

A SUFFICIENCY THEOREM FOR THE PLATEAU PROBLEM

BY

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1. Introduction. The Plateau problem in three-space is considered in several papers by Schwarz. In the second half of one of these papers [XI]⁽¹⁾, in dealing with the problem of minimizing the area of a surface through a single boundary, he states a necessary condition that the second variation be positive. A sufficient condition is also given for positive second variation. The criterion here depends on the spherical representation of the minimal surface considered. These results are reported in an excellent review of the whole question by Radó [IX] and may be stated as follows. A sufficient condition for positive second variation is that the spherical representation of the minimal surface lie within a hemisphere. A necessary condition is that this representation must not properly contain a total hemisphere; in this case the second variation could be made negative. The gap between these conditions is perhaps most easily illustrated in connection with the catenoid. That the catenoid minimizes among surfaces of revolution under certain restrictions is reported and discussed in detail by Bliss [II], as a problem in the plane. The condition which is sufficient there is that no point occur, conjugate to the initial point, and preceding the final point on the arc of the catenary which generates the surface. It is clear that the spherical representation of any catenoid is likely to be an equatorial zone, neither lying in a hemisphere nor properly containing a hemisphere; and we can readily have this equatorial zone representation when the catenoid does actually minimize, as we shall see, under much wider conditions.

In the first part of the Schwarz paper referred to there is another sufficient condition given for positive second variation. It is a condition on the tangent plane of the minimal surface. This seems a more suggestive result, and it is stated and treated by Bianchi [I]. Considering a family of neighboring surfaces obtained by normal variation, the result, in effect, is that a given minimal surface minimizes in this family, if there is a point in space through which no tangent plane to the bounded minimal passes. Bianchi's treatment can be followed up in the $(n-1)$ -dimensional case, and for a quite general type of boundary. For $(n-1)$ -dimensional hypersurfaces in n space, it is then possible to construct a field in §6, and for the multiple integral problem obtain a sufficiency proof. The result is: *If there exists one point in space through which no tangent hyperplane passes, then the minimal hypersurface is minimizing.*

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(1) Roman numerals in brackets refer to the list of references at the end of the paper.

2. **The $n-1$ curvatures K_1, \dots, K_{n-1} of a V_{n-1} in S_n .** We consider a hypersurface V_{n-1} embedded in a Euclidean space S_n . The coordinates y^i ($i=1, \dots, n$) of points in V_{n-1} are given in terms of $n-1$ parameters t^a ($a=1, \dots, n-1$) by equations of the form

$$(2.1) \quad y^i = y^i(t).$$

The t^a are to be independent and real, and the functions $y^i(t)$ are to be real and single-valued over some open region R in t -space. We assume also that the $y^i(t)$ are of class C'' over R , that is, that they have continuous first and second derivatives in R . We shall write y_a^i for $\partial y^i / \partial t^a$, and we assume the matrix $\|y_a^i\|$ to be of rank $n-1$ at all points in R . The $n-1$ vectors y_a^i ($i=1, \dots, n$ for each a) are then independent, and determine the tangent hyperplane at a point of V_{n-1} . The direction Jacobians of the normal to the hypersurface will be A_i , the determinants resulting from the deletion of the i th row from the matrix $\|y_a^i\|$ and multiplying by $(-1)^{i-1}$. It follows that $A_i y_a^i = 0$, summation on i implied. This states the orthogonality of the normal to the tangent hyperplane, and is conveniently seen by considering that $A_i y_a^i$ is an n by n determinant with two columns alike.

