THE L_2-SYSTEM OF A UNIMODULAR GROUP. I

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Introduction. Let G be a locally compact topological group which is unimodular, that is, the left and right Haar measures coincide. We propose, in this and subsequent papers, to investigate the structure of its "L_2-system." By the "L_2-system" we mean the (complex) Hilbert space formed from the complex-valued measurable functions f on G for which $|f|^2$ is integrable with respect to the Haar measure, and with a product defined in terms of convolution. This product is not always defined (for that reason we call it an L_2-system rather than an L_2-algebra) but there are dense subsets (for example, $L_1 \cap L_2$) on which multiplication always is defined so it is "almost" an algebra.

Our theorems depend only on certain key properties of these L_2-systems so we shall investigate general systems with these properties, calling them *H-systems*. We do this not only for generality but because when decomposing an L_2-system into minimal parts we expect those parts to be H-systems though they need not be the L_2-system of any group.

This paper contains the definitions of an H-system and an L_2-system; the proof that an L_2-system is an H-system; theorems on idempotents and generation of ideals by idempotents; a functional calculus of self-adjoint elements in an H-system; a simple structure theorem for abelian H-systems; and an example of the L_2-system of a special group, which has an interesting structure. Using the theorems of this paper we believe we shall be able to obtain the complete structure of H-systems—the ultimate theorems presumably asserting that every such system is a "direct integral" of simple systems, and that every simple system is a certain type of full matrix system—but with continuous rather than discrete matrices. At present these theorems are not proved but before even considering them carefully it seems necessary to develop the material of this paper as a foundation.

One might ask why we choose to investigate this L_2-system, in which multiplication is only partially defined, when there are other systems (for example, L_1—the integrable functions) which are actually algebras and whose structure is still unknown. Our belief is that in spite of having only a partially defined multiplication our system may be easier to study, in somewhat the same way that certain questions are easier to answer about unbounded hypermaximal operators on Hilbert space than about bounded operators on a general Banach space. That there is a relation between that situation and ours is shown by the fact that left (or right) multiplication by a fixed element in an L_2-system is, for many choices of the element, an unbounded hyper-

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maximal operator. It also seems that the problem of determining the structure of this L_2-system is simply not as deep a problem as that of determining the structure of some of the other systems.

We consider only unimodular groups because for the others it seems that each element of the system, instead of having a single "adjoint" (adjoints are defined below) would have two adjoints, one associated with left multiplication, the other with right multiplication. Until more is known about the unimodular case that seems too complicated to consider.

1. Definitions of L_2- and H-systems. Proof that an L_2-system is an H-system.

DEFINITION. An *H-system* is a set H of elements such that:

(1) H is a complex Hilbert space.

(2) A partial multiplication is defined in H, that is, for certain pairs, x and y, of elements in H is defined an element of H, called the *product* of x and y and denoted by xy.

(3) The set $A=[x \mid xy$ and yx are defined for every $y\in H]$ is dense in H, and is an algebra over the complex numbers (except that it need not be finite-dimensional) under the multiplication and linear operations in H. The elements of A are called *bounded* and A is called the *bounded algebra* of H.

(4) $xA=0$ implies $x=0$, and $Ax=0$ implies $x=0$, for each $x\in H$.

(5) For each $x\in H$ is defined an element $x^*\in H$, called the *adjoint* of x, such that $\|x\|=\|x^*\|$. If $x\in A$ then $x^*\in A$, and if we define the operators

L_x: $y\to xy$ defined for all y such that xy is defined in H,

R_x: $y\to yx$ defined for all y such that yx is defined in H,

l_x: $a\to xa$ defined for $a\in A$,

r_x: $a\to ax$, defined for $a\in A$,

then $L_{x^*}=L_x^*=l_x^*$ and $R_{x^*}=R_x^*=r_x^*$ (where T^* is the adjoint operator in the sense of operators on Hilbert space of the operator T, as defined in [IX, p. 42](1)).

Remarks on the definition. Perhaps the strangest part of this definition is the part of (5) asserting that $L_{x^*}=l_x^*$ and $R_{x^*}=r_x^*$. We include it because we need it for our proofs (and it holds in an L_2-system) but possibly it could be proved from the other axioms. Without it there would be, a priori, a possibility that the adjoint of l_x or r_x might have a larger domain of definition than L_x^* or R_x^* while with it we have that L_x^* is completely determined by l_x, that is, if we know how elements left- or right-multiply elements of A we know how they left- or right-multiply all elements of H, including *which* elements they do left- or right-multiply. It tells us we have $(xy, z)=(y, x^*z)$ and $(yx, z^*)=(y, zx^*)$ whenever the products involved are defined and makes the following a sufficient criterion that the product xy be defined: the existence of

(1) Roman numbers in brackets refer to the bibliography at the end of the paper.

a $z(=xy)$ such that $(z, a)=(x, zy^*)$ or such that (z, x^*a) for all $a\in A$. We shall use this criterion repeatedly.

In one way it is reasonable that we should include a condition of this kind, for it seems that we should have sufficient relation between multiplication and addition in our system. Within A we secure this when we demand that A be an algebra but that exerts no control over multiplication of elements not in A. This condition in (5) controls the relation of multiplication and addition throughout H by tying it up closely to the situation in A.

There are plenty of examples of H-systems. The L_2-system which we shall presently define is one example. Another is obtained by taking any measure space and defining H to be the Hilbert space L_2 with respect to the measure, with the ordinary point-wise product of two functions as the product in the system. Multiplication is not always defined because the product of two functions in L_2 need not be in L_2. In a sense this system is the "direct integral" or "L_2-integral" of complex number fields "with respect to" the given measure. We shall show that this is the most general kind of abelian H-system. The Plancherel theorem for abelian groups strongly includes the fact that the L_2-system (in the sense defined below) of such a group is such an L_2-integral of complex number fields and the general structure theorem at which we are aiming would be a generalization (for unimodular groups) of just this aspect of the Plancherel theorem. Another example of an H-system is an H^*-algebra [I]. Then we can form an L_2-integral of such H^*-algebras in the same way that the preceding system was considered as an L_2-integral of complex number fields, that is, we can consider functions $f(x)$ on a measure space X whose values for each x lie in some H^*-algebra H_x, then define the product of two such functions by pointwise multiplication, and consider the Hilbert space formed by those which are measurable and with $\|f(x)\|^2$ having a finite integral with respect to the given measure. We can go further than this by taking L_2-integrals of certain continuous matrix systems more general than the systems just mentioned, the continuous matrix systems themselves being H-systems. An example of one of these continuous matrix systems is given in the last section of this paper but the details about them will be reserved for later papers.

Now we want to define the L_2-system of a unimodular group G. In the remainder of this section we shall be considering functions on G and we use Weil's definitions [X, chap. II] of the spaces L and L_p over G, and also the notation $\| \ \|_p$ for the L_p-norm. Whenever we speak of convergence or topology without explicitly saying otherwise it will be understood we refer to that given by the L_2-metric.

