \dots = 0,$$

the trivial solution;

(ii) it is necessary that

$$(11) \quad \phi_1 \neq 0,$$

whereas

(iii) it is sufficient that $\phi_1 \neq 0$ and

$$(12) \quad \sum_{m=1}^{\infty} \sum_{j,k=m}^{\infty} |{}^* \phi_j \phi_k|^2 < \infty$$

(and (12) implies both

$$(13) \quad \sum_{n=1}^{\infty} |*\phi_n|^2 < \infty$$

and (5) itself); in particular,

(iv) it is sufficient that $\phi_1 \neq 0$ and, for some $\epsilon > 0$,

$$(14) \quad \phi_n = O(n^{-1/2-\epsilon}) \quad \text{and} \quad *\phi_n = O(n^{-1/2-\epsilon})$$

as $n \rightarrow \infty$; what is more,

(v) it is sufficient that $\phi_1 \neq 0$ and

$$(15) \quad \sum_{m=1}^{\infty} \left| \sum_{j,k=m}^{j>N} *\phi_j \phi_k \right|^2 \rightarrow 0$$

as $N \rightarrow \infty$; but

(vi) it is insufficient that just (5) and (11) or, for that matter,

$$(16) \quad \sum_{m=1}^{\infty} |\phi_m| < \infty$$

and (11) be satisfied; in fact, the choice $\phi_1 = 1, \phi_2 = -2, \phi_3 = \phi_4 = \dots = 0$ in (6), that is, the trigonometric polynomial

$$(17) \quad \phi(t) = \sin t - 2 \sin 2t,$$

supplies a counter-example.

The matrix of the linear system (8) happens to be precisely the D -matrix which Toeplitz [7] has associated with the Dirichlet series (3).

Proof of (i). The Fischer-Riesz assignment

$$(18) \quad x(t) \sim \sum_{m=1}^{\infty} x_m \sin mt$$

establishes a one-to-one correspondence between all functions $x(t)$ of class (L^2) on $(0, \pi)$ and all sequences x_1, x_2, \dots satisfying (9). Since (6) implies that

$$(19) \quad \phi(nt) \sim \sum_{m=1}^{\infty} \phi_n \sin mnt,$$

it is clear from the polarized form of Parseval's relation that the n th of the equations (8) is satisfied by the coefficients of (18) if and only if the (non-Hermitian) scalar product of $x(t)$ and $\phi(nt)$ vanishes. Consequently, no sequence x_1, x_2, \dots satisfying both (9) and each of the equations will be distinct from the trivial sequence $x_1 = 0, x_2 = 0, \dots$ if and only if the function set (7) has the property that a function $x(t)$ of class (L^2) cannot be orthogonal to each of the functions (7) unless $x(t) = 0$ (almost everywhere). Hence, (i) follows from a well known property of Hilbert's space, namely, from the fact

that a set of functions of class (L^2) forms a basis if and only if no function of class (L^2) which is orthogonal to all functions contained in the set is distinct from the function 0 (almost everywhere).

Proof of (ii). If $\phi_1=0$, then the system (8) is readily seen to have the solution

$$x_1 = 1, \quad x_2 = x_3 = \dots = 0.$$

Hence, (ii) follows from (i)

Proof of (iii). Clearly, (1) defines one of the sequences $\phi_1, \phi_2, \dots, \ast\phi_1, \ast\phi_2, \dots$ in terms of the other if and only if $\phi_1 \neq 0$ and/or $\ast\phi_1 \neq 0$. If (12) is assumed, then, since the finite sum representing the m th term of the series (12) is of the form $|\ast\phi_1\phi_m|^2 + \{ \dots \}$, where $\{ \dots \}$ is a set of non-negative values,

$$\sum_{m=1}^{\infty} |\ast\phi_1\phi_m|^2 < \infty.$$

Hence, (5) follows from $\ast\phi_1 \neq 0$, which is assumption (11) of (ii) (and (13) is now clear from the fact that both (1) and (12) are symmetric in $\phi, \ast\phi$).

In view of (i), the assertion of (iii) is equivalent to the statement that (8), (9) and (12) imply (10). In order to verify this implication, replace n in (8) by nk , where $k=1, 2, \dots$, and multiply the result by $\ast\phi_k$. Then summation with regard to k gives the infinity of equations

$$\sum_{k=1}^{\infty} \ast\phi_k \sum_{m=1}^{\infty} \phi_m x_{mkn} = 0 \quad (n = 1, 2, \dots).$$

However, since (9), (12) and Schwarz's inequality imply that

$$\sum_{m=1}^{\infty} |x_{mn}| \sum_{jk=m} |\ast\phi_j\phi_k| < \infty \quad (n = 1, 2, \dots),$$

these equations can be rearranged into

$$\sum_{m=1}^{\infty} x_{mn} \sum_{jk=m} \ast\phi_j\phi_k = 0 \quad (n = 1, 2, \dots).$$

Hence, (10) follows from (1).

Proof of (iv). If ϵ in (14) is replaced by 2ϵ , and if (14) is assumed, it is clear that

$$\sum_{jk=m} |\ast\phi_j\phi_k| = O(m^{-1/2-2\epsilon}) \sum_{jk=m} 1 = O(m^{-1/2-2\epsilon})d(m),$$

where $d(m)$ denotes the number of the divisors of m . Since, as is well known, $d(m) = O(m^\epsilon)$ holds for every $\epsilon > 0$, this means that the m th term of the series (12) is the square of $O(m^{-1/2-2\epsilon})O(m^\epsilon) = O(m^{-1/2-\epsilon})$. It follows therefore

from $\sum (m^{-1/2-\epsilon})^2 < \infty$ that (iv) is implied by (iii).