In general, repetition in any index will imply summation. We note $A_i A_i > 0$ because of the rank of $\|y_a^i\|$, and define $H = (A_i A_i)^{1/2} > 0$. If the direction cosines of the normal are Y_i , then we may list for reference

$$(2.2) \quad Y_i = A_i / H, \quad Y_i Y_i = 1, \quad Y_i y_a^i = 0.$$

If we write g_{ab} for $y_a^i y_b^i$ and g for $|g_{ab}|$, it follows easily that $g = H^2 > 0$. For $|Y_i, y_a^i| = Y_i A_i = H$, and the square of this determinant is g . We list

$$(2.3) \quad g_{ab} = g_{ba} = y_a^i y_b^i, \quad g = |g_{ab}|, \quad g = H^2 > 0.$$

It is desirable also to have the quantities g^{ab} defined by $g g^{ab}$ = the cofactor of g_{ab} in g . These quantities g^{ab} are symmetric and satisfy, by definition, the relations

$$(2.4) \quad g^{ab} g_{ac} = \delta_c^b \quad [\delta_c^b = 1, b = c; \delta_c^b = 0, b \neq c].$$

We shall represent $\partial^2 y^i / \partial t^a \partial t^b$ by y_{ab}^i . Differentiating $Y_i y_a^i = 0$ with respect to t^b we get

$$Y_i y_{ab}^i + Y_{ib} y_a^i = 0.$$

The symmetric quantities r_{ab} are defined by

$$(2.5) \quad r_{ab} = r_{ba} = Y_i y_{ab}^i = -Y_{ib} y_a^i.$$

We introduce also the quantities $r_b^a = g^{ac} r_{cb}$. From this definition it follows that

$$(2.6) \quad g_{ad} r_b^a = g_{ad} g^{ac} r_{cb} = \delta_d^c r_{cb} = r_{db}.$$

To see the geometric meaning of these r_b^a , we might write the preceding $g_{ad}r_b^a - r_{ab} = 0$ as

$$(2.7) \quad y_d^i(y_a^i r_b^a + Y_{ib}) = 0 \quad (d = 1, \dots, n - 1),$$

using (2.3) and (2.5). From $Y_i Y_i = 1$ we have $Y_i Y_{ib} = 0$, and since $Y_i y_a^i = 0$ as in (2.2), we can set up

$$(2.8) \quad Y_i(y_a^i r_b^a + Y_{ib}) = 0.$$

There are now, in (2.7) and (2.8), n equations in the n quantities $y_a^i r_b^a + Y_{ib}$ ($i = 1, \dots, n$). The determinant of the set is $|Y_i, y_a^i| = Y_i A_i = H > 0$ as seen before. Hence

$$(2.9) \quad Y_{ib} = -y_a^i r_b^a.$$

In other words, $Y_i Y_{ib} = 0$ states that Y_{ib} is orthogonal to Y_i , and hence can be expressed linearly in terms of the $n - 1$ independent tangential vectors y_a^i . It is clear that (2.9) achieves this, and so the r_b^a of (2.6) are exhibited as the appropriate coefficients. Considering now the determinant $|r_b^a|$, we shall define in terms of it the $n - 1$ curvatures of V_{n-1} .

DEFINITION 2.1. *The c th curvature K_c of V_{n-1} at P is the sum of all principal minors of $|r_b^a|$ of order c .*

These curvatures can be deduced as analogues of the two curvatures in the 3-space case. They are invariants with regard to the parameters t^a . We might set out explicitly

$$K_1 = r_a^a, \quad K_2 = (1/2)(r_a^a r_b^b - r_b^a r_a^b).$$

Summation is implied as a and b range over 1 to $n - 1$. In passing, we note that $K_{n-1} = |r_b^a|$.

Certain relations between the quantities so far introduced are useful in the sequel. A certain amount of manipulation is required in deriving them, and this will be omitted [VII]. We shall simply set

$$(2.10) \quad V_i = 2HK_2 Y_i - (\partial/\partial t^a)(H g^{ab} Y_{ib})$$

and state the following lemma.

LEMMA 2.1. *At each point of a hypersurface S of class C''' , the following identities hold, K_1 and K_2 being first and second curvatures of S ,*

$$(2.11) \quad Y_i V_i = HK_1^2, \quad y_a^i V_i = H \partial K_1 / \partial t^a \quad (a = 1, \dots, n - 1).$$