The L_2-system of G is to be the L_2-space (with respect to Haar measure on G) with convolution for multiplication, but it is necessary to explain what we mean by "convolution for multiplication." The most obvious possibility would be to mean: for f and g in L_2 we can define a function h (using unimodularity and the Schwarz inequality) by

$$h(x) = \int f(y)g(y^{-1}x)dy$$

so the "product" of f and g might be this h if it is in L_2 and otherwise be undefined. This however will not be our definition. Our definition will be a slight variation of, and equivalent to, that in [X]; it will go as follows: for fixed $f\in L_2$ and each $g\in L$ define $h=fg$ by the above convolution formula and denote the operator, $g\rightarrow fg$, by L_f'. We extend L_f' operatorially from the subspace L of L_2 to a larger class of functions, getting an operator L_f, and then define fh if and only if h is in the domain of L_f, by $fh=L_fh$. The way of extending L_f' to L_f is this: consider L_{f^*}' (where $f^*(x)=f(x^{-1})$) and define L_f to be the Hilbert space adjoint of L_{f^*}'. In this way we come out with $L_{f^*}=L_f^*$ and, in case $f=f^*$, with $L_f=L_f^*$—so we can apply the spectral theorem to these left-multiplication operators. As just explained, this process emphasizes left-multiplication over right-multiplication but that is not really the case. If we take $h\in L_2$, define R_h' for $g\in L$ by $R_h' g=gh$ where gh is defined through the above convolution formula, and extend R_h' operatorially to an R_h in the same fashion that L_f' was extended to L_f, then we show that f is in the domain of R_h if and only if h is in the domain of L_f, and in this case $L_fh=R_hf$.

In [X] fg is defined through the convolution formula only for f and g both in L, and is extended operatorially beyond that point. Lemma 1 below—whose proof is just a specialization of a proof in [X; p. 48] (we include it for completeness)—shows the two definitions are the same. In the next few lemmas and Theorem 1 we develop enough properties of the multiplication-to-be in an L_2-system so we can define an L_2-system. Then with a few more lemmas we prove that an L_2-system is an H-system. Following sections will then be devoted exclusively to general H-systems except for an example of an interesting L_2-system in the last section.

LEMMA 1.1. *If $f\in L_2$ and $g\in L$ then for every x the y-function $f(y)g(y^{-1}x)$ is in L_1, the function h defined by*

$$h(x) = \int f(y)g(y^{-1}x)dy \tag{*}$$

is in L_2 and $\|h\|_2\leqq\|f\|_2\|g\|_1$. Similarly if $f\in L$ and $g\in L_2$ then this integral exists, gives a function in L_2, and in this case $\|h\|_2\leqq\|f\|_1\|g\|_2$. In either case we shall denote this h by fg.

Proof. We shall use here the well known fact that the convolution of two functions in L_1 exists and defines a function in L_1; this of course follows almost immediately from the Fubini theorem plus the fact that the Haar measure of a product group is the product measure of the separate Haar measures.

That for each x the y-function $f(y)g(y^{-1}x)$ is in L_1 is immediate from unimodularity and the Schwarz inequality. We prove spearately that $h\in L_2$ and $\|h\|_2\leqq\|f\|_2\|g\|_1$.

Proof that $h\in L_2$. Let C be a compact set outside which g vanishes, so $g(y^{-1}x)$ vanishes outside xC^{-1}, let $M=\sup_y |g(y)|$, and denote the characteristic function of a set A by ϕ_A. Then

$$\begin{aligned} |h(x)|^2 &= \left|\int f(y)g(y^{-1}x)dy\right|^2 = \left|\int_{xC^{-1}} f(y)g(y^{-1}x)dy\right|^2 \\ &\leqq \int_{xC^{-1}} |f(y)|^2dy\cdot\int_{xC^{-1}} |g(y^{-1}x)|^2dy \\ &\leqq \int |f(y)|^2\phi_{xC^{-1}}(y)dy\cdot m(xC^{-1})M^2 \\ &\leqq \int |f(y)|^2\phi_C(y^{-1}x)dy\cdot mC\cdot M^2. \end{aligned}$$

Since $|f|^2$ and ϕ_C are both in L_1 their convolution is in L_1, so this shows $|h|^2$ is dominated by a function in L_1. Since measurability of h follows in standard ways we have $h\in L_2$.

Proof that $\|h\|_2\leqq\|f\|_2\|g\|_1$. For fixed y we shall denote the x-function $f(xy)$ by f_y; it is again in L_2 and $\|f\|_2=\|f_y\|_2$. For any $k\in L_2$ we have

$$\begin{aligned} |(h, k)| &= \left|\int\left\{\int f(y)g(y^{-1}x)dy\right\}\bar{k}(x)dx\right| \\ &= \left|\int\left\{\int f(xy)g(y^{-1})dy\right\}\bar{k}(x)dx\right| \\ &= \left|\int\left\{\int f(xy)\bar{k}(x)dx\right\}g(y^{-1})dy\right| \\ &= \left|\int (f_y, k)g(y^{-1})dy\right| \\ &\leqq \int \|f_y\|_2\cdot\|k\|_2|g(y^{-1})|dy \\ &= \|f\|_2\|k\|_2\|g\|_1. \end{aligned}$$

Taking $k=h$ then gives $\|h\|_2\leqq\|f\|_2\|g\|_1$. (The interchange of order of integration is justified here because the function $|f(y)g(y^{-1}x)k(x)|$ clearly has a finite integral if we integrate first on y, then on x.) The corresponding statement with $f\in L$ and $g\in L_2$ is proved in the same way, or even follows from this if we use this on the group obtained from G by reversing the multiplication of G.

This lemma makes clear that Weil's definition of $h=fg$, starting with f and g both in L and extending operatorially, gives the same h as ours for $f\in L_2$, $g\in L$ and for $f\in L$, $g\in L_2$. For it shows h is continuous in $f\in L_2$ for fixed $g\in L$, hence if we let $f_n\rightarrow f$, $f_n\in L$, then the corresponding $h_n=f_ng$ approach our fg.

Definition. *For each $f\in L_2$ we define operators L'_f and R'_f on L by*

$$L'_fg = fg, \qquad R'_fg = gf \qquad\qquad (g \in L).$$

We define the adjoint f^* of f by: $f^*(x)=\bar{f}(x^{-1})$. We define operators L_f and R_f on L_2 by

$$L_f = (L'_{f^*})^*, \qquad R_f = (R'_{f^*})^*.$$

*Here the * on an operator denotes its Hilbert space adjoint* [IX, p. 42], *and L is considered as a subspace of L_2 so the operators L'_f and R'_f are defined on a dense subspace of L_2.*

We shall define our most general product in an L_2-system in terms of these operators L_f and R_f but before defining fh to be either L_fh or R_hf we want to prove these two are equal.

Lemma 1.2. *The operators L'_f, L'_{f^*} are adjoint to each other, that is, for all g, $h\in L$ we have $(L'_fg, h)=(g, L'_{f^*}h)$.*

Proof. This is proved by a well known calculation. See [X, chap. III].

Lemma 1.3. *If L_0 is a subset of L, every $l\in L_0$ satisfies $\|l\|_2<K$ for some constant K, $p_n\in L_2$, $\|p_n\|_2\rightarrow 0$ as $n\rightarrow\infty$, and $f\in L_2$, then $(fl, p_n)\rightarrow 0$ uniformly in $l\in L_0$, as $n\rightarrow\infty$.*

Proof. First note that

$$\text{(a)} \qquad \overset{l}{p}_n(y) = \int l(y^{-1}x)p_n(x)dx \rightarrow 0, \qquad \text{uniformly in } y \text{ and } l, \text{ as } n \rightarrow \infty,$$

because

$$\left|\int l(y^{-1}x)p_n(x)dx\right|^2 \leqq \int |l(y^{-1}x)|^2dx\cdot \int |p_n(x)|^2dx \leqq K\|p_n\|_2.$$

Then note that

$$\text{(b)} \qquad \begin{aligned}(fl, p_n) &= \int \left\{\int f(y)l(y^{-1}x)dy\right\} p_n(x)dx \\ &= \int \left\{\int l(y^{-1}x)p_n(x)dx\right\} f(y)dy.\end{aligned}$$

Together (a) and (b) clearly imply the lemma.

