THE REPRESENTATION OF ABSTRACT MEASURE FUNCTIONS

BY

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1. Introduction.

1.1. Several authors have discussed abstract "measures" and "integrals," the values of which need not be numerical\(^{1}\). Our object here is to show that, in a sense, such abstract measures and integrals are often equivalent (under fairly general hypotheses) to combinations of ordinary numerical ones. We shall here consider only the theory of measure, to which the theory of abstract-valued integration can in most cases be reduced. The main theorem (3.1 (f) and §19) can be stated, roughly, as follows: a field of sets with an abstract measure on it is isomorphic to an ideal in the direct product of two such fields, on one of which the measure is an ordinary numerical one, while on the other factor the measure is trivial. (Precise definitions of the concepts involved will be given later.) As a consequence of this, the abstract measure can be replaced, to within isomorphism, by one whose values are (real-valued) numerical functions on a certain set, modulo a certain class of null functions.

We conclude the paper with a few deductions from this theorem, reserving however the application to abstract-valued integration for a subsequent paper.

1.2. Formulation of the problem. Instead of dealing directly with a field \(F\) of sets, on which a suitable abstract-valued measure \(\lambda\) is defined, it is more convenient to work with the Boolean algebra \(E\) of the sets of \(F\) modulo null sets—that is, with the algebra formed by the equivalence classes \(\{x\}\) of sets belonging to \(F\), where \(y \in \{x\}\) means \(\lambda(x + 2y) = 0\)\(^{2}\), the "measure" on \(E\) being given by setting \(\lambda(\{x\}) = \lambda(x)\). (That this relation is an equivalence relation, that \(\lambda(x) = \lambda(y)\) whenever \(y \in \{x\}\), and that the equivalence classes \(\{x\}\) do form a Boolean \(\sigma\)-algebra\(^{3}\) are in effect included in our postulates on \(\lambda\).) Thus we provisionally define an abstract measure algebra to be an ordered pair \((E, \lambda)\), where \(E\) is a Boolean \(\sigma\)-algebra and \(\lambda\) is an abstract measure-function on \(E\), satisfying "reasonable" postulates (as yet unspecified). Two abstract measure algebras \((E_1, \lambda_1)\) and \((E_2, \lambda_2)\) are defined to be isomorphic if there exists an isomorphism of the algebras \(E_1\) and \(E_2\), say \(\phi(x_1) = x_2\), such

\(^{1}\) We shall not be directly concerned here with abstract-valued integrals or with the closely related real-valued integrals of abstract-valued functions. For abstract-valued measures, see for example [4], [5], [10], and [14, §8] in the bibliography at the end of the paper.

\(^{2}\) See 1.3 for the notations.

\(^{3}\) See, for example [2, pp. 29, 88].

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that $\lambda_2(\phi(x_i)) = \lambda_2(\phi(y_i))$ if and only if $\lambda_1(x_i) = \lambda_1(y_i)$. If we further have $\lambda_2(\phi(x_i)) = \lambda_1(x_i)$ for a suitable isomorphism $\phi$, then $(E_1, \lambda_1)$ and $(E_2, \lambda_2)$ are said to be \textit{isometric}.

We shall here be interested in abstract measure algebras only to within isomorphism; and this permits a further reduction of the problem. In $(E, \lambda)$ we define the \textit{induced} equivalence relation, $\sim$, by writing $x \sim y$ to mean $\lambda(x) = \lambda(y)$. The properties of $(E, \lambda)$ to within isomorphism are evidently determined completely by this equivalence relation. Accordingly we define an \textit{abstract measure algebra} to be either an ordered pair $(E, \lambda)$ as before, or an ordered pair $(E, \sim)$, where $E$ is a Boolean $\sigma$-algebra and $\sim$ is an equivalence relation on $E$ satisfying the postulates to be given below. If $\sim$ is the equivalence relation induced by $\lambda$, we say that $(E, \lambda)$ and $(E, \sim)$ are \textit{naturally isomorphic}. The definition of isomorphism between two abstract measure algebras is now clear; for example, $(E_1, \sim)$ and $(E_2, \sim)$ will be isomorphic if and only if there exists a mapping $\phi$ of $E_1$ on $E_2$ such that (i) $\phi$ is an algebraic isomorphism and (ii) $\phi(x_i) \sim \phi(y_i)$ if and only if $x_i \sim y_i$.

1.3. \textit{Notations}. As far as practicable, we adhere to the following notations. $E$ denotes a Boolean $\sigma$-algebra having $o$ as zero element and $e$ as unit element. Small letters $a, b, x, y$, and so on, usually denote elements of $E$, except that $m, n, i, j$ usually denote positive integers; capital letters $X, Y$, and so on, usually denote subsets of $E$ (or of some other set). Phrases like “$x \in E$, $X \subseteq E$” are generally omitted when the meaning is clear. The partial ordering relation in $E$ is written $\leq$, as usual; thus $x < y$ means $x \leq y$ and $x \neq y$. (But $X \subseteq Y$ does not exclude the equality of $X$ and $Y$.) The supremum of two elements of $E$ is written $x \lor y$, and of an infinite collection is written $\bigvee x_i$; infima are written $x \land y$. We use $aX$ to denote the set of all elements of the form $ax$, where $x \in X$.

If $xy = o$, we say that $x$ and $y$ are disjoint. A subset of $E$ is said to be disjoint if every two of its elements are disjoint.

The complement of $x$ is denoted by $\neg x$, and $x - y$ is used as an abbreviation for $x(-y)$. The symmetric difference of $x$ and $y$ is the element $(x - y) \lor (y - x)$; we shall usually denote this by $x + y$ (“addition mod 2”).

$E$ will generally be assumed to satisfy the \textit{countable chain condition}, namely, there does not exist an uncountable disjoint subset of $E$. As is well known, this will imply that $E$ is \textit{complete} (that is, that for every $X \subseteq E$ both $\bigvee \{x \mid x \in X\}$ and $\bigwedge \{x \mid x \in X\}$ exist).

The words “algebra,” “sub-algebra,” used unqualified always refer to Boolean $\sigma$-algebras. A Boolean algebra in which only finite operations are to be considered will be called a “finitely additive algebra.”

For any $a \in E$, the \textit{principal ideal} $E(a)$ of $a$ in $E$ is the set of all $x \in E$ satisfying $x \leq a$. $E(a)$ is evidently itself an algebra with $\leq$, $\lor$, $\land$, having the same meaning as in $E$, but having $a$ as unit element instead of $e$. A sub-algebra of $E(a)$ will be called a sub-algebra of $E$ \textit{relative to} $a$. Similarly, if $F$ is a
subalgebra of £, and \( b \in F \), \( F(b) \) denotes the principal ideal of \( b \) in \( F \); it is clear that \( F(b) = bF \), but the notation \( F(b) \) is used to emphasize that \( b \in F \).

Given \( X \subseteq E(a) \), the sub-algebra relative to \( a \) which it generates is called the **Borel field of \( X \) relative to \( a \)**, and is written \( B_a(X) \); it is thus the smallest set \( B \subseteq E(a) \) which satisfies (i) \( B \supseteq X \), (ii) if \( x \in B \), then \( a - x \in B \), (iii) if \( x_n \in B \) (\( n = 1, 2, \ldots \)) then \( \forall x_n \in B \). If in particular \( a = e \), we omit the phrase "relative to \( a \),” and write the Borel field of \( X \) as \( B(X) \).

The usual notations of set theory are employed for subsets of \( E \) (or of other sets), the union of \( X \) and \( Y \) being written \( X \cup Y \), and their intersection \( X \cap Y \), and so on. The empty set is denoted by \( 0 \). The cardinal number of a set \( X \) is written \( |X| \).

Greek letters are generally used as follows: \( \alpha, \beta, \) and so on, are ordinal numbers; \( \lambda \) is an abstract measure, and \( \mu \) is an ordinary numerical one; \( \theta, \phi, \psi \), are mappings; \( \rho, \sigma, \tau \), are non-negative real numbers not exceeding \( 1 \), \( \rho \) usually being rational; and \( \epsilon \) is a positive number less than \( 1 \).

**Sequences** (indicated by the notation \( \{x_n\} \)) can usually be either finite or (countably) infinite, except where the distinction is emphasized (for example, by writing \( n = 1, 2, \ldots, \infty \)).

Where the abstract measure or equivalence relation intended is clear, we often write \( (E, \lambda) \) or \( (E, \sim) \) simply as \( E \).

2. **Postulates.**

2.1. Since the equivalence relation \( \sim \), rather than the abstract measure \( \lambda \), will be used in what follows, it is simpler to state our postulates in terms of \( \sim \). (It would be easy but tedious to deduce them from "reasonable” postulates on \( \lambda \); and conversely the main theorem of this paper, stated in 3.1(f), will show that every equivalence relation satisfying our postulates is in fact induced by such a "reasonable" \( \lambda \).) The postulates for the abstract measure algebra \( (E, \sim) \) are:

*0* \( E \) is a Boolean \( \sigma \)-algebra satisfying the countable chain condition, and \( \sim \) is an equivalence relation on \( E \).

1. If \( x = \vee x_n, \ y = \vee y_n \), where \( x_m x_n = \phi = y_m y_n \) (\( m \neq n ; m, n = 1, 2, \ldots \)), and if, for each \( n, x_n \sim y_n \), then \( x \sim y \).

2. If \( x \sim x' \), and if \( y \leq x \) is given, there exists \( y' \leq x' \) such that \( y \sim y' \).

Before stating the last postulate, we need the following definition. An element \( x \) is **bounded** if, whenever \( y \leq x \) and \( y \sim x \), we have \( y = x \).

3. If \( x \sim y \) there exist bounded elements \( x_n, y_n \) (\( n = 1, 2, \ldots \)) such that \( x = \vee x_n, y = \vee y_n, x_m x_n = \phi = y_m y_n (m \neq n) \), and \( x_n \sim y_n \).

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(4) The set of values taken by \( \lambda \) would be, for example, a lattice-ordered Abelian semi-group. Postulate I, for example, would be replaced by the requirement that \( \lambda \) be countably additive. One could deduce from [6, Theorem 10] that there is no loss in assuming the values of \( \lambda \) to be equivalence classes of numerical functions; but we shall derive independently a more precise result than this. For the device of replacing \( \lambda \) by \( \sim \), cf. [10] and (though with different objectives) [4] and [12, Ch. VI].
2.2. Numerical case. To illustrate the meaning of these postulates, consider a "numerical measure algebra" \((E, \mu)\), in which \(\sim\) is induced by an ordinary countably additive numerical function \(\mu\), taking non-negative real values and possibly the value \(\infty\). Suppose that there exists \(x \neq o\) for which \(\mu(x) < \infty\). If the postulates 0–III are satisfied it readily follows that the bounded elements are precisely those of finite measure, and thence that \(\mu\) is \(\sigma\)-finite (that is, \(e = \bigvee a_n\) where \(\mu(a_n) < \infty\)). Further, \(\mu\) must be "reduced" (that is, vanishes only for \(o\)). Conversely, if \(\mu\) is a numerical countably additive measure on \(E\), reduced and \(\sigma\)-finite, and if II holds, then elements of finite measure are certainly bounded, and postulates 0, I and III readily follow. Postulate II, though not automatically satisfied, can easily be seen to hold if either (i) \(E\) is non-atomic \((\xi)\), or (ii) \(E\) is generated by a number of atoms all of equal measure \((\xi)\). One would thus not expect II to be unduly restrictive in practice \((\xi)\). There are, of course, important numerical measures which satisfy neither 0 nor III—for example, Hausdorff \(m\)-dimensional measure in Euclidean \(n\)-space \((n > m)\) \((\xi)\). In what follows, the phrase "numerical measure algebra" will mean (except where the contrary is explicitly stated) one in which the postulates are satisfied, and in which the measure takes at least one finite nonzero value.

We note that, for numerical measure algebras, isomorphism implies isometry, to within a nonzero constant numerical factor.

Remark. It might seem preferable, in the interests of simplicity, to replace postulate III by the stronger requirement that all elements be bounded —corresponding in the numerical case to the restriction that \(\mu(e) < \infty\). But this would be undesirable, apart from the loss of generality, since it turns out that even then it would be necessary to introduce numerical measure algebras in which some elements have infinite measure and are therefore unbounded.

2.3. Elementary properties. The following properties of an abstract measure algebra are easy deductions from the postulates.

(1) \(e = \bigvee a_n\), where the elements \(a_n\) are bounded and disjoint.
(2) If \(x\) is bounded and \(y \leq x\), then \(y\) is bounded.
(3) \(o\) is bounded; conversely, if \(x \sim o\) and \(x\) is bounded, then \(x = o\).
(4) If \(x \sim o\), then \(x = o\). (From III, (3) and I.)

\((\xi)\) An atom is an element \(x \in E\) such that (i) \(x \neq o\), (ii) if \(o < y \leq x\), then \(y = x\).
\((\xi)\) Conversely, it is easy to see that a numerical measure algebra which satisfies the postulates is of one of the following four types: (i) non-atomic; (ii) generated by \(n\) atoms of equal measure; (iii) the direct sum of a non-atomic algebra of finite total measure and of an algebra generated by a finite number of atoms; (iv) entirely generated by atoms and of finite total measure—the possibilities in this case can be further analyzed.

\((\xi)\) Postulate II can be omitted if the values of the abstract measure \(\lambda\) are assumed to lie in a vector \(\sigma\)-lattice, for it can then be shown that the abstract measure algebra can be imbedded in one which satisfies all the postulates.

\((\xi)\) Examples of abstract measures not satisfying the postulates are: Choquet's \([4; 5]\), and Löwner's measure in Hilbert space (Ann. of Math. vol. 40 (1939) pp. 816–833).
(5) Two given elements have maximal equivalent sub-elements. More precisely; given \( a, b \in E \), there exist \( x \leq a, y \leq b \) such that (i) \( x \sim y \), (ii) if \( h \leq a - x, k \leq b - y \), and \( h \sim k \), then \( h = k = o \).

(The elements \( x \) and \( y \) are constructed by a transfinite induction argument—or equivalently by using Zorn’s lemma—the transfinite induction necessarily terminating countably because of the countable chain condition. This type of argument is often called the “method of exhaustion.”)

3. Fundamental examples.

3.1. We shall here define some particular abstract measure algebras which will play a fundamental part in the sequel.

(a) A numerical measure algebra \((J, \mu)\) (cf. 2.2). In particular, important measure algebras satisfying the postulates are: the measure algebra \(I^1\) (= \((I^1, \mu)\)) of Lebesgue measurable sets, modulo null sets, in the unit interval \((0, 1)\); and the direct product \(I^m\) of \(m\) copies of \(I^1\) (cf. [8, p. 419]), \(m\) being any cardinal number. If \(m \leq \aleph_0\), \(I^1\) and \(I^m\) are well known to be isometric. Other measure algebras which will arise in the sequel are: \(K\), the algebra generated by \(\aleph_0\) atoms all of measure 1, and \(L\), consisting of just two elements of measures 0 and 1 respectively.

(b) A “trivial” abstract measure algebra, \(U\). By definition this is one in which no two distinct elements are equivalent. Obviously any algebra \(U\) satisfying the countable chain condition can be regarded as a trivial abstract measure algebra.

(c) If \((E, \sim)\) is any abstract measure algebra, so is the relative algebra (principal ideal) \(E(a)\) (cf. 1.3), in which the equivalence relation is defined in the “natural” way; \(x \sim y\) in \(E(a)\) (where \(x, y \leq a\)) if and only if \(x \sim y\) in \(E\). The same is true, more generally, of any relative sub-algebra, provided that it satisfies postulate II.

(d) If \((E_1, \sim)\) and \((E_2, \approx)\) are abstract measure algebras (satisfying our postulates, as always), then so is their “free direct sum.” This is defined as follows. The algebra is the “direct sum” \(E_1 \oplus E_2\)—that is, the set of ordered pairs \((x_1, x_2)\) with \(x_1 \in E_1, x_2 \in E_2\), the ordering relation being given by: \((x_1, x_2) \leq (y_1, y_2)\) if and only if we have both \(x_1 \leq y_1\) and \(x_2 \leq y_2\). The equivalence relation in the direct sum is then given by: \((x_1, x_2)\) and \((y_1, y_2)\) are equivalent if and only if we have both \(x_1 \sim y_1\) and \(x_2 \approx y_2\).

REMARK. This “free direct sum” is to be distinguished from the ordinary “direct sum” of two numerical measure algebras. There the algebra is the same \((E_1 \oplus E_2)\), but the equivalence relation is different, being induced by the measure \(\mu\), where \(\mu(x_1, x_2) = \mu_1(x_1) + \mu_2(x_2)\), \(\mu_1\) and \(\mu_2\) being the measures in \(E_1\) and \(E_2\).

(e) A more complicated example is the “direct product” \(J \otimes U\) of a numerical measure algebra \(J = (J, \mu)\) (of type (a) above) and a trivial algebra \(U\) (of

(9) A numerical measure algebra can be trivial in this sense, though only if it is generated by atoms. Examples: \(L\), and the algebra generated by a sequence of atoms of measures \(3^{-n}\).
type (b)). This may be characterized as follows. Suppose \((E, M)\) is an abstract measure algebra in which the values of the abstract measure \(M\) are non-negative numerical \textit{functions} on some set, modulo some class of null functions (for example, continuous functions on some space modulo functions which vanish except for sets of first category). Suppose \(E\) contains two subalgebras \(J, U\), such that:

(i) \(E = B(J \cup U)\).
(ii) If \(x \in J\), \(M(x)\) is a constant function, \(= \mu(x)\) say.
(iii) If \(x \in U\), \(M(x) = \mu(e)\) times the characteristic function of some set (with the convention that \(\infty \cdot 0 = 0\)).

(\(E, M\)) is clearly isomorphic with a numerical measure algebra \((\mathbb{I}, \mu)\)\(^{(18)}\); and it is not hard to show that the sub-algebra \((U, M)\) is trivial. It further follows that, for any \(x \in J\) and \(u \in U\), \(M(xu) = \mu(x)M(u)\). We say that \((E, M)\) is the direct product of \((\mathbb{I}, \mu)\) and \(U\).

It will be proved in the next section that two given algebras \((\mathbb{I}, \mu)\) and \(U\) always have a direct product. This product is moreover unique to within isomorphism; we shall not prove this explicitly, but it follows by essentially the same arguments as in §17.

\(\rightarrow\) Finally, it is evident that a principal ideal (that is, relative algebra) in the free direct sum of two direct products of type (e) will be an abstract measure algebra. Our main theorem is that the converse is true; every abstract measure algebra is isomorphic to one of this type. (In fact, every abstract measure algebra is isomorphic to a principal ideal in a single direct product \(J \otimes U\) of type (e) except that \(J\) and \(J \otimes U\) need not satisfy postulates II and III above.) Thus the four concepts \((E, \lambda)\), \((E, \sim)\), abstract measure algebra of type (i), and \((E, \lambda)\) in which the values of \(\lambda\) are numerical functions (modulo null functions), are all equivalent to within isomorphism.

We shall also be able to deduce some simpler characterizations in some important special cases (§20).

4. Construction of the direct product \(J \otimes U\).

4.1. In this section we show that a given numerical measure algebra \(J = (\mathbb{I}, \mu)\) and a given trivial abstract measure algebra \(U\) (satisfying our postulates) always have a direct product in the sense of 3.1(e). To save the trouble of proving explicitly that the direct product in the sense of 3.1(e) is unique (to within isomorphism), we shall use the unqualified term “direct product” in the rest of the paper to refer to the particular direct product to be constructed in this section. As the construction will depend on the notion of the “representation space” of an algebra, we begin by outlining the relevant properties.

4.2. Representation spaces. Let \(E\) be any Boolean \(\sigma\)-algebra, with or with-\(^{(18)}\) Provided \(\mu\) is not always 0 or \(\infty\).
out a measure. Then there exists a compact (=bicom pact) totally disconnected Hausdorff space \( R \), the representation space of \( E \), such that \( E \) is isomorphic to the Boolean algebra formed by the "open-closed" (=simultaneously open and closed) subsets of \( R \). Let \( \ast \) denote the open-closed subset of \( R \) which corresponds to \( x \in E \) in this isomorphism. Then we have

1. The sets \( \ast \) form a basis for the open sets of \( R \).
2. For finite operations the "expected" relations hold; that is, \( o \ast = 0, e \ast = R, (-x) \ast = R - x \ast, x \leq y \Leftrightarrow x \ast \subseteq y \ast, (x \vee y) \ast = x \ast \cup y \ast, \) and \( (xy) \ast = x \ast \cap y \ast \). For infinite operations the correspondence is more complicated, but it is easily seen from the laws for finite operations that

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- (2) \( (\bigvee x_n) \ast = \bigcup x_n^\ast \cup z, (\bigwedge x_n) \ast = \bigcap x_n^\ast - t, \) where \( z \) and \( t \) are nowhere dense closed subsets of \( R \) (and are thus of the first category).

The Borel field \( \mathcal{B}(R) \) of \( R \) is defined (cf. 1.3) to be the smallest family \( \mathcal{B}' \) of subsets of \( R \) such that (i) \( \mathcal{B}' \) contains all open-closed sets, (ii) if \( X \subseteq \mathcal{B}' \), \( R - X \subseteq \mathcal{B}' \), (iii) if \( X_n \subseteq \mathcal{B}' \) \( (n = 1, 2, \cdots) \), \( \bigcup X_n \subseteq \mathcal{B}' \). We have:

- (3) Any given \( X \subseteq \mathcal{B}(R) \) can be written in one and only one way as \( X = x^\ast + z, \) where \( x^\ast \) is open-closed and \( z \) is of the first category\(^{12}\).

- (4) From (2) it is clear that the family \( \mathcal{B}' \) of all sets so expressible as \( x^\ast + z \) satisfies (i), (ii) and (iii); hence it contains \( \mathcal{B}(R) \). To see that the expression is unique, we note that if \( x^\ast + z = y^\ast + t \) (where \( x^\ast, y^\ast \) are open-closed and \( z, t \) are of the first category) then \( z + t = x^\ast + y^\ast \), an open set; and since a non-empty open set is not of the first category we must have \( z + t = 0 = x^\ast + y^\ast \), that is, \( z = t \) and \( x^\ast = y^\ast \).

- Now assume that \( E \) satisfies the countable chain condition, and is thus complete. From (1) and (2) it now follows that

- (4) If \( G \) is an open subset of \( R \), so is its closure \( \overline{G} \). As a corollary,
- (5) If \( G_1, G_2 \) are disjoint open subsets of \( R \), so are \( \overline{G_1} \) and \( \overline{G_2} \).

We shall be interested in numerical functions \( f \) on \( R \) whose values are between 0 and \( \infty \) inclusive; for brevity we refer to them simply as "functions." The continuity of functions is defined in the obvious way\(^{13}\), and a function \( f \) is said to be continuous on \( X \subseteq R \) if \( f \), considered only for the points of \( X \), is continuous relative to the subspace \( X \).

- (6) LEMMA. Given a function \( f \) on \( R \) which is continuous on \( R - H \), where \( H \) is of the first category, there exists one and only one function \( g \), continuous on \( R \), such that \( g(p) = f(p) \) for each \( p \in R - H \).

Given \( p \in H \), we assert \( \lim sup f(q) - \lim inf f(q) \) as \( q \rightarrow p \) through points

\(^{11}\) See [15, Theorem 67].

\(^{12}\) Thus \( E \) is \( \sigma \)-isomorphic, in a natural way, to the algebra formed by \( \mathcal{B}(R) \) modulo sets of first category—proving the known theorem (cf. [9]) that the notions "Boolean \( \sigma \)-algebra" and "\( X \)-field of sets modulo a \( \sigma \)-ideal" are equivalent.

\(^{13}\) If \( f(p) = \infty \), continuity of \( f \) at \( p \) means that, given any \( n \), there exists a neighborhood of \( p \) throughout which \( f > n \).
of \( R - H \)\(^{(14)}\). For if not, there exist (finite) real numbers \( \rho, \sigma \) such that \( \lim \sup f(q) > \rho > \sigma > \lim \inf f(q) \). Define \( K_1 = \{ q : q \in R - H \text{ and } f(q) > \rho \} \), \( K_2 = \{ q : q \in R - H \text{ and } f(q) < \sigma \} \). From the continuity of \( f \), \( K_1 \) and \( K_2 \) are open relative to \( R - H \); hence \( K_i = G_i \cap (R - H) \) where \( G_i \) is open in \( R \) \((i = 1, 2)\).

Since \( K_1 \cap K_2 = 0 \), we have \( G_1 \cap G_2 \subseteq H \); and since a non-empty open set is not of the first category in \( R \), it follows that \( G_1 \cap G_2 \circlearrowleft 0 \). Hence (by (5)) \( \overline{G_1 \cap G_2} = 0 \), contradicting \( p \in \overline{K_1 \cap K_2} \subseteq \overline{G_1 \cap G_2} \).

Define \( g \) by: \( g(p) = f(p) \) for \( p \in R - H \), and \( g(p) = \lim f(q) \) for \( p \in H \) (where \( q \to p \) in \( R - H \)). It is easy to see that \( g \) is continuous on \( R \). Thus the required function exists; and its uniqueness is immediate.

(7) \textbf{Lemma.} \( \{ f_n \} \) be a given sequence of continuous functions on \( R \) such that, for each \( p \in R \), \( f_1(p) \leq f_2(p) \leq \cdots \). Then there exists one and only one continuous function \( f \) on \( R \) such that \( f(p) = \lim f_n(p) \) for each \( p \in R - H \), \( H \) being a set of the first category.

Let \( f'(p) = \lim f_n(p) \); from the previous lemma it is enough to show that \( f' \) is continuous on \( R - H \) where \( H \) is of the first category. We can suppose that \( f_n \) and \( f' \) are all uniformly bounded (on replacing \( f_n \) by \( f_n/(1 + f_n) \)).

For each positive integer \( k \), let \( \{ x^*_i \} \) \((i = 1, 2, \cdots)\) be a maximal disjoint collection (necessarily countable) of nonempty open-closed subsets of \( R \), each having the property that there exists an integer \( n = n(k, i) \) such that \( f'(p) \leq f_n(p) + 1/k \) whenever \( p \in x^*_i \). Write \( G_k = \bigcup x^*_i \); clearly \( G_k \) is open. We assert \( G_k = R \). For suppose not. Define \( F_n \) to be the set of all points \( p \in R \) such that \( f_n(p) \leq f_m(p) + 1/k \) whenever \( m \geq n \); evidently \( F_n \) is closed, and \( U_n F_n = R \). Hence one of the sets \( (R - G_k) \cap F_n \) \((n = 1, 2, \cdots)\) must be somewhere dense; and so (from (4)) must contain a nonempty open-closed set \( y^*_k \). But \( y^*_k \) is disjoint from every \( x^*_i \) \((i = 1, 2, \cdots)\), and yet satisfies the requirement which was imposed on the sets \( x^*_i \); and this contradicts the maximality of the collection \( \{ x^*_i \} \).