Going back to (2.2) and (2.3) we shall deduce at this point one further relationship. From $Y_i Y_i = 1$, on differentiating with regard to y_b^j , we get $Y_i Y_{ib}^j = 0$. The same treatment of $A_i = Y_i H$ yields

$$A_{ib}^j = Y_{ib}^j H + Y_i H_b^j.$$

Multiplying this by Y_i , summing on i , and using $Y_i Y_{ib} = 0$, we have

$$Y_i A_{ib}^i = H_b^i$$

From $H^2 = g$, we may write $2HH_b^j = 2(\text{cofactor of } g_{ab} \text{ in } g)y_a^j = 2gg^{ab}y_a^j$. Thus we have $H_b^j = Hg^{ab}y_a^j$. Multiplication now by Y_j and summation gives $Y_j H_b^j = 0$; the same treatment by y_c^j using (2.4) gives the second result $H_b^j y_c^j = Hg^{ab}g_{ac} = H\delta_c^b$. We have derived

$$(2.12) \quad Y_j H_b^j = 0, \quad y_c^j H_b^j = H\delta_c^b$$

3. Analytic formulation of the problem. Let R be a region of points $(y, A) = (y^1, \dots, y^n, A_1, \dots, A_n)$ such that if (y, A) is in R then $(y, A) \neq (y, 0)$ and the element (y, kA) is in R for every constant $k > 0$. An element (y, A) in R will be called *admissible*. An *admissible region* T in t -space will be open and connected, and have a boundary B piecewise of class C' , non intersecting and non singular, representable parametrically in the form

$$t^a = t^a(p^1, \dots, p^{n-2}),$$

the intersection of these pieces of B being such that Green's theorem is applicable on $T+B$. By an *admissible hypersurface* S will be meant one that is defined by functions

$$y^i = y^i(t^1, \dots, t^{n-1}) \quad (t \text{ on } T)$$

which are continuous on $T+B$ where T is admissible, and are such that $T+B$ can be subdivided into a finite number of admissible subregions and their boundaries, on each of which they are of class C' . The corresponding elements (y, A) are to be admissible, where the A_i are the direction Jacobians of the normal to S . The image of B in y -space is called the boundary L of S . The problem with which we shall be concerned is that of minimizing the integral

$$(3.1) \quad I = \int_T H dt_1 dt_2 \dots dt_{n-1} = \int_T H dt$$

in the class of admissible hypersurfaces S having a common boundary L .

The special function H in the integrand clearly satisfies the homogeneity condition $H(kA) = kH(A)$, known to be a necessary and sufficient condition that the corresponding I be independent of the parametrization. We take $k > 0$ to preserve the orientation of S . We are considering the generalization of the area integral in 3-space and might refer to it as the $(n-1)$ -dimensional volume integral.

4. The first variation. A set of functions $\xi^i(t)$ constitutes an *admissible* variation of S if the ξ^i are of class C' on T and vanish on its boundary B . For any set ξ^i we may write $\xi^i = q(t)Y_i + q^a(t)y_a^i$. The n functions of t , q and q^a ($a = 1, \dots, n-1$) are well defined since $|Y_i, y_a^i| \neq 0$. If $\eta^i = qY_i$ and $\zeta^i = q^a y_a^i$ then $\xi^i = \eta^i + \zeta^i$. We shall refer to η^i as a *normal* variation and ζ^i as a *tangential*

variation. It turns out that we may restrict our consideration to normal variations η^i with no loss in generality. This will be indicated for the first variation of I , but merely stated for the second variation.

The first variation I_1 of I on S is

$$I_1(\xi) = \int_T (H_i \dot{\xi}^i + H_a^i \dot{\xi}_a^i) dt,$$

or

$$(4.1) \quad I_1(\xi) = \int_T H_a^i \dot{\xi}_a^i dt$$

where ξ^i is an arbitrary admissible variation in the function y^i , and it is noted that H is independent of y^i .

Since $\xi^i \equiv 0$ on B , an application of Green's theorem as stated by Carson [IV] yields for S of class C'' on $T+B$,

$$(4.2) \quad I_1(\xi) = \int_T (-\xi^i \partial H_a^i / \partial t^a) dt.$$

For a minimizing hypersurface, we must have $I_1(\xi) = 0$ for arbitrary ξ^i , hence the necessary conditions for a minimum for I , namely the *Euler equations*,

$$(4.3) \quad P_i \equiv -\partial H_a^i / \partial t^a = 0$$

are to hold at each point of S . Hypersurfaces satisfying (4.3) at each point are called *minimals*. It is readily shown that $P_i y_b^i = 0$, and since $Y_i y_b^i = 0$, we can write

$$(4.4) \quad P_i = P Y_i, \quad P_i Y_i = P$$

and have the single condition $P = 0$ equivalent to (4.3).