THEOREM 1. $L_f^* = L_{f^*}$.

Proof. L_f^* of course denotes the Hilbert space adjoint of L_f. By Lemma 1.2 we have $L_f \supseteq L_f'$ so, taking adjoints, $L_f^* \subseteq L_{f^*}$. It remains to show $L_{f^*} \subseteq L_f^*$. For this we must show that if g is in the domain of L_{f^*} and k is in the domain of L_f then $(L_f k, g) = (k, L_{f^*} g)$. To this end choose $g_n \to g$, $g_n \in L$, and $k_n \to k$, $k_n \in L$. Then

$$\begin{aligned}
| (L_f k, g) - (k, L_{f^*} g) | & \\
\leqq | (L_f k, g) - (L_f k, g_n) | &+ | (L_f k, g_n) - (k, L_{f^*} g_n) | \\
+ | (k, L_{f^*} g_n) - (k_m, L_{f^*} g_n) | &+ | (k_m, L_{f^*} g_n) - (L_f k_m, g_n) | \\
+ | (L_f k_m, g_n) - (L_f k_m, g) | &+ | (L_f k_m, g) - (k_m, L_{f^*} g) | \\
+ | (k_m, L_{f^*} g) - (k, L_{f^*} g) | &.
\end{aligned}$$

Of the seven terms on the right side the 1st and 7th obviously tend to 0 as m, n, the 2nd, 4th and 6th are 0 by definition of L_f and L_{f^*} together with the fact that $L_f \supseteq L_f'$, and the third and fifth tend to 0 by Lemma 1.3. Hence the theorem is proved.

LEMMA 1.4. *If f and g are in L_2 and $h \in L$ then $(g, L_{f^*} h) = (f, R_{g^*} h)$.*

Proof. This follows from an obvious direct calculation, using the fact that $L_f \supseteq L_f'$ and $R_f \supseteq R_f'$.

LEMMA 1.5. *Let f and g be in L_2. Then f is in the domain of R_g if and only if g is in the domain of L_f, and in this case $L_f g = R_g f$.*

Proof. The proof is immediate from the definitions of L_f and R_g, using Lemma 1.4.

DEFINITION. *The L_2-system of a unimodular locally compact topological group G is the L_2-space formed with respect to the Haar measure of G, with convolution for multiplication. The phrase "with convolution for multiplication" shall mean: for f and g in L_2, fg is defined if and only if f is in the domain of R_g and g is in the domain of L_f, and is defined by $fg = L_f g = R_g f$. We shall use the phrases "f left-multiplies g" or "g right-multiplies f" to mean the product fg is defined.*

THEOREM 2. *If G is a unimodular locally compact topological group then its L_2-system is an H-system.*

Proof. Properties (1) and (2) in the definition of an H-system obviously hold. To prove the others we need the following lemmas.

LEMMA 1.6. *If f left-multiplies g then f^* right-multiplies g^* and $(fg)^* = g^* f^*$.*

Proof. Let $k = fg$, and it will be sufficient to show $(k^*, l) = (g^*, lf)$ for all $l \in L$. From the formulas for fl and lf when $l \in L$ we see by an obvious calculation that $(fl)^*=l^*f^*$. Now let $l_n\to g$, $l_n\in L$, and we shall prove both: (i) $(l_n^*f^*, l)\to(k^*, l)$ for all $l\in L$, and (ii) $(l_n^*f^*, l)\to(g^*, lf)$ for all $l\in L$. This will prove the lemma.

Proof of (i). First we have $(fl_n, l)\to(fg, l)$ for all $l\in L$, because $(fl_n, l) =(l_n, f^*l)$ and $(fg, l)=(g, f^*l)$. If h_n, $h\in L_2$ and $(h_n, l)\to(h, l)$ for all $l\in L$ then we also have $(h_n^*, l)\to(h^*, l)$ for all $l\in L$ because $(h_n, l)=\overline{(h_n^*, l^*)}$ and $(h, l) =\overline{(h^*, l^*)}$ (these last equalities are clear from the integral formula expression for the inner product). Hence we have $((fl_n)^*, l)\to(k^*, l)$ for all $l\in L$. Since $(fl_n)^*=l_n^*f^*$ we have $(l_n^*f^*, l)\to(k^*, l)$ for all $l\in L$.

Proof of (ii). (ii) is clear because $(l_n^*f^*, l)=(l_n^*, lf)$.

LEMMA 1.7. *If f and g both left-multiply and right-multiply every element in L_2 then $L_{fg}=L_fL_g$. In particular fg also left-multiplies and right-multiplies every element in L_2.*

Proof. First we prove that $(fg)l=f(gl)$ for all $l\in L$. If $g\in L$ this is clear from the defining integral formulas. Now let $l_n\to g$, $l_n\in L$. By Theorem 2.26, p. 61 of [IX], and Lemma 1.6 we have $fl_n\to fg$, then $(fl_n)l\to(fg)l$, and in the same way $f(l_nl)\to f(gl)$. Hence $(fg)l=f(gl)$ for all $l\in L$.

To show $(fg)h$ is defined for all $h\in L_2$ and equals $f(gh)$ take $l_n\to h$, $l_n\in L$, and note that

$$(f(gh), l) = \lim\,(f(gl_n), l) = \lim\,((fg)l_n, l) = \lim\,(l_n, (fg)^*l) = (h, (fg)^*l),$$

which proves the lemma.

It follows that property (3) in the definition of an H-system holds in an L_2-system. To prove (4) note that $fA=(0)$ implies $fl=0$ for all $l\in L$. It follows in the usual way that the integral of f over any measurable set is 0, and hence that $f=0$; similarly for $Af=(0)$ implying $f=0$. Property (5) follows from the fact that L_f and R_f were defined as adjoints of the operators L'_{f^*} and R'_{f^*} acting on L, when we note that $L\subseteq A$.

2. **Some elementary lemmas about H-systems.** We now let H be a general H-system and A the bounded algebra of H, as explained in the definition of an H-system. If xy is defined we shall say "x left-multiplies y" and "y right-multiplies x." We remark that the adjoint x^* of x is clearly unique and that for x, y, H, and λ, μ complex numbers we have $(\lambda x+\mu y)^*=\bar\lambda x^*+\bar\mu y^*$.

LEMMA 2.1. $(x, y)=(y^*, x^*)$.

Proof. This follows from the assumption that $\|x\|=\|x^*\|$ by expressing the inner product in terms of the norm in the usual way.

LEMMA 2.2. *For $a\in A$, L_a and R_a are bounded operators on H, so ax and xa are continuous in x for fixed $a\in A$.*

Proof. Since we have assumed, for $a\in A$, that L_a and its adjoint L_{a^*}, and also R_a and its adjoint R_{a^*}, are everywhere defined this follows from Theorem 2.26 of [IX, p. 61].

LEMMA 2.3. *If x left-multiplies y then x^* right-multiplies y^* and $(xy)^* = y^*x^*$.*

Proof. It is sufficient to show that for $a \in A$ we have $((xy)^*, a) = (y^*, ax)$. This follows from: $((xy)^*, a) = (a^*, xy) = (a^*y^*, x) = (y^*, ax)$.

LEMMA 2.4. *If $a \in A$ and $x, y \in H$ then $(y, x^*a) = (x, ay^*)$.*

Proof. If x and y are in A this is trivial. To prove it for general x, y let $b_n \to x$, $b_n \in A$, and $c_n \to y$, $c_n \in A$, so also $b_n^* \to x^*$, $b_n^* \in A$ and $c_n^* \to y^*$, $c_n^* \in A$. Applying Lemma 2.2 the result follows.

