RECTIFICATIONS TO THE PAPERS SETS OF UNIQUENESS AND SETS OF MULTIPLICITY, I AND II(1)

BY
R. SALEM

I. In the first paper the proof of Theorem 3 is incorrect and should be omitted. Thus the main result of the paper should be modified as follows. Let S denote the set of all positive algebraic integers $\theta > 1$ whose conjugates (except θ itself) all have an absolute value less than 1. Let P denote the perfect set of the Cantor type and of constant ratio of dissection constructed on $(0, 2\pi)$ with the ratio of dissection $(\xi, 1-2\xi, \xi)$ where $0<\xi<1/2$. Then, if $1/\xi$ is not an algebraic integer of the class S, the set P is a set of multiplicity. But it remains unknown whether, if $1/\xi$ belongs to S, the set P is necessarily a set of uniqueness.

The parallel results of the second paper have to be modified in the same way.

II. The result of the first paper according to which there exist perfect sets of uniqueness with Hausdorff dimensionality arbitrarily close to 1 remains valid, with a different proof which is as follows. Consider a symmetrical perfect set of order d (d being a positive integer) and of constant ratio of dissection, whose points are given by the formula(2)

$$x = 2\pi\left[\frac{\epsilon_1}{d}(1-\xi) + \frac{\epsilon_2}{d}\xi(1-\xi) + \cdots + \frac{\epsilon_k}{d}\xi^{k-1}(1-\xi) + \cdots\right]$$

where $0<\xi<1/(d+1)$ and the ϵ_i take the values $0, 1, \cdots, d$. It has been proved(2) that this set is a set of uniqueness if $\xi = 1/q$, q being an integer. Take $q=d+2$; we have a set of uniqueness with Hausdorff dimensionality $\log(d+1)/\log(d+2)$, which is arbitrarily close to 1 if d is large enough.

III. As indicated above, we do not know if every perfect set of the Cantor type and of constant ratio of dissection $(\xi, 1-2\xi, \xi)$ is a set of uniqueness whenever $1/\xi$ belongs to the class of algebraic integers S. We can, however, give a sufficient condition such that, when it is satisfied by a number of the class S, the corresponding set P is a set of uniqueness. Let $\theta = 1/\xi$ belong to S, let k be the degree of θ, and let $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ be its conjugates ($|\alpha_i| < 1$). Let $P(z)$ be the irreducible polynomial of degree k with roots $\theta, \alpha_1, \cdots, \alpha_{k-1}$, and let $Q(z)$ be the reciprocal polynomial. If $R(z)$ is any polynomial with rational integral coefficients and of degree not exceeding $k-1$, one has

(1) See Trans. Amer. Math. Soc. vol. 54 (1943) pp. 218–228 and vol. 56 (1944) pp. 32–49.

(2) See Salem, *On singular monotonic functions of the Cantor type*, Journal of Mathematics and Physics vol 21 (1942) pp. 69–82, particularly p. 71.

$$\frac{R(z)}{Q(z)} = \sum_0^\infty c_n z^n = \frac{\lambda}{1-\theta z} + \sum_{i=1}^{k-1} \frac{\mu_i}{1-\alpha_i z}$$

where the c_n are integers, the expansion being valid for $|z| < 1/\theta$. By changing suitably the sign of $R(z)$ we can assume $\lambda > 0$. We have thus $\lambda\theta^m = c_m + \delta_m$ where $\delta_m = -\sum_{i=1}^{k-1} \mu_i \alpha_i^m$ tends to zero as $m \to \infty$. Consider now the perfect set P whose points are given by

$$\begin{aligned} x &= 2\pi[\epsilon_1(1-\xi) + \epsilon_2\xi(1-\xi) + \cdots + \epsilon_k\xi^{k-1}(1-\xi) + \cdots] \\ &= 2\pi(\theta - 1)[\epsilon_1/\theta + \epsilon_2/\theta^2 + \cdots] \end{aligned}$$

where $\epsilon_i = 0$ or 1 and θ belongs to S, and consider the homothetic set Q:

$$y = 2\pi\left[\frac{\epsilon_1}{\theta} + \frac{\epsilon_2}{\theta^2} + \cdots\right].$$

One has

$$\begin{aligned} \lambda\theta^m y &= 2\pi\lambda\left[\frac{\epsilon_{m+1}}{\theta} + \frac{\epsilon_{m+2}}{\theta^2} + \cdots\right] + 2\pi[\epsilon_m\lambda + \epsilon_{m-1}\lambda\theta + \cdots + \epsilon_1\lambda\theta^{m-1}] \\ &\equiv 2\pi\lambda\left[\frac{\epsilon_{m+1}}{\theta} + \cdots\right] + 2\pi[\epsilon_m\delta_0 + \epsilon_{m-1}\delta_1 + \cdots + \epsilon_1\delta_{m-1}] \pmod{2\pi}. \end{aligned}$$

Hence if the condition

$$\sum_0^\infty |\delta_m| + \frac{\lambda}{\theta - 1} < 1 \tag{1}$$

is satisfied, the set of points $\lambda\theta^m y$ has a contiguous interval of length larger than a fixed positive number, independent of m. The same is true for the set of points $c_m y$, since $\delta_m \to 0$. Thus, under the condition (1), the set Q is of the type H and consequently its homothetic P is a set of uniqueness. In order to satisfy (1) it suffices to be able to choose $\lambda, \mu_1, \cdots, \mu_{k-1}$ such that

$$\frac{|\lambda|}{\theta - 1} + \frac{|\mu_1|}{1 - |\alpha_1|} + \cdots + \frac{|\mu_{k-1}|}{1 - |\alpha_{k-1}|} < 1.$$