It is also seen readily that $\zeta^i P_i = 0$ and hence

$$(4.5) \quad I_1(\xi) = I_1(\eta).$$

Noting that

$$I_1(\eta) = \int_T H_a^i (q^a Y_i + q Y_{ia}) dt$$

we find, using (2.12), that the first term on the right vanishes, and the second admits the form $-qHK_1$. The following lemma then holds.

LEMMA 4.1. *The first variation $I_1(\xi)$ of I on S vanishes for all admissible variations $\xi^i \equiv 0$ on B if and only if it vanishes for all normal variations η^i having $\eta^i \equiv 0$ on B . For a normal variation $\eta^i = q Y_i$, the first variation takes the form*

$$(4.6) \quad \int_T (-qHK_1)dt$$

where $K_1=r_a^a$ is the first curvature of S .

We notice that P might be expressed as

$$P = -\partial(Y_i H_a^i)/\partial t^a + H_a^i Y_{i,a}$$

and again from (2.12), and the derivation of (4.6), we get that $P=0$ is equivalent to

$$(4.7) \quad K_1 = 0.$$

5. The second variation. We write $\partial \xi^i/\partial t^a$ as ξ_a^i and $\partial^2 H/\partial y_a^i \partial y_b^j$ as H_{ab}^{ij} . Since H is independent of y^i , the second variation of I is

$$(5.1) \quad I_2(\xi) = \int_T H_{ab}^{ij} \xi_a^i \xi_b^j dt = \int_T 2\Omega(\xi)dt.$$

The *Jacobi equations* are

$$(5.2) \quad J_i(\xi) = -\partial/\partial t^a \Omega_a^i(\xi) = 0$$

where $\Omega_a^i(\xi) = \partial \Omega/\partial \xi_a^i$. We state the following lemma.

LEMMA 5.1. *On a minimal, the quantities $J_i(\zeta)$ are identically zero, where $\zeta^i = q^c y_c^i$.*

We have

$$(5.3) \quad \Omega_a^i(\zeta) = H_{ab}^{ij} \zeta_b^j = q_b^c y_c^j H_{ab}^{ij} + q^c y_{cb}^j H_{ab}^{ij}.$$

Differentiating (2.12) with respect to y_a^i yields

$$(5.4) \quad y_c^j H_{ab}^{ij} = H_a^i \delta_c^b - H_b^i \delta_c^a.$$

Use of this justifies

$$\Omega_a^i(\zeta) = \partial(q^c H_a^i)/\partial t^c - q_b^a H_b^i$$

and a few more steps yield

$$\partial \Omega_a^i(\zeta)/\partial t^a = -\partial/\partial t^c (q^c P_i).$$

This implies that $J_i(\zeta) = 0$ when $P_i = 0$. By inspection $J_i(\xi)$ equals $J_i(\zeta) + J_i(\eta)$, hence when S is a minimal we have $J_i(\xi) = 0$ equivalent to $J_i(\eta) = 0$. We use this in the following theorem.

THEOREM 5.1. *On a minimal, the second variation I_2 has the same value for $\xi^i = \zeta^i + \eta^i$ as for the corresponding normal variation η^i , provided $\xi^i \equiv 0$ on B ,*

and hence also $\eta^i \equiv 0$ on B . For a minimal the Jacobi equations $J_i(\xi) = 0$ and $J_i(\eta) = 0$ are equivalent.