The following lemma is perhaps the most important one of this section for it gives us the best associativity law we can hope to have in an H-system.

LEMMA 2.5. *If xy, $(xy)z$, and yz are all defined then $x(yz)$ is defined and equals $(xy)z$.*

Proof. We consider several cases. In each case it is sufficient to prove that for all $a \in A$, $((xy)z, a) = (yz, x^*a)$.

Case I: $y, z \in A$.

$$\begin{aligned}((xy)z, a) &= (xy, az^*) = (x, (az)^*y^*) = (x, a(z^*y^*)) = (x(z^*y^*)^*, a) \\ &= (x(yz), a) = (yz, x^*a).\end{aligned}$$

Case II: $z \in A$.

$$((xy)z, a) = (xy, az^*) = (y, x^*(az^*)) = (y, (x^*a)z^*) = (yz, x^*a).$$

Case III: *general case.* Choose $b_n \to z$, $b_n \in A$. Since xy is defined we know y^*x^* is defined, by Lemma 2.3, so by Case II we can conclude that $y^*(x^*a)$ is defined and equals $(y^*x^*)a$, for $a \in A$. Then

$$\begin{aligned}((xy)z, a) &= (xy, az^*) = \lim (xy, ab_n^*) = \lim (y, x^*(ab_n^*)) \\ &= \lim (y, (x^*a)b_n^*) = \lim (yb_n, x^*a) = \lim (b_n, y^*(x^*a)) \\ &= (z, y^*(x^*a)) = (yz, x^*a).\end{aligned}$$

3. **Functions of a self-adjoint element and existence of idempotents.** In this section we develop a functional calculus for self-adjoint elements in an H-system, that is, if x is self-adjoint we define $F(x)$ for certain complex-valued functions $F(\lambda)$ of a real variable λ. This whole procedure is very like the functional calculus for self-adjoint operators on Hilbert space, and in fact we shall make it follow from that theory. We could define these functions of x by repeating any of the standard developments of functions of a self-adjoint

operator but it will be shorter to take that theory for granted and make ours follow from it. This can be done by applying that theory to L_x or R_x and then using the close relation between x and L_x to obtain functions of x, $F(x)$, from functions of L_x, $F(L_x)$.

The method for getting functions of x from functions of L_x is suggested by the following fact: if $P(\lambda)$ is any polynomial in λ with a zero constant term (we have to assume a zero constant term because H has no identity) and x is a bounded element in H then we can of course form the function of x, $P(x)$, by replacing λ with x. But we can also look at this slightly differently by noting that if $P'(\lambda)=P(\lambda)/\lambda$ then $P(x)=P'(L_x)x$, that is, $P(x)$ is obtained by applying the operator function of L_x, $P'(L_x)$, to the element x. Since we have much more general functions of L_x defined through the operational calculus of operators on Hilbert space we can use this latter to get a more general definition of functions of x, and it works also without assuming x bounded. Given any function $F(\lambda)$ and letting $F'(\lambda)=F(\lambda)/\lambda$ we define $F(x)$ to be $F'(L_x)x$ whenever the operator $F'(L_x)$ is defined at x.

With this method we can define $F(x)$ for every function $F(\lambda)$ which is in L_2 with respect to a certain measure on the real line, namely, the measure defined for all Borel sets B by

$$mB = \int_B \frac{1}{\lambda^2}\, d\|E_\lambda x\|^2$$

where E_λ is the spectral family of the operator L_x. And the mapping $F(\lambda) \to F(x)$ will be an isomorphism of the H-system formed from this L_2-space onto the subsystem of H generated by x. In particular, taking $F(\lambda)$ to have only the values 0 and 1—so it is real and idempotent—the corresponding $F(x)$ will be a self-adjoint idempotent in H. Then because the function $F(\lambda)=\lambda$, which corresponds to x, can be approximated by step functions it follows that x can be approximated by linear combinations of self-adjoint idempotents, and finally by writing an arbitrary (that is, not necessarily self-adjoint) z in the form $z=x+iy$, with x and y self-adjoint, it follows that every element can be approximated by linear combinations of self-adjoint idempotents. All the operator theory we need for this is contained in Theorem 6.1 of [IX, p. 222].

Definition. *Let x be a self-adjoint element of H (that is, $x=x^*$) so L_x is a self-adjoint operator, and let E_λ be the spectral family of L_x. Define a measure m for all Borel sets B on the real line by*

$$mB = \int_B \frac{1}{\lambda^2}\, d\|E_\lambda x\|^2$$

and let F be the L_2-space with respect to that measure, that is, F consists of equivalence classes of complex-valued m-measurable functions F for which $\int|F(\lambda)|^2 dm$ is finite, two such functions being equivalent if and only if they *differ only on a set of m-measure* 0. *For each such function F we define the element $F(x)$ of H by*

$$F(x) = F'(L_x)x,$$

where $F'(\lambda)$ denotes the function $F(\lambda)/\lambda$.

To justify this definition we need to know that x is in the domain of $F'(L_x)$. The domain of $F'(L_x)$ is the set of all $y \in H$ such that $\int |F'(\lambda)|^2 d\|E_\lambda y\|^2$ is finite, so we need that $\int |F'(\lambda)|^2 d\|E_\lambda x\|^2$ is finite, but this is precisely the same as saying that $\int |F(\lambda)|^2 dm$ is finite. This justifies our definition and also shows that our way of defining $F(x)$ does not extend to more general $F(\lambda)$.

Our aim in this section is to prove the following three theorems:

THEOREM 3. *If $F \in F$ the operator $y \rightarrow F(x)y$ includes the operator $F(L_x)$.*

THEOREM 4. *The mapping of F into H defined by: $F(\lambda) \rightarrow F(x)$ is an isomorphism of F onto the subsystem of H generated by x. By an isomorphism we mean: it preserves sums, scalar multiples, products when they are defined, adjoints, and norms. In particular, in the H-system F the product of two elements F and G is defined if and only if the corresponding product is defined in H.*

THEOREM 5. *Every element of H is a limit of linear combinations of self-adjoint idempotents. If the element x is self-adjoint the idempotents can be chosen to commute with each other and with x, and to be mutually orthogonal, and each of them can be obtained by left-multiplying x with a suitable function of x.*

The proofs of these theorems will be carried out through a sequence of lemmas. In some places we use a small amount of integration theory where the measure has projection operators on Hilbert space as values. This could be reduced to ordinary numerical integration in the usual way, through taking inner products, but that would be really more of a complication than a simplification. A *self-adjoint idempotent* is defined as an element $e \neq 0$ such that $e = e^*$, and ee is defined and equals e. It is easy to see that such an e is in A. For it is clearly sufficient to show that $\|ea\| \leq \|a\|$ for all $a \in A$. Since we can write $a = ea + (a - ea)$ and these two summands are clearly orthogonal we have the desired result. By *mutually orthogonal* idempotents we mean idempotents e and f such that $(e, f) = 0$. If they are self-adjoint then it is trivial that this is equivalent to $ef = fe = 0$.