Proof of (v). The proof of this refinement of (iii) is similar to that of (iii) itself (and relates to the proof of (iii) itself in about the same way as Mertens' multiplication theorem does to that of Cauchy).

First, if n in (8) is replaced by jn , it follows that the equations

$$\sum_{j=1}^N {}^* \phi_j \sum_{m=1}^{\infty} \phi_m x_{mjn} = 0 \quad (n = 1, 2, \dots)$$

hold for arbitrary values n and N . Since (6) and (9) imply, by Schwarz's inequality, the absolute convergence of the series occurring in these equations, it is allowed to rearrange the latter into

$$\sum_{m=1}^{\infty} x_{mn} \sum_{jk=m}^{i \leq N} {}^* \phi_j \phi_k = 0.$$

On the other hand, from (1),

$$x_n = \sum_{m=1}^{\infty} x_{mn} \sum_{jk=m} {}^* \phi_j \phi_k.$$

Hence, by subtraction,

$$x_n = \sum_{m=1}^{\infty} x_{mn} \sum_{jk=m}^{i > N} {}^* \phi_j \phi_k.$$

Consequently, if ϵ_N denotes the series (15), then, by Schwarz's inequality,

$$|x_n|^2 \leq \epsilon_N \sum_{m=1}^{\infty} |x_m|^2,$$

since $\sum_{m=1}^{\infty} |x_{mn}|^2 \leq \sum_{m=1}^{\infty} |x_m|^2$.

If n is fixed and $N \rightarrow \infty$, then, since the assumption of (v) is that $\epsilon_N \rightarrow \infty$, the last formula line implies that $x_n = 0$. Hence, (v) follows from (i).

Proof of (vi). According to (6), the equations belonging to the function (17) are

$$x_n - 2x_{2n} = 0 \quad (n = 1, 2, \dots)$$

and are therefore satisfied by placing $x_n = 2^{-i}$ or $x_n = 0$ according as n is the i th power of 2 or is not of the form 2^i , where $i = 0, 1, \dots$. Since this sequence x_1, x_2, \dots satisfies (9) but is not the trivial solution, (vi) follows from (i).

2. As a further application of (i), the following characterization of the trigonometric sequence

$$(20) \quad \sin t, \sin 2t, \dots, \sin nt, \dots$$

will now be deduced:

(vii) On the interval $(0, \pi)$, the pure harmonic system (20) is the only complete orthogonal system of the form (7), where $\phi(t)$ is odd, of period 2π and of class (L^2) .

Needless to say, (vii) is meant in the sense that two systems (7), belonging to $\phi(t) = \phi_I(t)$ and $\phi(t) = \phi_{II}(t)$, are considered as identical if

$$(21) \quad \phi_I(t) \doteq \text{const. } \phi_{II}(t) \text{ almost everywhere.}$$

Proof of (vii). If z^* denotes the complex conjugate of z , the orthogonality of the sequence (7) means that

$$\int_0^\pi \phi(jt)\phi^*(kt)dt = 0 \quad \text{if } j \neq k.$$

In view of (19) and of the polarized form of Parseval's relation, this is equivalent to

$$\sum_{m=1}^{\infty} \phi_{km/(j,k)} \phi_{jm/(j,k)}^* = 0 \quad \text{if } j \neq k,$$

where the subscripts are the ratios of the products km , jm to the greatest common divisor, (j, k) , of j and k . In particular, if j is a multiple of k , say $j = nk$, then the proviso $j \neq k$ becomes satisfied when $n \neq 1$. Accordingly, since $(j, k) = (nk, k) = k$,

$$\sum_{m=1}^{\infty} \phi_m \phi_{mn}^* = 0 \quad \text{if } n > 1.$$

Since an orthogonal system is an (L^2) -basis if and only if it is complete, (i) is applicable under the assumptions of (vii). In particular, $\phi_1 \neq 0$, by (ii). But $\phi_1 \neq 0$ and the last formula line imply that the system (8) is satisfied by

$$x_1 = - \sum_{m=2}^{\infty} \phi_m \phi_m^* / \phi_1, \quad x_2 = \phi_2^*, \quad x_3 = \phi_3^*, \dots$$

According to (5), this solution of (8) fulfils (9). Hence, it must be the trivial solution, by (i). In other words, every x_n in the last formula line is 0. Since this means that $\phi_2^* = \phi_3^* = \dots = 0$, hence $\phi_2 = \phi_3 = \dots = 0$, the expression (6) reduces to

$$\phi(t) \sim \phi_1 \sin t.$$

In view of the completeness of (20) on $(0, \pi)$, this means that (21) holds for

$$\phi_I(t) = \phi(t), \quad \phi_{II}(t) = \sin t, \quad \text{and} \quad \text{const.} = \phi_1.$$

This proves (vii).

If the system (7) is not an (L^2) -basis on $(0, \pi)$, it can be an orthogonal

system distinct from (20). If it is an orthogonal system on $(0, \pi)$, then the implication

$$(22) \quad \left\{ \sum_{j=1}^i C_j \phi(jt) = 0 \text{ p.p.} \right\} \rightarrow \{C_1 = 0, \dots, C_i = 0\}, \text{ unless } \phi(t) = 0 \text{ p.p.,}$$

is trivial for every $i < \infty$, even though the system (7) need not be complete, that is, an (L^2) -basis on $(0, \pi)$. On the other hand, (22) is not evident if the system (7) is an (L^2) -basis but not orthogonal. But it will now be shown that (22) is then true. As a matter of fact, (22) is true even if (7) fails to be an (L^2) -basis. Still more is contained in the following assertion:

(viii) *If an odd function $\phi(t)$, of class (L^2) and of period 2π , is not 0 almost everywhere, then, whether the corresponding sequence (7) is or is not an (L^2) -basis on $(0, \pi)$, none of the functions (7) is a limit, in the mean (L^2) on $(0, \pi)$, of linear combinations of all the other functions (7).*

This implies, of course, that the functions (7) are linearly independent (L^2) on $(0, \pi)$, which means that (22) is true on $(0, \pi)$.