Since \( G_k = R \), the set \( H = U_k (R - G_k) \) is of the first category. Suppose \( p \in H \); then, for every \( k \), \( p \) belongs to some \( x^*_i \). Writing \( n \) for \( n(k, i) \), we see from the continuity of \( f_n \) that there is an open-closed set \( y^* \) such that \( p \in y^* \subseteq x^*_i \) and, whenever \( q \in y^* \), \( |f_n(q) - f_n(p)| < 1/k \). Thus \( |f'(q) - f'(p)| < 3/k \) for all \( q \in y^* \). That is, \( f' \) is continuous at every point of \( R - H \), and is a fortiori continuous on \( R - H \). Q.E.D.

\textbf{Remark.} Arguments similar to the foregoing—which is evidently analogous to the proof of Egoroff's theorem—will occur again; we shall refer to them as "Egoroff arguments."

4.3. Now let \((J, \mu)\) be a numerical measure algebra, and let \( U \) be a trivial abstract measure algebra. (It is assumed that neither \( J \) nor \( U \) consists of \( o \) only.) Let \( R \) and \( S \) be the representation spaces of \( J \) and \( U \) respectively, and

\(^{(14)}\) Convergence is in the sense of Moore and Smith; cf. [3].
let \( T = R \times S \) (the space formed by the ordered pairs \((r, s)\), \(r \in R, s \in S\), with the obvious definition of the topology). Define the restricted Borel field \( \mathcal{B}_r(T) \) to be the Borel field generated (in the algebra of subsets of \( T \)) by the "rectangles" \( x^* \times u^* \) and \( u^* \) being open-closed subsets of \( R \) and \( S \) respectively. For each restricted Borel set \( H \in \mathcal{B}_r(T) \) and for each \( s \in S \) define \( K(H, s) \) to be the set of all points \( r \in R \) such that \((r, s) \in H\). It is easily seen that \( K(H, s) \in \mathcal{B}(R) \) (as follows, for example, by the same reasoning as in 8.5 below). Hence (from (3) above) each \( K(H, s) \) can be uniquely expressed as \( x^* + z \), where \( x^* \) is open-closed in \( R \) and \( z \) is of the first category in \( R \). The set \( x^* \) corresponds to a unique element \( x \in J \), with measure \( \mu(x) \). Thus we can define a function \( M'_H \) on \( S \) by setting

\[
(8) \quad M'_H(s) = \mu(x), \text{ where } x^* + z = K(H, s).
\]

We next show that

\[
(9) \text{There exists a function } M_H, \text{ differing from } M'_H \text{ on a set of first category (at most) in } S, \text{ and continuous on } S. \text{ (Obviously } M_H \text{ will then be unique.)}
\]

Let \( \{a_n\} \) be a disjoint sequence of elements of \( J \) such that \( \mu(a_n) < \infty \) and \( V_{a_n} = j \), the unit-element in \( J \). (The existence of such a sequence follows from the fact that \( J \) satisfies our postulates; cf. 2.3(1).) For each \( H \in \mathcal{B}_r(T) \), define \( H_n = H \cap (a_n^* \times S) \); thus the sets \( H_n \) belong to \( \mathcal{B}_r(T) \), are disjoint (for fixed \( H \)), and their union is \( H \)—a set of first category (from (2)). Hence \( M'_H = \sum_n M'_{H_n} \). By (7) it will thus suffice to prove that, for each \( n \), \( M'_{H_n} \) differs from a continuous function only on a set of first category in \( S \). Consider the family \( \mathcal{B}' \) of those sets \( H \in \mathcal{B}_r(T) \) for which this is true. Clearly \( \mathcal{B}' \) contains the "rectangles" \( x^* \times u^* \); also if \( H' \supset H'' \) and \( H', H'' \in \mathcal{B}' \) then \( H' - H'' \in \mathcal{B}' \) (for \( M'_{H' - H''} = M'_{H'} - M'_{H''} \) since these functions are finite); and finally if \( \{H_n\} \) is a disjoint sequence of sets in \( \mathcal{B}' \) it readily follows from (7) again that \( \bigcup H_n \in \mathcal{B}' \). Hence \( \mathcal{B}' \supset \mathcal{B}_r(T) \), and (9) is established.

We now define \( J \otimes U \) to consist of the algebra \( \mathcal{B}_r(T) \) modulo "null sets"—those for which \( M_H = 0 \). Formally, we write \( H \equiv H' \) to mean that \( H, H' \in \mathcal{B}_r(T) \) and \( M_{H + H'} = 0 \) for every \( s \in S \)—that is, that, except for a first category set of \( s \)'s, each set \( K(H, s) + z \) \( K(H', s) \) is of the first category in \( R \)—and write \( \{H\} = \text{set of all sets } H' \text{ such that } H' \equiv H \). Algebraic operations on the equivalence classes \( \{H\} \) (\( H \in \mathcal{B}_r(T) \)) are defined in the obvious way; it is easily seen that they form a Boolean \( \sigma \)-algebra. On this we define an abstract measure by setting \( M\{H\} = M_H \) (which also = \( M'_{H'} \) if \( H' \equiv H \); thus the values of the "measure" are continuous functions on \( S \). This measure is evidently countably additive, provided that we define \( \sum_{n=1}^\infty M\{H_n\} \) to mean not the ordinary sum of these functions, but the continuous function which differs from this sum on a set of the first category (at most).

An alternative procedure would be to redefine the "measure" of \( \{H\} \) to be the set of all functions which differ from \( M\{H\} \) on an at most first category subset of \( S \); addition would have then have its natural meaning. This is in fact the procedure which was used in the formulation in 3.1(e); however, the
choice of $M\{H\}$ to be a single continuous function is more convenient in proving that $J \otimes U$ has the desired properties.

4.4. It is easy to see that $J \otimes U$ (assuming it for the moment to satisfy our postulates) is in fact (to within isomorphism) a "direct product" of $(J, \mu)$ and $U$ in the sense of 3.1(e). For $J \otimes U$ contains a sub-algebra $J'$ consisting of all elements of the form $\{x^* \times S\}$ ($x^*$ open-closed in $R$), and a sub-algebra $U'$ consisting of the elements $\{R \times u^*\}$ ($u^*$ open-closed in $S$). The function $M\{x^* \times S\}$ has evidently the constant value $\mu(x)$, and similarly $M\{R \times u^*\}$ has the value $\mu(j)$ for $s \in u^*$ (where $j$ denotes the unit element of $J$) and the value 0 for $s \in S - u^*$. Thus $(J', M)$ is isomorphic to the given measure algebra $(J, \mu)$, and $(U', M)$ is isomorphic to the given trivial algebra $U$. All that remains to be verified is that $J \otimes U = B(J' \cup U')$. To see this, consider the family $B'$ of all sets $H \in B_0(T)$ for which the elements $\{H\}$ belong to $B(J' \cup U')$. Clearly $B'$ contains the "rectangles" $x^* \times u^*$, and contains $T-H$ and $U \cap H_n$ if it contains $H$ and $H_n$ ($n = 1, 2, \cdots$). Hence $B' \supset B_0(T)$, and so $B' = B_0(T)$, from which it follows that $B(J' \cup U') = J \otimes U$.

4.5. We must now verify that $J \otimes U$ satisfies our postulates (2.1), $\sim$ being, of course, the equivalence relation induced by the abstract measure $M$.

(0) All that has to be proved here is that $J \otimes U$ satisfies the countable chain condition. Suppose there exists an uncountable disjoint collection of elements $\{H_n\}$, all different from the zero element. As in the proof of (9) (4.2), we choose a disjoint sequence $\{a_n\}$ in $J$ with $\mu(a_n) < \infty$ and $a_n = j$. For some $n$, there will be uncountably many nonzero elements $\{H_n \cap (a_n^* \times S)\}$, and from these we obtain a transfinite sequence of strictly increasing elements $\{L_\beta\}$, $\beta < \omega_1$, $\{L_1\} < \{L_2\} < \cdots < \{(a_n^* \times S)\}$. For each $\beta < \omega_1$, we have $M\{L_\beta\} < M\{L_{\beta + 1}\}$ for at least one $s \in S$; and since these functions are continuous there exists a nonempty open-closed set $u_\beta \subset S$ such that $M\{L_\beta\} < \rho_\beta < \sigma_\beta < M\{L_{\beta + 1}\}$, where $\rho_\beta$, $\sigma_\beta$ are fixed positive numbers, for all $s \in u_\beta^*$. For some positive integer $m$ there will be uncountably many values (say) $\gamma$ of $\beta$ for which $\sigma_\beta - \rho_\beta > \mu(a_n)/m$. Thus no $s \in U_{u_\beta^*}$ can belong to more than $m$ sets $u_\gamma^*$. Let $v^*(s)$ denote the intersection of all sets $u_\gamma^*$ containing $s$; thus $v^*(s)$ is open-closed in $S$, and distinct sets $v^*(s)$ are disjoint. Since $U$ satisfies the countable chain condition, the distinct sets $v^*(s)$ must be countable in number; and since each $v^*(s)$ is contained in $u_\gamma^*$ for at most $m$ values of $\gamma$, the $\gamma$'s are countable—a contradiction.

(I) This is trivial here, since $M$ is countably additive.

(II) Given $\{H\}$, $\{H'\} \in J \otimes U$ such that $M\{H\} = M\{H'\}$, and given $L' \in B_0(T)$ such that $\{L'\} \leq \{H'\}$, we must show that there exists $L \in B_0(T)$ such that $L \subset H$ and $M\{L\} = M\{L'\}$. The proof is built up by first considering two special cases.

(i) Suppose $J$ has no atoms. We need the following lemma:

(10) Given $H \in B_0(T)$, and given any continuous function $\phi$ on $S$, not identically zero, such that $0 \leq \phi \leq M\{H\}$ (for all $s \in S$), there exists $L_1 \in B_0(T)$
such that $L_1 \subset H$, $\{L_i\} \neq \emptyset$, and $M\{L_i\} \leq \phi$.

For, from the continuity of $\phi$, there exists a nonempty open-closed set $u^* \subset S$ and an $\epsilon > 0$ such that $\phi(s) \geq \epsilon$ whenever $s \in u^*$. Let $\{c^n\}$ be a maximal disjoint collection of nonzero elements of $J$ each of measure less than $\epsilon$; since $J$ is non-atomic, it is easily seen that $\forall c^n = j$. Write $L^* = H \cap (c^n \times u^*)$. Since $\sum M\{L^n\} = M\{H\} \geq \phi$ (except on a first category set), there exists an $n$ for which $M\{L^n\}$ is not identically 0, and we have only to take $L_1 = L^n$.

The usual "exhaustion" argument now shows that, under the hypotheses of the lemma, there exists $L \in B_*(T)$ such that $L \subset H$ and $M\{L\} = \phi$. (Take $L$ to be a maximal $L_1$, modulo null sets; if the function $\phi' = \phi - M\{L\}$ were not identically 0, with the convention $\infty - \infty = 0$, the lemma applied to the set $H - L$ and function $\phi'$ would enable $L$ to be enlarged.) In particular, on taking $\phi = M\{L'\}$, we see that II is established in this case.

(ii) Suppose $J$ is generated by a number of atoms, all of equal measure. The proof proceeds as in case (i), the only difference being that we now restrict the continuous function $\phi$ to have only integral multiples of the measure of each atom (or $\infty$) as values.

(iii) In the general case, there are at most $N_0$ atoms in $J$; let the distinct values of their measures be enumerated as $\mu_1, \mu_2, \ldots$, and so on. Write $b_n = \supremum$ of all atoms in $J$ of measure $\mu_n$, and $b_0 = j - \forall b_n$. For each $X \in B_*(T)$ write $X^n = X \cap (b_n \times S)$. It is not hard to see, from the fact that $J$ itself satisfies II, that if $M\{H\} = M\{H'\}$ then $M\{H^n\} = M\{H'^n\}$ for each $n$ ($n = 0, 1, \cdots$) (13). Now, given $\{L'\} \subseteq \{H'\}$, we apply the argument in case (i) to obtain $L^n \subset H^n$ such that $M\{L^n\} = M\{L'^n\}$, and similarly apply the argument in case (ii) to obtain $L^n \subset H^n$ such that $M\{L^n\} = M\{L'^n\}$ ($n \geq 1$).

Write $L = \bigcup L^n$ ($n \geq 0$); evidently II follows.

(iii) Given $H, H' \in B_*(T)$ such that $M\{H\} = M\{H'\}$, we decompose them into equivalent bounded elements as follows. As before, let $\{a_n\}$ be a disjoint sequence of elements of $J$ such that $\mu(a_n) < \infty$ and $V a_n = j$. Write $X_n = H \cap (a_n \times S)$, $Y_n = H' \cap (a_n \times S)$. Define $H_1 = X_1$. From postulate II, there exists $H^1 \subset H'$ such that $M\{H^1\} = M\{H_1\} \leq \mu(a_1)$; hence $M\{H - H_1\} = M\{H^1 - H_1\}$. Define $H'_1 = Y_1 \cap (T - H_1)$; from II again there exists $H_2 \subset H - H_1$ such that $M\{H_2\} = M\{H'^1\} < \infty$, so that $M\{H - (H_1 \cup H_2)\} = M\{H' - (H_1 \cup H'_1)\}$. Define $H_3 = X_3 \cap (T - (H_1 \cup H_2))$; and so on. We thus obtain disjoint sequences $\{H_n\}$, $\{H'_n\}$ such that $H = \bigcup H_n$, $H' = \bigcup H'_n$, and $M\{H_n\} = M\{H'_n\} < \infty$, so that the elements $\{H_n\}, \{H'_n\}$ are bounded, as required.

Remark. The construction of $J \otimes U$ will apply even if $J = (J, \mu)$ is an arbitrary numerical measure algebra in which $\mu$ is $\sigma$-finite. Naturally, if $J$ does not satisfy the postulates, $J \otimes U$ need not either.

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(13) Let $K(H, s) = s^* + \sigma$, $K(H', s) = s'^* + \sigma'$; then (ignoring a first category set of $s$'s) $\mu(x) = \mu(x')$. If $x \geq \mu$ an atom of measure $\mu_1$, $x' \geq \mu$ an element of $J$ of measure $\mu_1$, which must also be an atom. Iteration of this argument gives $\mu(b_{n'}x) = \mu(b_{n'}x') (n = 0, 1, \cdots)$, and the result follows.
4.6. We conclude this section by noting, for future use, that the elements of $J \otimes U$ are (roughly speaking) approximable by countable sums of rectangles. More precisely:

(11) **Lemma.** Suppose $\mu(j) < \infty$. Then, given $H \subseteq \mathcal{B}_n(T)$ and $\varepsilon > 0$, there exist sequences $x_n^*, u_n^*$ of open-closed subsets of $R$ and $S$ respectively, such that

(i) the sets $u_n^*$ are disjoint and cover $S$ except for a set of first category, and (ii) we have

$$M \big\{ H + z \cup (x_n^* \times u_n^*) \big\} < \varepsilon \quad (\text{for all } z \in S).$$

Consider the family $\mathcal{B}'$ of sets $H \subseteq \mathcal{B}_n(T)$ for which this is true (for every $\varepsilon$). Clearly $\mathcal{B}'$ contains the "rectangles" $x^* \times u^*$. It is also easy to see that if $H, H' \in \mathcal{B}'$ then so do $T-H$ and $H \cup H'$; thus $\mathcal{B}'$ is a finitely additive field, and to prove $\mathcal{B}' = \mathcal{B}_n(T)$ it will suffice to prove that if $H_n \subseteq \mathcal{B}'$ and $H_1 \subseteq H_2 \subseteq \cdots$, then $\cup H_n \subseteq \mathcal{B}'$. This follows by an "Egoroff argument" (cf. the proof of 4.2(7)).

5. **Invariant elements.**

5.1. **Definition.** Returning now to an abstract measure algebra $(E, \sim)$ satisfying the postulates of 2.1, we define an element $u$ to be **invariant** if, whenever $x \sim u$, we have $x \leq u$. (Thus if $u$ is invariant and bounded, we have further $x = u$.) Clearly no two distinct invariant elements can be equivalent. (The term "invariant" will be justified in 20.4(1).)

The invariant elements will play a fundamental part in the analysis of the structure of $(E, \sim)$. In fact, if $E$ were a non-atomic numerical measure algebra, the only invariant elements would evidently be $0$ and $e$; and this suggests, as is in fact the case, that the "trivial" factors in the desired "product" representation of $E$ (cf. 3.1(f)) in the general case, will be formed from invariant elements.

The set of all invariant elements of $E$ will be written $U$.

5.2. If $u$ is invariant and $x \sim x' \leq u$, then $x \leq u$.

Write $y = x - u$; by II(2.1) there exists $y' \leq x$ such that $y \sim y'$. Now let $z = (u - y') \setminus y$. From I(2.1) $z \sim (u - y') \setminus y = u$; hence, from the definition of invariance, $z \leq u$. A fortiori $y \leq u$, and so $y = 0$ and $x \leq u$.

5.3. If $u$ is not invariant, there exist nonzero $x$, $y$ such that $x \leq u$, $y \leq -u$, and $x \sim y$.

For there exists $x_1 \sim u$ such that $x_1 - u \neq 0$. Define $y = x_1 - u$. By II(2.1) there exists $x \leq u$ such that $x \sim y$ (for $y \leq x_1 \sim u$).

5.4. **The invariant elements form a complete Boolean sub-algebra of $E$.**

It is trivial that $0$ and $e$ are invariant (2.3(4)). Thus we have only to prove (a) if $u$ is invariant so is $-u$, and (in view of the countable chain condition) (b) if $u_n$ is invariant ($n = 1, 2, \cdots$) then so is $\bigvee u_n$. Now (a) is an immediate consequence of 5.3 and 5.2. To prove (b), let $u = \bigvee u_n$, and suppose

\(^{(*)}\) This lemma can easily be modified so as to apply even if $\mu(j) = \infty$. Both lemma and proof can also be reformulated in terms of the properties given in 3.1 (e).
that \( u \) is not invariant. By 5.3, there exist \( x, y \) such that \( x \leq u, y \leq -u, \) and \( x \sim y \neq o. \) For some \( n \) we must have \( xu_n \neq o, \) and from postulate II we then have that there exists \( y' \leq y \) such that \( y' \sim xu_n. \) From 5.2, \( y' \leq u_n \leq u; \) but \( y' \leq y \leq -u, \) and so \( y' = o — that is, xu_n = o \) (from 2.3(4)), a contradiction.

5.5. Definition. For each \( x \in E \) we can (from 5.4) define the invariant closure \( \hat{x} \) of \( x \) to be the smallest invariant element containing \( x. \) To simplify the printing of complicated expressions, we shall sometimes also write \( \text{Cl}(x) \) instead of \( \hat{x}. \) This closure operation will then have the usual properties familiar in topology (\( \hat{o} = o, \hat{e} = e, \text{Cl}(\text{Cl}(x)) = \text{Cl}(x) \geq x, \text{Cl}(x) \vee \text{Cl}(y) = \text{Cl}(x \vee y), \) and \( x \) is invariant if and only if \( \hat{x} = x), \) as well as some others, notably:

5.6. For any collection of elements \( \{x_a\}, \text{Cl}(\bigvee x_a) = \bigvee \hat{x}_a. \)

For it is trivial that \( \text{Cl}(\bigvee x_a) \geq \bigvee \hat{x}_a, \) while conversely \( \bigvee \hat{x}_a \) is invariant (5.4) and greater than or equal to \( \bigvee x_a, \) and so greater than or equal to \( \text{Cl}(\bigvee x_a). \)

5.7. For any \( x, \hat{x} = \bigvee \{y | y \sim y' \leq x\}. \)

Let \( x^* \) denote the element on the right. Any invariant \( u \geq x \) will satisfy \( u \geq x^* \) (from 5.2), and so \( \hat{x} \geq x^*. \) All that remains to be proved is that \( x^* \) is invariant; for clearly \( x^* \geq x. \) If \( x^* \in U, \) then (5.3) there exist \( z, t \) such that \( z \leq x^*, t \leq -x^*, z \sim t, \) and \( z \neq o. \) Hence for some \( y \sim y' \leq x \) we have \( zy \neq o; \) and postulate II(2.1) then gives the existence of \( p \leq t \) and \( q \leq y' \) such that \( p \sim zy \sim g. \) Since \( q \leq x, \) this shows that \( p \leq x^*; \) but \( p \leq t \leq -x^*, \) and so \( p = o. \) That is (2.3(4)), \( zy = o, \) a contradiction.

5.8. If \( x \sim x', \) then \( \text{Cl}(x) = \text{Cl}(x'). \)

This is immediate from 5.7 and postulate II(2.1).

5.9. \( \hat{x} = o \) whenever \( h, k \) are such that \( h \leq x, k \leq y, \) and \( h \sim k, \) then \( h = k = o. \)

The implication \( \Rightarrow \) is trivial, since \( h \leq \hat{x} \) and also \( h \leq \hat{y} \) (5.7). Conversely, if nonzero \( h, k \) exist such that \( h \leq x, k \leq y, \) and \( h \sim k, \) we have, on writing \( u = h = k \) (from 5.8), \( u \leq \hat{x} \).

5.10. If \( u \) is invariant, \( \text{Cl}(xu) = \text{Cl}(x)u. \)

Trivially \( \text{Cl}(xu) \leq \text{Cl}(x)u. \) Write \( p = \text{Cl}(x)u - \text{Cl}(xu), \) and suppose \( p \neq o. \) Since \( p \leq x, \) 5.7 shows that \( py \neq o \) for some \( y \sim y' \leq x. \) Write \( q = py; \) then, by postulate II, there exists \( q' \leq x \) such that \( q \sim q'. \) Since \( q' \sim q \leq p \leq u, \) 5.2 shows that \( q' \leq u; \) thus \( q' \leq xu, \) and so (5.7) \( q \leq \text{Cl}(xu). \) But \( q \leq p \leq -\text{Cl}(xu), \) so that \( q = o, \) a contradiction.

6. Bounded elements. In this section we develop further fundamental properties of the invariant and other elements of \((E, \sim)\) which depend on the notion of boundedness. (One example of the way in which this property is of service has already been furnished by the proof of 2.3(4).)

6.1. ("Cancellation law"). If \( a, a' \) are both bounded and equivalent, and if \( b \leq a, b' \leq a', \) and \( b \sim b', \) then \( a - b \sim a' - b'. \)

Remark. As will be proved later, it suffices that \( a \) be bounded; for the
boundedness of $a'$ will then follow from $a' \sim a$ (6.4).

From 2.3(5) and 5.9, there exist $x \leq a - b$, $y \leq a' - b'$, such that (i) $x \sim y$, (ii) on writing $p = (a - b) - x$, $q = (a' - b') - y$, we have $pq = o$. We shall prove $p = o$; a similar argument will then show that $q = o$, whence the result will follow.

Write $h = pa'$; thus $hq = o$, and accordingly (since the elements $b'$, $y$, $q$ are disjoint and $b' \vee y \vee q = a'$) $h \leq b' \vee y \sim b \vee x$. Thus (postulate II, 2.1) there exists $k \leq b' \vee x$ such that $k \sim h$. Since $p \vee k \leq a' - a$, there exists $m \leq a'$ such that $m \sim p \vee k$. Now, $k \leq \bar{h}$ (from 5.2, since $k \sim h \leq \bar{h}$), and therefore $k \vee h \leq \bar{h}$; and since $\bar{h}$ is invariant, it follows (from 5.2 again) that $m \leq \bar{p}$. Thus $m \leq k \sim h$, so (from II again) there exists $n \leq k$ such that $n \sim m \sim k \vee p$. But $k \vee p$ is bounded (from 2.3(2), for $a$ is bounded). Hence $n = k \vee p$, so that $k \sim p \vee k$; and since $k \sim p \vee k$, it follows (6.2) that $p = o$; Q.E.D.

6.2. Given $y \sim x = V x_n$, where the sequence $\{x_n\}$ is disjoint, there exists a disjoint sequence $\{y_n\}$ such that $y = V y_n$ and $y_n \sim x_n$ ($n = 1, 2, \cdots$).

First assume that $x$ and $y$ are both bounded. In this case, postulate II gives the existence of $y_1 \sim y$ such that $y_1 \sim x_1$; and from 6.1 we have $x - x_1 \sim y - y_1$. A straightforward induction, repeating this argument, gives a disjoint sequence of elements $y_n \leq (y - (y_1 \vee \cdots \vee y_{n-1}))$ such that $y_n \sim x_n$. Write $y' = V y_n$; thus $y' \leq y$ and (from postulate I) $y' \sim x' \sim y$. Since $y$ is bounded, we have $y' = y$, and the result is established in this case.

In the general case, we apply postulate III (2.1) to write $x = V f_n$, $y = V g_n$, where $f_n \sim g_n$ and the sequences $\{f_n\}$, $\{g_n\}$ are each disjoint and bounded. Write $h_{mn} = x_m f_n$. The case already established enables us to write $g_n = V k_{mn}$ where $k_m \sim h_{mn}$ and $k_m k_m' = o$ ($m \neq m'$). Define $y_m = V k_m n$; from postulate I we have $y_m \sim V k_{mn} x_m$; the elements $y_m$ are clearly disjoint; and finally $V y_m = V x_n k_{mn} = V g_n = y$.

6.3. A necessary and sufficient condition that $x$ be unbounded is that there exist an infinite disjoint sequence $\{z_n\}$, with $z_n \leq x$, $z_n \neq o$, and $z_1 \sim z_2 \sim \cdots \sim z_n \sim \cdots$.

The sufficiency is trivial, since $V \{z_n | n \geq 1\}$ and $V \{z_n | n \geq 2\}$ are (from postulate I) equivalent, so that $V z_n$ (and so, a fortiori, $x$) is not bounded.