The fact that $Y_i \xi^i = Y_i \eta^i$ implies $\xi^i \equiv 0$ on B if and only if $\eta^i \equiv 0$ on B . We see, referring to (5.3), that

$$2\Omega(\xi) = 2\Omega(\eta) + 2\Omega(\zeta) + 2\eta^i \Omega_a^i(\zeta).$$

Since $\Omega(\xi)$ is homogeneous of degree two in the ξ_a^i , we may replace $2\Omega(\zeta)$ by $\zeta_a^i \Omega_a^i(\zeta)$, resulting in the identity

$$2\Omega(\xi) = 2\Omega(\eta) + \partial/\partial t^a (\xi^i + \eta^i) \Omega_a^i(\zeta).$$

An application of Green's theorem justifies

$$(5.5) \quad I_2(\xi) = I_2(\eta)$$

and completes the theorem.

We turn now to the special forms of I_2 and the Jacobi equations, for normal variations $\eta^i = qY_i$. We obtain

$$2\Omega(\eta) = H_{ab}^{ij} (q_a Y_i + q Y_{ia}) (q_b Y_j + q Y_{jb}) = q^2 N + 2qq_a Q^a + q_a q_b R^{ab}$$

where

$$(5.6) \quad N = H_{ab}^{ij} Y_{ia} Y_{jb}, \quad Q^a = H_{ab}^{ij} Y_i Y_{jb}, \quad R^{ab} = H_{ab}^{ij} Y_i Y_j.$$

On reconsidering these in some detail [VII], they permit of the following forms:

$$(5.7) \quad N = 2HK_2, \quad Q^a = 0, \quad R^{ab} = Hg^{ab}$$

and hence we have the following theorem.

THEOREM 5.2. *For normal variations $\eta^i = qY_i$, the second variation $I_2(\eta)$ of I takes the form*

$$(5.8) \quad I_2(\eta) \equiv \mathfrak{J}_2(q) = \int_T H(2K_2 q^2 + g^{ab} q_a q_b) dt.$$

We may remark here that the Jacobi equation arising from the form of the second variation given in (5.8) is

$$(5.9) \quad J(q) \equiv 2HK_2 q - \partial/\partial t^a (Hg^{ab} q_b) = 0.$$

This is equivalent to the set $J_i(\eta) = 0$ in Theorem 5.1, provided we are on a minimal. For it can be shown that then the $J_i(\eta)$ are not independent, but satisfy $\gamma_a^i J_i(\eta) = 0$. It would follow that the $J_i(\eta)$ are proportional to Y_i , and indeed

$$(5.10) \quad J_i(\eta) = JY_i, \quad Y_i J_i = J.$$

The result is that $J_i = 0$ and $J = 0$ are equivalent on a minimal, the J found here being the same as in (5.9).

At this stage, we will set up a Clebsch transformation of the second variation. We prove the following theorem.

THEOREM 5.3. *If there exists a solution $U(t)$ of class C'' of the Jacobi equation (5.9) which does not vanish anywhere on $T+B$, then the second variation $\mathfrak{J}_2(q)$ on S is expressible in the form*

$$(5.11) \quad \mathfrak{J}_2(q) = \int_T Hg^{ab}[q_a - U_a(q/U)][q_b - U_b(q/U)]dt$$

for every admissible variation q having $q \equiv 0$ on B . The form $Hg^{ab}\pi_a\pi_b$ being positive definite, then $\mathfrak{J}_2(q) > 0$ for every admissible variation $q \neq 0$ having $q \equiv 0$ on B .

Let U be a solution of the Jacobi equation satisfying the above hypotheses. Define v by $q = Uv$. Since $q \equiv 0$ on B , then $v \equiv 0$ on B . We get, since g^{ab} is symmetric,

$$2\Omega(Uv) = Uv^2(2HK_2U) + 2vv_aU(Hg^{ab}U_b) + v^2U_a(Hg^{ab}U_b) + Hg^{ab}U^2v_av_b.$$

Since U satisfies (5.9) we may replace $2HK_2U$ in the first term on the right by $\partial/\partial t^a(Hg^{ab}U_b)$, and then have

$$2\Omega(Uv) = \partial/\partial t^a(Hg^{ab}U_bUv^2) + Hg^{ab}U^2v_av_b.$$

An application of Green's theorem then yields

$$\mathfrak{J}_2(q) = \int_T Hg^{ab}v_av_bU^2dt$$

since $v \equiv 0$ on B . Noting that $q_a = U_av + Uv_a$ we finally write $\mathfrak{J}_2(q)$ in the form (5.11).