Proof of (viii). Let h be the index of the first non-vanishing coefficient of (6), that is, let

$$(23) \quad \phi(t) \sim \sum_{m=h}^{\infty} \phi_m \sin mt, \quad \text{where } \phi_h \neq 0.$$

This h exists, since $\phi(t)$ is not 0 almost everywhere.

Let i be a fixed positive integer, denoting the index of that of the functions (7) for which the impossibility of the (L^2) -approximation by linear combinations of the functions $\phi(nt)$, where $n \neq i$, should be proved. If $i = 1$, the assertion of (viii) is true. In fact, $\phi(it)$ then becomes the function (23), and it is clear from (23) that $\sin ht$ is orthogonal to each of the functions $\phi(2t)$, $\phi(3t)$, \dots but is not orthogonal to $\phi(t) = \phi(it)$. Hence, if $i = 1$, the assertion of (viii) follows from standard properties of Hilbert's space. It will therefore be assumed that $i > 1$.

With reference to any set of $i - 1$ data C_1, \dots, C_{i-1} (a set which, because of $i > 1$, is not vacuous), consider the $i - 1$ linear equations

$$(24) \quad \sum_{m < i/n} \phi_{hm} y_{nm} = C_n, \quad n = 1, \dots, i - 1,$$

for the $i - 1$ unknowns y_1, \dots, y_{i-1} . The fixed divisor, h , of the subscript of ϕ in (24) is the integer defined by (23). The number of the summation values m occurring in the n th of the $i - 1$ equations (24) is $[i/n]$ or $[i/n] - 1$ according as $[i/n]$ is less than or equal to i/n , that is, according as i is not or is a multiple of n . The matrix of (24) is triangular and all of its diagonal elements are $\phi_h \neq 0$. Hence, (24) has a unique solution for arbitrary C_n .

Choose C_n to be $-\phi_{ih/n}$ or 0 according as the quotient ih/n is or is not

an integer, and denote by y_1, \dots, y_{i-1} the values which are then defined by (24); so that

$$\sum_{m < i/n} \phi_{hm} y_{nm} + \phi_{hi/n} = 0, \quad n = 1, \dots, i - 1,$$

if the last ϕ denotes 0 when its subscript is not an integer. In terms of this set of $i - 1$ values y_j , define an infinite sequence x_1, x_2, \dots as follows: x_k is 0 unless k is a multiple of h , whereas in the remaining case the complex conjugate of x_k is $y_{k/h}$, 1 or 0 according as k is less than, equal to or greater than the integer ih . Then (18) defines a function $x(t)$ of class (L^2) ; in fact, (18) becomes a trigonometric polynomial. If it is verified that the (Hermitian) scalar product of this trigonometric polynomial and of the n th of the functions (7) is 0 or distinct from 0 according as $n \neq i$ or $n = i$, then the assertion of (viii) for the i th of the functions (7) follows for the same reason as it did in the case $i = 1$ (the latter has been considered before (24) and is excluded now).

If the polarized form of Parseval's relations is applied to (19) and to the complex conjugate of the trigonometric polynomial (18) in the same way as at the beginning of the proof of (vii), it is seen from the definition of x_1, x_2, \dots in terms of the $i - 1$ values y_j assigned by the last formula line that the scalar product of $x(t)$ and $\phi(nt)$ is a sum of 0's (hence 0) or the expression on the left of the last formula line (hence 0) according as $n > i$ or $n < i$, and that it has the value ϕ_h if $n = i$. But $\phi_h \neq 0$, by (23). Hence, $x(t)$ and $\phi(nt)$ are orthogonal or not according as $n \neq i$ or $n = i$. This proves (viii).

3. In (i)-(vi), the sequence (7) is meant to be an (L^2) -basis on $(0, \pi)$ if, corresponding to every $x(t)$ of class (L^2) on $(0, \pi)$, it is possible to find a $K = K(\epsilon)$ and $K(\epsilon)$ constants $C_n(\epsilon)$ satisfying

$$(25) \quad \int_0^\pi \left| x(t) - \sum_{n=1}^{K(\epsilon)} C_n(\epsilon) \phi(nt) \right|^2 dt < \epsilon,$$

when $\epsilon > 0$ is given arbitrarily. In contrast to this definition of a mere (L^2) -basis (7) on $(0, \pi)$, let (7) be called a *Fourier (L^2) -basis* on $(0, \pi)$ if, corresponding to every $x(t)$ of class (L^2) on $(0, \pi)$, the $K(\epsilon)$ constants $C_n(\epsilon)$ occurring in (25) can be chosen independent of ϵ . In other words, (7) will be called a *Fourier (L^2) -basis* on $(0, \pi)$ if there belongs to every $x(t)$ of class (L^2) on $(0, \pi)$ a sequence of constants c_n satisfying

$$(26) \quad \int_0^\pi \left| x(t) - \sum_{n=1}^k c_n \phi(nt) \right|^2 dt \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where every c_n is independent of k .

The difference between the notion of an (L^2) -basis (7) and the more restricted notion of a *Fourier (L^2) -basis* (7) (which, in view of (vii), are beforehand identical only in the classical case) has been emphasized in [10, p. 764].