Conversely, suppose $x$ is unbounded; then there exists $z_1 \leq x$ such that $z_1 \neq o$ and $x \sim x - z_1$. The elements $z_n$ are now defined by induction. When $z_n$ has been defined, in such a way that $z_n \sim z_1$, $z_n \leq x - (z_1 \vee \cdots \vee z_{n-1}) = y_{n-1}$, say, and $y_n \sim x$, we apply 6.2 to the decomposition $x = z_1 \vee y_1$ to obtain an "equivalent" decomposition $y_n = z_{n+1} \vee y_{n+1}$ such that $z_{n+1} y_{n+1} = o$, $z_{n+1} \sim z_1$, and $y_{n+1} \sim y_1 = x - z_1 \sim x$. Thus the inductive hypotheses are maintained, and the desired sequence $\{z_n\}$ is thereby constructed.

6.4. If $x \sim x'$ and $x$ is bounded, then so is $x'$.

This follows from 6.3 and 6.2, since if $x'$ were unbounded, the decomposition $x' = V z_n \vee (x' - \vee z_n)$ given by 6.3 could (6.2) be imitated in $x$, contradicting the boundedness of $x$. 
Remark. From these results one can show that if \( x, y \) are both bounded, then so is \( x \vee y \); we omit the proof, since the result will not be needed.

6.5. Let \( \{x_n\} \) be a sequence of bounded elements whose invariant closures are disjoint (that is, \( \bar{x}_m \bar{x}_n = 0 \) if \( m \neq n \)). Then \( \forall x_n \) is bounded.

Suppose \( \forall x_n = x \sim x - y \) for some \( y \leq x \); we must prove \( y = 0 \). Now, for each \( n \) we have \( x_n \leq x \sim x - y \); hence (postulate II) there exists \( x'_n \leq x \sim y \) such that \( x'_n \sim x_n \). Now if \( m \neq n \) we have \( x'_n x_m \leq \bar{x}_n \) (from 5.7), and also \( x'_n x_m \leq x_m \leq \bar{x}_n \); hence \( x'_n x_m = 0 \) \( (m \neq n) \). It follows that \( x'_n \leq x_n \), and so (since \( x_n \) is bounded) \( x'_n = x_n \); that is, \( x_n y = 0 \) for each \( n \), so that \( y = x y = 0 \).

6.6. If \( u \) is invariant and \( x \sim y \), then \( u x \sim u y \).

First suppose \( x \) and \( y \) are both bounded. Then (postulate II) there exists \( z \leq y \) such that \( z \sim u x \). From 5.2, \( z \leq u \), so \( z \leq u y \). Similarly there exists \( t \leq u x \) such that \( t \sim u y \). By II again, there exists \( t' \leq z \) such that \( t' \sim t \sim u y \). But \( u y \) is bounded \( (2.3(3)) \), so \( t' \sim u y \); and consequently \( z = u y \), whence the result.

The extension to the general case now follows from postulate III; we have \( x = \forall x_n, y = \forall y_n \), where each of the sequences \( \{x_n\}, \{y_n\} \) is disjoint, and where \( x_n, y_n \) are bounded and equivalent. The preceding case gives \( u x_n \sim u y_n \); whence (from postulate I) \( u x \sim u y \).

7. Equivalence classes.

7.1. Notation. We write \([x]\) for the "equivalence class" \( \{x'|x' \sim x\} \), and write \([x]\leq[y]\) (or \([y]\geq[x]\)) to mean that there exist \( x' \in [x] \) and \( y' \in [y] \) such that \( x' \leq y' \). Thus (from II, 2.1) there will then exist, for each \( y' \in [y] \), an \( x' \in [x] \) such that \( x' \leq y' \). It follows immediately that if \([x]\leq[y]\) and \([y]\leq[z]\) then \([x]\leq[z]\) (so that \( \leq \) is in fact a partial ordering of the equivalence classes), and that if \([x]\leq[y]\) and \( x \) is bounded then \([x]=\[y]\). (This holds even if \( x \) is unbounded; but this fact will not be needed.) We further write \([x]<[y]\) (or \([y]>[x]\)) to mean that \([x]\leq[y]\) and \([x]\neq[y]\). Of course, two given equivalence classes will in general be incomparable.

Note that \([x]\leq[y]\) implies \( \bar{x}\leq\bar{y} \) (from 5.8). (The converse is false, in general.)

7.2. If \( x_1 \geq x_2 \geq \cdots \) is a decreasing sequence of bounded elements such that \([x_n]\geq[y]\) for all \( n \), then \([\forall x_n]\geq[y]\).

Since \([x_1]\geq[y]\), there exist \( x'_1 \sim x_1, y' \sim y \), such that \( x'_1 \geq y' \). Now elements \( x'_2, x'_1, \cdots \), such that \( x'_1 \geq x'_2 \geq \cdots \geq y' \) and \( x'_n \sim x_n \), are defined by induction, as follows. When \( x'_n \) has been defined, postulate II gives the existence of \( y^{n+1} \sim x_{n+1} \sim y' \); and from 6.4 and 6.1 \( x'_n - y' \sim x_n - y^{n+1} \geq x_n - x_{n+1} \). Hence (II) there exists \( z_{n+1} \leq x'_n - y' \) such that

\(^{(1)}\) There will also exists, for each \( x' \in [x] \), a \( y' \in [y] \) such that \( x' \leq y' \); but the proof is not easy, except in the case in which \( x \) is bounded.

\(^{(2)}\) One could go on to make \([x]\) play the role of an "abstract measure" \( \lambda(x) \), by defining addition, and so on, suitably, along the lines of \( [12] \); cf. also §12. We shall not use this procedure, since a much more specific abstract measure will be constructed later.
$z_{n+1} - z_n = x_{n+1} - x_n$. Define $x'_{n+1} = x'_{n+1} - z_{n+1}$; thus $x'_{n+1} \geq y'_{n+1}$, and (from 6.1)

$x'_{n+1} - x_n = (x_{n+1} - x_n) = x_{n+1}$.

Now $x'_{i+1} - x'_{i} = V(x'_{i+1} - x_{n+1}) = V(x_{i+1} - x_n)$; hence (6.1 and 6.4) $V(x'_{i+1} - x'_{i}) \geq y'_{i+1} - y'_{i}$; and the result is proved.

7.3. If $x_1 \leq x_2 \leq \cdots$ is an increasing sequence such that $[x_n] \leq [y]$ for all $n$, and if $y$ is bounded, then $[Vx_n] \leq [y]$.

This is proved by an argument similar to the preceding (19).

7.4. If $[x] \leq [y]$, and $u \in U$ (the set of invariant elements), then $[ux] \leq [uy]$.

This is an immediate consequence of 6.6.

7.5. Given $a, b \in E$, there exist "maximal" equivalent elements $x \leq a$, $y \leq b$, such that (i) $\text{Cl}(a - x)$, $\text{Cl}(b - y) = o$, (ii) given any other equivalent elements $x' \leq a$, $y' \leq b$, we have $[x'] \leq [x]$ and $[y'] \leq [y]$.

The existence of $x \leq a$ and $y \leq b$ such that $x \sim y$ and (i) holds is given by 2.3(5), in view of 5.9. We prove that they satisfy (ii). Write $z = x \text{Cl}(x' - x)$ and $t = x' \text{Cl}(x' - x)$; thus $t \leq x' \text{Cl}(x' - x)$. Since $t \leq y' \leq x'$, there exists $s \sim t$ such that $s \leq y' \leq b$. But $s \leq \text{Cl}(x' - x) \leq \text{Cl}(a - x)$, so $s(b - y) = o$, and therefore $s \leq y$. Thus $s \leq y \text{Cl}(x' - x) \sim \text{Cl}(x' - x)$ (from 6.6) = $z$, so that there exists $s' \leq z$ such that $s' \sim s \sim t$. From postulate I we now have $x' = t \vee x'(x - z) \sim s' \vee x'(x - z) \leq x$, and so $[x'] \leq [x]$. Similarly $[y'] \leq [y]$.

7.6. Given elements $a, b \in E$, of which at least $a$ is bounded, there exists $u \in U$ such that (i) $[au] \leq [bu]$; (ii) if $v$ is a nonzero invariant element disjoint from $u$, then $[av] < [bv]$.

Let $x, y$ be the elements given by 7.5. Write $u = - \text{Cl}(a - x)$; thus $u$ is invariant (5.4). Further, $ua = ux \sim uy$ (from 6.6) $\leq ub$, establishing (i). Finally, if $v$ is invariant and $uv = o$, we have $v \leq \text{Cl}(a - x)$ and therefore $v \text{Cl}(b - y) = o$; hence $bv = yv \sim xv \leq av$, which shows that $[bv] \leq [av]$, and that if $[bv] = [av]$ then $xv = av$ — that is, $v(a - x) = o$, since $a$ is bounded. But this would imply $o = \text{Cl}(v(a - x)) = v \text{Cl}(a - x)$ (from 5.10) = $v$. Hence if $v \neq o$ we must have $[bv] < [av]$.

8. Relativization.

8.1. Given any $a \in E$, it was pointed out in 3.1(c) that the "relative algebra" $(E(a), \sim)$, where $E(a)$ consists of all elements $x \leq a$, satisfies our postulates. We shall here derive, for later use, some relations between this relative algebra $E(a)$ and $E$.

An element $v \leq a$ which is invariant in $E(a)$ (that is, in $(E(a), \sim)$) will be said to be invariant relative to $a$, and the set of elements invariant relative to $a$ will be denoted by $U_a$. For the other notations used in this section, we refer to 1.3.

8.2. A necessary and sufficient condition that an element $v \leq a$ be invariant relative to $a$ is that there exist $u \in U$ such that $v = au$. Further, $u$ can be

(19) The restriction of boundedness can be removed in 7.3, but not in 7.2—just as in the familiar numerical case.
chosen so that \( u \leq a \); and if this is done, the correspondence between \( u \) and \( v \) is 1-1.

The condition is sufficient; for if \( v = au \), where \( u \in U \), and if \( v \sim v' \leq a \), then \( v' \leq u \) (from the invariance of \( u \)), and so \( v' \leq v \)—that is, \( v \in U_a \).

The condition is necessary. For if \( v \in U_a \), define \( u = v \). Trivially \( v \leq au \); let \( w = au - v \), and suppose \( w \neq 0 \). Then, since \( w \leq v \), 5.7 shows that, for some \( x \sim x' \leq v \), we have \( wx = y \) (say) \( \neq 0 \). From II, there exists \( y' \leq x' \) such that \( y \sim y' \); and since \( y' \leq v \) and \( y \leq w \leq a \), 5.2 applied to \( E(a) \) shows that \( y \leq v \). But \( y \leq w \leq -v \); hence \( y = 0 \), a contradiction. Thus \( v = au \), where \( u = v \), so that \( u \in U \) and \( u \leq a \).

To show that the correspondence is 1-1, we must show that if \( au_1 = au_2 \) (where \( u_1, u_2 \in U(a) \)) then \( u_1 = u_2 \); and, writing \( u = u_1 + u_2 \), it will suffice to show that if \( au = 0 \) (where, from 5.4, \( u \in U(a) \)) then \( u = 0 \). But we have \( o = Cl(au) = Cl(a)u \) (5.10) = \( u \).

The result just established can be restated as:

8.3 Corollary. The algebra \( U_a = aU \), and is isomorphic to the principal ideal \( U(a) \) (of all elements of \( U \) which are \( \leq a \)) under the correspondence \( v \mapsto u = v, u \mapsto yv = au \).

8.4. An element of \( E(a) \) is bounded relative to \( a \) (that is, in \( E(a) \)) if and only if it is bounded.

This is trivial from the definition of boundedness.

8.5. For any set \( P \subseteq E \), the Borel field relative to a generated by \( aP \) (in symbols, \( B_a(aP) \)) is \( aB(P) \).

Write
\[
B' = aB(P) = \text{set of all elements } ax, \text{ where } x \in B(P).
\]
Clearly \( B' \supset aP \); and if \( y \) and \( y_n \) belong to \( B' \), then so do \( a - y \) and \( y_n \); hence, \( B' \supset B_0(aP) \). Now write
\[
B'' = B_0(aP) \cap E(-a) = \text{set of all elements expressible as } y \forall z \text{ where } y \in B_0(aP) \text{ and } z \in E(-a).
\]
It is easy to see that \( B'' \supset P \), and that if \( x \) and \( x_n \) belong to \( B'' \) then so do \( -x \) and \( z \forall x_n \). Hence \( B'' \supset B(P) \); so \( aB(P) \subseteq aB'' = B_0(aP) \), completing the proof.

9. Indecomposable elements.

9.1. Definitions. An element \( x \) is decomposable if there exist \( y_1, y_2 \) such that \( 0 < y_i \leq x \) \((i = 1, 2)\), \( y_1 \sim y_2 \), and \( y_1y_2 = 0 \). Otherwise \( x \) is said to be indecomposable. Clearly \( o \) is indecomposable, and so is each atom of \( E \); there may, of course, be other indecomposable elements. The indecomposable elements are the easiest to deal with, and we shall dispose of them in this section; this will in effect reduce the problem of representing \( E \) to the case in which all nonzero elements are decomposable.

9.2. The following properties are easy consequences of the above definition.

(1) If \( x \) is indecomposable, and \( y \leq x \), then \( y \) is indecomposable.

(2) Indecomposable elements are bounded. (From 6.3.)

(3) If \( x \sim x' \) and \( x \) is indecomposable, then so is \( x' \). (From 6.2.)
(4) If $x$ is indecomposable, and $y \leq x$, then $y = xy$.

For trivially $y \leq xy$. Write $z = xy - y$; if $z \neq 0$, 5.7 shows that $zy' \neq 0$ for some $y' \leq y$, and then II gives the existence of $z_1 \leq z$ and $y_1 \leq y$ such that $z_1 \sim zy' \sim y_1$. The definition of indecomposability gives $z_1 = o$, whence $zy' = o$, a contradiction.

(5) If $x$ is indecomposable and $y_1 \vee y_2 \leq x$, and if $y_1 \sim y_2$, then $y_1 = y_2$.

For, from (4) and 5.8, $y_1 = x\bar{y}_1 = x\bar{y}_2 = y_2$.

(6) If $x$ is indecomposable, $y \vee z \leq x$, and $yz = o$, then $\bar{y}z = o$. (From 5.9.)

9.3. Let $\{x_n\}$ be a sequence of indecomposable elements such that $x_n \leq o$ whenever $m \neq n$. Then $Vx_n$ is indecomposable.

Let $y \vee z \leq Vx_n$, and suppose $y \sim z$ and $yz = o$. (We must prove $y = o$.) Let $y_n = yx_n$; thus the elements $y_n$ are disjoint, and $Vy_n = y$. Hence (6.2) we can write $z = Vz'_n$, where the elements $z'_n$ are disjoint and $z'_n \sim y_n$. Let $z_{nm} = z'_n x_m$; from 6.2 again we can write $y_n = Vy_{nm}$, where $y_{nm} \sim z_{nm}$. Now, if $m \neq n$, we have $y_{nm} \leq x_m$ (from 5.7, since $y_{nm} \sim z_{nm} \leq x_m$), and so $y_{nm} \leq x_n \bar{x}_m = o$. And if $m = n$, $y_{nm} \vee z_{nm} \leq x_m$, which is indecomposable; hence again $y_{nm} = o$. Hence $y = V_{m,n} y_{nm} = o$.

9.4. If $b_1 \leq b_2 \leq \cdots$ is an increasing sequence of indecomposable elements, then $Vb_n$ is indecomposable.

Write $x_1 = b_1$, $x_n = b_n - b_{n-1}$; thus $x_n$ is indecomposable (9.2(1)), and $x_m x_n = o$ if $m \neq n$. Since $x_m \vee x_n \leq b_{m+n}$, which is indecomposable, 9.2(6) shows that $\bar{x}_m \bar{x}_n = o$. So $Vx_n$ is indecomposable (9.3); and $Vx_n = Vb_n$.

9.5. Given any collection $C$ of indecomposable elements, there exists an indecomposable element $b$ such that (i) $b \leq V\{c | c \in C\}$, (ii) for each $c \in C$, $[c] \leq [b]$.

By transfinite induction, using 9.4, we see that there is a maximal indecomposable $b$ satisfying (i). We prove that $b$ satisfies (ii) also. In fact, given $c \in C$, there exist (7.5) elements $b_1 \leq b$, $c_1 \leq c$, such that $b_1 \sim c_1$ and $Cl(c - c_1)$ \cdot $Cl(b - b_1) = o$. Also (9.2(6)) $Cl(c_1) Cl(c - c_1) = o$, and $b_1 = c_1$ (5.8), so that $Cl(b_1) Cl(c - c_1) = o$. Again, from 9.2(6), we have $Cl(b_1) Cl(b - b_1) = o$. Hence 9.3 shows that the element $p = Cl(b_1) \vee Cl(b - b_1) \vee Cl(c - c_1)$ is indecomposable. But $b \leq p \leq V\{c | c \in C\}$. Hence $b = p$, proving that $c - c_1 = o$, that is, that $c \sim b_1 \leq b$. Q.E.D.

9.6. Notation. Let $e^0$ denote the supremum of all the indecomposable elements. (Naturally, $e^0$ may be $o$, and also need not itself be indecomposable.)

**Theorem.** There exists a disjoint sequence $\{b_n\}$ of indecomposable elements, such that (i) $Vb_n = e^0$, and (ii) for each $m$, $n$, either $[b_m] \leq [b_n]$ or $[b_n] \leq [b_m]$. Further, (iii) $e^0$ is invariant.

We first define a transfinite sequence $\{b_\alpha\}$ of disjoint indecomposable elements, as follows. From 9.5 applied to the collection $C_1$ of all indecomposable elements, there exists an indecomposable element $b_1 \leq e^0$ such that $[c] \leq [b_1]$ for every indecomposable $c$. When $b_\beta$ has been defined for all $\beta < \alpha$, we define $C_\alpha =$ set of all indecomposable elements $c$ satisfying $c \leq e^0 - V_{\beta < \alpha} b_\beta$. By 9.5,
there exists an indecomposable \( b_a \leq e^a - V_{\beta < a} b_\beta \) such that \([c] \leq [b_a]\) for every \(c \in C_a\). From postulate 0(2.1) we have \(b_{a_0} = 0\) for some \(a_0 < \omega_1\), and the process is terminated. \(C_{a_0}\) must then consist of 0 only, so that the element \(d = e^a - V_{\alpha < a_0} b_\alpha\) must be 0 (for otherwise, since \(d \leq e^a\), there would exist an indecomposable \(c\) such that \(cd \neq 0\); and, from 9.2(1), \(cd \subseteq C_{a_0}\)). On renumbering the elements \(b_\alpha (\alpha < a_0)\) into a simple sequence, we see that (i) and (ii) are satisfied.

To prove (iii), suppose \(y \sim x \leq e^a\); it will suffice to prove \(y \leq e^a\). From (i), \(x = V_0 b_\gamma;\) hence (6.2) we can write \(y = V_n y_n\) where \(y_n \sim x b_\beta\). But \(x b_\beta\) is indecomposable, since \(b_\beta\) is (9.2(1)); hence so is \(y_n\) (9.2(3)), so that \(y_n \leq e^a\) for each \(n\). It follows that \(y \leq e^a\).

9.7. Let \(K\) (as in 3.1(a)) be the numerical measure algebra generated by \(N_0\) atoms \(p_n\) \((n = 1, 2, \ldots)\) each of measure 1; that is, the elements of \(K\) are the subsets of \(U(p_n)\), and the measure of any such set is the number of elements \(p_n\) in it.

**THEOREM.** The relative algebra \((E(e^a), \sim)\) is isomorphic to a principal ideal in the direct product \(K \otimes U(e^a)\).

(Here \(U(e^a)\) denotes, as usual, the principal ideal in \(U\) of \(e^a\)—which belongs to \(U\), as just shown.)

Let \(R, S\) denote the representation spaces of \(K, U(e^a)\), respectively (cf. §4). The open-closed subset \(p_n^*\) of \(R\) which corresponds to \(p_n \subseteq K\) is now easily seen to consist of a single point (say) \(q_n\). Write \(Q = R - U(q_n);\) thus (from 4.2(2)) \(Q\) is of the first category in \(R\).

Now let \(b_\alpha\) be the sequence of indecomposable elements given by 9.6. For each \(x \leq e^a\), define \(\psi(x)\) to be the subset \(U_n (q_n, (\text{Cl}(x b_\beta))^*)\) of \(T = R \times S;\) clearly \(\psi(x) \subseteq B_{\beta}(T)\). And define \(\theta(x)\) to be the element \(\{\psi(x)\}\) of \(K \otimes U(e^a)\), that is, the class of all sets \(H \subseteq B_{\beta}(T)\) such that \(M\{H + \psi(x)\}\) is identically 0, \(M\) being the continuous-function-valued “measure” defined in §4. We shall show that \(\theta\) is the desired isomorphism.

(1) For any sequence \(\{x_n\}\) with \(x_n \leq e^a, \theta(Vx_n) = V\theta(x_n)\).

For we have \(\text{Cl}(V_n x_n b_n) = V_n \text{Cl}(x_n b_n)\) (5.6), and so (4.2(2)) \((\text{Cl}(V_n x_n b_n))^* = U_m (\text{Cl}(x_n b_n))^* + Y_n,\) where \(Y_n\) is of the first category in \(S\). Hence \(\psi(V x_n) = U \psi(x_n) + U (q_n, Y_n),\) so that (the sets \(q_n, Y_n\) being null) \(\theta(V x_n) = V \theta(x_n)\).

(2) If \(x \leq e^a, \theta(e^a - x) = \theta(e^a) - \theta(x)\).

For we have \(\text{Cl}((e^a - x) b_n) \cap (x b_n) = 0\) from 9.2(6); and trivially \(\text{Cl}((e^a - x) b_n) \cap (x b_n) = \text{Cl}(b_n)\). Hence

\[
\psi(e^a) - \psi(x) = U (q_n, (\text{Cl}(b_n))^*) - U (q_n, (\text{Cl}(x b_n))^*)
= U (q_n, (\text{Cl}((e^a - x) b_n))^*) = \psi(e^a - x),
\]

and (2) follows.

(3) If \(x, y \leq e^a\) and \(\theta(x) = \theta(y)\), then \(x = y\).
From (1) and (2), it will suffice to prove (on writing $z = x + 2y$) that if $\theta(z) = 0$ then $z = 0$. We are given, then, that $M\{U(q_n, (C_l(zb_n))^*)\}$ is identically 0; hence, for each $n$, $M\{(q_n, (C_l(zb_n))^*)\} = 0$, and since this function is 1 for each $s \in (C_l(zb_n))^*$, it follows that $(C_l(zb_n))^* = 0$. That is, $Cl(zb_n) = o$ for each $n$. Thus $z = \sqrt{zb_n} = o$.

(4) If $\{H\} \in K \otimes U(e^o)$ and $\{H\} \leq \theta(e^o)$, there exists $x \leq e^o$ such that $\theta(x) = \{H\}$.

We first note that, given any $H \in \mathcal{B}_{c}(T)$, there exists $H' \subseteq \{H\}$ and expressible as $H' = U(q_n, u_n^*)$ ($u_n^*$ open-closed in $S$). (This is substantially the result of 4.6 in the present case, and follows by the same reasoning applied here.) Now if $\{H\} \leq \theta(e^o) = \{U(q_n, b_n^*)\}$, the corresponding $H'$ must satisfy $u_n \leq b_n$ for each $n$. Define $x = \sqrt{b_n}$. Then $x \leq e^o$, and (since the elements $b_n$ are disjoint) we have $Cl(xb_n) = Cl(b_n, u_n) = (Cl(b_n), u_n)$. Thus $\psi(x) = H'$, so that $\theta(x) = \{H\}$.

Properties (1)–(4) show that $\theta$ is an algebraic isomorphism between $E(e^o)$ and the principal ideal of $\theta(e^o)$ in $K \otimes U(e^o)$. All that remains is to show that $\theta$ is equivalence-preserving (both ways).

(5) If $x, y \in e^o$ and $x \sim y$, $M\{\theta(x)\} = M\{\theta(y)\}$.

First suppose $x \leq b_n$ and $y \leq b_m$. We have $x = \sqrt{b_n}$, say (5.8); and clearly $\psi(x) = U(q_n, u_n^*)$ where $u_k = o$ if $k \neq n$, and $u_n = u$. Hence, from the definition of $M$, $M\{\theta(x)\} = M\{\psi(x)\} = 0$ for $s \notin u$, and 1 for $s \in u$. The same applies to $M\{\theta(y)\}$, which thus equals $M\{\theta(x)\}$.

In the general case, 6.2 shows that we can write $y = \sqrt{y_n}$, where the sequence $y_n$ is disjoint and $y \sim zb_n$. Now write $y_{nm} = y_n b_m$; by 6.2 again we can write $xb_n = \sqrt{y_{nm}}$ where the elements $y_{nm}$ are disjoint and $x \sim y_{nm}$. Since $y_{nm} \leq b_m$ and $x_{nm} \leq b_n$, the case already established shows that $M\{\theta(x_{nm})\} = M\{\theta(y_{nm})\}$. But $\sqrt{y_{nm}} x_{nm} = y_n$ and $\sqrt{y_{nm}} y_{nm} = y$; hence, from the countable additivity of $M$, $M\{\theta(x)\} = M\{\theta(y)\}$.

(6) If $x, y \leq e^o$ and $M\{\theta(x)\} = M\{\theta(y)\}$, then $x \sim y$.

First suppose $\theta(x) \leq \{(q_n, b_n^*)\}$ and $\theta(y) \leq \{(q_m, b_m^*)\}$. Then $xb_n = o$ unless $k = n$, so that $x \leq b_n$; and similarly $y \leq b_m$. The equality of the measure functions now gives $x = \sqrt{y}$. Now (9.6) we can assume without loss of generality that $[b_n] \leq [b_m]$; and, on applying postulate II, we obtain $x \sim x' \leq b_m$. Hence $x' = x' b_m$ (from 9.2(4)) = $\sqrt{b_m}$ (from 5.8) = $\sqrt{b_m} = y$ (from 9.2(4) again). That is, $y = x' \sim x$.

The extension to the general case now follows by an argument similar to that used in proving (5). (We apply 6.2 to $K \otimes U(e^o)$.) The proof of the theorem is thus complete.