LEMMA 3.1. *If $a \in A$ then $(E_\lambda x)a = \int_{-\infty}^{\lambda} \mu dE_\mu a$.*

Proof. Since R_a commutes with L_x it commutes with E_λ, hence

$$(E_\lambda x)a = E_\lambda(xa) = E_\lambda(L_x a) = E_\lambda \int_{-\infty}^{\infty} \mu dE_\mu a = \int_{-\infty}^{\lambda} \mu dE_\mu a.$$

LEMMA 3.2. *If $a \in A$ and $F \in F$ then a is in the domain of $F(L_x)$ and $F(x)a = F(L_x)a$.*

Proof. First we show a is in the domain of $F(L_x)$:

$$\int |F(E_\lambda)^2 d\|E_\lambda a\|^2 = \int \left|\frac{F(\lambda)}{\lambda}\right|^2 d\int_{-\infty}^{\lambda} u^2 d\|E_\mu a\|^2$$
$$= \int |F'(\lambda)|^2 d\|E_\lambda(xa)\|^2$$
$$= \int |F'(\lambda)|^2 d\|(E_\lambda x)a\|^2$$
$$\leqq C\int |F'(\lambda)|^2 d\|E_\lambda x\|^2\|a\|^2$$
$$< \infty.$$

Now we show $F(x)a = F(L_x)a$:

$$F(x)a = \left\{\int \frac{F(\lambda)}{\lambda} dE_\lambda x\right\} a$$
$$= \int \frac{F(\lambda)}{\lambda} dE_\lambda(xa)$$
$$= \int \frac{F(\lambda)}{\lambda} d\int_{-\infty}^{\lambda} \mu dE_\mu a$$
$$= \int F(\lambda) dE_\lambda a$$
$$= F(L_x)a.$$

LEMMA 3.3. *If $F \in F$ then $\overline{F}(x) = F(x)^*$.*

Proof. From Theorem 6.1 of [IX, p. 222] we have, for $a, b \in A$,

$$(F(L_x)a, b) = (a, \overline{F}(L_x)b)$$

and then by the preceeding lemma we have the corresponding fact with $F(L_x)$ and $F(L_x)$ replaced by $F(x)$ and $\overline{F}(x)$. This implies $\overline{F}(x) = F(x)^*$.

Proof of Theorem 3. Let y be in the domain of $F(L_x)$. Then, since $a \in A$ is necessarily in the domain of $\overline{F}(L_x)$ and $\overline{F}(L_x)$ is adjoint to $F(L_x)$ (by Theorem 6.1 of [IX, p. 222]) we have

$$(F(L_x)y, a) = (y, \overline{F}(L_x)a) = (y, \overline{F}(x)a) = (y, F(x)^*a)$$

which shows $F(x)y$ is defined and equals $F(L_x)y$.

LEMMA 3.4. *The mapping of F into H taking $F(\lambda) \rightarrow F(x)$ is linear and isometric.*

Proof. Linearity is obvious and isometry is shown by

$$\begin{aligned}\|F(x)\|^2 = \|F'(L_x)x\|^2 &= \int |F'(\lambda)|^2 d\|E_\lambda x\|^2 \\ &= \int |F(\lambda)|^2 d\int_{-\infty}^{\lambda} \mu^{-2} d\|E_\mu x\|^2 \\ &= \int |F(\lambda)|^2 dm..\end{aligned}$$

LEMMA 3.5. *If F, G and $K=FG$ are in F then for every $a\in A$ the elements*

$$G(X)(F(X)a),\ F(X)(G(X)a),\ K(X)a$$

are defined and equal.

Proof. If T is an operator we shall denote the domain of T by $D(T)$. Since $a\in D(K(L_x))$ and $a\in D(F(L_x))$ (by Lemma 3.2) we have $F(L_x)a\in D(G(L_x))$ and $G(L_x)F(L_x)a=K(L_x)a$ by Theorem 6.1, part (6), of [IX, p. 222]. Then from part (7) of that same theorem we have, since $a\in D(G(L_x)$, that $G(L_x)a \in D(F(L_x))$ and

$$G(L_x)F(L_x)a = K(L_x)a = F(L_x)G(L_x)a.$$

Then, by Theorem 3, $G(x)$ left-multiplies $F(x)a$, $F(x)$ multiplies $G(x)a$, and $G(x)(F(x)a)=F(x)(G(x)a)=K(x)a$.

Proof of Theorem 4. We have proved in Lemmas 3.3 and 3.4 that the mapping $F(\lambda)\rightarrow F(x)$ is a linear isometry and preserves adjoints. From Lemma 3.5 it follows that whenever the product FG of F and G in F is defined in F so is $F(x)G(x)$ defined, and equal to $FG(x)$. By the isometry property we know that only the 0 element of F goes into the 0 of H, so this mapping is an isomorphism of F into a subsystem of H, and it is trivial that it is actually an isomorphism onto the subalgebra generated by x.

Proof of Theorem 5. Theorem 5 holds for a self-adjoint x essentially because the $E_\lambda - E_\mu$ (where E_μ is the spectral family of L_x) occur not only as operators on H but as idempotents in H, provided $0<\mu<\lambda$ or $\mu<\lambda<0$. That is, there exists an idempotent $e\in H$ such that $ey=(E_\lambda-E_\mu)y$ for all $y\in H$. This follows from our functional calculus since if we take $F(\lambda)$ to be the function taking the value 1 on the interval (λ, μ) and the value 0 elsewhere it is clear that $F(x)=e$ will have the desired properties.

Given a self-adjoint x, to approximate it by sums $\sum\lambda_i e_i$ with the e_i self-adjoint and mutually orthogonal idempotents, consider the correspondence between the subsystem generated by x in H and F. We know x corresponds to the function $F(\lambda)=\lambda$, so we first approximate this function by step functions $\sum\lambda_i f_i(\lambda)$, where $f_i(\lambda)$takes only the values 1 and 0, $f_i f_j=0$, and each f_i vanishes on some neighborhood of $\lambda=0$. This approximation is to be in the sense of the L_2-metric given by the measure m. Letting $e_i=f_i(x)$ it follows from the isomorphism of F with the subsystem of H generated by x that the corresponding $\sum \lambda_i e_i$ approximate x, and that all the properties mentioned in Theorem 5 hold.

If z is a non-self-adjoint element of H then in the usual way we can write z in the form $z = x + iy$, where x and y are self-adjoint. Then by first approximating x and y separately and adding the approximating sums we get such a sum approximating z.

4. **Generation of ideals by subsets of A and by idempotents.** In this section we derive results that tend to reduce the study of ideals in an H-system to the study of ideals in the algebra A, and to a study of the self-adjoint idempotents in A. For example, we show that every left ideal in H is generated by the elements of A contained in it, and that every self-adjoint element in the ideal can be approximated by idempotents in the ideal. We are also concerned with choosing a maximal abelian family of self-adjoint idempotents and the generation of ideals by the idempotents in such a family. Such families clearly exist because once we know that self-adjoint idempotents exist an application of the Hausdorff principle gives maximal families of them. One of the things we shall prove is that for such a family the collection of left ideals generated by all subfamilies of it contains all 2-ideals. This is an analogue of the familiar theorem about the uniqueness of the decomposition into 2-ideals, for a semi-simple finite-dimensional algebra. Of course we have to be a little bit careful how ideals are defined since not any two elements can be multiplied, but we prove the only two reasonable ways of defining this concept are really the same.

DEFINITION. *A left ideal in H is a closed linear subspace L such that if $l \in L$ and $a \in A$ then $al \in L$. Similarly we define a right ideal as a closed linear subspace R such that $RA \subseteq R$, and a 2-ideal as a set which is both a left and right ideal.*

It is trivial that if L is a left (or right, or 2) ideal so is $L^\perp$; also that if L is a left ideal then L^* (the set of all adjoints of elements in L) is a right ideal and conversely.

LEMMA 4.1. *If L is a left ideal and x an element of H such that $Ax \subseteq L$ then $x \in L$.*

Proof. Write $x = x_1 + x_2$ with $x_1 \in L$, $x_2 \in L^\perp$. Then $ax = ax_1 + ax_2$ and $ax_1 \in L$, $ax_2 \in L^\perp$. Since $ax \in L$ and $ax_1 \in L$ we have $ax_2 \in L \cap L^\perp$ so $ax_2 = 0$ for all $a \in A$. Hence $x_2 = 0$ and $x = x_1 \in L$.