The fact that the g_{ab} as introduced are coefficients of a positive definite form implies the same for the g^{ab} . Looking at (5.11), we have the second variation certainly positive. The possibility that it be zero would imply, for every a ,

$$q_a/q = U_a/U$$

which, in turn, implies $q = cU$ where c is constant. The hypotheses $q \equiv 0$ on B , $U \neq 0$ on B rule this out.

Having Theorem 5.3, we now exhibit certain particular solutions of (5.9). It will be seen that $J(Y_i)$ is the V_i of (2.10). On a *minimal*, (2.11) yields, for minimals of class C''' ,

$$(5.12) \quad V_i Y_i = 0, \quad V_i \dot{y}_a = 0.$$

Since $|Y_i, \dot{y}_a| \neq 0$, we have $V_i = 0$, that is, Y_i is a solution of (5.9). Another solution is $\dot{y}^i Y_i$. For $\partial(\dot{y}^i Y_i)/\partial t^b$ is $\dot{y}^i Y_{ib}$ since $\dot{y}_b^i Y_i = 0$. It follows that

$$J(\dot{y}^i Y_i) = 2HK_2 \dot{y}^i Y_i - \dot{y}^i (\partial/\partial t^a)(Hg^{ab} Y_{ib}) - Hg^{ab} Y_{ib} \dot{y}_a^i.$$

Knowing that Y_i satisfies (5.9) we get

$$J(y^i Y_i) = -Hg^{ab} Y_{ib} y_a^i = Hg^{ab} y_c^i r_b y_a^i = Hr_b^c \delta_c^b = HK_1 = 0.$$

Because Y_i and $y^i Y_i$ are solutions of (5.9) we can state that $W = (c^i - y^i) Y_i$ is a solution for any fixed point c^i in y -space. This completes the following lemma.

LEMMA 5.2. *For a minimal hypersurface S of class C''' the Jacobi equation (5.9) has as particular solutions*

$$(5.13) \quad Y_i; \quad y^i Y_i; \quad W = (c^i - y^i) Y_i.$$

Let us suppose, in passing, that the hyperspherical representation of S lies within a hemi-hypersphere. Then there exists a unit vector c^i such that the angle between c^i and Y_i is always less than 90° over S . Hence $c^i Y_i > 0$ over S . We then have a solution of the Jacobi equation of class C'' which is nonvanishing over $T+B$, and by Theorem 5.3 the second variation is certainly positive. We state here the following lemma.

LEMMA 5.3. *If the hyperspherical representation of a given minimal hypersurface S lies within a half hypersphere, then the second variation $\mathfrak{J}_2(q)$ in (5.8) is positive for all admissible variations $q \neq 0$ on T , having $q \equiv 0$ on B .*

Where z^i are the current coordinates, the tangent hyperplane to S is

$$(5.14) \quad (z^i - y^i) Y_i = 0.$$

For any point c^i , the expression $(c^i - y^i) Y_i$ is the perpendicular length from the hyperplane to c^i . Now suppose there is a point c^i in y -space such that no tangent hyperplane to S passes through c^i . For such a point $W = (c^i - y^i) Y_i$ is nonvanishing on $T+B$, and hence again we have a solution of (5.9) of the type required in Theorem 5.3. Hence

LEMMA 5.4. *If there exists a point c^i in y -space through which no tangent hyperplane to a given minimal hypersurface S of class C''' passes, then the second variation $\mathfrak{J}_2(q)$ in (5.8) is positive for all admissible variations $q \neq 0$ on T , having $q \equiv 0$ on B .*