Remark. In the present case, the direct product (as constructed in §4) can be simplified. It is not hard to see that the abstract measure-algebra $K \otimes U(e^o)$ is isomorphic with the algebra of sequences $\{u_n\}$ ($u_n \subseteq U(e^o)$), in which the algebraic operations are defined "coordinatewise" (cf. 3.1(d)), but in which the equivalence relation is induced by the continuous numerical
function $M\{u_n\}$ on $S$, defined by: the value of $M\{u_n\}$ at $s \in S$ is (except for a first category set of $s$'s) the number of values of $n$ for which $s \in u_n^*$.  

10. **Reduction to the bounded case.**

10.1. The ideal $E(e^o)$ having been disposed of by the theorem just established, we next study the ideal (that is, relative algebra) $E(-e^o)$, in which (from the definition of $e^o$) no nonzero element can be indecomposable. In effect this means that the indecomposable elements have been eliminated. In the present section we shall make a further reduction which will in effect eliminate the unbounded elements.

10.2. **Definitions.** We write $e'$ to denote the supremum of all bounded invariant elements in $E(-e^o)$, and write $e'' = (-e^o) - e'$. Thus $e', e'$ and $e''$ are disjoint, and $e'' \setminus e' \setminus e'' = e$.

10.3. **Theorem.** $e'$ is bounded and invariant.

That $e'$ is invariant follows from 5.4. Further, the countable chain condition (postulate 0) gives $e' = \bigvee u_n$ for some sequence $\{u_n\}$ of bounded invariant elements. Define $v_n = u_1, v_n = u_n - \bigvee_{i=1}^{n-1} u_i$; thus $e' = \bigvee v_n$ where each $v_n$ is bounded (2.3(2)) and invariant (5.4), and where the sequence $\{v_n\}$ is disjoint. Hence $e'$ is bounded, from 6.5.

10.4. **Lemma.** There exists a sequence $\{z_n\}$ ($n = 1, 2, \ldots$ to $\infty$) of disjoint equivalent bounded elements such that $e'' = z_n$ for each $n$.

We can obviously assume $e'' \neq 0$. Then since $e''$ is invariant (from 9.6(iii) and 10.3) and disjoint from $e'$, $e''$ is unbounded, and consequently (6.3) there exists an infinite sequence of disjoint equivalent nonzero elements (say) $t_n \leq e''$. From postulate III (2.1) there exists $p_{11} \leq t_1$ such that $p_{11}$ is bounded and nonzero; and then from postulate II there exists $p_{1n} \leq t_n$ such that $p_{1n} \sim p_{11}$. Thus the elements $p_{1n}$ are disjoint, equivalent, bounded and nonzero, and $\bigvee p_{1n} \leq e''$. Write $q_2 = \text{Cl}(\bigvee p_{1n}) \leq e''$; thus $e'' - q_2$ is invariant and therefore unbounded (unless it is 0). Thus we obtain (by repeating the argument) a disjoint sequence $\{p_{2n}\}$ of bounded equivalent nonzero elements less than or equal to $e'' - q_2$. The argument is repeated transfinitely; when $p_{2n}$ has been defined for all $n$ and all $\beta < \alpha$, we set $s_\alpha = \text{Cl}(\bigvee p_{2n})$ ($\beta < \alpha$, $n = 1, 2, \ldots$); if $e'' - q_\alpha$ is not 0, it is invariant and unbounded, and so there exists a disjoint infinite sequence $\{s_\alpha\}$ of bounded equivalent nonzero elements less than or equal to $e'' - q_\alpha$. The process must terminate countably; and, on defining $z_n = \bigvee_{\alpha} p_{2n}$, the elements $z_n$ are disjoint, equivalent (postulate I), bounded (6.5), and satisfy $\text{Cl}(\bigvee z_n) = e''$. But $\text{Cl}(\bigvee z_n) = \text{VCl}(z_n)$ (5.6) = $\text{Cl}(z_n)$ for each $n$ (from 5.8).

10.5. **Lemma.** Given $u \in U$ such that $0 < u \leq e''$, there exists $v \in U$ such that $0 < v \leq u$ and $v = \bigvee s_n$, where the elements $s_n$ are disjoint, equivalent, and bounded ($n = 1, 2, \ldots$, to $\infty$).
Let \( \{z_n\} \) be the sequence given by 10.4. Then the elements \( uz_n \) are disjoint, equivalent (6.6) and bounded; also they are nonzero, since \( \text{Cl}(uz_n) = u \). The collection \( \{uz_n\} \) can thus be extended to a maximal collection of disjoint equivalent (and thus also bounded and nonzero) elements less than or equal to \( u \); and this collection (from postulate 0) will consist of exactly \( \aleph_0 \) elements (say) \( t_n \) \((n=1, 2, \ldots, \infty)\). Let \( y_t, y' \) be maximal equivalent elements such that \( y_t \leq t_1 \) and \( y' \leq u - Vt_1, \) so that (7.5) \( \text{Cl}(t_1 - y_t) \leq u - Vt_1 - y' = 0. \) Define \( v = \text{Cl}(t_1 - y_t) \); we shall show that \( v \) has the required properties.

Trivially, \( v \) is invariant and \( v \leq t_1 \leq u. \) Also, \( v \neq 0, \) since otherwise \( t_1 = y_t \sim y' \), contradicting the maximality of the collection \( \{t_n\} \) (for \( y' \) could then be adjoined to it).

Now, from postulate II, there exists \( y_n \leq t_n \) such that \( y_n \sim y_t \), and (since then \( y_n \sim y' \)) there exists \( x_n \leq y_n \) such that \( x_n \sim y'v \). Define \( s_1 = y'v \vee (t_1 - y_t) \) and (if \( n > 1 \)) \( s_n = x_{n-1} \vee (t_n - y_n) \); thus the elements \( s_n \) \((n=1, 2, \ldots, \infty)\) are disjoint and equivalent (from 6.1 and postulate III). Further, \( s_n \sim x_n \vee (t_n - y_n) \leq t_n, \) so that \( s_n \) is bounded (6.4). We have \( t_n - y_n \leq v, \) from 5.7, and \( x_n \leq v, \) from 5.2; thus \( Vs_n \leq v. \) On the other hand, \( v(u - Vt_n - y') = 0, \) so that \( v \leq Vt_n \vee y', \) and therefore \( v \leq y'v \vee V \{ (t_n - y_n) \vee (y_n - x_n) \vee x_n \} v; \) but \( (y_n - x_n)v \sim (y' - y)v; \) (from 6.6) = 0, and it follows that \( v \leq Vs_n. \) Thus \( v = Vs_n, \) as required.

10.6. Theorem. We can write \( e'' = Vf^n \) \((n=1, 2, \ldots, \infty)\), where the elements \( f^n \) are disjoint, equivalent, bounded, and satisfy \( \text{Cl}(f^n) = e'' \) for each \( n. \)

Let \( \{v_m\} \) be a maximal disjoint (and thus countable) collection of nonzero invariant elements less than or equal to \( e'' \), each of which can be written as \( v_m = \sqrt{v_m} \) \((m=1, 2, \ldots, \infty)\), where the elements \( s_m \) are, for each \( m, \) disjoint, equivalent and bounded. Let \( u = e'' - Vv_m; \) then \( u \) is invariant (5.4), so that if \( u \neq 0 \) an application of 10.5 would at once contradict the maximality of the collection \( v_m. \) Hence \( Vm, s_m = Vv_m = e'' \). Now define \( f^n = Vm, s_m; \) thus \( Vf^n = e'', \) and the elements \( f^n \) are disjoint, equivalent (postulate I) and bounded (from 6.5). Lastly, \( e'' = V\text{Cl}(f^n) = \text{Cl}(f^n) \) for each \( n, \) from 5.8.

Corollary. For each \( n, \) the algebra \( Uf^n \) of invariant elements relative to \( f^n \) is isomorphic to the ideal \( U(e'') \).

(From 8.3.)

10.7. Convention. The considerations in what follows (till §19) are going to be applied to the relative abstract measure algebras \( E(e') \) and \( E(f^n) \). Thus we shall assume, throughout §§11–18, that all elements (except \( o\) are decomposable), and that all elements are bounded.

11. Decomposition into homogeneous parts.

11.1 Before the desired direct product representation can be obtained, two further reductions are necessary. In this section we carry out the first
of these, which will have the effect of enabling us to assume that \( E \) is, in a
sense to be defined, "homogeneous" with respect to its invariant elements.

**Definitions.** Given \( x \in E \), and given any nonempty subset \( P \subseteq E \), con-
sider all sets \( Q \subseteq E \) such that \( E(x) \subseteq xB(P \cup Q) \)—so that (8.5) \( E(x) = xB(P \cup Q) = B_x(xP \cup xQ) \). The smallest cardinal \( m \) of such a \( Q \) is called the
**order** of \( x \) over \( P \). (Thus, for example, the order of \( o \) over any \( P \) is 0.) If \( x \neq o \), and every \( y \) such that \( o < y \leq x \) has order \( m \) over \( P \), \( x \) is said to be homo-
geneous of order \( m \) over \( P \). Clearly, if \( o < y \leq x \) and \( x \) is homogeneous of order \( m \) over \( P \), then so is \( y \).

11.2. If \( x \) has order \( m \) over \( P \), and \( y \leq x \), then the order of \( y \) over \( P \) is not
greater than \( m \).

For \( E(x) = xB(P \cup Q) \), where \( |Q| = m \). Given \( z \leq y \), we have \( z \in E(x) \), and so \( z = xt \) where \( t \in B(P \cup Q) \). Hence \( z = yt \); and thus \( E(y) \subseteq yB(P \cup Q) \).

11.3. If \( x \) is homogeneous of order \( m \) over \( P \), where \( m \) is infinite, and if
\( a \geq x \), then \( x \) is homogeneous of order \( m \) over \( aP \), and further there exists a
set \( Q \subseteq E(x) \), with \( |Q| = m \), such that \( E(x) = xB(Q \cup aP) \).

We have \( E(x) = xB(P \cup Q') \) for some \( Q' \subseteq E \) with \( |Q'| = m \). Write \( Q = xQ' \); then \( xB(P \cup Q') = xB(Q \cup aP) \), by 8.5, since both equal \( B_x(Q \cup xP) \). This
proves the last part of the statement, and shows that \( x \) has order not greater
than \( m \) over \( aP \). To complete the proof, it will suffice to show that if \( o < y \leq x \) than the order of \( y \) over \( aP \) is not less than \( m \). Suppose not; then there exists
\( Q'' \subseteq E \), with \( |Q''| < m \), such that \( E(y) = yB(Q'' \cup aP) \subseteq yB(Q'' \cup P \cup \{a\}) \).
The homogeneity of \( x \) over \( P \) now gives \( |Q'' \cup \{a\}| \geq m \); and since \( m \) is infinite
this is a contradiction.

11.4. If \( m \) is an infinite cardinal, and if \( x_n \) is homogeneous of order \( m \) over
\( P \) \((n = 1, 2, \ldots)\), then so is \( \forall x_n \).

Let \( \forall x_n = x \); we must prove (i) \( x \) is of order not greater than \( m \) over \( P \), and (ii) if \( o < y \leq x \), \( y \) is of order not less than \( m \) over \( P \).

We have \( E(x_n) = x_nB(P \cup Q_n) \) where \( |Q_n| = m \). Let \( UQ_n \cup U(x_n) = Q \); thus
\( |Q| = m \) also. Given \( y \leq x \), write \( y_n = yx_n \in E(x_n) \); thus \( y_n = x_nz_n \) where
\( z_n \in B(P \cup Q_n) \subseteq B(P \cup Q) \). Since \( x_n \subseteq Q \subseteq B(P \cup Q) \), this gives that each \( y_n \),
and thus also \( y = Vy_n \), belongs to \( B(P \cup Q) \), which proves (i).

For (ii), we again write \( y_n = yx_n \); then for some \( n \) we have \( y_n \neq o \). If \( y \) has
order less than \( m \) over \( P \), then so does \( y_n \) (11.2); but this contradicts the
homogeneity of \( x_n \).

11.5. If \( x \) is a nonzero element of finite order over \( P \), there exists a nonzero
element \( y \leq x \) of order 0 over \( P \).

We have (say) \( E(x) = xB \{ P \cup (a_1) \cup (a_2) \cup \cdots \cup (a_n) \} \). Consider the \( 2^n \)
elements of the form \( b_1b_2 \cdots b_n \), where each \( b_i \) is either \( a_i \) or \( -a_i \), and all
combinations are taken; enumerate them as \( t_1, t_2, \ldots, t_N \) \((N = 2^n)\). Let \( B' \)
denote the set of all elements expressible as \( \bigvee x_j \) \((x_j \in B(P)) \). It is easy to
verify that \( B' \) contains, and thus coincides with, \( B \{ P \cup (a_1) \cup \cdots \cup (a_n) \} \).
Thus \( E(x) = xB' \), so for at least one \( j \) we have \( xt_j = y \), say, \( \neq o \). Now if \( z \in E(y) \),
we have \( z \in E(x) = x \bar{B} \), so \( z = \bigvee_{i=1}^{N} x_i \Delta x \) (\( x_i \in \bar{B}(P) \)); but \( z \leq y \leq t_j \), and \( t_i \Delta t_j = 0 \) unless \( i = j \). Hence \( z = x_i \Delta x = x_j y \); and this proves that \( E(y) \subseteq \bar{B}(P) \)—that is, \( y \) is of order 0 over \( P \).

11.6. If \( y \) is of order 0 over \( U \), then \( y \) is indecomposable and therefore zero.

If \( y \) were decomposable, there would exist (9.1) \( z_1, z_2 \leq y \) such that \( z_1 \sim z_2 \), \( z_1 \Delta z_2 = 0 \), and \( z_i \neq 0 \) (\( i = 1, 2 \)). Since \( E(y) = y U \), by hypothesis, we have \( z_i = y u_i \) (\( u_i \in U \)). Thus \( y z_i = y (y u_i) \) (from 5.10) = \( y u_i = z_i \), so that \( z_1 = y z_1 = y z_2 \) (5.8) = \( z_2 \), whence a contradiction. Since all nonzero elements are decomposable (in accordance with the convention introduced in 10.7), \( y = 0 \).

Corollary. Every nonzero element has infinite order over \( U \).

(From 11.5 and 11.6.)

11.7. Theorem. There exists a disjoint sequence \( \{ a_n \} \) (finite or infinite) such that \( V a_n = e \) and each \( a_n \) is homogeneous of infinite order over \( U \).\(^{(20)} \)

Let \( m_1 \) be the smallest of the orders of nonzero elements \( x \) over \( U \); thus \( m_1 \) is infinite, as just shown, and from 11.2 each \( x \) of order \( m_1 \) over \( U \) is homogeneous. Let \( a_1 = \bigvee \{ x \mid x \text{ is of order } m_1 \text{ over } U \} \). The countable chain condition shows that \( a_1 \) is the supremum of a countable set of such elements \( x \); and 11.4 then shows that \( a_1 \) is homogeneous of order \( m_1 \) over \( U \). The construction is iterated transfinitely, \( a_\alpha \) being defined as the supremum of all nonzero elements less than or equal to \( \bigvee_{\beta < \alpha} a_\beta \) (if any) having the smallest possible order \( m_\alpha \) over \( U \); as before, \( a_\alpha \) is homogeneous of order \( m_\alpha \) over \( U \).

The construction terminates for some \( a_\alpha < \omega_1 \) for the elements \( a_\alpha \) are disjoint and nonzero, and we have \( e = \bigvee_{\alpha < \omega_1} a_\alpha \). All that remains is to reorder the elements \( a_\alpha \) (\( \alpha < \omega_1 \)) into a simple sequence.

12. Scalar multiplication of equivalence classes.

12.1 We could now consider the relative abstract measure algebras \( E(a_\alpha) \), in which all nonzero elements are now decomposable, bounded, and (from 11.3 and 8.3) homogeneous over the algebra of relatively invariant elements. However, though it would be possible to prove a representation theorem for each \( E(a_\alpha) \), it would not be easy to extend the representation to all of \( E \); this is because (roughly speaking) the elements \( a_\alpha \) are not related in any convenient way to their invariant closures in \( E \). In the next section we shall replace them by elements which are "comparable," in the desired way, with their invariant closures in \( E \). Here we shall define and develop the necessary properties of this comparability. It would be possible to elaborate these notions further into a systematic arithmetic of equivalence classes (cf. footnote 17), but we shall confine ourselves to the properties actually needed in the sequel.

12.2. Given \( x \neq 0 \), and a positive integer \( n \), there exist \( n \) disjoint equiva-
lent nonzero elements $y_i \leq x$ ($i = 1, \ldots, n$).

For $x$ is decomposable (cf. 10.7), so there exist $z_1, z_2 \leq x$ such that $z_1 \neq 0$, $z_1 z_2 = 0$, and $z_1 \sim z_2$. Since $z_1$ is decomposable, there exist $z_{11}, z_{12} \leq z_1$ such that $z_{11} \neq 0$, $z_{11} z_{12} = 0$, and $z_{11} \sim z_{12}$; and from 6.2 there exist disjoint elements $z_{21}, z_{22} \leq z_1$ such that $z_{21} \sim z_{22}$. Proceeding in this way we obtain, for each $m$, $2^m$ disjoint equivalent nonzero elements less than or equal to $x$; and the result follows.

12.3. In 12.2 we can further suppose $\forall y_i = x$.

This follows from 12.2 by the usual “exhaustion” argument.

12.4. If $y_1, y_2, \ldots, y_n$ are disjoint equivalent elements, and if $z_1, z_2, \ldots, z_n$ are disjoint equivalent elements such that $\forall y_i \sim \forall z_i$, then $y_i \sim z_i$.

Let $p_i, q_i$ be maximal equivalent sub-elements of $y_i, z_i$ (7.5). Since all elements are now bounded (10.7), it readily follows from 7.5(ii) that the $2n$ elements $p_i, q_i$ are all equivalent. Write $h_i = y_i - p_i, k_i = z_i - q_i$; thus $h_i k_i = 0$, and (6.1) we have $h_1 \sim h_2 \sim \cdots \sim h_n, k_1 \sim k_2 \sim \cdots \sim k_n$, and $\forall h_i, \forall k_i$. Thus (from 5.6 and 5.8) $\forall (h_i) = \forall (k_i)$. Write $h_i = a_i, k_i = b_i$; thus $a_i b_i = 0$, and (5.6) we have $a_1 \sim a_2 \sim \cdots \sim a_n, b_1 \sim b_2 \sim \cdots \sim b_n$, and $\forall a_i, \forall b_i$. Thus $y_i = p_i, q_i = z_i$.

12.5. Definitions. Given any $x$ and any positive integer $n$, 12.3 shows that, for each $x' \in [x]$ (that is, $x' \sim x$) we can write $x' = \bigvee_i y_i$, where the $n$ elements $y_i$ are disjoint and equivalent; and 12.4 then shows that the equivalence class $[y_i] ([y_i])$ is uniquely determined. We write $[y_i] = (1/n)!x_i$, and more generally $[y_i \vee y_2 \vee \cdots \vee y_m] = (m/n)!x$, for each $m \leq n$. Thus $(m/n)!x$ is uniquely determined by the equivalence class $x$ and the integers $m, n (0 \leq m \leq n \leq 1)$. Clearly $(0/n)!x = o, (n/n)!x = x$, $(m/n)!o = o$.

The following properties follow easily from this definition, together with the results in §7.

(1) If $\rho \geq 1, n \geq 1$, and $0 \leq m \leq n$, then $(m/p)!x = (m/n)!x$.

Thus $(m/n)!x$ depends only on $[x]$ and the value of the proper fraction $m/n$. If $m/n = \rho$, we write $(m/n)!x$ as $\rho[x]$. In the following statements, $\rho$, $\rho_1$, and so on denote non-negative rational numbers not exceeding 1.

(2) If $\rho_1 < \rho_2$, then $\rho_1!x \leq \rho_2!x$, with equality only if $[x] = o$.

(Immediate on writing $\rho_1$ and $\rho_2$ with a common denominator.)

(3) If $[x] < [y]$, then $\rho[x] \leq \rho[y]$, with equality only if $\rho = 0$.

(4) If $[y] = \rho_1!x$ and $[z] = \rho_2!x$, where $y \vee z \leq x, x \neq 0$, and $yz = 0$, then

\[ \rho_1 + \rho_2 \leq 1 \text{ and } [y \vee z] = (\rho_1 + \rho_2)!x. \]

(5) If $[y_n] = \rho[x_n]$, where the elements $y_n$ are disjoint, and the elements $x_n$ are disjoint, then $[y_n] = \rho[x_n]$.

(6) If $[y] = \rho_1!x$, then $\rho_2[y] = (\rho_1 \rho_2)!x$.

The next step will consist in extending this definition of “multiplication” of equivalence classes by (rational) scalars, to allow the multipliers to be irrational. Several lemmas are needed.

12.6. If $[x] \leq \rho[e]$ for arbitrarily small values of $\rho$, then $x = o$.

For, from (2) above, we have $[x] \leq (1/2^n)!e$ for every $n$. Now (12.3) we
can write \( e = f_1 \lor g_1 \) where \( f_1, g_1 \) are disjoint and equivalent. Similarly, \( g_1 = f_2 \lor g_2 \) where \( f_2, g_2 \) are disjoint and equivalent; and so on. The disjoint sequence \( \{f_n\} \) is such that \( [f_n] = (1/2^n)[e] \); hence there exists \( x_n \leq f_n \) such that \( x_n \sim x \). Then \( V^+ x_n \sim V^+ x_n \); but all elements are now bounded \((10.7)\), and so \( x_1 = o \), whence \( x = o \).

12.7. If \( [y_n] \leq (1/2^n)[x] \) \((n = 1, 2, \cdots)\) then \( [Vy_n] \leq [x] \).

This is proved by essentially the same argument as in 12.6, since there is no loss in assuming the elements \( y_n \) to be disjoint.

12.8. Suppose \( x \) and \( y \) are such that, for each positive integer \( n \), there exist \( p_n, q_n \in E \) such that \( [p_n] \leq (1/2^{n+1})[e] \), \( q_n \leq (1/2^{n+1})[e] \), and \( [x - p_n] \leq [y] \).

There is no loss in assuming \( p_n \leq x \). Then there exist \( x_n \sim x \) and \( r_n \sim p_n \) such that \( r_n \leq x_n \) and \( x_n - r_n \leq y \lor q_n \). Thus \( x_n \leq y \lor r_n \lor q_n \leq y \lor f_n \), where \( f_n = V\{r_m \lor q_n | m \geq n\} \). Now \( [r_n \lor q_n] \leq (1/2^n)[e] \), from 12.5(4); hence \( [f_n] \leq (1/2^{n-1})[e] \), as readily follows from 12.7. Thus \((12.6) \land f_n = o \). But \((7.2)\) we have \([x] \leq [\Lambda(y \lor f_n)] = [y \lor \Lambda f_n] = [y] \).

12.9. Given \( x \in E \) and a real number \( \sigma \) such that \( 0 \leq \sigma \leq 1 \), there exists \( y \in E \) having the following property: Given any positive rational number \( \epsilon < 1 \), there exist (i) a rational number \( \rho \) \((0 \leq \rho \leq 1)\) such that \( |\rho - \sigma| \leq \epsilon \), (ii) an element \( d \in E \) such that \([d] \leq \epsilon[e] \) and \([y + 2d] = \rho[x] \).

As in the proof of 12.6, we construct a disjoint sequence of elements \( f_n \leq x \) such that, on writing \( g_n = x - V\{f_m | m \leq n\} \), we have \([f_n] = [g_n] = (1/2^n)[x] \). Expand \( \sigma \) as a binary decimal, say \( \sigma = \sum_{n=0}^{\infty} m_n/2^n \), where each \( m_n \) is 0 or 1, and define \( t_n = f_n \) if \( m_n = 1 \), and \( t_n = o \) otherwise. Write \( y = V t_n \).

To satisfy the requirements, we have only to choose \( n \) so that \( 2^{-n} < \epsilon \), and take \( d = V\{t_m | m > n\} \) and \( \rho = \sum_{k=0}^{n} m_k/2^k \). Clearly \( |\rho - \sigma| < 2^{-n} \) and (from 12.5(4)) \([y + 2d] = \rho[x] \). Finally, the fact that \([d] \leq (1/2^n)[e] \) follows from 12.7.

Remark. The proof shows that a little more is true than was stated; we can further make \( d \leq y, \rho \leq \sigma \), and \([d] \leq \epsilon[x] \).

12.10. Given \( x \in E \) and \( \sigma \) \((0 \leq \sigma \leq 1)\), suppose that \( y' \) also has the same property as \( y \) in 12.9; then \( y' \sim y \).

For each \( n \) we can write
\[
[y + 2d_n] = \rho_n[x], \quad |\rho_n - \sigma| < 1/2^{n+1}, \quad [d_n] \leq (1/2^{n+2})[e],
\]
\[
[y' + 2d'_n] = \rho'_n[x], \quad |\rho'_n - \sigma| < 1/2^{n+2}, \quad [d'_n] \leq (1/2^{n+2})[e],
\]
and can suppose without loss that \( \rho'_n \geq \rho_n \) for infinitely many values of \( n \).

Thus we find \( d''_n \leq y' + d'_n \) such that \([d''_n] = (\rho'_n - \rho_n)[x] \leq (1/2^{n+1})[e] \), and then have \((6.1) [y' + y'' + d'_n - d''_n] = \rho_n[x] \). Hence \((y' + y'' + d'_n - d''_n) \sim y + y'' + d''_n \) for arbitrarily large \( n \); and from 12.8 it readily follows that \([y] \leq [y'] \leq [y] \).

Since \( y \) is bounded, we obtain \([y'] = [y] \).

12.11. Definition. If \( x, \sigma \) and \( y \) are related as in 12.9, we write \([y] = \sigma[x] \); from 12.9 and 12.10, this relation determines one and only one equivalence.
class \([y]\) when \([x]\) and \(\sigma (0 \leq \sigma \leq 1)\) are given. Further, if \(\sigma\) is rational this definition reduces to that of 12.5, since in 12.9 we can then take \(\rho = \sigma\).

Throughout the rest of this section, \(\sigma, \sigma_1, \) and so on will denote real numbers between 0 and 1 (inclusive), and \(\epsilon\) denotes a real number such that \(0 < \epsilon < 1\). Most of the proofs will be omitted, since they involve no new ideas.

12.12. If \(y \leq x\) and \([y] = \sigma [x]\), then \([x - y] = (1 - \sigma) [x]\).

This is immediate from the way in which \(y\) was constructed in 12.9, if we note that \(\nu / \beta = x\) there (since \(hgn = o\), from 12.6).