LEMMA 4.2. *If $y_n \to y$, xy_n and xy are all defined, then $(xy_n, a) \to (xy, a)$ for all $a \in A$.*

Proof. Because $y_n \to y$ we have $y_n^* \to y^*$ and then $ay_n^* \to ay^*$ by Lemma 2.2. Hence $(xy_n, a) = (x, ay_n^*) \to (x, ay^*) = (xy, a)$.

A reasonable alternative to our definition of a left ideal might have been this: a closed linear subspace L such that if $l \in L$ and xl is defined then $xl \in L$.

The following lemma shows this definition is equivalent to that given above. This lemma even shows more for it contains a statement asserting the existence of certain products.

LEMMA 4.3. *Let L be a left ideal and denote the projection in L of any $x \in H$ by x'. Then if xy is defined so is xy' and $xy' = (xy)'$.*

Proof. First note that if a, $b \in A$ then $(ab)' = ab'$. This is clear because writing $b = b' + b''$ $(b'' \in L^\perp)$ we have $ab = ab' + ab''$, and $ab' \in L$, $ab'' \in L^\perp$, since L and $L^\perp$ are left ideals.

Next suppose $x \in H$ and $b \in A$ and we shall prove the lemma in this case by showing $((xb)', c) = (b', x^*c)$ for all $c \in A$. For this let $a_n \to x$, $a_n \in A$, so also $a_n^* \to x^*$ and then $a_n^* c \to x^* c$ by Lemma 2.2. Then $((xb)', c) = \lim\ ((a_n b)', c) = \lim\ (a_n b', c) = \lim\ (b', a_n^* c) = (b', x^* c)$.

Now consider the general case, x and y in H, and again it is sufficient to show $((xy)', c) = (y', x^*c)$ for all $c \in A$. This is shown by $((xy)', c) = (xy, c') = (y, x^*c') = (y, (x^*c)') = (y', x^*c)$.

THEOREM 6. *If L is a left ideal in H then L equals the closure of $(L \cap A)^n$ for every positive integer n.*

Proof. This is clear for $L = (0)$ so assume $L \neq (0)$, and consider first the case $n = 1$. We show first that L contains a nonzero element of A. Every $a \in A$ can be expressed as $a = a' + a''$ with $a' \in L$, $a'' \in L^\perp$, and for some a we must have $a' \neq 0$ since otherwise we would have $L^\perp = H$ and $L = (0)$. We do not know $a' \in A$ but by Lemma 4.3 we do know xa' is defined for all $x \in H$. We define $b = a'^* a'$ and we have $b \neq 0$ for if $b = 0$ then $bc = 0$ for all $c \in A$, hence $0 = (bc, c) = (a'^* a' c, c) = (a'c, a'c)$, that is, $\|a'c\|^2 = 0$ for all $c \in A$—a contradiction. Since b is self-adjoint (using Lemma 2.3) we know that among the functions of it can be found a self-adjoint idempotent. Moreover this idempotent can be obtained by left-multiplying b with some element of H (as stated in Theorem 5) so it is in L by Lemma 4.3 and is in A because it is self-adjoint and idempotent. If we define $L_1 =$ the closure of $L \cap A$ then clearly L_1 is a left ideal and L_1 must equal L for otherwise $L \cap L_1^\perp$ would contain no elements of A but 0.

Now consider a general n. Let L_1 be the closure of $(L \cap A)^n$ and since $L_1 \subseteq L$ it is sufficient to prove that $L \cap L_1^\perp = (0)$. Then from the case $n = 1$ it is sufficient to prove $L \cap L_1^\perp \cap A = (0)$. So let $a \in L \cap L_1^\perp \cap A$ and it is sufficient to prove $a = 0$. Consider $(aa^*)^n a$, which belongs both to $(L \cap A)^n$ and L_1, and hence is 0. If $a \neq 0$ then $a^*a \neq 0$ for $a^*a = 0$ would imply $(a^*ab, b) = \|ab\|^2 = 0$ for all $b \in A$, hence $aA = 0$—which shows $a^*a \neq 0$. Applying this a sufficient number of times clearly implies $(aa^*)^n a \neq 0$—a contradiction.

LEMMA 4.4. *If L is a left ideal then every self-adjoint element of L can be approximated by a linear combination of self-adjoint idempotents in L.*

Proof. The proof is the same as that used in the above theorem, for given a self-adjoint x we consider the approximating sums $\sum \lambda_i e_i$, noting that each e_i and hence each sum can be obtained by left-multiplying x with an appropriate function of x.

THEOREM 7. *Every 2-ideal I is self-adjoint, that is, contains with each x its adjoint x^*.*

Proof. It is sufficient, by Theorem 6, to show that if $a \in A \cap I$ then $a^* \in I$. If $a \in A \cap I$ then $(Aa^*, I^\perp) = (A, I^\perp a)$ and $I^\perp a = (0)$ since every element of $I^\perp a$ would have to be both $I^\perp$ and I. Hence $Aa^* \subseteq I$, and then by Lemma 4.1 $a^* \in I$.

LEMMA 4.5. *If I is a 2-ideal then every $x \in I$ is a limit of linear combinations of self-adjoint idempotents, each of which is in I.*

Proof. Using Theorem 7 it is easy to see that a 2-ideal is an H-system when considered by itself, so this follows from Theorem 5. Or the proof of Theorem 5 can be repeated using Theorem 7 to gain the added information that all adjoints involved are in I.

LEMMA 4.6. *If L is a left ideal, e a self-adjoint idempotent, e_1 the projection of e into L, and e_2 the projection of e into L, then*

(1) *$ee_1 = e_1$ and $ee_2 = e_2$,*

(2) *$e_1 e_2^*$ is defined and equals* 0,

(3) *$e_1 e_2^*$ is defined, equals $e_1 e = ee_1^*$, and is self-adjoint.*

Proof. We have $e_1 + e_2 = e = e^2 = e(e_1 + e_2) = ee_1 + ee_2$. Because L and $L^\perp$ are left ideals we have $ee_1 \in L$, $ee_2 \in L^\perp$, hence $ee_1 = e_1$ and $ee_2 = e_2$. For (2) note that for all $a \in A$, $(0, a) = 0 = (e_1, ae_2)$. For (3), $e_1 e_1^*$ is defined because $e_1 e_2^*$ and $e_1 e$ are both defined, and $e_1^* = e - e_2^*$. Then $e_1 e = e_1(e_1^* + e_2^*) = e_1 e_1^* + 0 = e_1 e_1^*$, showing $e_1 e$ self-adjoint. Hence $e_1 e = (e_1 e)^* = ee_1^*$.

LEMMA 4.7. *The projection of a self-adjoint idempotent into a 2-ideal is again a self-adjoint idempotent.*

Proof. Let I be the 2-ideal, e the self-adjoint idempotent, and write $e = e_1 + e_2$, with $e_1 \in I$, $e_2 \in I$. Then $e_1 + e_2 = e = e^2 = (e_1 + e_2)(e_1 + e_2) = e_1^2 + e_1 e_2 + e_2 e_1 + e_2^2$, and because $e_1 e_2 = e_2 e_1 = 0$ we have $e_1 = e_1^2$ and $e_2 = e_2^2$, so the projection e_1 is idempotent. Because $e_1 + e_2 = e = e^2 = (e_1 + e_2)e = e_1 e + e_2 e$ we have $e_1 e = e_1$ and since $e_1 e$ is self-adjoint this proves e_1 self-adjoint.