Due to the criteria (i)–(vi) for mere (L^2)-bases, it will now be possible to study this anomaly in some detail, if use is made of the criterion proved in [10] for Fourier (L^2)-basis. The latter criterion becomes equivalent to (xiv) below by virtue of the following fact:

(ix) *If $\phi(t)$ is an odd function, of class (L^2) and of period 2π , satisfying $\phi_1 \neq 0$ in (6), and if $x(t)$ is a fixed function, of class (L^2) on $(0, \pi)$, corresponding to which there exists at least one sequence c_1, c_2, \dots satisfying (26), then there exists just one such sequence for the given $x(t)$; in fact, the values of c_1, c_2, \dots are then determined by*

$$(27) \quad x_n = \sum_{d|n} c_d \phi_{n/d} \quad (\phi_1 \neq 0)$$

or, equivalently, by

$$(28) \quad c_n = \sum_{d|n} x_d * \phi_{n/d} \quad (*\phi_1 \neq 0),$$

where $x_n, \phi_n, *\phi_n$ are defined by (18), (6), (1) respectively.

Proof of (ix). This is the same as in [10]. In fact, (18) means that

$$\pi x_{n/2} = \int_0^\pi x(t) \sin ntdt.$$

Hence, (27) follows from (26) and (19) by strong convergence. This proves (ix), since (28) is equivalent to (27) by virtue of (1).

For certain, rather than for all, functions $x(t)$ of class (L^2), the question as to the existence of a sequence c_1, c_2, \dots satisfying (26) has an arithmetical origin and goes back to Riemann. In this connection, cf. [4], where other references are given.

Besides (ix), the following fact which proves to be a mere restatement of the corollary, (22), of (viii), will also be needed:

(x) *If $\phi(t)$ is an odd function, of period 2π and of class (L^2), and if $x(t)$ is any function of class (L^2) on $(0, \pi)$, then, unless $\phi(t) = 0$ (almost everywhere), there belongs to every k just one set of k constants $c_1 = c_1(k), \dots, c_k = c_k(k)$ minimizing the integral*

$$(29) \quad \int_0^\pi \left| x(t) - \sum_{n=1}^k c_n \phi(nt) \right|^2 dt.$$

Proof of (x). The integral (29) is a quadratic polynomial in the k variables c_1, \dots, c_k and their complex conjugates. Hence, (x) will be proved if it is shown that the quadratic part of this polynomial, namely, the non-negative definite Hermitian form

$$\int_0^\pi \left| \sum_{n=1}^k c_n \phi(nt) \right|^2 dt,$$

is positive definite unless $\phi(t)$ vanishes almost everywhere. But the discriminant of this Hermitian form is the Gram determinant

$$(30) \quad \det \left(\int_0^\pi \phi_m(t) \phi_n^*(t) dt \right) \quad (m, n = 1, \dots, k),$$

where $\phi_n^*(t)$ denotes the complex conjugate of

$$\phi_n(t) = \phi(nt).$$

On the other hand, k functions $\phi_n(t)$ of class (L^2) on $(0, \pi)$ are known to be linearly independent (almost everywhere) if and only if the determinant (30) vanishes. Hence, (x) follows from (22),

4. Since (7) cannot be a Fourier (L^2) -basis without being an (L^2) -basis, (ii) implies that $\phi_1 \neq 0$ is necessary for a Fourier (L^2) -basis. Criteria for a Fourier (L^2) -basis, which correspond to the criteria (i), (iii), (iv), (v) for a mere (L^2) -basis, can be formulated as follows:

Let $\phi(t)$ be an odd function of period 2π and of class (L^2) , that is, let (6) define a sequence ϕ_1, ϕ_2, \dots satisfying (5). Then, in order that the functions (7) form a Fourier (L^2) -basis on $(0, \pi)$,

(xi) it is necessary and sufficient that $\phi_1 \neq 0$ and that the sequence $^*\phi_1, ^*\phi_2, \dots$, which is then defined by (1), be such that, if $x = (x_1, x_2, \dots)$ is any fixed point of Hilbert's space,

$$(31) \quad \Phi_k(x) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

where $\Phi_k(x)$ denotes, for $k = 1, 2, \dots$, the Hermitian form

$$(32) \quad \Phi_k(x) = \sum_{m=1}^{\infty} \left| \sum_{n=1}^{\infty} x_n \sum_{\substack{nd > k \\ nd | m}} ^*\phi_d \phi_{m/(nd)} \right|^2$$

(with the understanding that the convergence of each of the series (32) at every point of Hilbert's space or, equivalently, the boundedness of each of bilinear forms

$$(33) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_n y_m \sum_{\substack{nd > k \\ nd | m}} ^*\phi_d \phi_{m/(nd)},$$

where $k = 1, 2, \dots$, is part of the criterion);

(xii) it is necessary that

$$(34) \quad ^*\phi_n \rightarrow 0 \text{ as } n \rightarrow \infty;$$

what is more,

(xiii) it is necessary that

$$(35) \quad \sum_{n=1}^{\infty} \left| \sum_{\substack{d > k \\ d | n}} ^*\phi_d \phi_{n/d} \right|^2 \rightarrow 0 \text{ as } k \rightarrow \infty$$

(and that the series (35) be convergent for every k);

(xiv) it is sufficient that $\phi_1=0$ and that the Dirichlet series (3), (4) be convergent in the half-plane $\sigma > 0$ and be such as to represent there functions $\Phi(s)$, $^*\Phi(s)$ satisfying

$$(36) \quad \text{l.u.b.}_{\sigma > 0} \left| \Phi(s) \right| < \infty$$

and

$$(37) \quad \text{l.u.b.}_{\sigma > 0} \left| ^*\Phi(s) \right| < \infty, \quad \text{that is,} \quad \text{g.l.b.}_{\sigma > 0} \left| \Phi(s) \right| > 0, \quad \text{cf. (2).}$$

Proof of (xi). It is clear from (ix) and (x) that, if $x(t)$ is a fixed function of class (L^2) on $(0, \pi)$, that is, if x_1, x_2, \dots in (18) is a fixed sequence satisfying (9), then (26) holds for *some* sequence c_1, c_2, \dots if and only if zero is the limit, as $k \rightarrow \infty$, of the quadratic expression which results if c_n is substituted from (28) into (9). But the expression which thus results for a fixed k is seen to be the Hermitian form (32). Hence, (xi) follows from the fact that the definition of a Fourier (L^2)-basis involves every point $x = (x_1, x_2, \dots)$ of Hilbert's space.