12.13. If \([y] = \sigma [x]\), and \(u\) is invariant, then \([uy] = \sigma [ux]\).

Immediate from 6.6 and the definitions—12.5 and 12.11.

12.14. If \(\sigma_1 \leq \sigma_2, \sigma_1 [x] \leq \sigma_2 [x]\).

This follows from 12.8 by an argument similar to that in 12.10.

**Corollary.** If \(\sigma_1 < \sigma_2, \sigma_1 [x] < \sigma_2 [x]\) unless \(x = o\).

For we can choose rational \(\rho_1, \rho_2\) such that \(\sigma_1 < \rho_1 < \rho_2 < \sigma_2\); the corollary now follows from 12.14 and 12.5(2).

12.15. If \([p] = \sigma_1 [x], [q] = \sigma_2 [x], pq = o, \rho \vee q \leq x, \) and \(x \neq o\), then \(\sigma_1 + \sigma_2 \leq 1\) and \([p \vee q] = (\sigma_1 + \sigma_2) [x]\).

That \(\sigma_2 \leq 1 - \sigma_1\) follows easily from 12.12. The remainder of the assertion follows from the definition of \((\sigma_1 + \sigma_2) [x]\) and from the rational case (12.5(4)).

12.16. If \(\{y_n\}\) is a disjoint sequence such that \(y_n \leq x \neq o\) and \([y_n] = \sigma_n [x]\), then \(\sigma = \sum \sigma_n \leq 1, \) and \([Vy_n] = \sigma [x]\).

An easy induction based on 12.15 gives \(\sum^n \sigma_n \leq 1, \) for each \(m, \) so that \(\sigma \leq 1\). It also gives \([Vy_n] = (\sum^n \sigma_n) [x] \leq \sigma [x]\) (12.14), for each \(m;\) hence \((7.3) [Vy] \leq \sigma [x]\). On the other hand, we can assume \(\sigma > 0;\) let \(\epsilon\) be any positive number less than \(\sigma.\) If \(N\) is large enough, we have (on using 12.15 and 12.14) \([Vy_n] \geq [V\sigma_n] = (\sum^n \sigma_n) [x] \geq (\sigma - \epsilon) [x];\) and it readily follows from 12.8 that \([Vy_n] \geq \sigma [x].\) Thus, since all elements are now bounded, \([Vy_n] = \sigma [x]\).

12.17. If \(\{y_n\}\) and \(\{z_n\}\) are two sequences, each disjoint, such that \([y_n] = \sigma [z_n] \) \((n = 1, 2, \ldots)\), then \([Vy_n] = \sigma [Vz_n].\)

We may suppose \(y_n \leq z_n.\) Given any rational \(\epsilon > 0\) (and \(< 1)\) we choose a rational number \(\rho\) so that \(\sigma - \epsilon < \rho < \sigma,\) and then (12.9, Remark) choose \(d_n \leq y_n\) so that \([d_n] \leq (\epsilon/2^n) [\epsilon]\) and \([y_n - d_n] = \rho_n [z_n],\) where \(\rho < \rho_n < \sigma\) and \(\rho_n\) is rational. The rest is routine.

12.18. If \(x \geq y \geq z,\) and \(x \neq o,\) and if \([y] = \sigma_1 [x]\) and \(z = \sigma_2 [x],\) then \(\sigma_1 \geq \sigma_2\) and \([y - z] = (\sigma_1 - \sigma_2) [x].\)

12.19. If \(y_1 \leq y_2 \leq \cdots \leq x,\) and if \([y_n] = \sigma_n [x]\) and \(x \neq o,\) then \(\sigma_1 \leq \sigma_2 \leq \cdots \leq 1,\) and \([Vy] = (\lim \sigma_n) [x].\)

(From 12.16 and 12.18 applied to the relation \(V_y = V(y - y_{n-1}).\))

A similar result holds for decreasing sequences.

12.20. If \([y] = \sigma_1 [x],\) then \(\sigma_1 [y] = (\sigma_1 \sigma_2) [x].\)

This will be needed only in the case in which \(\sigma_1\) is rational, when it fol-
lows fairly easily from 12.5(6) and the definition (12.11). The extension to the general case can then be derived, though with more trouble.

13. **Decomposition into parts comparable with their closures.**

13.1. We begin with a fundamental result which, roughly speaking, asserts that the abstract measures can be approximated by "step functions."

**Theorem.** Given \( x \in E \) and a positive integer \( n \), there exist \( n \) disjoint invariant elements \( u_1, u_2, \ldots, u_n \) such that (i) \( \forall u_i = e \), and (ii) \( ((i-1)/n) [u_i] \leq [xu_i] \leq (i/n) [u_i] \) (1 \( \leq i \leq n \)).

By 12.3, we can write \( e = t_1 \lor t_2 \lor \cdots \lor t_m \), where the elements \( t_i \) are disjoint and equivalent. For each \( i \), consider the elements \( x \) and \( t_i \lor t_2 \lor \cdots \lor t_i \); from 7.6 there exists \( w_i \in U \) such that

(a) \([xw_i] \leq [w_i (t_i \lor \cdots \lor t_i)]\), (b) if \( v \in U (-w_i) \) then \([xv] \geq [t_i (t_i \lor \cdots \lor t_i)]\), with equality only if \( v = o \). Now \( [t_i (t_i \lor \cdots \lor t_i)] = ((i)/n) [e] \), by definition; hence, from 12.13, properties (a) and (b) may be restated as:

(a') \([xw_i] \leq ((i)/n) [w_i] \), (b') if \( v \in U (-w_i) \) then \([xv] > ((i)/n) [v] \) unless \( v = o \).

We assert: \( o \leq w_1 \leq w_2 \leq \cdots \leq w_n = e \). For suppose \( i < j \) and let \( v = w_j - w_i \); then \( \forall v \neq o \), we have at once from 12.13 that \([xv] > ((j)/n) [v] > ((i)/n) [v] (12.5(2)) \geq [v] \), a contradiction. Thus \( v = o \) and \( w_i \leq w_j \). The fact that \( w_n = e \) is immediate from (b').

Write \( w_0 = o \), and define \( u_i = w_i - w_{i-1} \) (1 \( \leq i \leq n \)). Thus the elements \( u_1, \ldots, u_n \) are invariant and disjoint, and \( \forall u_i = e \). From (a'), (b') and 12.13 we have \( ((i-1)/n) [u_i] \leq [xu_i] \leq (i/n) [u_i] \) as desired; and have in fact more — that \( ((i-1)/n) [u_i] < [xu_i] \) if \( i < n \), unless \( u_i = o \).

13.2 **Lemma.** Given \( x \neq o \), there exists \( y \in E \) such that \( o < y \leq x \) and \([y] = \rho [y] \) for some rational \( \rho \) (0 \( < \rho \leq 1 \)).

We apply 13.1, and note that if \( n \) is large enough \( u_i \neq e \), since otherwise \([x] \leq (1/n) [e] \) for arbitrarily large \( n \), which is impossible (12.6). Thus for some \( i > 1 \) we have \( u_i \neq o \); and since \((i-1)/n) [u_i] \leq [xu_i] \) there exists \( y \leq xu_i \) such that \([y] = ((i-1)/n) [u_i] \). Since \( i > 1 \) and \( u_i \neq o \) we have \( y \neq o \); and, from 12.13, \([y] = \rho [y] \) (where \( \rho = (i-1)/n \), that is, \([y] = \rho [y] \)).

13.3. **Theorem.** Given \( x \neq o \), there exists a sequence \{ \( z_n \) \} (finite or infinite) of disjoint nonzero elements such that (i) \( \forall z_n = x \), (ii) \([z_n] = \rho_n [\tilde{z}_n] \), where \( 0 < \rho_n \leq 1 \) and \( \rho_n \) is rational.

Let \{ \( z_n \) \} be a maximal disjoint collection of nonzero elements \( \leq x \) satisfying (ii). From 13.2 it readily follows that \( \forall z_n = x \), proving the theorem.

**Remark.** The theorem also holds if \( x = o \) provided we adopt suitable conventions about "empty" sequences.

13.4. **Theorem.** There exists a disjoint sequence \{ \( e_n \) \} (finite or infinite) such that (i) \( \forall e_n = e \), (ii) each \( e_n \) is homogeneous of infinite order \( m_n \) over \( U \), (iii) for each \( n \), \([e_n] = \rho_n [\tilde{e}_n] \), where \( \rho_n \) is rational and 0 \( < \rho_n \leq 1 \).
From 11.7 there exist disjoint (nonzero) elements $a_n$ such that $V a_n = e$ and $a_n$ is homogeneous of infinite order over $U$. Applying 13.3, we write $a_n = \bigvee m e_{nm}$ where (for each $n$) the elements $e_{nm}$ are disjoint, nonzero, and $[e_{nm}] = \rho_{nm} [e_{nm}]$. From the definition of homogeneity (11.1), each $e_{nm}$ is homogeneous of infinite order over $U$. We have only to renumber the elements $e_{nm}$ into a single sequence $\{ e_n \}$.

This theorem provides the desired decomposition of $e$ ($= e'$ or $f^n$; cf. 10.7). In the next sections (§§14–17) we shall consider the relative algebras $\{ E(e_n), \sim \}$, showing that each is isomorphic with the direct product of a numerical measure algebra with a trivial abstract measure algebra. It will then be relatively easy (§§18, 19) to derive a product representation for all of $E$.

14. Structure of $E(e_n)$; separable case.

14.1. Let $e_n$ be one of the elements in 13.4, and suppose that its order over $U$ is exactly $\aleph_0$. In this section we shall analyze the structure of the relative algebra $E(e_n)$; and, as all considerations here will be relative to $e_n$, we shall write $E(e_n)$ simply as $E$, $e_n$ as $e$, and the relative sub-algebra of relatively invariant elements, $U e_n$, as $U$. From 11.3 and 8.3, we can thus assume that every nonzero $x \in E$ has order $\aleph_0$ over $U$, and is also decomposable and bounded (10.7). We shall deduce the following theorem:

**Theorem.** There exists a sub-algebra $P$ of $E$ such that

- (i) $E = \mathcal{B}(P \cup U)$,
- (ii) $(E, \sim)$ is naturally isomorphic to a numerical measure algebra $(P, \mu)$, in such a way that, for each $p \in P$, $[p] = \mu(p) [e]$,
- (iii) $(P, \mu)$ is isometric to $P^1$ (cf. 3.1(a)).

14.2. Since the order of $e$ over $U$ is now $\aleph_0$, there exists a countably infinite set $B = \{ b_n \} \ (n = 1, 2, \ldots)$ such that $E = \mathcal{B}(B \cup U)$. The first step in the proof consists in replacing the elements $b_n$ by others with improved properties; and this is done in two stages.

14.3. The elements $c(\tau)$. Let $\tau$ denote any fraction of the form $i/4^n \ (n \geq 0, \ 0 \leq i \leq 4^n)$. We define $c(\tau)$ by induction, as follows: When $n = 0$, we set $c(0) = 0$ and $c(1) = e$. When $c(i/4^n)$ has been defined (for a particular $n \geq 0$ and for all $i$ such that $0 \leq i \leq 4^n$), we choose elements $f(i/4^n)$, $g(i/4^n)$, such that $f(i/4^n) \leq b_{n+1} \{ c(i/4^n) - c((i-1)/4^n) \} = h(i/4^n)$, say, $g(i/4^n) \leq (-b_{n+1}) \{ c(i/4^n) - c((i-1)/4^n) \} = k(i/4^n)$, and $[f(i/4^n)] = (1/2) [h(i/4^n)]$, $[g(i/4^n)] = (1/2) [k(i/4^n)]$, using 12.3, and then define (for $1 \leq i \leq 4^n$)

$$c((4i - 4)/4^{n+1}) = c((i - 1)/4^n),$$

$$c((4i - 3)/4^{n+1}) = c((i - 1)/4^n) \lor f(i/4^n),$$

$$c((4i - 2)/4^{n+1}) = c((i - 1)/4^n) \lor g(i/4^n),$$

and

$$c((4i - 1)/4^{n+1}) = c((4i - 2)/4^{n+1}) \lor g(i/4^n).$$
It readily follows by induction that
(1) If $\tau_1 \leq \tau_2$ then $c(\tau_1) \leq c(\tau_2)$, and that
(2) $[c((i/4^n) - c((i-1)/4^n)] \leq (1/2^n)[e].$

Again, we have
\[
\forall i \in \{c((4i - 2)/4^n + 1) - c((4i - 4)/4^n + 1)\} = V h(i/4^n)
\]
\[
= b_{n+1} V\{c(i/4^n) - c((i - 1)/4^n)\}
\]
\[
= b_{n+1};
\]
hence, on writing $C = \text{set of all elements } c(\tau)$, we have $B \subseteq B(C)$, and therefore
(3) $E = B(C \cup U)$.

14.4. The elements $d_{\rho}$. Now let $\rho$ be any rational number such that $0 \leq \rho \leq 1$. Choose any $y_{\rho}$ such that $[y_{\rho}] = \rho [e]$; an application of 7.6 to the elements $c(\tau)$, $y_{\rho}$, then gives an invariant element $u_{\rho}$ such that (on using 12.13)
(4) $[c(\tau) u_{\rho}] \leq \rho [u_{\rho}]$ whenever $u \subseteq U(u_{\rho})$, and $[c(\tau) v] \geq \rho [v]$ whenever $v \subseteq U(-u_{\rho})$, unless $v = o$.

It readily follows (by the same argument as in 13.1) that
(5) $u_{\rho \rho} \geq u_{\rho \rho}$ if $\tau_1 \leq \tau_2$; $u_{\rho 0} = e$; $u_{\rho 0} = o$ unless $\rho = 1$; and $u_{11} = e$.

Similarly
(6) $u_{\rho 1} \leq u_{\rho 2}$ if $\rho_1 \leq \rho_2$; $c(\tau) u_{10} = o$; and $u_{11} = e$.

Now define, for each $n$,
$$d_{\rho n} = \bigvee \{c(\tau) u_{\rho} \mid \tau = i/4^n, 0 \leq i \leq 4^n\},$$
and define
$$d_{\rho} = \bigvee \{c(\tau) u_{\rho}, \text{ all } \tau\}.$$

It readily follows that $d_{\rho 1} \leq d_{\rho 2} \leq \cdots$, and $V_n d_{\rho n} = d_{\rho}$. Further, from (6) we have
(7) $d_{\rho 1} \leq d_{\rho 2}$ if $\rho_1 \leq \rho_2$; and $d_0 = o$, $d_1 = e$.

We next show that
(8) $[d_{\rho}] = \rho [e]$.

Since $c((i+1)/4^n) \geq c(i/4^n)$, we have (on writing $u_{\rho} = o$ if $\tau > 1$, so that (5) still holds)
$$d_{\rho n} = \bigvee_{i=0}^{4^n} c((i/4^n) v_{\rho i}, \text{ where } v_{\rho i} = u((i/4^n) + u((i+1)/4^n) o).$$

From (4) and 12.5(5) it follows that $[d_{\rho n}] \leq [V_i v_{\rho i}] = \rho [e];$ and therefore (7.3) we have $[d_{\rho}] \leq \rho [e]$. 
To obtain the opposite inequality, we note that (from (7)) we may assume \( \rho < 1 \), so that now \( \nu = 0 \); thus \( \nu = 0 \), so that

\[
d_{\rho} \leq \sum_{i=0}^{\rho-1} c((i+1)/4^{\rho}) \nu = q_{\rho}, \quad \text{say.}
\]

But \( q_{\rho} - d_{\rho} = \sum_{i=0}^{\rho-1} \left[ c((i+1)/4^{\rho}) - c(i/4^{\rho}) \right] \nu \); and from (2) and 12.13 it now follows that \( [q_{\rho} - d_{\rho}] \leq (1/2^{\rho}) \nu \). A fortiori, therefore, \( q_{\rho} \geq \rho \). But from (4) we have \( [c((i+1)/4^{\rho}) \nu] \geq \rho \nu \), whence \( q_{\rho} \geq \rho \nu \). It readily follows from 12.8 that \( [d_{\rho}] \geq \rho \); and thus \( [d_{\rho}] = \rho \), Q.E.D.

Let \( D \) denote the set of all elements \( d_{\rho} \). We shall now prove that \( C \subseteq B(D \cup U) \). For convenience, we define \( d_{\rho} = 0 \) for \( \rho < 0 \), and \( d_{\rho} = e \) for \( \rho > 1 \); thus (7) is still maintained.

By 13.1 there exist, for each \( (i,j,4^{\rho}) \) and each \( n > 0 \), \( n \) disjoint invariant elements \( w_{\rho n} \) (1 \( \leq i \leq n \)) such that

\[
\bigvee_{i} w_{\rho n} = e,
\]

and

\[
(i-1)/n \leq c(\tau) \leq (i,n) w_{\rho n} \quad (1 \leq i \leq n).
\]

From 12.13 and (4) we see that, on writing \( \rho = (i-2)/n \) (where it is assumed for the moment that \( i \geq 2 \)), we have \( w_{\rho n} \nu = 0 \), and therefore (from (5)) \( w_{\rho n} \nu = 0 \) if \( \tau \neq \tau' \). Hence

\[
d_{\rho} w_{\rho n} \leq \bigvee_{\tau} c(\tau) w_{\rho n} \nu_{\tau} \leq \bigvee_{\tau} c(\tau) \leq c(\tau),
\]

so that

\[
d_{\rho} w_{\rho n} \leq c(\tau) w_{\rho n} \tag{11}
\]

where \( \rho = (i-2)/n \), a result which holds for all \( i \geq 0 \) since \( d_{\rho} = 0 \) if \( \rho < 0 \). It follows that

\[
c(\tau) \geq \bigvee_{0} d_{\rho} w_{\rho n} \tag{12}
\]

To obtain the reverse inequality, let \( c(\tau) \) denote the element on the right of (12). Since \( [c(\tau) w_{\rho n}] \leq (i/n) [w_{\rho n}] \) (from (10)), while if \( i \geq 2 \), \( [d_{\rho} w_{\rho n}] = (i-2)/n \) \( [w_{\rho n}] \) (from (8) and 12.13), we readily obtain \( [c(\tau) w_{\rho n} - d_{\rho} w_{\rho n}] \leq (2/n) [w_{\rho n}] \), a result which also holds (trivially) for \( i = 0, 1 \). Hence, on summing over \( i \), we obtain (using (9) and 12.17)

\[
(2/n) [e] \leq [c(\tau) - \bigvee_{i} d_{\rho} w_{\rho n}] \leq [c(\tau) - c(\tau)];
\]

and from 12.6 it follows that \( c(\tau) - c(\tau) = 0 \). Thus \( c(\tau) = c(\tau) \in B(D \cup U) \), and (3) now gives
14.5. The finitely additive algebra generated by $D$. Consider the set $F$ of all elements $x$ expressible as

$$x = (d_{p_1} - d_{p_2}) \lor (d_{p_3} - d_{p_4}) \lor \cdots \lor (d_{p_{2n}} - d_{p_{2n}}),$$

where $1 \geq p_1 \geq p_2 \geq \cdots \geq p_{2n} \geq 0$, and the numbers $p_i$ are rational. It is easy to see that, if $x$ and $y \in F$, then so do $x \lor y$ and $-x$; thus $F$ is a finitely additive sub-algebra of $E$. Clearly $\mathcal{B}(D) \subseteq F \subseteq D$ (for $d_0 = o=(7)$).

From (8), 12.18 and 12.16, we obtain

$$\text{(14)} \quad \text{If } x \in F, \quad [x] = \rho [e],$$

where $\rho = \sum_{i=0}^{n} (p_{2i+1} - p_{2i})$ (the numbers $p_i$ arising from any expression of $x$ as above, so that $\rho$ is rational and $0 \leq \rho \leq 1$). Note that (from 12.14) the number $\rho$ in (14) is determined uniquely by $x$—unless $e = o$, which we shall assume not to be the case.

14.6. The $\sigma$-algebra $P'$. Now consider the set $P'$ of elements $y$ which have the following property: For each $\epsilon$ ($0 < \epsilon < 1$) there exists $z \in E$ such that $y + 2z \in F$ and $[z] < \epsilon [e]$.

It is easy to verify that, if $y$ and $y_1$ ($1 \leq i \leq n$) belong to $P'$, then so do $-y$ and $y_1$; thus $P'$ is at least a finitely additive sub-algebra of $E$. Now, we have

$$\text{(15)} \quad \text{If } y \in P', \text{ there exists } \sigma (0 \leq \sigma \leq 1) \text{ such that } [y] = \sigma [e].$$

For there exist elements $z_n$ such that $y + 2z_n \in F$ and $[z_n] \leq \epsilon_n [e]$, where $\epsilon_n = 1/(n+1)$. Thus, writing $x_n = y + 2z_n$, we have (14) $[x_n] = \rho_n [e]$, where $\rho_n$ is rational. Further, since $x_m x_n \in F$, we have $[x_m x_n] = \rho_m [e]$, say. Thus

$$[x_n - x_m x_n] = (\rho_n - \rho_m) [e], \quad [x_m - x_m x_n] = (\rho_m - \rho_m) [e],$$

and so

$$[x_m + z_n] = [x_m + z_n] \leq [z_m + z_n] \leq (\rho_m + \epsilon_n) [e], \quad \text{and so} \quad (\rho_n - \rho_m) + (\rho_m - \rho_m) \leq \epsilon_n + \epsilon_n \quad \text{(12.5(2))}.$$}

Thus, since each term on the left is non-negative, we have $\rho_m \leq \rho_n \leq \rho_m + \epsilon_n + \epsilon_n$ and $\rho_m \leq \rho_n \leq \rho_m + \epsilon_n + \epsilon_n$, whence $|\rho_m - \rho_n| \leq \epsilon_n + \epsilon_n$. The sequence $\{\rho_n\}$ thus converges to a limit $\sigma$ ($0 \leq \sigma \leq 1$) as $n \to \infty$; and the definition of $\sigma [e]$ (12.11) shows at once that $[y] = \sigma [e]$.

To prove that $P'$ is a $\sigma$-subalgebra of $E$, we have only to show that if $y_n \in P'$ and $y_1 \leq y_2 \leq \cdots$, then $\forall y_n \in y, \mbox{ say, } \in P'$. Now from (15) we have $[y_n] = \sigma_n [e]$ and so (from 12.19) $y = \sigma [e]$ where $\sigma = \lim \sigma_n$. Thus, given $\epsilon$, there exists $n$ such that $|y - y_n| = (\sigma - \sigma_n) [e] \leq (\epsilon/2) [e]$. Since $y_n \in P'$, there exists $z$ such that $[z] \leq (\epsilon/2) [e]$ and $y_n + 2z \in F$. Let $z' = y + 2y_n + 2z$; then, since $y + 2z' \in F$ and $[z'] \leq [e]$, we have $y \in P'$.

14.7. Since $P' \supset F$, we have $P' \supset \mathcal{B}(D) = P$ (in fact $P' = P$, but this is not needed); and (15) then shows that to each $y \in P$ corresponds a unique real number $\mu(y)$, such that $0 \leq \mu(y) \leq 1$ and $[y] = \mu(y) [e]$. Properties 12.16, 12.14, show that $\mu$ is a countably additive numerical measure on $P$, vanishing only for $o$, and such that $\mu(y_1) = \mu(y_2)$ (where $y_1, y_2 \in P$) if and only if $y_1 \sim y_2$. That is, the sub-algebra $(P, \sim)$ of $E$ is naturally isomorphic to the numerical measure algebra $(P, \mu)$. We have already shown (13) that $E = \mathcal{B}(P \cup U)$. 
Finally, it is not hard to see that \((P, \mu)\) is isometric with \(I_1\), the algebra of Lebesgue-measurable sets modulo null sets in the unit interval; one makes the interval \((0, \mu)\) correspond to the element \(d_\mu\), and extends this correspondence in succession to \(F\) and \(P\). (We omit the details.) Thus the theorem stated in 14.1 is now proved.

15. **Structure of \(E(e_n)\); inseparable case.**

15.1. Now let \(e_n\) be one of the elements in 13.4 which has uncountable order over \(U\); we analyze the structure of the relative algebra \(E(e_n)\). As in the previous section, all considerations will be relative to \(e_n\); thus we write \(E(e_n)\) as \(E\), \(e_nU\) as \(U\), and so on, and can now assume that every nonzero element of \(E\) has order \(m > \aleph_0\) over \(U\), and is also decomposable and bounded. The fundamental theorem is the same as in the separable case (14.1), except that the measure-algebra is different.

**Theorem.** There exists a sub-algebra \(P\) of \(E\) such that

(i) \(E = \mathcal{B}(P \cup U)\),

(ii) \((P, \sim)\) is naturally isomorphic to a numerical measure algebra \((P, \mu)\), in such a way that, for each \(p \in P\), \([p] = \mu(p)[e]\),

(iii) \((P, \mu)\) is isometric to \(I^m\) (cf. 3.1(a)).

15.2. We shall require the following lemma.

**Principal Lemma.** Let \(S\) be a subalgebra of \(E\) such that \(a) S \supset U\), \(b)\) every nonzero element of \(E\) is of infinite order over \(S\). Then, given any \(x \in E - S\), there exists a sub-algebra \(Q\) of \(E\) such that \((i) x \in \mathcal{B}(Q \cup S)\), \((ii) (Q, \sim)\) is naturally isomorphic to a numerical measure algebra \((Q, \mu)\), in such a way that, for each \(q \in Q\) and \(s \in S\), \([qs] = \mu(q) [s]\), and \((iii) (Q, \mu)\) is isometric to \(I^1\).

This lemma will be proved in §16. Taking it for granted, we now deduce the theorem.

15.3. There exists a set \(A \subseteq E\), with \(|A| = m\), such that \(E = \mathcal{B}(A \cup U)\). Well-order the elements of \(A\) as \(a_\alpha\), \(1 \leq \alpha < \Omega\), where \(\Omega\) is the first ordinal of power \(m\). From the lemma (applied with \(S = U\) and \(x = a_1\)), there exists a sub-algebra \(Q_1\) such that \(a_1 \in \mathcal{B}(Q_1 \cup U)\) and for each \(q \in Q_1\) and \(u \in U\) we have \([qu] = \mu_1(q) [u]\); further \((Q_1, \mu_1)\) is isometric to \(I^1\). Now suppose that sub-algebras \(Q_\beta\) have been defined for all \(\beta < \alpha\) (where \(\alpha < \Omega\), in such a way that:

(i) \(a_\beta \in \mathcal{B}(U_{\gamma \leq \beta} Q_\gamma \cup U)\);

(ii) \((Q_\beta, \sim)\) is naturally isomorphic to a numerical measure algebra \((Q_\beta, \mu_\beta)\) in such a way that, on defining \(S_\beta = \mathcal{B}(U_{\gamma < \beta} Q_\gamma \cup U)\), we have \([bc] = \mu_\beta(b) [c]\) for all \(b \in Q_\beta\) and \(c \in S_\beta\); and

(iii) \(Q_\beta\) is isometric either to \(I^1\) or to the trivial measure algebra \(L\) consisting of \(\sigma\) and \(e\) only (these elements having measures 0 and 1 respectively).