Now, denoting by $T(z)$ the polynomial reciprocal to $R(z)$, one has

$$\begin{aligned} \lambda &= \lim_{z \to 1/\theta} (1 - \theta z)\frac{R(z)}{Q(z)} = \lim (1 - \theta z)\frac{z^{k-1}T(1/z)}{z^k P(1/z)} \\ &= \lim\left(\frac{1}{z} - \theta\right)\frac{T(1/z)}{P(1/z)} = \frac{T(\theta)}{P'(\theta)} \end{aligned}$$

and similarly $\mu_i = T(\alpha_i)/P'(\alpha_i)$. Thus condition (1) is equivalent to

$$\left|\frac{T(\theta)}{P'(\theta)}\right|\frac{1}{\theta-1}+\sum_{i=1}^{k-1}\left|\frac{T(\alpha_i)}{P'(\alpha_i)}\right|\frac{1}{1-|\alpha_i|}<1. \tag{2}$$

One has $T(z)=a_0+a_1z+\cdots+a_{k-1}z^{k-1}$, the a_i being unknown integers. The absolute value of the determinant of the linear forms $T(\theta)$, $T(\alpha_1)$, $\cdots$, $T(\alpha_{k-1})$ is $|D|^{1/2}$, D being the discriminant of the polynomial $P(z)$. On the other hand

$$|P'(\theta)\cdot P'(\alpha_1)\cdots P'(\alpha_{k-1})|=|D|.$$

Hence by the theorem of Minkowski on sums of absolute values of linear forms, it is certainly possible to determine the integers $a_0, a_1, \cdots, a_{k-1}$, not all zero, satisfying (2), if

$$\left(\frac{4}{\pi}\right)^s\frac{k!}{|D|^{1/2}}\frac{1}{(\theta-1)(1-|\alpha_1|)\cdots(1-|\alpha_{k-1}|)}<1$$

where $2s$ is the number of imaginary roots of $P(z)$. In this case (1) can be satisfied by a convenient choice of $R(z)$ and the set corresponding to θ is a set of uniqueness.

IV. As an application, consider the case where θ is a quadratic irrational root of $P(z)=z^2-pz+q$ (p, q, rational integers). One has here $p>0$ and $P(-1)>0$, $P(1)<0$, hence $1+p+q>0$, $1-p+q<0$, that is to say $p>|q+1|$. Also $D=p^2-4q>0$, $s=0$, and the set corresponding to the larger root θ (supposed to be larger than 2 is a set of uniqueness if $(p^2-4q)^{1/2}(\theta-1)(1-|\alpha|)>2$, α being the root with absolute value less than 1. Suppose first $q>0$, hence $\alpha>0$. Then the condition is equivalent to $(p^2-4q)^{1/2}(p-q-1)>2$. One has $p\geqq q+2$, $(p^2-4q)^{1/2}\geqq(q^2+4)^{1/2}$, thus the condition is certainly satisfied. Suppose now $q<0$, hence $\alpha<0$. The condition becomes $(p^2-4q)^{1/2}(\theta-1)(1+\alpha)>2$, that is to say

$$(p^2-4q)^{1/2}[(p^2-4q)^{1/2}+q-1]>2. \tag{3}$$

If $q=-1$, the left-hand side is $(p^2+4)^{1/2}[(p^2+4)^{1/2}-2]$ and is larger than 2 if $p\geqq 2$. On the other hand the case $p=1$ is impossible, since it leads to $\theta<2$.

If $q\leqq -2$, one has $|q+1|=|q|-1$, and thus $p\geqq|q|$ and the left-hand side of (3) is larger than

$$(q^2+4|q|)^{1/2}[(q^2+4|q|)^{1/2}-(|q|+1)]$$
$$=(q^2+4|q|)^{1/2}\frac{2|q|-1}{(q^2+4|q|)^{1/2}+|q|+1}>|q|-\frac{1}{2}\cdot$$

Thus if $q\leqq -3$, the inequality (3) is satisfied.

If $q=-2$, the left-hand side of (3) becomes $(p^2+8)^{1/2}[(p^2+8)^{1/2}-3]$ and is larger than 2 if $p\geqq 3$. The case $p=1$ is impossible since $p>|q+1|$. Remains the case $q=-2$, $p=2$, which is the only one in which the inequality (3) is not satisfied. However, it is possible to satisfy (2). For

$$P(z) = z^2 - 2z - 2, \qquad \theta = 3^{1/2} + 1, \qquad \alpha = 1 - 3^{1/2},$$
$$P'(\theta) = 2(3^{1/2}), \qquad P'(\alpha) = -2(3^{1/2})$$

and we have to find $T(z) = Az + B$, with A, B integers such that

$$\frac{|T(\theta)|}{6} + \frac{|T(\alpha)|}{2(3^{1/2})(2 - 3^{1/2})} < 1,$$

which can be done by taking $A = 1$, $B = 1$.

Thus when $\theta \in S$ is quadratic, the set is a set of uniqueness.

We add, as a final remark, that if $\theta \in S$ there exists always an exponent $p_0 = p_0(\theta)$ such that the set corresponding to $\xi = \theta^{-p}$ is a set of uniqueness when the integer $p > p_0$. This follows immediately from the argument used above, taking $\lambda = 1$.

Massachusetts Institute of Technology,
Cambridge, Mass.