LEMMA 4.8. *Let L_1 and L_2 be left ideals. Then L_1 is orthogonal to L_2 if and only if for every $a_1 \in L_1 \cap A$ and every $a_2 \in L_2 \cap A$ we have $a_1 a_2^* = 0$.*

Proof. If L_1 is orthogonal to L_2 then for any $a \in A$ and a_1, a_2 as mentioned we have $0 = (a_1, aa_2) = (a_1 a_2^*, a)$, hence $a_1 a_2^* = 0$. If we always have $a_1 a_2^* = 0$ then $(a_1, aa_2) = (a_1 a_2^*, a) = 0$ showing $L_1 \cap A$ orthogonal to $(L_2 \cap A)^2$. By

Theorem 6 this implies the orthogonality of L_1 and L_2.

LEMMA 4.9. *If e and f are commutative self-adjoint idempotents, I_1 and I_2 are orthogonal 2-ideals, and e_i and f_i are the projections of e and f into I_i then all the elements e_i, f_i, e, f commute with each other.*

Proof. This proof follows the stereotype set above.

In the following E will denote a maximal abelian family of self-adjoint idempotents in H. As we mentioned in the introduction to this section the existence of such an E is trivial. We shall be concerned with left ideals generated by subsets of E, and if $E_1 \subseteq E$ we shall denote the left ideal generated by E_1 by $L(E_1)$.

THEOREM 8. *If E is a maximal abelian family of self-adjoint idempotents then every 2-ideal I has the form $L(E_1)$ for some $E_1 \subseteq E$.*

Proof. First we must prove that I contains some $e \in E$. If e is any element of E write $e = e_1 + e_2$ with $e_1 \in I$, $e_2 \in I^{\perp}$. Then e_1 and e_2 are self-adjoint idempotents and commute with E and with each other by Lemmas 4.7 and 4.9 so we only need show $e_1 \neq 0$ for some $e \in E$. If every $e \in E$ had $e_1 = 0$ then by choosing a self-adjoint idempotent $f \in I$ we would have f orthogonal to E. But if f is orthogonal to E then we have $fe = ef = 0$ for all $e \in E$, so that $f \in E$—a contradiction.

Now let $E_1 = E \cap I$ and we shall prove $I = L(E_1)$. Since $L(E_1) \subseteq I$ it is sufficient to prove $L = I \cap L(E_1)^{\perp}$ is (0). If not then it contains a self-adjoint idempotent f, and we shall show $f \in E$ by showing f is orthogonal to E; this will be a contradiction and hence show $L = (0)$. Consider any $e \in E$ and express $e = e_1 + e_2$ with $e_1 \in I$, $e_2 \in I^{\perp}$. Then $(f, e_2) = 0$ because $f \in I$, and $e_2 \in I^{\perp}$, and f is orthogonal to e_1 because $e_1 \in I \cap E_1 \subseteq L(E_1)$. Hence f is orthogonal to e.

LEMMA 4.10. *If E_1 and E_2 are subsets of E then $L(E_1)$ is orthogonal to $L(E_2)$ if and only if $L(E_1) \cap L(E_2) = (0)$.*

Proof. Clearly the orthogonality of $L(E_1)$ and $L(E_2)$ implies their intersection is (0). For the other direction it is sufficient, since $L(E_1)$ consists of all limits of finite expressions $\sum x_i e_i$ with $e_i \in E_1$ and $L(E_2)$ consists of all limits of expressions $\sum y_i e_i$ with $e_i \in E_2$, to prove that $(xe, ye') = 0$ for all $e \in E_1$ and $e' \in E_2$. Since $ee' = e'e = 0$ we have $(xe, ye') = (xee', y) = (0, y) = 0$.

DEFINITION. *If e_1, e_2 are self-adjoint idempotents we define $e_1 \leqq e_2$ to mean that $e_1 e_2 = e_2 e_1 = e_1$.*

If e_1 and e_2 are self-adjoint idempotents with $e_1 \leqq e_2$ then for any x we have $\|x - xe_2\| \leqq \|x - xe_1\|$ since $x - xe_1 = (x - xe_2) + (xe_2 - xe_1)$ and trivial calculations show these summands are orthogonal.

The following theorem shows that we have something like an approximate identity in an H-system, and that furthermore this approximate identity can be chosen from any maximal abelian family of self-adjoint idempotents E.

We say "something like" because all it says is that for each x we can find a sequence of elements $\{e_n\}$ in E such that $x=\lim xe_n$. If the H-system is separable than a single sequence can be found to work for all x but this is definitely not the case if H is nonseparable. This theorem really shows a little more for it says that if x is in one of the ideals $L(E_1)$ with a subset E_1 of E which is closed under finite sups (in the partial ordering just defined) then the e_n can be chosen in E_1.

THEOREM 9. *If E is a maximal abelian family of self-adjoint idempotents, E_1 is a subset of E closed under finite sups in the partial ordering defined above, and $x\in L(E_1)$ then there exists an increasing sequence of elements $e_n\in E_1$ such that $x=\lim xe_n$.*

Proof. Let K be the set of $x\in L(E_1)$ for which such a sequence of e_n's exists. It is of course sufficient to show K is a left ideal. It is clear that K has the algebraic properties necessary for being an ideal (though to show it closed under finite sums requires the use of Lemma 4.10) so we only need prove K closed. Let then $x^m\to x$, $x^m\in K$, and $x^m=\lim x^m e_k^m$. Then for each m choose n_m such that $\|x^m-x^m e_{n_m}^m\|<1/m$ and define $e^m=\sup(e_{n_1}^1,\cdots,e_{n_m}^m)$. It is easily seen that $x^m e_m\to x$, hence $xe_m\to x$.

5. **Abelian H-systems.** We mentioned previously that one example of an H-system can be obtained by taking any measure space X and forming the (complex) L_2-space with respect to this measure, defining the product in this system in terms of point-wise multiplication of functions. In particular, if f, $g\in L_2$ then fg is defined in the system if and only if this point-wise product is in L_2. Since we have reserved the name L_2-system for a different kind of H-system we shall call a system of this type a *scalar-system*, since the functions considered take only complex numbers, or scalars, as values. Now we prove that every abelian H-system is of this type. We define an *abelian* H-system to be one in which the bounded algebra A is abelian, and it is clear this is equivalent to saying that whenever xy is defined then yx is defined and equal to it.

THEOREM 10. *Every abelian H-system is a scalar system. It has an identity if and only if the measure space involved has finite measure.*

Proof. The second part of the theorem is immediate once the first half has been proved. To prove the first half consider the family of all self-adjoint idempotents. This is a Boolean ring (under the operations of product and join, if we define the product to be the ordinary algebraic product and define the join of two idempotents e and f to be $e+f-ef$) which is "almost" a σ-measure ring (that is, a σ-ring with a countably additive measure) if we define the measure of an idempotent e by $me=\|e\|^2$. We say "almost" a σ-measure ring because countable unions of such idempotents will not in general be in the system. But we form the enlarged Boolean ring obtained by adjoining all such unions to the system and this will be a σ-measure ring if we define the measure of any new element to be ∞. A more careful way of handling this situation is to say that in the enlarged Boolean ring, which is clearly a σ-ring, the idempotents of H form a ring on which we have a measure. This measure is countably additive on this ring, because if $e_1, \cdots, e_n, \cdots$ are in H and $e_ie_j=0$ for $i\neq j$ then e_i is orthogonal to e_j so that by Hilbert space trivia we have $m(\bigcup e_i)=\|\bigcup e_i\|^2=\sum\|e_i\|^2=\sum me_i$, and by the Kolmogoroff extension theorem [VII] this measure can be extended to the whole σ-ring to obtain a σ-measure ring. Once this is done we apply a known theorem [II, III] asserting that such a σ-measure ring is the σ-ring of measurable sets mod null sets in some measure space. This way we obtain the measure space and it remains to identify the elements of H with functions in L_2 on this measure space in such a way that sums, scalar multiples and products are preserved. We shall outline this procedure but shall not elaborate the details for they are quite standard. Having established a correspondence between idempotents and sets (or really, equivalence classes of sets) one extends linearly to a correspondence between linear combinations of idempotents and step functions on the measure space. Then since every element in H is a limit of linear combinations of idempotents and every function in L_2 over this measure space is a limit (in the L_2-metric) of step functions we make limits of corresponding sequences correspond, thus setting up the complete correspondence and proving the theorem.