Proof of (xii). The inequality

$$\left| ^*\phi_{k+1}\phi_1 \right|^2 \leq \sum_{n=1}^{d > k} \left| \sum_{d|n} ^*\phi_d\phi_{n/d} \right|^2$$

is obvious for arbitrary values $^*\phi_n, \phi_n$ and for every k . Hence, if (xiii) is granted for a moment, it follows from (35) that

$$\left| ^*\phi_{k+1}\phi_1 \right|^2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In view of $\phi_1 \neq 0$, this proves (34).

Proof of (xiii). According to (xi), condition (31) is necessary and sufficient, if x_1, x_2, \dots in (31) is an unspecified sequence satisfying (9). Hence, there results a necessary condition if (38) is required for the particular sequence

$$x_1 = 1, \quad x_2 = x_3 = \dots = 0.$$

But (32) shows that (31) then becomes precisely (35), the assertion of (xiii).

The proof of the remaining criterion, (xiv), will be omitted, since (xiv) is contained in the following theorem, proved in [10]:

(xv) Let ϕ_1, ϕ_2, \dots , where $\phi_1 \neq 0$, be a sequence of numbers for which both Dirichlet series $\Phi(s)$, $^*\Phi(s)$, defined by (3), (4) and (1), are convergent in the half-plane $\sigma > 0$ and satisfy both (36) and (37). Then (6) defines a function of class (L^2) corresponding to which the sequence (7) is a Fourier (L^2)-basis on $(0, \pi)$.

In fact, if $x(t)$ is any function of class (L^2) on $(0, \pi)$, the sequence c_1, c_2, \dots

satisfying (26) is uniquely determined by (28), where the data x_1, x_2, \dots are given by (18).

In addition, if x_1, x_2, \dots is any sequence possible in (18), that is, any sequence satisfying (5), then

$$(38) \quad \sum_{m=1}^{\infty} |c_m|^2 < \infty$$

holds by virtue of (29). Conversely, if c_1, c_2, \dots is any sequence satisfying (38), then (9) holds by virtue of (28) and belongs, via (18), to a function $x(t)$, of class (L^2) , satisfying (26).

It remains undecided whether the sufficient criterion supplied by (xiv) for a Fourier (L^2) -basis is necessary as well. The purpose of the following considerations is to fill into this gap so much that the notions of an (L^2) -basis and of a Fourier (L^2) -basis become provably distinct.

5. First, the proof of (xv) indicates that the assumptions of (xiv) can hardly be relaxed. In this regard, the following negation should be mentioned:

(xvi) Corresponding to every $\epsilon > 0$, there exists a sequence of numbers $\phi_1 \neq 0, \phi_2, \phi_3, \dots$ having the following properties: The Dirichlet series (3), (4) are convergent in the respective half-plane $\sigma > 0, \sigma > \epsilon$ and satisfy (36) and

$$(39) \quad \text{l.u.b.}_{\sigma > \epsilon} |\ast \Phi(s)| < \infty, \quad \text{that is,} \quad \text{g.l.b.}_{\sigma > \epsilon} |\Phi(s)| > 0,$$

cf. (2), but (6) defines a periodic continuous function for which the sequence (7) fails to be an (L^2) -basis on $(0, \pi)$.

In other words, the difference between $\epsilon = 0$ and any $\epsilon > 0$ is so great that, while the former case necessitates, by (xiv), a Fourier (L^2) -basis, the latter is compatible with no (L^2) -basis.

Proof of (xvi). Define (6) by

$$\phi(t) = \sin t - q \sin 2t,$$

where q is a constant. Thus (17) results if $q = 2$. Correspondingly, the equations (8) belonging to an arbitrary q result if the 2 is replaced by q in the equations given in the proof of (vi). Thus it is clear that, for the same reasons as in the case $q = 2$ of (vi), the sequence (7) fails to be an (L^2) -basis on $(0, \pi)$ if $q > 1$. On the other hand, from (3), (6) and the last formula line,

$$\Phi(s) = 1 - q/2^s;$$

so that (3) is convergent everywhere and satisfies (27). Finally, from (2),

$$\ast \Phi(s) = (1 - q/2^s)^{-1} = \sum_{k=0}^{\infty} q^k / 2^{ks}.$$

This is an ordinary Dirichlet series which is absolutely convergent in the half-plane $\sigma > \log_2 q$. Hence, (xvi) follows by letting $q \rightarrow 1 + 0$.