The sufficiency condition for positive second variation in Lemma 5.4 is much wider than the one in Lemma 5.3 and we can see as follows that the wider one includes Lemma 5.3. Let $S+L$ be enclosed in a hypersphere, and through the center of the hypersphere construct a vector pointing in the direction c^i , where c^i is chosen as in the proof of Lemma 5.3. Going along the negative direction of this vector we certainly will come to a point through which no tangent hyperplane of $S+L$ can pass. Specifically, if the angle between c^i and the normal Y_i to S is not greater than $\alpha < 90^\circ$, form a hypercone

with vertex at the center of the enclosing sphere, with its axis along the negative direction of c^i , and with its semi-vertical angle equal to α . Where hypercone and hypersphere intersect, form a hypercone tangential to the hypersphere, which will have its vertex at P on c^i in the negative direction from the center. Any point on c^i further along than P in this negative direction will serve for Lemma 5.4. It is readily seen that no tangent hyperplane could pass through such a point without having its normal make an angle greater than α with c^i . Hence we have established, calling such a point k^i , the following lemma.

LEMMA 5.5. *If the hyper-spherical representation of a given minimal hypersurface S lies within a half hypersphere, there is a point k^i in y -space through which no tangent hyperplane to S passes.*

The Jacobi necessary condition has been discussed in very general form by Reid [X] and under classical assumptions by Raab [VIII]. A form which suits our present purpose could be stated as the following lemma.

LEMMA 5.6. *If T_0 is a proper subregion of T with total boundary B_0 , then along a minimizing hypersurface of class C''' , there can exist no solution $U(t)$ of the Jacobi equation (5.9) such that $U \neq 0$ on T_0 , $U \equiv 0$ on B_0 , and not all U_a are zero at some point P_0 of B_0 interior to T .*

The necessary condition, stated by Schwarz, is that if the spherical representation of a minimal surface properly contains a total hemisphere, then the surface is not minimizing. The analogue here would be to assume a total half hypersphere contained properly, and then to state the existence of a solution $c^i Y_i$ of the Jacobi equation, where c^i is a unit vector perpendicular to the hyperplane defining the covered half hypersphere. This solution $U = c^i Y_i$ vanishes on the total boundary of a proper subregion T_0 of T . Not all U_a can be zero at any point of S at which $U = 0$ and the $(n-1)$ th curvature $K_{n-1} \neq 0$. For we would be requiring the unit vector c^i to satisfy simultaneously

$$U = c^i Y_i = 0, \quad U_a = c^i Y_{ia} = 0$$

which means c^i is to be orthogonal to the normal Y_i , and to the tangent hyperplane, if the Y_{ia} are independent vectors. We had in (2.9) that $Y_{ia} = -y_b^i r_a^b$, hence the Y_{ia} are dependent only if the determinant $|r_b^a| = K_{n-1}$ is zero. It follows that if at any point of B_0 interior to T , $K_{n-1} \neq 0$, then Lemma 5.6 applies and S is not minimizing. Neither could we have W of Lemma 5.2 vanishing on B_0 with $K_{n-1} \neq 0$ at a point of B_0 interior to T . For

$$W = (c^i - y^i) Y_i = 0, \quad W_a = (c^i - y^i) Y_{ia} = 0$$

simultaneously imply $K_{n-1} = 0$ at the point, assuming c^i is not on S and so

$c^i - y^i \neq 0$. Hence there must not be a point c^i in space, not on S , such that along B_0 all tangent hyperplanes pass through c^i , and such that at a point of B_0 interior to T , $K_{n-1} \neq 0$.

6. Sufficient condition for a minimum in the Plateau problem. The theorem of this section should be read with the references to Schwarz and Bianchi on page 192 in mind.

THEOREM 6.1. *Let M be a minimal hypersurface $y^i(t)$ of class C'' , not intersecting itself, and defined on a neighborhood of $T+B$, its assigned boundary L in y -space having B for its projection in t -space. If there is a point P in y -space through which there passes no tangent hyperplane to M , then there is a neighborhood F of M in y -space, such that the inequality $I(S) > I(M)$ holds for every admissible hypersurface $S \neq M$ in F having boundary L .*