Define \(Q_\alpha\) as follows: If \(a_\alpha \in S_\alpha\), take \(Q_\alpha\) to consist of the elements \(\sigma\) and \(e\) only, with \(\mu_\alpha(\sigma) = 0\) and \(\mu_\alpha(e) = 1\). The properties of the transfinite sequence are clearly maintained.
If \( a_\alpha \in S_a \), we first note that every nonzero element \( x \in E \) is of infinite order (and in fact of order \( m \) over \( S_a \); in fact, if \( E(x) = xB(X \cup S_a) \), there exists an at most countable subset \( A_\beta \) of \( Q_\beta \) (from the fact that \( Q_\beta \) is isomorphic to \( I^1 \) or \( L \)), and we have \( E(x) = xB(U_{\beta < \alpha} A_\beta \cup U \cup X) \) so that \( \bigcup_{\beta < \alpha} A_\beta \cup X \geq m \) and therefore \( |X| \geq m \). Hence the lemma (15.2) can be applied, taking \( S = S_\alpha \) and \( x = a_\alpha \); the sub-algebra given by the lemma is taken to be \( Q_a \).

Thus \( Q_a \) is defined for all \( \alpha < \Omega \); and since \( a_\alpha \in S_{\alpha + 1} \) we readily obtain

\[ (1) \quad E = B(S_\alpha \cup U), \quad S_\alpha = B(UQ_\alpha). \]

It follows (by the same order argument as before) that \( Q_a \) is isometric to \( I^1 \) for \( m \) values of \( \alpha \).

Suppose \( x \in Q_\alpha \cap Q_\beta \ (\alpha \neq \beta) \). Since \( [x] = \mu_\alpha(x) [e] \) and also \( [x] = \mu_\beta(x) [e] \), we have \((12.14) \mu_\alpha(x) = \mu_\beta(x); \) hence we may omit the suffix \( \alpha \), and write \( \mu_\alpha(x) = \mu(x) \). Again, we may suppose \( \beta < \alpha \); then \( x \in S_\alpha \), so that \((15.3(i)) \ [x] = \mu(x) [x]. \) By 12.14, either \( x = 0 \) or \( \mu(x) = 1 \); in the latter case \( [x] = [e] \), and so \( (e \) being bounded) \( x = e \). Thus:

\[ (2) \quad \alpha \neq \beta, \quad Q_\alpha \cap Q_\beta = \{0, e\}. \]

15.4. The desired sub-algebra \( P \) of \( E \) will actually be \( S_a \); but it is more convenient to define it in another way. Let

\[ D = \text{set of all elements } d \text{ expressible as } d = b_1 b_2 \cdots b_i, \quad \text{where } b_i \in Q_\alpha \text{ and } \alpha_1 > \alpha_2 > \cdots > \alpha_k; \]

\[ F = \text{set of all elements } f \text{ expressible as } f = d_1 \vee d_2 \vee \cdots \vee d_m, \quad \text{where } d_i \in D \text{ and the elements } d_i \text{ are disjoint;} \]

\[ G = \{g \mid g = \bigvee d_i, \quad d_i \in D \} \quad (i = 1, 2, \cdots); \]

\[ P = \{p \mid p = \bigwedge g_n, \quad g_n \in G, \quad g_1 \geq g_2 \geq \cdots \}. \]

Clearly \( D \subseteq F \subseteq G \subseteq P \). The first step consists in proving:

\[ (3) \quad \text{If } x \in P, \quad x = \mu(x) [e] \text{ for some real number } \mu(x) \quad (0 \leq \mu(x) \leq 1). \quad (\mu(x) \text{ is then unique—from 12.14, Corollary.)} \]

For suppose first \( x = d \in D \), say \( x = b_1 b_2 \cdots b_k \) as above. Then, since \( b_2 \cdots b_k \in S_{\alpha_1}\), \( [d] = \mu(b_1) [b_2 \cdots b_k] \) (from 15.3(ii)) = \( \mu(b_1) \mu(b_2) [b_3 \cdots b_k] = \mu(b_1) \mu(b_2) \cdots \mu(b_k) [e] \), similarly; thus \( (3) \) holds in this case, with \( \mu(d) = \prod \mu(b_i) \).

Next, suppose \( x = f \in F \). Then \( f = \bigvee_i d_i \), where the elements \( d_i \) are disjoint and belong to \( D \); by the preceding, \( [d_i] = \mu(d_i) [e] \); and then \((12.16) \) \( [f] = \mu(f) [e] \) where \( \mu(f) = \sum \mu(d_i) \).

Before considering \( G \), we note that \( F \) is a finitely additive algebra. In fact, it is clear that if \( d_1, d_2 \in D \), then \( d_1 d_2 \in D \); and it readily follows that if \( f, f' \in F \), then \( ff' \in F \). Again, given \( d \in D \), we see that \( -d \in F \) as follows: We have \( d = b_1 b_2 \cdots b_n \), say, where \( b_i \in Q_\alpha \); consider the \( 2^k \) elements \( d_i \) \((1 \leq i \leq 2^k)\) of the form \( c_0 c_2 \cdots c_0 \) where each \( c_i \) is either \( b_i \) or \( -b_i \). They are disjoint, and we may suppose \( d = d_i \), say; then \( -d = \bigvee d_i \) \((2 \leq i \leq 2^k)\), showing that \( -d \in F \) if \( d \in D \). From the fact that \( F \) is closed under (finite) infima, it now follows
that \(-f \in F\) if \(f \in F\); for we have \(f = \bigvee d_i\) (\(d_i \in D\)), and consequently \(-f = \bigwedge (-d_i) \in F\).

It follows that each \(g \in G\) is expressible as \(g = \bigvee f_n\), where \(f_n \in F\) and the elements \(f_n\) are disjoint. (For if \(g = \bigvee d_i\), \(d_i \in D\), we take \(f_n = \bigvee d_i - \bigvee_{i=1}^{-1} d_i\).) From the preceding, we have \([f_n] = \mu(f_n)[e]\), and therefore (12.16) \([g] = \mu(g)[e]\) where \(\mu(g) = \sum_{i=1}^{\infty} \mu(f_n)\).

Finally, given \(p \in P\), we have \(p = \bigwedge g_n\), where \(g_1 \geq g_2 \geq \cdots\), and, as just shown, \([g_n] = \mu(g_n)[e]\). By 12.19, \([p] = \mu(p)[e]\) where \(\mu(p) = \lim_{n \to \infty} \mu(g_n)\).

Next,

(4) If \(p_n \in P\) \((n = 1, 2, \cdots)\), then \(\bigvee p_n \in P\).

For, from the definition of \(P\), we have \(p_n = \bigwedge m g_{nm}\), where \(g_{ni} \geq g_{n2} \geq \cdots\) and \(g_{nm} \in G\); and it has just been shown that \(\mu(p_n) = \lim_{m \to \infty} \mu(g_{nm})\). Hence for each \(k\) we can choose \(m = m(k, n)\) so that (i) \(\mu(g_{nm}) \leq \mu(p_n) + 1/2^k\), (ii) for each \(k\) and \(n\), \(m\) is the least such number. Write \(h_k = \bigwedge g_{nm}\), \(h_k \in G\) (from the definition of \(G\)), and (from (ii)) \(h_k \geq h_{k-1} \geq \cdots\). Further, 12.7 gives that \([h_k - \bigvee p_n] \leq (1/k)[e]\), and so (12.6) \(\bigvee p_n = h_k\). Thus \(\bigvee p_n \in P\).

Again,

(5) If \(p \in P\), then \(-p \in P\).

For we have \(p = \bigwedge g_n\), where \(g_1 \geq g_2 \geq \cdots\) and \(g_n \in G\). Thus \(-p = \bigvee (-g_n)\), and by (4) it will suffice to prove that \(-g_n \in P\), for each \(n\). But \(g_n = \bigvee d_{ni}\), say, where \(d_{ni} \in D\). Hence \(-g_n = \bigwedge (-d_{ni}) = \bigwedge x_{nm}\), where \(x_{nm} = \bigwedge (-d_i)\) \((1 \leq i \leq m)\). Thus \(x_{n1} \geq x_{n2} \geq \cdots\), and \(x_{nm} \in F\) (for \(F\) is finitely additive) \(\subset G\), so that \(-g_n \in P\) by definition.

(4) and (5) show that \(P\) is a \((\sigma-)\) sub-algebra of \(E\); and (3), together with the properties in §12, shows that \(\mu\) is a countably additive measure on \(P\), \((P, \mu)\) being naturally isomorphic to \((P, \sim)\). Since \(P \supset D \supset Q_a\) (1) gives \(E = B(P \cup U)\). Finally, the fact that \((P, \mu)\) is isometric to \(\mathbb{R}\) is an immediate consequence of the way in which the values of \(\mu\) were derived successively on \(UQ_a, D, F, G\), and \(P\); this shows (virtually by definition) that \((P, \mu)\) is (to within isometry) the direct product of all the measure algebras \(Q_a\), and those \(Q_a's\) for which \(Q_a = L\) can obviously be disregarded.

16. Inseparable case continued; proof of the principal lemma.

16.1. Let \(S\) be a sub-algebra of \(E\) (= \(E(e_n)\), as in §15) such that (a) \(S \supset U\), (b) every nonzero element of \(E\) is of infinite order over \(S\). The first part of the proof of 15.2 (16.2–16.13) consists essentially in showing that \(S\) has properties similar to those of \(U\). Once this is done, the desired numerical measure algebra is constructed in much the same way as in the "separable case" (§14).

16.2. Notation. For any \(X \subset E\), write \(X' = X - (o)\). Thus, if \(z \in S', S'(z)\) denotes \(\{y \in S' \text{ and } o < y \leq z\}\).

Throughout 16.2–16.10 we suppose a fixed element \(b \in E\) to be given, and we define, for each \(z \in S'\),
\[ \sigma(z) = \inf \{ \eta \mid 0 \leq \eta \leq 1 \text{ and } \eta[z] \geq [b\eta] \}, \]
\[ \tau(z) = \sup \{ \eta \mid 0 \leq \eta \leq 1 \text{ and } \tau[z] \leq [b\eta] \}. \]

Thus (from 12.14 and 12.19) we have
\[ \sigma(z)[z] \geq [b\tau]\geq \tau(z)[z] \]

and
\[ 0 \leq \tau(z) \leq \sigma(z) \leq 1. \]

In the next few paragraphs, we derive some properties of the function \( \sigma \); similar results and arguments will hold throughout for \( \tau \). (The immediate goal is 16.7.)

16.3. If \( z_n \in S' \) (\( n = 1, 2, \cdots \)), and the elements \( z_n \) are disjoint, then \( \sigma(\bigcup z_n) \leq \sup \sigma(z_n) \).

For otherwise \( \sigma(\bigcup z_n) > \eta \geq \sigma(z_n) \) for all \( n \); and then \( \eta[z_n] \geq [b\eta] \), whence \( \eta[\bigcup z_n] \geq [b(\bigcup z_n)] \) and therefore \( \eta \geq \sigma(\bigcup z_n) \)—a contradiction.

16.4. If \( z \in S' \) and \( \sigma(z) > \eta \geq 0 \), there exists \( z_0 \in S'(z) \) such that, for all \( y \in S'(z_0) \), \( \sigma(y) \geq \eta \).

For if this is false, transfinite induction gives a countable set of disjoint elements \( z_n \) such that \( Vz_a = z \) and \( \sigma(z_n) < \eta \). From 16.3, \( \sigma(z) \leq \eta \)—a contradiction.

16.5. Given \( z_0 \in S' \) and \( \epsilon > 0 \), there exists \( y \in S'(z_0) \) such that, whenever \( y_1, y_2 \in S'(y) \), we have
\[ |\sigma(y_1) - \sigma(y_2)| < \epsilon \quad \text{and} \quad |\tau(y_1) - \tau(y_2)| < \epsilon. \]

Write \( \eta = \sup \{ \sigma(z) \mid z \in S'(z_0) \} \), and choose \( z_1 \in S'(z_0) \) such that \( \sigma(z_1) > \eta - \epsilon \). We can suppose \( \eta \neq 0 \) (else we merely take \( y = z_0 \)), and thus that \( \eta - \epsilon > 0 \). On applying 16.4 to \( z_1 \), we obtain (say) \( y^1 \in S'(z_1) \subset S'(z_0) \) having the desired property as far as the function \( \sigma \) is concerned. An "exhaustion" argument then gives \( z_0 = V\gamma^1 \), where \( \gamma^1 \in S'(z_0) \) and \( |\sigma(y_1) - \sigma(y_2)| < \epsilon \) whenever \( y_1, y_2 \in S' \). An entirely similar argument applied to the function \( \tau \) will give \( z_0 = V\gamma^1 \), where \( \gamma^1 \in S'(z_0) \) and \( |\tau(y_1) - \tau(y_2)| < \epsilon \) whenever \( y_1, y_2 \in S' \). Since \( z_0 = V_{m,n} \gamma^{m,n} \), at least one of the elements \( \gamma^{m,n} \) is nonzero, and gives the desired element \( y \).

16.6. Given \( z_0 \in S' \) and \( \epsilon > 0 \), there exists \( z \in S'(z_0) \) such that \( |\sigma(y_1) - \tau(y_2)| < \epsilon \) whenever \( y_1, y_2 \in S'(z) \).

By 16.5, there exists \( y \in S'(z_0) \) such that, whenever \( y_1, y_2 \in S'(y) \), \( |\sigma(y_1) - \sigma(y_2)| < \epsilon/3 \) and \( |\tau(y_1) - \tau(y_2)| < \epsilon/3 \). We can suppose \( yb = 0 \) (for otherwise we merely take \( z = y \)). Now (13.1) there exist, for each positive integer \( n \), \( n \) disjoint invariant elements \( u_n (1 \leq i \leq n) \in U \) such that \( V_i u_n = \epsilon \) and
\[ ((i - 1)/n)[u_{ni}] \leq [ybu_{ni}] \leq (i/n)[u_{ni}]; \]
and similarly there exist \( n \) disjoint elements \( v_{nj} \in U \) (\( 1 \leq j \leq n \)) such that
There will exist integers \( n_0 \) and \( i_0 \), with \( n_0 \geq i_0 \geq 2 \), such that \( u_{n_0} \neq o \); for otherwise \( [y_b] \leq (1/n) [e] \) for all \( n \), and thus (12.6) \( y_b = o \), contrary to our supposition. Choose a positive integer \( p \) so large that \( p(i_0 - 1) > 2 + 6/e \).

Write \( N = n_0 p \), and consider the elements \( u_{n_0} (1 \leq i \leq N) \). At least one of them, say \( u_{n_0} i_o \), meets \( u_{n_0} i_o = e \); and from (1), 12.13 and 12.14 we have \( \eta(n) = i/o(n_0) \), and so \( \eta(n) > 2 + 6/e \). Similarly \( u_{n_0} i_o \) meets some \( v_{n_0} i_o \), write \( u_{n_0} i_o v_{n_0} i_o \; u \), and \( v_{n_0} i_o = z \). Thus \( z \in S \) and \( z \leq y \leq z_0 \). From (1), (2) and 12.13 we have

\[
\left( \frac{(k - 1)}{N} \right)[u] \leq [z] \leq \left( \frac{(k)}{N} \right)[u].
\]

Since \( u \neq o \), it follows (12.14, Corollary) that \( j \geq k - 1 > 1 + 6/e \). Thus \( z \neq o \). Further, on using 12.5(6) we obtain \( \left( \frac{(k - 1)j}{N} \right) \leq (k - 1) / \leq \sigma(z) \). Again, \( \sigma(z) \leq k/(j - 1) \); for this is trivial if \( j > k - 1 \), and otherwise we have \( [z] \leq \left( \frac{(j - 1)}{N} \right) \leq (k - 1)/(j - 1) \leq \sigma(z) \). Thus \( 0 \leq \sigma(z) - \sigma(z) \leq k/(j - 1) - (k - 1)/j < \epsilon/3 \) (from the choice of \( p \)). Finally, if \( y_1, y_2 \in S'(z) \), we have \( y_1, z \in S'(y) \) and so \( (\sigma(y) - \sigma(z)) < \epsilon/3 \); similarly \( (\sigma(y) - \sigma(z)) < \epsilon/3 \), and the result follows.

16.7. Given \( \epsilon > 0 \), there exists a disjoint sequence \( \{ z_n \} \) such that \( z_n \in S' \), \( \forall z_n = e \), and \( |\sigma(y_1) - \sigma(y_2)| < \epsilon \) whenever \( y_1, y_2 \in S'(z_n) \).

This follows from the preceding by “exhaustion.”

16.8. Given a real number \( \eta \) such that \( 0 < \eta < 1 \), there exists \( z \in S \) such that \( \eta[y] \leq [b y] \) for all \( y \in S(z) \), and \( \eta[z] \geq [b z] \) for all \( t \in S(-z) \).

On applying the preceding result, with \( \epsilon = 1/n \), we obtain elements \( z_n \in S' \) such that \( z_n, z_n = o \) if \( i \neq j \), \( \forall z_n = e \), and \( |\sigma(y) - \sigma(z)| \leq 1/n \) whenever \( y \in S'(z_n) \). We can further suppose that each \( z_n \in S' \) is less than or equal to some \( z_n \), since otherwise we replace (for each \( n \)) the elements \( z_n \) by all the nonzero elements of the form \( z_{i_1} z_{i_2} \cdots z_{i_n} \). Now let \( I_n = \{ t \in S(z) \} \) denote the set of values of \( i \) (possibly empty) for which it is true that \( \sigma(y) \leq \eta \) whenever \( y \in S(z) \), and define

\[
\mathcal{S}_o = \mathcal{S} = \mathcal{S} \{ z_n \mid i \in I_n \}.
\]

Let \( z = -\mathcal{S}_o \). Thus \( z_n \in S \), \( z \leq S \), and \( s_1 \leq s_2 \leq \cdots \). Hence if \( t \in S(-z) \), we have \( t = \mathcal{S}_o \), where \( t_n = 1(s_n - s_{n-1}) \) (\( n = 1, 2, \cdots \)); thus \( \sigma(t_n) \leq \eta \), and therefore (16.3) \( \sigma(t) \leq \eta \), so that \( \eta[t] \geq [b] \).

On the other hand, if \( y \in S(z) \) then \( y_s = o \) for each \( n \), and therefore \( y = \mathcal{S} \{ y_s \mid i \in I_n \} \). For each \( z_n \) occurring here, there exists (say) \( r_n \in S(z_n) \) such that \( \sigma(r_n) > \eta \). Hence \( \tau(y_{i_n}) > \eta - 1/n \), and therefore (from the property
of \( \tau \) analogous to 16.3) \( \tau(y) \geq \eta - 1/n, \) so that \( \tau(y) \geq \eta \) and \( \eta[y] \leq [by]. \)

16.9. There exists a greatest \( z \) having the properties stated in 16.8.

Let \( z_0 \) be the supremum of all \( z \)'s for which 16.8 holds. Then, from the countable chain condition, we have \( z_0 = \bigvee z_n \) \((n = 1, 2, \ldots)\), where 16.8 holds for each \( z_n. \) If \( y \in S(z_0), \) we have \( y = \bigvee y_n \) where \( y_1 = y_{21} \) and \( y_n = y(z_n - (z_1 \lor \cdots \lor z_{n-1})). \) Thus \( \eta[y_n] \leq [by_n], \) and it readily follows (cf. 12.17) that \( \eta[y] \leq [by]. \) If \( t \in S(-z_0), \) then \( t \in S(-z_1), \) and so \( \eta[t] \geq [bt]. \) Thus 16.8 holds for \( z_0. \)

16.10. For each positive integer \( n, \) there exist \( n \) disjoint elements \( s_1, s_2, \ldots, s_n \in S \) such that (i) \( \forall s_i = e, \) (ii) whenever \( z \in S(s_i), \) \((i-1)/n) [z] \leq [zb] \leq (i/n) [z] \((1 \leq i \leq n). \)

(Compare 13.1.)

Using 16.8 and 16.9, define \( z_i \) to be the greatest element of \( S \) such that \( (i/n) [y] \leq [by] \) whenever \( y \in S(z_i), \) and \( (i/n) [t] \geq [bt] \) whenever \( t \in S(-z_i). \)

Thus if \( i < j \) we have \( (i/n) [s_j - z_i] \geq [b(s_j - z_i)] \geq (j/n) [s_j - z_i], \) so that \( z_j - z_i = 0; \) that is, \( e \geq z_1 \geq z_2 \cdots \geq z_n. \) We have only to define \( s_i = -z_1, \) \( s_i = z_{i-1} - z_i \) \((1 < i < n), \) and \( s_n = z_{n-1}. \)

(The maximality of \( z_i \) has not been used, but will be convenient in the next paragraph.)

16.11. Given \( c, d \in E, \) there exists \( z_0 \in S \) such that \( [cy] \leq [dy] \) whenever \( y \in S(z_0), \) and \( [et] \geq [dt] \) whenever \( t \in S(-z_0). \)

(Compare 7.6.)

Write \( N = 2^n; \) then, for each \( n \) \((= 1, 2, \ldots)\) the preceding (applied with \( b = c \)) gives \( N \) disjoint elements \( s_n \) \((1 \leq i \leq N) \in S \) such that \( e = \bigvee s_{ni} \) and \((i-1)/N) [z] \leq [zc] \leq (i/N) [z] \) for all \( z \in S(s_{ni}). \) Similarly there exist \( 2^n \) disjoint elements \( t_{nj} \) \((1 \leq j \leq N) \in S \) such that \( e = \bigvee t_{nj} \) and \((j-1)/N) [z] \leq [zd] \leq (j/N) [z] \) for all \( z \in S(t_{nj}). \) The method of construction used in 16.10 further shows that \( S(s_{ni} \lor (j-1) \lor S(t_{nj}), s_t = s_{ni}, \) with a similar result for the \( t \)'s.

Thus, on writing \( f_n = \bigvee \{s_{ni}t_{nj} | j \geq i\}, \) we have \( f_1 \geq f_2 \cdots \geq \cdot \cdot \cdot . \) Now define \( z_0 = Kf_n. \) It is not hard to verify that \( z_0 \) has the desired properties.

16.12. Given \( b \in E', \) there exists \( y \in E'(b) \) such that, for each \( z \in S, \) \([yz] \leq [bz]/2. \)

(Here we use for the first time the assumption that every element of \( E' \) has infinite order over \( S. \))

Since \( b \) is not of order \( 0 \) over \( S, \) there exists \( h \leq b \) such that \( h \in bS. \)

Applying 16.11 to \( c = h, \) \( d = b - h, \) we obtain \( z_0 \in S \) such that \([sh] \leq [s(b-h)] \) if \( s \in S(z_0), \) and \([th] \geq [t(b-h)] \) if \( t \in S(-z_0). \) Thus \([sh] \leq (1/2) [sb] \) and \([t(b-h)] \leq (1/2) [tb]. \) Define \( y = z_0h \lor (-z_0)(b-h). \) Thus clearly \( y \leq b; \) and also \( y \neq 0, \) since otherwise we obtain \( b - z_0 = h - z_0 \) and \( kh = 0, \) whence \( h = b - z_0 \in bS, \) a contradiction. Finally, given \( z \in S, \) we can write \( yz = sh \lor t(b-h) \) where \( s = z_0z \) and \( t = z-z_0. \) Hence \([yz] = (1/2) [sb \lor tb] \leq (1/2) [bz]. \)

By iteration we now obtain:

16.12'. Corollary. Given \( b \in E' \) and \( n > 0, \) there exists \( y_n \) such that \( o < y_n \)
$\leq b$ and, for each $z \in S$, $[y_n z] \leq (1/2^n)[y z]$.

16.13. Let $b \in E, z \in S,$ and $\eta (0 < \eta < 1)$ be such that $[b y] \geq \eta [y]$ whenever $y \in S(z)$. Then there exists $b' \leq b$ such that $[b' y] = \eta [y]$ whenever $y \in S(z)$.

We can clearly suppose $z \neq o; thus bz \neq o,$ and so (16.12') there exists $c \in E'(bz)$ such that, for each $y \in S, [c y] \leq \eta [b y]$. Thus $c$ certainly satisfies:

(i) $0 < c \leq b z,$ (ii) for each $y \in S(z), [c y] \leq \eta [b y]$. An easy "exhaustion" argument now shows that there exists a maximal $c$, say $b'$, satisfying (i) and (ii).

Then $b'$ is the desired element. For suppose not; then there exists $y \in S(z)$ such that $[b' y] < \eta [y]$. From 16.10 we obtain, for each $n, y$ disjoint elements $s_n \in S (1 \leq i \leq n)$ such that $\forall_{s_m = e}$ and, for all $s \in S(s_m),$

$$[(i - 1)/n][s] \leq [sb'] \leq (i/n)[s].$$

Suppose first that, for every $n$, we have $y_{s_n} = o$ whenever $(i + 1)/n < \eta$. It readily follows that $[b' y] \geq (\eta - 2/n)[y]$ for every $n$, and so $[b' y] \geq \eta [y]$, contrary to hypothesis. Thus we may choose $n$ and $i$ so that $(i + 1)/n < \eta$ and $y_{s_n} = t, say, \neq o$. Then, for every $z \in S(t)$, we have $[s b'] \leq (i/n)[s] \leq (\eta - 1/n)[s], whence [s(b - b')] \geq (1/n)[s].$

It follows that $(b - b') \neq o, so that (16.12') there exists $d \in E'(b - b')$ such that $[d s] \leq (1/n)[t(b - b')] \leq (1/n)[t s]$ whenever $s \in E$. Define $c' = b' \vee d$; thus $0 < c' \leq b$. Further, given any $y' \in S(z)$, we have $c'y' = b'y' \land b'y'(-t) \lor dy', a supremum of three disjoint elements in which $[b'y'] \leq (\eta - 1/n)[y'],

[b'y'(-t)] \leq \eta [y'(-t)] \quad (\text{from property (ii) of } b'),$

and $[dy'] \leq (1/n)[y'],$

from which we obtain $[c'y'] \leq \eta [y']$, for all $y' \in S(z)$. But this contradicts the maximality of $b'$.