6. **An example.** We discuss here an example of an L_2-system which is a "full matrix system" of continuous matrices, and which shows better than any previous examples the sort of continuous matrix system into which a general H-system can probably be decomposed. This example is due to Murray and von Neumann [VI, p. 788] and our whole discussion merely amounts to pointing out that their theory can be reformulated in our terms to show that the L_2-system of a certain group G is such a continuous matrix system. It is a discrete group G in which the ring generated (in the sense of Murray and von Neumann) by all operators $U_a: f(x)\rightarrow f(xa)$ ($f\in L_2$, that is, $\sum_x|f(x)|^2$ is finite) is an approximately finite factor of type II_1. If M denotes their ring then our L_2-system is just the completion $Q(M)$ of M with respect to the metric $[[\ ]]$, and as such has been considered by Murray and von Neumann [V]. However they did not point out that $Q(M)$ could be considered as an algebra of continuous matrices; we shall point out that their theory shows that $Q(M)$ in this case is just such a matrix system. First we define this $Q(M)$ as an H-system, then we use their theory to show that it is just the $Q(M)$ of the ring M defined above through the operators U_a.

We define here a particular H-system, to be denoted by H_0, then afterwards we consider its relation to the group G. Let X be the unit square of points (x, y). We shall put a measure on X and the Hilbert space of this particular system will be the L_2-space with respect to this measure. To define the measure m consider, for each pair of positive integers k and p, the segment s_{kp} in which the line: $x+y=k/2^p$ cuts the unit square. A subset of s_{kp} is to be measurable if and only if it is linearly Lebesgue measurable considered as a subset of the real line, and the measure of such a set shall be its Lebesgue linear measure divided by $2^{1/2}$. (The factor $2^{1/2}$ is used simply to normalize the system. With it so normalized the identity will have norm 1.) All countable unions of such linearly measurable sets on segments s_{kp} will be the collection of measurable sets and having prescribed the measure within each s_{kp} it is defined for all such measurable sets through the assertion that it is to be a countably additive measure. This defines m and as we said above the Hilbert space of the H-system H_0 is to be the one formed from the set of complex-valued measurable functions (for this measure m) for which $|f|^2$ is m-integrable. Now we have to define multiplication and we shall do this through consideration of a special subalgebra B, which is the union of a sequence of subalgebras $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots$ where each B_n is isomorphic to the full matrix algebra of all $2^n \times 2^n$ matrices with complex elements. For each positive integer n, B_n is defined by: B_n is to consist of those functions $f=f(x, y)$ in H for which both: (1) $f(x, y)$ vanishes except on the $2^{n+1}-1$ segments $s_{1,n}, \cdots, s_{2^{n+1}-1, n}$ and (2) if we divide the segment $s_{k,n}$ into successive intervals of measure $1/2^n$ then f is constant on each of these intervals. In other words, to define B_n write down a $2^n \times 2^n$ matrix in the unit square, each element of the matrix filling a square of size $2^{-n} \times 2^{-n}$. Then in each of these small squares take the main diagonal (from upper left to lower right in the small square) and consider the function taking the same value on that diagonal as the value of the matrix element filling that square, and taking the value 0 at all points not on the main diagonal of any of the small squares. We define multiplication in B_n to be the same as for the finite matrices from which it was formed, that is, if f, $g \in B_n$ we define their product $h=fg$ by: $h(x, y) = \sum_z f(x, z)g(z, y)$, where this sum is over all real c in the unit interval. Since this summand is 0 except for a finite number of z-values (for each x and y) this multiplication is well defined. Clearly we have $B_n \subseteq B_{n+1}$ and we define $B = \bigcup B_n$. Since multiplication is defined in each B_n it is now defined in B, and B is a subalgebra of H. We can give the complete definition of multiplication in H by extending the definition operatorially from B to H (since B is dense in H) but we shall not give the details of this for they go in a by now familiar way.

Having defined H_0 above we want to show that the theory of Murray and von Neumann implies it is the L_2-system of a particular group, namely the group G of all finite permutations of the integers. (A permutation of the integers is *finite* if and only if it moves but a finite number of integers.) We show this by showing that both H_0 and the L_2-system of this G are the completion of an approximately finite factor of type II_1 and since all such factors are equivalent, their completions are equivalent. Every part of this argument will depend on the work of Murray and von Neumann, though it is probably not too difficult to give a direct treatment.

First we want to point out how the H-system H_0 can conveniently be considered as the completion of a family of operators on Hilbert space, and for this we first consider B as operators. The matrices in B can be made into operators in the same way as discrete matrices usually are, but furthermore they can all be made into operators on "the same" Hilbert space by allowing them to act on "continuous" column vectors, that is, on functions on a measure space, which are written as columns alongside the matrices of B. More precisely, let V be the L_2-space formed from the unit interval with respect to Lebesgue measure. Then for any $f \in B$ and any $v \in V$ we define, for t in the unit interval,

$$(fv)(t) = \sum_x f(x, t)v(x).$$

Since $f \in B$ this sum is finite for each t, so clearly $fv \in V$ and the mapping $v \to fv$ realizes the f's in B as operators on the Hilbert space V. It is easily seen that the family of all these operators, when completed to a ring M in the sense of Murray and von Neumann, becomes a factor and approximate finiteness follows immediately if we use Definition 4.6.1 of [VI, p. 777], with our B_n for the N_n of that definition.

Since Murray and von Neumann show that all approximately finite factors are equivalent [Theorem XII, p. 778 of VI] it is sufficient to show that both: (1) each of the H-systems H_0 and $L_2(G)$ is the completion of an approximately finite factor with respect to Murray and von Neumann's $[[\ ]]$-norm, (2) in each case their $[[\ ]]$-norm is the same as our Hilbert space norm. We first consider (2). If we observe that in B we can define a trace by

$$Trf = \int f(x, x)dx$$

and that our L_2-norm is then defined by $\|f\|^2 = Trff^*$, where $f^*(x, y) = \bar{f}(y, x)$, then the uniqueness of the trace [V, p. 218] and the fact that their norm is also defined this way in terms of the trace implies (after properly extending our trace to M) that $\|f\| = [[f]]$. That these two norms are the same in $L_2(G)$ is proved in Lemma 5.3.6 [VI, p. 792] so point (2) above is taken care of. For point (1) we simply note that both H_0 and $L_2(G)$ are clearly the completion of the corresponding rings M (in the first case M is the ring generated by B, and in the second case the ring generated by the U_a's) in our $\|\ \|$-norm.

Added in proof. In a paper entitled *Unitary rings*, C. R. (Doklady) Acad. Sci. URSS. N.S. vol. 59 (1948) pp. 643–646, V. Rohlin has discussed what we call H-systems, giving for the abelian case much more complete results than ours.

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