The integral we are minimizing is

$$I = \int_T H dt.$$

Take P as origin and apply the transformation

$$(6.1) \quad \bar{y}^i(t, e) = e y^i(t), \quad e > 0.$$

Since no tangent to M can pass through the origin, there exists a neighborhood F of M determined by $|1 - e| \leq \epsilon$, $\epsilon > 0$, which is simply covered by a one-parameter family of hypersurfaces swept out by M under the similarity transformation (6.1). The Jacobian of the transformation (6.1) is

$$(6.2) \quad \begin{aligned} |\bar{y}_a^i(t, e), \bar{y}_e^i| &= |e y_a^i(t), y^i(t)| \\ &= \pm e^{n-1} A_i y^i = \pm e^{n-1} H y^i Y_i \\ &= \pm e^{n-1} H W, \end{aligned}$$

where W is defined as in (5.13), if we note that now c^i is the origin. Since no tangent hyperplane passes through the origin, W is nonvanishing over M , and since e and H are positive, the Jacobian is nonvanishing over F . Let us note the effect of this transformation (6.1) on the first curvature K_1 . We see the following in easy sequence. First $\bar{y}_a^i = e y_a^i$, and hence $\bar{A}_i = e^{n-1} A_i$, $\bar{H} = e^{n-1} H$. Then $\bar{Y}_i = Y_i$ and $\bar{r}_{ab} = -\bar{y}_a^i \bar{Y}_{ib} = e r_{ab}$. Also $\bar{g}_{ab} = e^2 g_{ab}$, $\bar{g} = e^{2n-2} g$, $\bar{g}^{ab} = g^{ab}/e^2$. Finally

$$\bar{K}_1 = \bar{g}^{ab} \bar{r}_{ab} = K_1/e.$$

It is clear that since $K_1 = 0$ for M , $\bar{K}_1 = 0$ for the one-parameter family M sweeps out as it moves over F under (6.1). In other words, F is simply covered by a family of *minimals* which defines a *field* over F , with slope functions $\bar{A}_i(y)$. Let Z_i be the corresponding unit vector at each point of F . The Weierstrass E function is then $H - A_i Z_i$ where the H and the A_i are those of any

admissible S in F through L . It can be shown [VII] that the integral

$$I^* = \int_T A_i Z_i dt$$

is invariant for S in F through L , and we have

$$I^*(M) = \int_T A_i Y_i dt = I(M).$$

It follows that

$$\begin{aligned} I(S) - I(M) &= I(S) - I^*(M) = I(S) - I^*(S) \\ &= \int_V (H - A_i Z_i) dv \\ &= \int_V E(y, \bar{A}, A) dv \end{aligned}$$

where S is defined in terms of parameters v over V . The inequality $H - A_i Z_i > 0$ or $H(1 - Y_i Z_i) > 0$ clearly holds, unless $Y_i \equiv Z_i$; since $Y_i Z_i = \cos \theta$ where θ is the angle between these unit vectors. We have then that

$$I(S) - I(M) > 0$$

unless the normal at every point of S is the same as the field normal. Under this condition, M also having the field normal, and M and S having a common boundary, it can be shown by an argument like that given by Bliss [III] that M and S are identical. The sufficiency proof is complete.

We conclude with a remark on the catenoid in 3-space, a surface obtained by rotating a catenary about its directrix. It arises in the study of surfaces of revolution of minimum area, where the problem is a simple integral problem of determining under certain conditions the arc to be rotated so as to achieve this minimum area. A thoroughgoing discussion is given by Bliss [II]. We propose to glance at the problem as a multiple integral one, in the light of the preceding sufficiency proof. Let A, B, C be points on a catenary in the xy -plane having the x -axis for directrix, and B for its vertex, the points being on the curve in the order indicated, and A and C such that tangents to the curve at A and C meet at 0 on the x -axis. Choose P and Q on the curve close to A and C respectively, and between A and C . Let the curve revolve in 3-space about the x -axis. Then the surface generated is known to have its first curvature $K_1 = 0$, and hence is minimal. Theorem 6.1 yields quickly that the surface generated minimizes among *all* admissible surfaces through the circles generated by P and Q , in a cone-shaped region F . For it is easily seen that the point 0 will do for choice of origin in xyz -space satisfying Theorem 6.1, since a little consideration shows that no tangent plane to the catenoid

through the circles generated by P and Q can pass through 0. Actually there is a small open region about 0, bounded by caps of two cones, through which no tangent plane passes.

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