16.14. Given $x \in S, there exists elements $b(\rho) \in E \ (\rho = \iota/2^n, 0 \leq i \leq 2^n, n = 0, 1, 2, \ldots)$ such that

(i) $[b(\rho) z] = \rho [z]$ whenever $z \in S,$

(ii) $b(\rho_1) \leq b(\rho_2) \ if \ \rho_1 \leq \rho_2; \ b(0) = e, \ and \ b(1) = e.$

(iii) $x \in B(S \cup B), where B = \bigcup_{b(\rho)}.$

For each $\rho = \iota/2^n \ (0 \leq i \leq 2^n, n \geq 0)$, we define $s(\rho)$ (using 16.9) to be the greatest element of $S$ such that

$$[x z] \geq \rho [z] \ whenever \ z \in S(s(\rho)), \ and$$

(1)

$$[x z] \leq \rho [z] \ whenever \ z \in S(-s(\rho)).$$

Thus (cf. 16.10) we have

(2) If $\rho_1 \leq \rho_2, e = s(0) \geq s(\rho_1) \geq s(\rho_2),$ and $s(1) \leq x.$

The elements $b(\iota/2^n)$ are defined by induction over $n$, starting with
Suppose that \( n \geq 1 \), and that \( b(i/2^m) \) has been defined for all \( m < n \) and \( i \leq 2^m \) in such a way that properties (i) and (ii) (of the statement of the present proposition) hold, and so that, in addition, we have
\[
(3) \quad b(\rho) s(\rho) \leq x, \quad \text{and} \quad x - s(\rho) \leq b(\rho)
\]
for all \( \rho = i/2^m, m < n \). (This will trivially be the case when \( n = 1 \).) We define \( b((2k-1)/2^n) = b(k/2^{n-1}) \), so that the definition is consistent. The definition of \( b((2k-1)/2^n) \) is more troublesome, and in giving it we suppose (to save suffixes) that \( k \) and \( n \) are fixed \( (1 \leq k \leq 2^{n-1}) \), and write \( \rho = (2k-1)/2^n \), and \( \rho^0 = k/2^{n-1} \). We use \( s_1, s_2 \) and \( s_3 \) to denote (arbitrary) elements of \( S(-s(\rho^0)) \), \( S(s(\rho^0) - s(\rho^+)) \) and \( S(s(\rho^+)) \), respectively.

First we show
\[
(4) \quad [s_1 x - b(\rho^-)] \leq (1/2^n) [s_1].
\]
For \( s_1 = s' \vee s'' \) where \( s' = s_1 - s(\rho^-) \) and \( s'' = s_1 \{ s(\rho^-) - s(\rho^0) \} \). Now \( [z'' x] \leq s_0 [z''] \) from (1); \( [z'' \vee b(\rho^-)] = [z'' b(\rho^-)] = \rho^- [z''] \) from (3) and (i); and \( z' x - b(\rho^-) = o \) from (3). (4) now follows easily.

A fortiori, \( [s_1 x \{ b(\rho^+) - b(\rho^-) \}] \leq (1/2^n) [s_1] \); and since \( [s_1 \{ b(\rho^+) - b(\rho^-) \}] = (1/2^{n-1}) [s_1] \) (from (i) and (ii)), we have \( [s_1 (-x) \{ b(\rho^+) - b(\rho^-) \}] \geq (1/2^n) [s_1] \). Here \( s_1 \) is any element of \( S(-s(\rho^0)) \). Thus, from 16.13, there exists \( y_0 \leq (-x) \{ b(\rho^+) - b(\rho^-) \} - s(\rho^0) \) such that \( [y_0 s_1] = (1/2^n) [s_1] \) for all \( s_1 \in S(-s(\rho^0)) \). Write \( y_1 = \{ b(\rho^+) - b(\rho^-) \} (-s(\rho^0)) - y_0 \). Thus for all \( s_1 \) we have (on using (i) and (ii))
\[
(5) \quad [y_1 s_1] = (1/2^n) [s_1].
\]

Similar arguments prove the existence of \( y_2 \leq x \{ b(\rho^+) - b(\rho^-) \} \{ s(\rho^0) - s(\rho^+) \} \) such that
\[
(6) \quad [y_2 s_2] = (1/2^n) [s_2] \quad (s_2 \in S\{ s(\rho^0) - s(\rho^+) \}),
\]
and of \( y_3 \leq \{ b(\rho^+) - b(\rho^-) \} s(\rho^+) \leq x \) such that
\[
(7) \quad [y_3 s_3] = (1/2^n) [s_3] \quad (s_3 \in S(\rho^+)).
\]

Define \( y = y_1 \vee y_2 \vee y_3 \), and \( b(\rho^0) = b(\rho^-) \vee y \). Clearly property (ii) is maintained. To establish (i), it will suffice to verify that \( [y s] = (1/2^n) [s] \) for all \( s \in S \). But we can write \( s = s_1 \vee s_2 \vee s_3 \); and it is easily verified that \( y s_i = y_i s_i \) \( (i = 1, 2, 3) \). Since the elements \( s_i \) are necessarily disjoint, the desired relation follows from (5), (6) and (7). Property (3) can likewise be verified, and the inductive definition of the elements \( b(\rho) \) is complete.

Finally, to prove (iii), we show
\[
(8) \quad x = \bigvee x_n, \quad \text{where} \quad x_n = \bigvee (i/2^n) s(i/2^n).
\]
From (3), \( x \geq x_n \) for every \( n \). Now
\[ x - x_n \leq x - x \left\{ \bigvee_{i=0}^{2^n-1} b(i/2^n) \{ s(i/2^n) - s((i+1)/2^n) \} \right\} \]
\[ = \bigvee_{i=0}^{2^n-1} \left\{ x - b(i/2^n) \{ s(i/2^n) - s((i+1)/2^n) \} \right\} \]
\[ \text{(since } x = \bigvee x \{ s(i/2^n) - s((i+1)/2^n) \}) \]
\[ = \bigvee_t, \text{ say.} \]

But from (1), \[ [x \{ s(i/2^n) - s((i+1)/2^n) \}] \leq ((i+1)/2^n) [s(i/2^n) - s((i+1)/2^n)]; \]
and, from (3) and (i), \[ [xb(i/2^n) \{ s(i/2^n) - s((i+1)/2^n) \}] = (i/2^n) [s(i/2^n) - s((i+1)/2^n)]. \]
Hence \[ [t_i] \leq (1/2^n) [s(i/2^n) - s(i+1)/2^n], \] and so (12.5(5)) \[ [x-x_n] \leq (1/2^n) [e]. \] That is, \[ [x-Vx_n] \leq (1/2^n) [e] \text{ for every } n; \text{ and therefore} \]
(12.6) \[ x = Vx_n, \text{ completing the proof.} \]

16.15. The "principal lemma" (15.2). In 16.14, let \( Q = \mathcal{B}(B); \text{ then } (Q, \sim) \]
is naturally isomorphic to a numerical measure algebra \( (Q, \mu), \text{ in such a way that,} \]
for each \( q \in Q \text{ and } s \in S, \) \[ [qs] = \mu(q) [s]. \] Further, \( (Q, \mu) \text{ is isometric to } I^1. \]

This following from 16.14 by essentially the same arguments as in the "separable case" (14.5 and 14.6), the elements \( b(i/2^n) \) here playing the role of the elements \( d_n \) in §14.

17. \( E(e_n) \) as a direct product.

17.1. As in §§14, 15, we consider the relative algebra \( E(e_n), e_n \) being any one of the elements of 13.4. The theorems of 14.1 and 15.1 show that, in every case, \( E(e_n) = \mathcal{B}(P \cup U), \) where \( P \) is a (relative) sub-algebra naturally isomorphic with \( I^1 \text{ or } I^m, \) in such a way that (from 12.13) we have the fundamental relation:

\[ [pu] = \mu(p) [u] \text{ whenever } p \in P \text{ and } u \in U. \]

We shall now deduce that \( E \text{ is isomorphic to the direct product } P \otimes U \) (cf. §4).

17.2. Definitions. Let \( R, S \) denote the respective representation spaces of \( P \text{ and } U; \) as in 4.2, we use \( p^*, u^* \) to denote the open-closed subsets of \( R, S \text{ respectively which correspond to } p \in P, u \in U. \) Let \( Q^* \) denote the class of all subsets \( Y \subseteq R \times S \) which are expressible in the form \( Y = \bigcup (p^* \times u^*), \) where the (open-closed) sets \( u^* \) are disjoint and their union is residual in \( S \) (that is, \( S - \bigcup u^* \) is of first category). Similarly let \( Q \) denote the class of all elements \( y \in E \) expressible as \( y = \bigvee p_i u_i, \) where \( p_i \in P, u_i \in U, \) the elements \( u_i \) are disjoint, and \( V u_i = e. \)

We observe that \( Q \) is a finitely additive sub-algebra of \( E. \) In fact, if \( y_1, y_2 \in Q \text{ and } y_j = \bigvee p_i u_i, \) then \( y_1(y_2) = (y_1 y_2) u_i u_j \in Q \) and \( y_1 \cup y_2 \text{ is finitely additive field of sets.} \)

If \( y \in Q \) is expressed as above in two ways, say \( y = \bigvee p_i u_i = \bigvee q_j v_j \) (where \( p_i, q_j \in P; u_i, v_i \in U; \) \( V u_i = e = V v_j, \) and each of the sequences \( u_i, v_j, \) is disjoint), then the "corresponding" sets \( \bigcup (p^* \times u^*), \) and \( \bigcup (q^* \times v^*) \) in \( Q^* \) are equal. For we have \( (p_i + q_j) u_i v_j = (y + y) u_i v_j = e, \) so (from the fundamental
property of \( P \) \( \mu(p_1+q_1)[u,v] = 0 \), and therefore \( p_i = q_i \) whenever \( u,v \neq 0 \).
Thus \( p_i^* = q_i^* \) whenever \( u_i^* \cap v_i^* \neq 0 \), so that
\[
U(p_i^* \times u_i^*) = U(p_i^* \times (u_i^* \cap v_i^*)) = U q_i^* \times v_i^*.
\]

Hence we may define a mapping \( \phi_1 \) of \( Q \) in \( Q^* \) by setting \( \phi(V p_i u_i) = U(p_i^* \times u_i^*) \). An argument similar to the preceding shows that \( \phi_1 \) is 1-1; and \( \phi_1 \) is evidently a finitely additive algebraic isomorphism between \( Q \) and \( Q^* \).

We have \( (M \) denoting the continuous-function "measure" in \( P \otimes U \)—cf. 4.3)

1. If \( Y = \phi_1(y) \) \( (\gamma \in Q, Y \in Q^*) \), then \( \gamma \subseteq \epsilon \Rightarrow M\{Y\} \leq \epsilon \) \( (0 < \epsilon < 1) \).

For suppose \( y = \bigvee p_i u_i \) in "normal form," as above. Then if \( \gamma \subseteq \epsilon \) we have (12.13) \( [u_i \gamma] \subseteq [u_i] \); but \( [u_i \gamma] = [u_i p_i] = \mu(p_i)[u_i] \), and thus (12.14) \( \mu(p_i) \subseteq \epsilon \) whenever \( u_i \neq 0 \). Thus \( M\{Y\} \leq \epsilon \) (for each \( s \in S \)). The converse implication is proved similarly.

17.3. The desired isomorphism between \( E \) and \( P \otimes U \) is now obtained by "extending" \( \phi_1 \), as follows. Suppose two elements \( x \in E \) and \( Y \in P \otimes U \) are so related, that to every \( \epsilon > 0 \) (and \( < 1 \)) there correspond \( y \in Q \) and \( \gamma \subseteq \epsilon \) such that \( \gamma \subseteq \epsilon \Rightarrow M\{X+y\} \leq \epsilon \), \( M\{X + \gamma\} \leq \epsilon \), and \( Y = \phi_1(y) \). We then write \( X = \phi(x) \). Thus if \( x \in Q \) then \( \phi(x) \) exists and equals \( \{\phi_1(x)\} \).

17.4. Let \( y = \bigvee y_n \), where \( y_n \in Q \) \( (n = 1, 2, \cdots) \). Then given \( \epsilon > 0 \) (and \( < 1 \)), there exists \( z \in Q \) such that \( [y+z^2] \subseteq \epsilon \).

We may suppose that \( y_1 \subseteq y_2 \subseteq \cdots \) (on replacing \( y_n \) by \( V_{y_{i+1}} y_i \)). By 7.6 there exists for each \( n \) an invariant element \( w_n \) such that (i) \( [y-y_n(w_n)] \subseteq \epsilon [w_n] \), (ii) if \( v \) is a nonzero invariant element less than or equal to \( -w_n \), \( [y-y_n] \subseteq \epsilon [w_n] \).
It readily follows that \( w_1 \subseteq w_2 \subseteq \cdots \); further, \( Vw_n = \epsilon \), since if \( \gamma = -Vw_n = v_0 \), we have \( [y-y_n] \subseteq [(y-y_n)w_0] \subseteq \epsilon [w_0] \) for every \( n \), whence (7.2) \( \gamma \subseteq \epsilon [v_0] \), and so (12.14, Corollary) \( v_0 = 0 \). Now write \( v_1 = w_1, v_n = w_n - w_{n-1} \) \( (n \geq 2) \), so that the elements \( v_n \) are disjoint and \( Vv_n = \epsilon \); and define \( z = Vv_n y_n \). Then \( y+z^2 = V(y+z_2)v_n = V(y-y_n)v_n \) where, since \( v_n \subseteq w_n \), we have \( [y-y_n] \subseteq \epsilon [v_n] \). Thus (12.17) \( [y+z^2] \subseteq \epsilon [v] \). Finally, since \( y_n \in Q \), we may write \( y_n = Vv_{n-1} u_{n-i} \) in "standard form"; and then \( z = Vv_n y_n y_{n-i} \in Q \).

17.5. Given \( x \in E \) and \( \epsilon > 0 \) (and \( < 1 \)), there exists \( y \in Q \) such that \( [x+y] \subseteq \epsilon [v] \).

Let \( B' \) be the set of those elements \( x \in E \) for which the above statement is true (for all \( \epsilon \)). Evidently \( B' \subseteq \bigcup P \cup U \); also if \( x \in B' \) so does \( -x \), and if \( x_n \in B' \) then \( Vx_n \in B' \), as readily follows from 17.4. Thus \( B' \subseteq B(P \cup U) = E \).

17.6. If \( y \in E \) and \( [y_m+z_m y_n] \subseteq (1/2^n) [\epsilon] \) whenever \( m \geq n \), there exists \( y \in E \) such that \( [y+z^2 y_n] \subseteq (1/2^{n-1}) [\epsilon] \) for all \( n \geq 2 \); further, this \( y \) is unique, and is given by \( y = \bigwedge n V_{y_{i+1}} y_i \) \( (= \lim sup y_i) \).
Remark. On complementation we obtain that \( y \) is also given by 
\[
y = \bigwedge_n \bigvee_{i \geq n} y_i = \liminf y_i, \]
thus the assertion is, roughly speaking, that “Cauchy sequences” converge.

Write 
\[
t_n = y_n + 2 \bigvee_{i < n} y_i = V_{i \geq n} (y_i - y_n) = V_{i \geq n} (y_{i+1} - y_{i^*}) \leq V_{i \geq n} (y_{i+1} + z_i). \]
Thus (12.7) \( |t_n| \leq (1/2^{n-1})[e] \). Now define 
\[
y = \bigwedge_n \bigvee_{i \geq n} y_i. \]
Then 
\[
y = \bigwedge_{n \geq N} \bigvee_{i \geq n} y_i = \bigwedge_{n \geq N} (y_i + z_i) \]
so that, for each \( N \), 
\[
y_N + z_N \leq \{ y_n \wedge \bigwedge_{n \geq N} (y_n - t_n) \} \leq [y_N \vee V_n \geq N t_n] - \{ y_n - V_n \geq N t_n \} = V_n \geq N t_n. \]
Hence (12.7 again) \( |y_N + y| \leq (1/2^{N-1})[e] \) for all \( N > 1 \). Finally, \( y \) is unique, since if \( z \) also has the same property we have \( |y + z| \leq (1/2^{n-1})[e] \) for all \( n \geq 3 \), and so (12.6) \( y = z \).

17.7. \( \phi \) is a finitely additive algebraic isomorphism between \( E \) and \( P \otimes U \).

Given \( x \in E \), we first show that \( \phi(x) \) exists. From 17.5, there exists a sequence \( y_n \in \mathbb{Q} \) with \( |x + 2y_n| \leq (1/2^{n+2})[e] \). By 17.2(1), the corresponding sets 
\[
Y_n = \phi_*(y_n) \]
that is, \( Y_n = \phi_*(y_n) \), satisfy 
\[
M\{ Y_n + z \} \leq 1/2^n \text{ whenever } m \geq n. \]
Hence, by 17.6 applied to the algebra \( P \otimes U \), there exists \( X \in P \otimes U \) such that 
\[
M\{ X + \{ Y_n \} \} \leq 1/2^{n-2}. \]
Clearly \( X = \phi(x) \), by definition (17.3).

A similar argument (starting from 4.6) shows that \( \phi^{-1}(X) \) exists for every \( X \in P \otimes U \). Finally, it has already been pointed out that \( \phi \) is 1-1, and the relations \( \phi(-x) = -\phi(x) \), \( \phi(x_1 \wedge x_2) = \phi(x_1) \wedge \phi(x_2) \) are obvious.

17.8. If \( y_n \in \mathbb{Q}, \phi(V_n) = \mathcal{V}_0(y_n). \)

It will suffice to prove this assuming \( y_1 \leq y_2 \leq \cdots \). Given \( e > 0 \), consider the element \( z \in \mathbb{Q} \) constructed in 17.4. Thus \( |y + z| \leq e[e] \), where \( y = V_n n \); further, the construction shows that 
\[
z = V_{y_n \vee v_n}, \text{ where } v_n \in \mathbb{U}, \forall v_n = e, \text{ and the elements } v_n \text{ are disjoint.} \]
Thus \( z_n \leq y_n \). Now write \( y_n = \phi_1(y_n), Z = \phi_1(z) \); we have \( Z \cap (R \times v_n^*) \subseteq Y_n \), so that \( Z \subseteq \bigcup Y_n \cup (R \times H) \) where \( H \) is of the first category in \( S \).

Now, we have from 17.4 that \( |y_n - z_n| \leq e[e] \); a fortiori, therefore, 
\[
|y_m v_n - z_n| \leq e[v_n] \quad \text{when } m \geq n, \text{ so that (17.2(1))} \quad M\{ Y_m \cap (R \times v_n^*) \} - \{ Z \cap (R \times v_n^*) \} \leq eM\{ R \times v_n^* \}. \]
The countable additivity of \( M \) then gives (on summing first over \( m \) and then over \( n \))
\[
M\{ U Y_n + Z \} \leq e. \]

Thus \( M\{ U Y_n + Z \} \leq e; \) and so \( \phi(y) = \{ U Y_n \} \) from the definition.

17.9. If \( x_n \in E, \phi(V x_n) = \mathcal{V}(x_n). \)

As before, we may suppose that \( x_1 \leq x_2 \leq \cdots \). Let \( e \) be given, and choose 
\( y_n \in \mathbb{Q} \) so that \( |x_n + 2 y_n| \leq (e/2^n)\{ e \} \) and \( M\{ \phi(x_n) + \phi(y_n) \} \leq e/2^n \) (using the definition of \( \phi \)). Write \( x = V x_n, \ y = V y_n \). It is easily verified that \( x + 2 y \leq V(x_n + 2 y_n), \) so that \( |x + 2 y| \leq e[e] \). Similarly \( M\{ X + 2 Y \} \leq e, \) where 
\[
X = \mathcal{V}(x_n), \quad Y = \mathcal{V}(y_n) \]
But 17.8 shows that \( Y = \phi(y) \); hence there exists 
\( z \in \mathbb{Q} \) such that \( |y + z| \leq e[e] \) and \( M\{ Y + 2 Z \} \leq 2e, \) where \( Z = \phi_2(z) \). Thus \( |x + z| \leq (2e)[e] \) and \( M\{ X + 2 Z \} \leq 2e, \) showing that \( X = \phi(x) \).

This completes the proof that \( \phi \) is an algebraic \((\sigma)\) isomorphism between \( E \) and \( J \otimes U \). To prove that \( \phi \) is an isomorphism of the abstract meas-
ure algebras, we must show that \( \phi \) is “measure-preserving” both ways.

17.10. If \( x_1 \sim x_2 \), \( M \{ \phi(x_1) \} = M \{ \phi(x_2) \} \).

Given \( \epsilon > 0 \) (and \( < 1 \)), we choose \( y_i \in Q \) \((i = 1, 2)\) so that \( [x_i + 2y_i, x_i y_i] \leq \epsilon \) and \( M \{ X_i + \{ Y_i \} \} \leq \epsilon \), where \( X_i = \phi(x_i) \) and \( Y_i = \phi(y_i) \). Let \( y_i = \sqrt[p]{p_n u_n} \), \( y_2 = \sqrt[q]{q_n v_n} \), where (as usual) \( p_n, q_n \in P \), \( u_n, v_n \in U \), \( \forall u_n = e = v_n \), and the sequences \( \{ u_n \}, \{ v_n \} \) are each disjoint. We can further suppose \( u_n = v_n \neq o \) (on replacing both sequences \( u_n, v_n \) by an enumeration of the nonzero elements \( u_n v_n \)).

First we show

\[(1) \mu(p_n) \leq \mu(q_n) + 2\epsilon.\]

For we can suppose \( \mu(q_n) + 2\epsilon \leq 1 \), else there is nothing to prove. Now, since \( x_1 \sim x_2 \), there exists \( t \leq x_2 \) such that \( t \sim x_1 y_i \). For each \( n \) we have
\[
[x_1 + x_2 y_i] = [u_n y_2] \leq [u_n y_2 \lor u_n (x_2 - y_2)],
\]
where \( [u_n y_2] = [u_n q_n] = \mu(q_n) [u_n] \), and \( [u_n (x_2 - y_2)] \leq \epsilon [u_n] \). Thus \( [u_n x_1 y_1] \leq (\mu(q_n) + \epsilon) [u_n] \). Since further \( [u_n (y_1 - x_1)] \leq \epsilon [u_n] \), we have \( [u_n y_1] \leq (\mu(q_n) + 2\epsilon) [u_n] \). But \( [u_n y_1] = [u_n p_n] = \mu(p_n) [u_n] \); and (1) follows.

A similar argument applies with \( p_n \) and \( q_n \) interchanged. Thus we evidently have

\[(2) | M \{ Y_1 \} - M \{ Y_2 \} | \leq 2\epsilon.\]

But \( M \{ X_1 \} = M \{ X_1 \{ Y_1 \} \} + M \{ X_1 - \{ Y_1 \} \} = M \{ Y_1 \} - M \{ \{ Y_1 \} - X_1 \} + M \{ X_1 - \{ Y_1 \} \} \), so that \( | M \{ X_1 \} - M \{ Y_1 \} | \leq \epsilon \). Hence \( | M \{ X_1 \} - M \{ X_2 \} | \leq 4\epsilon \), and therefore

\[M \{ X_1 \} = M \{ X_2 \} .\]

17.11. If \( M \{ \phi(x_1) \} = M \{ \phi(x_2) \} \), then \( x_1 \sim x_2 \).

Given \( \epsilon (0 < \epsilon < 1/9) \), we take \( y_1, y_2 \in Q \) so that \( [x_i + 2y_i] \leq \epsilon [e] \) and \( M \{ X_i + \{ Y_i \} \} \leq \epsilon \), where \( X_i = \phi(x_i) \) and \( Y_i = \phi(y_i) \). As in the previous argument, we can suppose \( y_i = \sqrt[p]{p_n u_n} \), \( y_2 = \sqrt[q]{q_n v_n} \), where \( p_n, q_n \in P \), \( u_n, v_n \in U \), \( u_n \neq o \), and the sequence \( u_n \) is disjoint. As at the end of 17.10, we have
\[| M \{ X_1 \} - M \{ Y_1 \} | \leq \epsilon \] and therefore \( | M \{ Y_1 \} - M \{ Y_2 \} | \leq 2\epsilon \); and it readily follows that \( | \mu(p_n) - \mu(q_n) | \leq 2\epsilon \), for each \( n \). We may therefore choose \( p_n' \in P(p_n) \), \( q_n' \in P(q_n) \) so that \( \mu(p_n') = \mu(q_n') \), \( \mu(p_n) - \mu(p_n') \leq 2\epsilon \), and \( \mu(q_n) - \mu(q_n') \leq 2\epsilon \). Define \( y_i' = \sqrt[p]{p_n' u_n} \) and \( y_i' = \sqrt[q]{q_n' v_n} \); thus \( y_i \sim y_i' \) and \( y_i' \leq y_i \), and
\[| y_i - y_i' | \leq (2\epsilon) [e] .\]

Since \( y_i \sim y_i' \), we can write \( y_i' = t' \lor z_i \), where \( t' \sim x_i y_i' \) and \( z_i \sim y_i' - x_i \), so that \( [z_i] \leq (3\epsilon) [e] \); and we then have \( y_i' = t x_i y_i' \lor z_2 \), where \( z_2 = z_i \lor (y_i' - x_2) \), so that \( [z_2] \leq (6\epsilon) [e] \). Now \( t x_i y_i' \) is less than or equal to \( x_2 \), and is equivalent to a sub-element of \( x_1 \) (for \( t \) is); hence if \( h_1, h_2 \) denote maximal equivalent sub-elements of \( x_1, x_2 \) respectively, 7.5 shows that there exists \( k_2 \geq k_2 \) such that \( t x_i y_i' \sim k_2 \). Then we have \( x_2 - h_2 \leq x_2 - h_2 - t x_i y_i' \leq (x_2 - y_i') \lor z_2 \), so that \( [x_2 - h_2] \leq (9\epsilon) [e] \). This holds for every \( \epsilon \) sufficiently small, so that \( x_2 = h_2 \).
Similarly $x_1 = h_1$, and therefore $x_1 \sim x_2$.