In order to illustrate the limitations imposed by (xvi) on $\phi(t)$, let (6) be

the generalized Bernoullian function

$$\phi_\lambda(t) = \sum_{m=1}^{\infty} m^{-\lambda} \sin mt,$$

where λ need not be a real (and still less an odd) integer. This is a convergent Fourier series (L) if $\Re\lambda > 0$. Clearly, (5) is satisfied if and only if $\Re\lambda > 1/2$, and this trivial necessary condition turns out to be sufficient in order that (7) be a mere (L_2)-basis on $(0, \pi)$. This fact, proved in the last of the papers listed in [9, pp. 572–573], can be concluded from (iii) also. On the other hand, in order to conclude from (xvi) that (7) is a Fourier (L^2)-basis on $(0, \pi)$, it is necessary (and sufficient) to replace $\Re\lambda > 1/2$ by $\Re\lambda > 1$. In fact, (6) and the last formula line show that the Dirichlet series (3) now becomes

$$\Phi(s) = \sum_{m=1}^{\infty} m^{-\lambda}/m^s = \zeta(s + \lambda);$$

so that, from (2),

$$*\Phi(s) = 1/\zeta(s + \lambda) = \sum_{m=1}^{\infty} \mu(m)/m^{s+\lambda},$$

where $\mu(m)$ is Möbius' factor. Hence the assumptions of (xvi) are satisfied or violated by both (3) and (4) according as $\Re\lambda > 1$ is or is not assumed.

For arithmetical reasons, the necessity of the latter limitation is a more involved question than what is needed in the proof of

(xvii) *There exist odd functions $\phi(t)$, of period 2π and of class (L^2), for which the sequence (7) is an (L^2)-basis, but fails to be a Fourier (L^2)-basis, on $(0, \pi)$; in fact, the trigonometric polynomial*

$$(40) \quad \phi(t) = \sin t - \sin 2t$$

is such a $\phi(t)$.

Proof of (xvii). According to (6) and (40),

$$\phi_1 = 1, \quad \phi_2 = -1, \quad \phi_3 = \phi_4 = \dots = 0.$$

Hence, the equations (8) reduce to $x_n - x_{2n} = 0$ ($n = 1, 2, \dots$). Accordingly, $x_n = x_{2n} = x_{4n} = x_{8n} = \dots$. Consequently, (10) is implied by (9). It follows therefore from (i) that (7) is an (L^2)-basis on $(0, \pi)$.

On the other hand, since the Dirichlet series (3) reduces to

$$\Phi(s) = 1/1^s - 1/2^s + 0 + 0 + \dots = 1 - 2^{-s},$$

(2) and (4) show that

$$(41) \quad \sum_{n=1}^{\infty} *\phi_n/n^s = (1 - 2^{-s})^{-1} = \sum_{k=0}^{\infty} 1/2^{ks}.$$

Since this implies that (34) is violated, it follows from (xii) that (7) cannot be a Fourier (L^2)-basis on $(0, \pi)$.

6. The sufficient criterion of (xiv) for a Fourier (L^2)-basis, when compared with the failure claimed by (xvi) for a mere (L^2)-basis, raises the question whether or not there exist sufficient criteria for an (L^2)-basis which involve, as (xiv) does, Dirichlet series only. It turns out that such criteria result if only *one* of the two conditions, (36) and (37), of (xiv) is required, except that, in the case of (36), a supplementary restriction is necessitated by (vi). This pair of sufficient criteria for (L^2)-bases is the content of (xxi) and (xix) below.

The proof will require the following refinement of (v):

(xviii) *The assertion of (v) remains true if its assumption (15), that is, the assumption*

$$\sum_{n=1}^{\infty} \left| \sum_{d|n}^{d>k} * \phi_d \phi_{n/d} \right|^2 = o(1),$$

where $k \rightarrow \infty$, is relaxed to

$$\sum_{n=1}^{\infty} \left| \sum_{d|n}^{d>k} * \phi_d \phi_{n/d} \right|^2 = O(1),$$

that is, to

$$(42) \quad \sum_{n=1}^{\infty} \left| \sum_{d|n}^{d \leq k} * \phi_d \phi_{n/d} \right|^2 = O(1).$$

Proof of (xviii). Let x_1, x_2, \dots , be any sequence satisfying (9) and (8). Then, if k and n are arbitrary,

$$\sum_{m=1}^k * \phi_m \sum_{j=1}^k \phi_j x_{mnj} = 0.$$

Clearly, this can be written in the form

$$\sum_{j=1}^{\infty} \left(\sum_{m|j}^{m \leq k} * \phi_m \phi_{j/m} \right) x_{jn} = 0.$$

In view of (1), this in turn is equivalent to

$$x_n + \sum_{j=k+1}^{\infty} \left(\sum_{m|j}^{m \leq k} * \phi_m \phi_{j/m} \right) x_{jn} = 0.$$

Hence, by Schwarz's inequality,

$$|x_n|^2 \leq C_k(n) \sum_{j=k+1}^{\infty} |x_{jn}|^2 \leq C_k(n) \sum_{m=k+1}^{\infty} |x_m|^2,$$

where

$$C_k(n) = \sum_{j=k+1}^{\infty} \left| \sum_{m|j}^{m \leq k} {}^* \phi_m \phi_{j/m} \right|^2 \leq \sum_{j=1}^{\infty} \left| \sum_{m|j}^{m \leq k} {}^* \phi_m \phi_{j/m} \right|^2.$$

But the last sum is identical with the sum on the left of (42). Hence, the last two formula lines and (42) imply that

$$|x_n|^2 \leq \text{const.} \sum_{m=k+1}^{\infty} |x_m|^2.$$

If $k \rightarrow \infty$ when n is fixed, it follows that $x_n = 0$, where $n = 1, 2, \dots$. In view of (i), this completes the proof of (xviii).

The two criteria announced before (xviii) are the third and the first of the following four assertions:

Let ϕ_1, ϕ_2, \dots be a sequence of numbers satisfying

$$\phi_1 \neq 0 \quad \text{and} \quad \sum_{m=1}^{\infty} |\phi_m|^2 < \infty,$$

and let ${}^* \phi_n, \Phi(s), {}^* \Phi(s), \phi(t)$ be defined by (1), (3), (4), (6) respectively. Then, in order that the functions (7) form an (L^2)-basis on $(0, \pi)$,

(xix) it is sufficient that

$$(43) \quad \sum_{m=1}^{\infty} |{}^* \phi_m|^2 < \infty$$

and that the Dirichlet series

$$(44) \quad \Phi(s) \text{ converges for } \sigma > 0, \quad \text{and} \quad \text{l.u.b.}_{\sigma > 0} |\Phi(s)| < \infty;$$

(xx) it is insufficient that (44), without (43), be satisfied;

(xxi) it is sufficient that the Dirichlet series

$$(45) \quad {}^* \Phi(s) \text{ converges for } \sigma > 0, \quad \text{and} \quad \text{l.u.b.}_{\sigma > 0} |{}^* \Phi(s)| < \infty;$$

(xxii) it is necessary that both Dirichlet series $\Phi(s), {}^* \Phi(s)$ be absolutely convergent for $\sigma > 1/2$ and that

$$(46) \quad \Phi(s) \neq 0 \quad \text{and} \quad {}^* \Phi(s) \neq 0 \quad \text{when } \sigma > 1/2.$$

The dissymmetry of (xix) and (xxi), emphasized by (xx), is apparent only, since what corresponds to the restriction (43) of (xix) in (xxi) is the restriction (7), assumed from the beginning. The symmetry in the duals (xix), (xxi) is carried even further by the following fact:

(xxiii) The assumptions of (xxi) imply that (43) is satisfied and that

$$(7^*) \quad {}^* \phi(t), {}^* \phi(2t), \dots, {}^* \phi(nt), \dots,$$

defined by (43) and

$$(6^*) \quad {}^*\phi(t) \sim \sum_{m=1} {}^*\phi_m \sin mt,$$

form an (L^2) -basis on $(0, \pi)$.

However, the symmetry is only in the wording, rather than in the proof, of (xix) and (xxi), since (xxi) turns out to be deeper than (xix).

Proof of (xix). According to Toeplitz [7], the assumption (44) is equivalent to the boundedness of the infinite bilinear “ D -form” which he associates with the Dirichlet series $\Phi(s)$. Since a bilinear form having the matrix A is bounded if and only if the Hermitian form belonging to AA^* (and/or A^*A) is, it follows from Toeplitz’s definition (loc. cit.) of the D -form of $\Phi(s)$ that (44) is equivalent to the existence of a constant satisfying

$$(47) \quad \sum_{n=1}^{\infty} \left| \sum_{d|n} \phi_{n/d} x_d \right|^2 \leq \text{const.} \sum_{m=1}^{\infty} |x_m|^2,$$

where (x_1, x_2, \dots) is any point of Hilbert’s space.

For a fixed k , choose $x_1 = {}^*\phi_1, \dots, x_k = {}^*\phi_k, x_{k+1} = x_{k+2} = \dots = 0$. Then, from (47),

$$\sum_{n=1}^{\infty} \left| \sum_{d|n} {}^*\phi_d \phi_{n/d} \right|^2 \leq \text{const.} \sum_{m=1}^k |{}^*\phi_m|^2.$$

It follows therefore from the assumption (43) (which has not been used thus far) that (42) is satisfied. In view of (xviii), this completes the proof of (xix).

Proof of (xx). According to (6) and (3), the Dirichlet series $\Phi(s)$ belonging to (17) is $\Phi(s) = 1 - 2/2^s + 0 + 0 + \dots$. Hence, (xx) follows from (vi).

Proof of (xxi). According to Bohr [1], the abscissa of uniform convergence of an ordinary Dirichlet series is identical with abscissa of bounded regular-analyticity of the function represented by the Dirichlet series (provided that the latter does not diverge everywhere). Hence, (45) implies that the function ${}^*\Phi(s) = {}^*\Phi(\sigma + it)$ of t is uniformly almost-periodic on every line $\sigma > 0$, and that the Fourier expansion of this almost-periodic function of t is supplied precisely by the Dirichlet series. Consequently, the Cesàro sums of this Fourier expansion are uniformly bounded for $-\infty < t < \infty$ when $\sigma > 0$ is fixed and, in view of the second of the assumptions (45), this uniform boundedness is uniform in $\sigma (> 0)$ as well; so that

$$(48) \quad |{}^*\Phi^k(s)| < \text{const.}, \quad \text{where } k = 1, 2, \dots \text{ and } \sigma > 0,$$

if ${}^*\Phi^k(s)$ denotes the Cesàro sum

$$(49) \quad {}^*\Phi^k(s) = \sum_{n_1=0}^k \dots \sum_{n_k=0}^k (1 - n_1/k) \dots (1 - n_k/k) {}^*\phi_j / j^s,$$

$$j = p_1^{n_1} \dots p_k^{n_k},$$

where the p 's denote primes and the n 's their multiplicities in j (cf. [2, pp. 73–75]). Let (49), a terminating Dirichlet series for every k , be denoted by

$$(50) \quad * \Phi^k(s) = \sum_{m=1}^{\infty} {}^k \phi_m / m^s$$

(so that ${}^k \phi_m = 0$ if m is large enough with reference to k).

According to (48), the Dirichlet series (50), where $k = 1, 2, \dots$, are (convergent and) uniformly bounded in the half-plane $\sigma > 0$. In view of Toeplitz's criterion for the bound of a D -form, this means (cf. [7]) that the sequence of the infinite bilinear D -forms belonging to the sequence of the Dirichlet series (50) is uniformly bounded (with respect to k); so that, corresponding to (47),

$$\sum_{n=1}^{\infty} \left| \sum_{d|n} {}^k \phi_{n/d} x_d \right|^2 \leq \text{const.} \sum_{m=1}^{\infty} |x_m|^2,$$

where the constant is independent of k and of the choice of the point (x_1, x_2, \dots) in Hilbert's space (cf. [5, p. 304]).

The last inequality, when applied to $x_n = \phi_n$, gives

$$(51) \quad \sum_{n=1}^{\infty} \left| \sum_{d|n} {}^k \phi_{n/d} \phi_d \right|^2 \leq \text{const.} \sum_{m=1}^{\infty} |\phi_m|^2 = \text{const.} < \infty,$$

by (5). Due to (51), the proof of (xxi) can now be completed along the lines of the proof of (xviii), as follows:

If x_1, x_2, \dots is a sequence satisfying (9) and (8), and if k and n are arbitrary, then

$$\sum_{m=1}^{\infty} {}^k \phi_m \sum_{j=1}^{\infty} \phi_j x_{jmn} = 0.$$

Since all but a finite number of the coefficients ${}^k \phi_1, {}^k \phi_2, \dots$ vanish if k is fixed, this identity can be rearranged into

$$\sum_{m=1}^{\infty} \left(\sum_{d|m} {}^k \phi_{m/d} \phi_d \right) x_{mn} = 0.$$

Hence, if h is arbitrary,

$$\sum_{m=1}^h \left(\sum_{d|m} {}^k \phi_{m/d} \phi_d \right) x_{mn} = - \sum_{m=h+1}^{\infty} \left(\sum_{d|m} {}^k \phi_{m/d} \phi_d \right) x_{mn}.$$

Consequently, by Schwarz's inequality,

$$\left| \sum_{m=1}^h \left(\sum_{d|m} {}^k \phi_{m/d} \phi_d \right) x_{mn} \right|^2 \leq C^k(h) \sum_{m=h+1}^{\infty} |x_{mn}|^2,$$

where

$$C^k(h) = \sum_{m=h+1}^{\infty} \left| \sum_{d|m} {}^k\phi_{m/d}\phi_d \right|^2 \leq \sum_{m=1}^{\infty} \left| \sum_{d|m} {}^k\phi_{m/d}\phi_d \right|^2.$$

Since the last series is the one occurring on the left of (51), the last two formula lines imply that

$$\left| \sum_{m=1}^h \left(\sum_{d|m} {}^k\phi_{m/d}\phi_d \right) x_{mn} \right|^2 \leq \text{const.} \sum_{m=h+1}^{\infty} |x_m|^2,$$

where the constant is independent of the three variables h, k, n .

On the other hand, a comparison of (50) with (49) shows that

$${}^k\phi_m \rightarrow {}^*\phi_m \text{ as } k \rightarrow \infty,$$

if m is fixed. Hence, if h and n are kept fixed in the last inequality, it follows, by letting $k \rightarrow \infty$, that

$$\left| \sum_{m=1}^h \left(\sum_{d|m} {}^*\phi_{m/d}\phi_d \right) x_{mn} \right|^2 \leq \text{const.} \sum_{m=h+1}^{\infty} |x_m|^2.$$

According to (1), this can be reduced to

$$|x_n|^2 \leq \text{const.} \sum_{m=h+1}^{\infty} |x_m|^2.$$

It follows therefore from (9), by letting $h \rightarrow \infty$ while n is fixed, that $x_n = 0$, that is, that (10) is a necessity. In view of (i), this completes the proof of (xxi).

Proof of (xxii). In view of Schwarz's inequality, it is clear from (5) and (3) that the Dirichlet series $\Phi(s)$ is absolutely convergent in the half-plane $\sigma > 1/2$. Since the constant term, ϕ_1 , of $\Phi(s)$ is supposed to be distinct from 0, it follows from (2) and from well known theorems on reciprocation, that the ordinary Dirichlet series of ${}^*\Phi(s)$ must converge absolutely in the half-plane $\sigma > 1/2$ if the function $\Phi(s)$ does not vanish there, that is, if the first of the two assertions (46) is true. On the other hand, the second of the assertions (46) is obvious from (2). Hence, (xxii) will be proved if it is shown that $\Phi(s)$ cannot vanish in the half-plane $\sigma > 1/2$. But this can be concluded, by an argument applied in [8] in a finer context, as follows:

Suppose that $\Phi(s)$ has a zero, s , in the half-plane $\sigma > 1/2$. Then, if $x_n = n^{-s}$, condition (9) is satisfied and, although (10) does not hold, the equations (8) are fulfilled. In fact, by the definition of x_n and of $\Phi(s)$,

$$\sum_{m=1}^{\infty} \phi_m x_{nm} = \sum_{m=1}^{\infty} \phi_m / (nm)^s = \Phi(s) / n^s = 0,$$

since $\Phi(s) = 0$. It follows therefore from (i) that (7) cannot be an (L^2)-basis on $(0, \pi)$. This proves (xxii).

Proof of (xxiii). It is sufficient to show that the assumptions of (xxi) imply (43). For, if this is shown, (xxiii) follows (xxi) for reasons of symmetry, the dual of (43) being (5). Accordingly, it is sufficient to ascertain that (45) implies (43).

As seen above, (44) implies (47). Hence, for reasons of symmetry, (45) implies

$$\sum_{n=1}^{\infty} \left| \sum_{d|n} * \phi_{n/d} x_d \right|^2 \leq \text{const.} \sum_{m=1}^{\infty} |x_m|^2.$$

If

$$x_1 = 1, \quad x_2 = x_3 = \dots = 0,$$

this gives

$$\sum_{n=1}^{\infty} |* \phi_{n/1} + 0 + 0 + \dots|^2 \leq \text{const.} (1^2 + 0 + 0 + \dots),$$

which proves (43).

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