The isomorphism between $(E, \sim)$ and $P \otimes U$ is now established.(21)

18. Imbedding of $E(e')$ in a direct product.

18.1. Having fully considered the relative algebras $E(e_n)$, we turn now to $E(e')$, where $e'$ is the greatest bounded invariant element disjoint from all indecomposable elements (10.2, 10.3). It was shown in 13.4 that $e' = \bigvee e_n$, that the elements $e_n$ are disjoint, and that $[e_n] = \rho_n [\bar{e}_n]$ for each $n$ (where $\rho_n$ is rational and $0 < \rho_n \leq 1$). In the preceding section it was shown that each principal ideal $(E(e_n), \sim)$ is isomorphic to the direct product (say) $P_n \otimes U_n$, where $P_n$ is a (relative) sub-algebra of $E(e_n)$ which is (to within isomorphism) a numerical measure algebra ($I^1$ or $I^m$), and $U_n$ is the (relative) algebra of relatively invariant elements in $E(e_n)$. Since $U_n = e_n U$ is isomorphic, in a natural way, to the principal ideal $U(\bar{e}_n)$ (8.3), we have an isomorphism $\phi_n$ mapping $(E(e_n), \sim)$ on $P_n \otimes U(\bar{e}_n)$.

To combine these $\phi_n$'s into an isomorphism of $E(e')$, we first define a numerical measure algebra $J'$ as follows: $J'$ is the direct sum of the measure algebras $\rho_n P_n$ obtained by multiplying all measures in $P_n$ by $\rho_n$. That is, $J'$ is the set of all sequences $x = (x_n), x_n \in P_n$, in which algebraic operations are defined in the obvious way (for example, $(x_n) \vee (y_n) = (x_n \vee y_n)$, and in which a (numerical) measure is defined by: $\mu'(x) = \sum \rho_n \mu_n(x_n), \mu_n$ being the measure in $P_n$. It is easy to see that $J'$ (= $J'(\bar{e}')$) is a non-atomic numerical measure algebra in which the measure is at most countably infinite, so that it satisfies our postulates. We use $J'$ to denote the unit-element in $J'$ (that is, the sequence $(e_n)$), and $j_n$ to denote the element of $J'$ whose “$n$th coordinate” is $e_n$ and whose other coordinates are all $0$. The ideal $J'(j_n)$ is thus isomorphic to $(P_n, \mu_n)$; and to save notation we shall usually not distinguish between them. Note that they are not isometric, in general; we have, for $x \in J'(j_n)$, $\mu'(x) = \rho_n \mu_n(x)$. We shall prove that $(E(e'), \sim)$ is isomorphic to a principal ideal in $J' \otimes U(\bar{e}')$.

All considerations in the present section will be relative to $e'$, so we write $e'$ as $e$ and $E(e')$ as $E$ in what follows—noting for future use that, since $e'$ is invariant, the relatively invariant elements form the ideal $U(e')$.

18.2. Let $R, S$ be the representation spaces of $J'$ and $U$ respectively. To the element $j_n \in J'$ there corresponds an open-closed subset $j_n^* \subset R$; and it is easy to see that $j_n^*$ is homeomorphic to the representation space $R_n$ of $J'(j_n)$ —that is, of $P_n$. Thus we can take $R_n = j_n^*$, so that $UR_n$ is residual in $R$. Similarly, we can take the representation space $S_n$ of $U(\bar{e}_n)$ to be the open-closed subset $\bar{e}_n^*$ of $S$ which corresponds to $\bar{e}_n \subset U$, and have that $US_n$ is residual in $S$. It is easy to see that the direct product $P_n \otimes U(\bar{e}_n)$ is isomorphic

(21) Substantially the same arguments prove that the “direct product” $J \otimes U$, as characterized in 3.1 (e), is isomorphic to the product constructed in §4. More precisely, the arguments apply provided that $J$ is non-atomic and of finite total measure; and the general case can be reduced to this one.
to the principal ideal of \( \{ R_n \times S_n \} \) in \( J' \otimes U \), and we identify them accordingly. We write \( M'\{H\} \) for the measure-function on \( J' \otimes U \), and \( M_n\{H\} \) for the measure-function on \( P_n \otimes U(\varepsilon_n) \), with the natural convention that \( M_n \) is always 0 for each \( s \in S - S_n \). Thus, if \( H \subseteq j_n^* \times e_n^* \), \( M'\{H\} = \rho_n M_n\{H\} \).

Note that, while the sets \( R_n \) are necessarily disjoint (for the elements \( j_n \) are), the sets \( S_n \) are not disjoint in general (for the same is true of the elements \( \varepsilon_n \)).

For each \( n \) we have an isomorphism \( \phi_n \) between \( E(e_n) \) and the ideal of \( \{ j_n^* \times e_n^* \} \) (\( = \{ R_n \times S_n \} \)) in \( J' \otimes U \). Define, for each \( x \in E \), \( \phi(x) = \mathcal{V}_n(x e_n) \).

It is immediate that \( \phi \) is an algebraic (\( \sigma \)-) isomorphism between \( E \) and the principal ideal of \( V\{ j_n^* \times e_n^* \} \) in \( J' \otimes U \). We have only to verify that \( \phi \) is "measure-preserving" both ways.

18.3. If \( x \subseteq e_n \), \( x' \subseteq e_m \), and \( x \sim x' \), then \( M'\{\phi(x)\} = M'\{\phi(x')\} \).

If \( m \neq n \) this follows from the corresponding property of \( \phi_n \); thus we can assume \( m \neq n \). Now, as was shown in §17, there exists \( y \subseteq e_n \) such that (i) \( [x + y'] \subseteq [e_n] \), (ii) \( M_n\{\phi_n(x) + \phi_n(y)\} \subseteq \varepsilon_n \), and (iii) \( y = \mathcal{V}_n \delta \) where \( \delta \subseteq P_n \), \( \delta_i \subseteq U(\varepsilon_n) = e_n U \), and the elements \( \delta_i \) are disjoint and nonzero. Thus \( \delta_i = e_n \delta_i \), say, where \( \delta_i \subseteq U(\varepsilon_n) \) (cf. 8.2) and the invariant elements \( \delta_i \) are disjoint; and, since \( \delta_i \subseteq e_n \), we have \( y = \mathcal{V}_n \delta \). Similarly there exists \( y' \subseteq e_m \) such that (i') \( [x' + y'] \subseteq [e_m] \), (ii') \( M_m\{\phi_m(x') + \phi_m(y')\} \subseteq \varepsilon_m \), and (iii') \( y' = \mathcal{V}_m \delta' \), where \( \delta_j \subseteq P_m \), \( \delta_j \subseteq U(\varepsilon_m) \), and the invariant elements \( \delta_j \) are disjoint. Now enumerate the nonzero elements \( u_i \), \( u_j \), \( u_i - \varepsilon_i \), \( u_j - \varepsilon_j \), into a sequence \( w_k \); thus the elements \( w_k \) are disjoint and invariant, and we may write \( y = \mathcal{V}_r w_k \), \( y' = \mathcal{V}_s w_k \), where \( r_k \subseteq P_n \) and \( s_k \subseteq P_m \).

We have \( [w_k y] = [r_k e_n w_k] = \mu_n(r_k) [e_n w_k] \), from the fundamental property of \( P_n \); but \( [e_n] = \rho_n [e_n] \), and therefore (12.13) \( [e_n w_k] = \rho_n [w_k] \) if \( w_k \subseteq e_n \). If \( w_k \) is not \( \subseteq e_n \), then \( w_k \not\subseteq -e_n \), by construction, so that there is no loss in supposing \( r_k = 0 \). Thus (using 12.20) we see that, in all cases, \( [w_k y] = \rho_n [\mu_n(r_k) w_k] \). Similarly \( [w_k y'] = \rho_m [\mu_m(s_k) w_k] \). But since \( x \sim x' \), we have \( [w_k x] = [w_k x'] \); and by arguments similar to those in 17.10 it follows that \( [\rho_n \mu_n(r_k) - \rho_m \mu_m(s_k)] \) \( \leq 2 \varepsilon \). Thus, \( [\rho_n M_n\{\phi_n(x)\} - \rho_m M_m\{\phi_m(x')\}] \) \( \leq 2 \varepsilon \), whence \( [\rho_n M_n\{\phi_n(x)\} - \rho_m M_m\{\phi_m(x')\}] \) \( \leq 4 \varepsilon \). That is, \( M'\{\phi(x)\} - M'\{\phi(x')\} \) \( \leq 4 \varepsilon \) for every \( \varepsilon \), so that \( M'\{\phi(x)\} = M'\{\phi(x')\} \).

18.4. If \( x \sim x' \), then \( M'\{\phi(x)\} = M'\{\phi(x')\} \).

From 6.2, we can write \( x' = V x_m \) where the elements \( x_m \) are disjoint, and \( x_m \sim x \). Similarly we can write \( x \sim x_m \) where the elements \( x_m \) are disjoint and \( x_m \sim x_m^* e_n \). From 18.3 we have \( M'\{\phi(x_m)\} = M'\{\phi(x_m^* e_n)\} \); and the countable additivity of \( M' \) then gives \( M'\{\phi(x)\} = M'\{\phi(x')\} \).

18.5. If \( M'\{\phi(x)\} = M'\{\phi(x')\} \) then \( x \sim x' \).

By an argument similar to that in 18.4, we see that it will be enough to prove this assuming \( x \subseteq e_n \) and \( x' \subseteq e_m \). Let \( y \), \( y' \) be maximal equivalent subelements of \( x \), \( x' \), respectively (7.5), and write \( x = x - y \), \( z = x' - y' \). By 18.4, \( M'\{\phi(y)\} = M'\{\phi(y')\} \); also here \( M'\{\phi(x)\} < \varepsilon \); hence the additivity of \( M' \).
gives \( M'\{\phi(z')\} = M'\{\phi(z')\} \). Now it is clear from the construction in §17 that \( M'\{\phi(z)\} \) has the value 0 for all \( s \in S - \bar{z}^* \); similarly \( M'\{\phi(z')\} = 0 \) outside \( \bar{z}^* \). Since \( \bar{z} \cap \bar{z}' = o \), and therefore \( M'\{\phi(z)\} = M'\{\phi(z')\} = 0 \) for all \( s \in S \). Since \( M_n\{\phi_n(z)\} = 0 \), we have \( z = o \); similarly \( z' = o \), so that \( x = y \sim y' = x' \).

The proof is now complete; but for later use we note the property:

18.6. If \( u \in U \), \( M'\{\phi(u)\} \) is the characteristic function of \( u^* \).

We have \( [\epsilon']w = [\epsilon']M' \), and so, since \( \phi \) is an isomorphism, \( M'\{\phi(e_nu)\} = \rho_n M'\{\phi(e_nu)\} \). But \( M'\{\phi(e_nu)\} = \rho_n M'_n\{\phi(e_nu)\} \), by definition; hence \( M'_n\{\phi(e_nu)\} = M_n\{\phi_n(e_nu)\} \). Since \( e_nu \) is invariant relative to \( e_n \), the construction in §17 shows that \( M_n\{\phi_n(e_nu)\} \) is the characteristic function of the open-closed subset of \( S_n \) which "corresponds" to \( e_nu \); and since the (absolute) invariant element corresponding to \( e_nu \) in the isomorphism of 8.3 is \( \bar{e}_n, u \), we have \( M'_n\{\phi(e_nu)\} = \text{characteristic function of } (\bar{e}_n u)^* \).

It readily follows that \( M'\{\phi(u(e_n-\bar{e}_1-\bar{e}_2-\cdots-\bar{e}_{n-1})\} = \text{characteristic function of } u^*(e_n-\bar{e}_1-\bar{e}_2-\cdots-\bar{e}_{n-1}) \); and on summing (and neglecting a subset of \( S \) of the first category) we obtain \( M'\{\phi(u)\} = \text{characteristic function of } u^* \).

19. The representation of \((E, \sim)\).

19.1 We begin by imbedding \( E(e') \) in a direct product, where \( e'' \) is the element defined in 10.2. It has been shown (10.6) that \( e'' = \bigvee f_n \), where the elements \( f_n \) are disjoint, bounded and equivalent, and where \( \bigvee f_n = e'' \), for each \( n \). Now \( E(f_n) \) has, relative to \( f_n \), all the properties which were used in studying \( E(e') \). Hence the imbedding obtained in §18 for \( E(e') \) applies also to each \( E(f_n) \).

Remembering that the algebra \( U(e) \) of invariant elements relative to \( f_n \) is isomorphic to \( U(f_n) \) (8.3)—that is, to \( U(e') \)—we thus have, for each \( n \), an isomorphism \( \psi_n \) between \((E(f_n), \sim)\) and a principal ideal in a direct product (say) \( J^* \otimes U(e') \), \( J^* \) being a non-atomic numerical measure algebra. Let \( J'' \) denote the "direct sum" of the measure algebras \( J^*\)—that is, \( J'' \) is the set of sequences \( x = (x^*) \), \( x^* \in J^* \), with algebraic operations defined "co-ordinate wise," and with a (numerical) measure \( \mu'' \) defined by: \( \mu''(x) = \sum \mu^*(x^*) \), \( \mu^* \) being the measure in \( J^* \). We now show that \( E(e'') \) is isomorphic to a principal ideal in \( J'' \otimes U(e') \).

As in 18.1 and 18.2, we can express the unit-element \( j'' \) of \( J'' \) as \( \bigvee j^* \), where the elements \( j^* \) are disjoint and the principal ideal \( J''(j^*) \) can be regarded as being \( J^* \). Also as before, the open-closed subset \( j^* \) of the representation space \( R' \) of \( J'' \) can be regarded as the representation space \( R^* \) of \( J^* \); and the principal ideal of \( \{R^* \times S\} \) in \( J'' \otimes U(e'') \), where \( S \) is the representation space of \( U(e'') \), can be regarded as being \( J^* \otimes U(e') \). We use \( M^* \) for the measure-function in \( J^* \otimes U(e') \), and \( M'' \) for the measure-function in \( J'' \otimes U(e'') \); thus \( M''(H) = \sum M^* \{H \cap (j^* \times S)\} \).

We define, for each \( \psi(x) = \bigvee \psi_n(f_n) \); trivially \( \psi \) is an algebraic (\( \sigma' \)) isomorphism between \( E(e'') \) and the principal ideal of \( \bigvee \psi_n(f_n) \) in
All that remains is to verify that ψ is "measure-preserving"; and as in doing this all considerations will be relative to ϵ", we may suppose ϵ"=ε in 19.2 and 19.3.

19.2. If x∼y, M"  \{ψ(x)\} = M"  \{ψ(y)\}.

By the same argument as in 18.4, it is enough to prove this when x≤f and y≤f. Now from 13.1 applied relative to f, (so that the "invariant elements" are of the form uf, u∈U), there exist, for any given integer N, elements u,i∈U (1≤i≤N) such that f = f when i≠j, Vui∈f (so that Vui = f = e), and ((i−1)/N)[ufm] ≤ [ux] ≤ (i/N)[ufm]. From 5.10 it follows that the elements ui are actually disjoint. Choose ti ≤ u,x with [ti] = ((i−1)/N)[ufm], and write z = u,x−ti and z = Vz,i. Thus x = Vti ∨ z, where the elements ti and z are disjoint, and (from 12.5(5)) [z] ≤ (1/N)[f].

Since y∼x, we can (6.2) write y = Vz,i ∨ z, where the elements t’ and z’ are disjoint, t’∼ti, and z’∼z.

Now uf is invariant relative to f; hence (16.6) M[ψm(uf)] is the characteristic function of the open-closed subset of S which "corresponds" to uf. Here uf was replaced by uf = 1 on the isomorphism between U/m and U (= U(e)); thus we have M[ψm(uf)] = 1 for s∈uf*, 0 for s∈S−uf*. Since ψm is an isomorphism, the relation [ti] = ((i−1)/N)[ufm] gives M"  \{ψ(ti)\} = ((i−1)/N)[ufm].

Thus from the additivity of M", we have [M"  \{ψ(x)\} − M"  \{ψ(y)\}] ≤ 1/N; and since N here was arbitrary, it follows that M"  \{ψ(x)\} = M"  \{ψ(y)\}.

19.3. If M"  \{ψ(x)\} = M"  \{ψ(y)\}, then x∼y.

This follows by the same argument as in 18.5. The desired isomorphic imbedding of (E(ϵ"), ∼) is thus established.

19.4. It is now easy to extend the imbedding to one of all of E. For convenience of exposition, we assume that none of the elements e", e', e'' (cf. 9.6, 10.2) is zero; the modifications needed otherwise are obvious. We have from 9.7 an isomorphism θ of E(e') on a principal ideal in K⊗U(e'), and from §18 an isomorphism φ of E(e') on a principal ideal in J/′⊗U(e′). We have just constructed an isomorphism ψ of E(e") on a principal ideal in J/′⊗U(e′). Further, the elements e", e', e'' are disjoint and invariant, and e"∨e'∨e"=e.

Now form the usual direct sum J=J′⊗J'' of the numerical measure algebras J', J'': thus J consists of the pairs (x', x'') (x'∈J', x''∈J''), with measure µ given by µ(x', x'') = µ(x') + µ''(x''), µ' and µ'' being the measures in J' and J''. As a non-atomic algebra of at most countably infinite measure, J satisfies our postulates. Let E denote the free direct sum (cf. 3.1(d)) of the
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direct products \( K \otimes U(e^o) \) and \( J \otimes U(-e^o) \). To \( x \in E \) we make correspond the pair \((\theta(xe^o), \phi(xe') \vee \psi(xe''))\) in \( \mathcal{E} \). (The products \( J' \otimes U(e') \) and \( J'' \otimes U(e'') \) are regarded as imbedded in \( J \otimes U(e' \setminus e'') \) in the obvious way.) This evidently produces an algebraic isomorphism between \( E \) and the principal ideal of the element \((\theta(e^o), \phi(e') \vee \psi(e''))\) in \( \mathcal{E} \). Further, if \( x \sim y \), then (since \( e^o, e' \) and \( e'' \) are invariant) 6.6 gives \( xe^o \sim ye^o, xe' \sim ye' \) and \( xe'' \sim ye'' \), so that the corresponding elements of \( \mathcal{E} \) are equivalent in \( \mathcal{E} \). Conversely, if \( x, y \in E \) correspond to equivalent elements in \( \mathcal{E} \), we have \( xe^o \sim ye^o \) and also \( M(\phi(xe')) + M(\psi(xe')) = M(\phi(ye')) + M(\psi(ye')) \), \( M \) denoting the measure function in \( J \otimes U(-e^o) \). But \( M(\phi(xe')) \) is a function on the representation space \( S \) of \( U(-e^o) \) which vanishes outside the open-closed subset \( e'* \) corresponding to \( e' \). Similarly \( M(\psi(xe'')) \) vanishes outside \( e''* \). Since \( e'* \cap e''* = \emptyset \), and \( e'* \cup e''* = S \), we obtain \( M'(\phi(xe')) = M'(\phi(ye')) \) and \( M''(\psi(xe'')) = M''(\psi(ye'')) \), so that \( xe' \sim ye' \) and \( xe'' \sim ye'' \). Hence, finally, \( x \sim y \), and the main theorem of this paper is proved: \((E, \sim)\) is isomorphic to a principal ideal in the free direct sum of \( K \otimes U(e^o) \) and \( J \otimes U(-e^o) \), each of the two summands being the direct product of a numerical measure algebra with a trivial abstract measure algebra.

19.5. The above argument would evidently also prove that \((E, \sim)\) is isomorphic to a principal ideal in the single direct product \((K \oplus J) \otimes U\). This way of stating the theorem has the drawback that the numerical measure algebra \( K \oplus J \) will in general not satisfy postulate II (for \( K \) has atoms of measure 1, and \( J \) has in general non-atomic elements of the same measure), so that \((K \oplus J) \otimes U\) will in general not satisfy our postulates. However, it is immediate from this alternative formulation of the theorem that \((E, \sim)\) is naturally isomorphic to an abstract measure algebra \((E, M)\) in which the values of the abstract measure \( M \) are equivalence classes of numerical functions (more precisely, non-negative real-valued continuous functions on the representation space of \( U \), modulo functions which vanish outside a set of first category), \( M \) being countably additive in the obvious sense. Incidentally this proves that any equivalence relation satisfying our postulates (2.1) is induced by a "reasonable" countably additive abstract measure \( \lambda \).

20. Corollaries and applications.

20.1. In this section we mention briefly some deductions from the preceding general theory. There is, of course, an application to the theory of abstract-valued integration, which will be treated in a subsequent paper. Of the deductions to be considered here, the decomposition theorem given in 20.4(2) below is probably the most significant.

20.2. Some special cases.

(1) Consider the elements \( a_n \) given by 11.7; thus they are disjoint, \( \forall a_n = e' \) (10.2), and \( a_n \) is homogeneous of infinite order \( m_n \) over \( U \). We observe that the arguments of §§14–17 apply to \( a_n \) as well as to \( e_n \). Thus each principal ideal \((E(a_n), \sim)\) is isomorphic to the direct product \( I^{m_n} \otimes U(a_n) \).
(2) In particular, if all nonzero elements of \( E \) are decomposable and bounded (that is, if \( e = e' \)), and if \( E \) is separable over \( U \) (that is, if \( e \) is of order \( \leq \mathbf{Q} \)), and thus of order exactly \( \mathbf{Q} \), over \( U \), \( (E, \sim) \) is isomorphic to \( I^1 \otimes U \)—giving a characterization of the direct product of \( I^1 \) with a \lq\lq trivial\rq\rq\, algebra.

20.3. The numerical case.

(1) Characterizations. It is easy to derive characterizations of numerical measure algebras (satisfying our postulates—notably II; cf. 2.2). From the remark in footnote 6, it is enough to restrict attention to the case in which \( E \) is non-atomic. We then have the theorem:

If \( E \) is a non-atomic algebra, a necessary and sufficient condition that \((E, \sim)\) be isomorphic to a numerical measure algebra is that \((E, \sim)\) satisfies the postulates of 2.1 and has \( o \) and \( e \) for its only invariant elements.

An immediate consequence of this is (cf. [10, p. 423]):

If \( E \) is non-atomic, \((E, \sim)\) (satisfying the postulates of 2.1) is isomorphic to a numerical measure algebra if and only if, for every \( x, y \in E \), either \([x] \leq [y]\) or \([y] \leq [x]\).

(2) Decomposition into homogeneous parts. From 20.2(1) we have at once the theorem (cf. [11]):

A non-atomic numerical measure algebra \((E, \mu)\) (with \( \sigma \)-finite measure, of course) is the direct sum of a countable number (at most) of homogeneous measure algebras \((E(a_n), \mu)\); and \((E(a_n), \mu)\) is isomorphic to \( I^\infty_n \) (and thus isometric with \( I^\infty_n \), to within a multiplicative nonzero constant).

20.4. Groups of measure-preserving automorphisms.

(1) Equivalence with respect to \( \Phi \). Let \( E \) be a given algebra (satisfying the countable chain condition), and let a group \( \Phi \) of automorphisms \( \phi \) of \( E \) be given. Write \( x \sim_\phi y \) to mean that there exist sequences \( \{x_n\}, \{y_n\} \), each disjoint, in \( E \), and automorphisms \( \phi_n \in \Phi \), such that \( x = \bigvee x_n \), \( y = \bigvee y_n \), and \( y_n = \phi_n(x_n) \).

One readily sees that \( \sim_\phi \) is an equivalence relation, and that \((E, \sim_\phi)\) satisfies all the postulates (2.1) except, in general, III. Postulate III will also be satisfied in at least one important case (cf. 20.4(2)).

It is easy to see that \( u \in E \) is invariant in \((E, \sim_\phi)\) if and only if \( \phi(u) \leq u \) for every \( \phi \in \Phi \)—that is, \( u \) is \lq\lq invariant under \( \Phi \rq\rq\, in the usual sense.

(2) Decomposition with respect to \( \Phi \).

Theorem. Let \((E, m)\) be a numerical measure algebra in which \( m \) is an arbitrary reduced \( \sigma \)-finite numerical measure (cf. 2.2; postulate II need not be satisfied); and let \( \Phi \) be a given group of measure-preserving automorphisms of \( E \). Then there exist disjoint nonzero elements \( c_n \in E \), each invariant under \( \Phi \), and sub-algebras \( A_n, B_n \) of \( E(c_n) \) and reduced \( \sigma \)-finite numerical measures \( \mu_n \) on \( A_n, \nu_n \) on \( B_n \), such that:

(i) \( \bigvee c_n = e \).

(ii) \( B_n \) consists of exactly those elements of \( E(c_n) \) which are invariant under \( \Phi \).

(iii) If \( x, y \in A_n \), \( \mu_n(x) = \mu_n(y) \) if and only if \( x \sim_\phi y \) (in \( E \)).
(iv) \((A_n, \mu_n)\) is isometric to one of the following:
(a) \(I^m_n\) (for some cardinal \(m_n\); cf. 3.1(a)),
(b) the direct sum of \(\aleph_0\) copies of \(I^m_n\),
(c) the algebra generated by a finite number of atoms each of measure 1,
(d) \(K\) (cf. 3.1(a)),
and no two of the measure algebras \((A_n, \mu_n)\) are isomorphic.

(v) \((E(c_n), m)\) is the "direct product" of \((A_n, \mu_n)\) and \((B_n, \nu_n)\); that is, \(E(c_n) = c_n \otimes (A_n \cup B_n)\) and, whenever \(x \in A_n\) and \(z \in B_n\), \(m(xz) = \mu_n(x)\nu_n(z)\) (with the convention \(0 \cdot \infty = 0 = \infty \cdot 0\)).

Remarks. Before sketching the proof, we note the following consequences:
(a) For any \(x, y \in A_n\) we have \([\mu_n(x) = \mu_n(y)] \leftrightarrow [x \sim y] \rightarrow [m(x) = m(y)]\).
Hence, if for some \(n\) the measure \(m\) is \(\sigma\)-finite on \(A_n\) (that is, \(c_n = \bigvee x_k\) where \(m(x_k) < \infty\) and \(x_k \in A_n\)), it follows that \(\mu_n\) is then a nonzero finite constant multiple of \(m\). If further \(m(c_n) < \infty\), the product formula shows that \(\mu_n\) and \(\nu_n\) are both nonzero finite constant multiples of \(m\). In particular, this is the case if \(w(e) < \infty\).

(b) On applying this theorem to the algebra of measurable sets modulo null sets of a separable measure space, one can derive sharpened forms of theorems of von Neumann [13 p. 617] and Halmos [7, Theorem 2] (22), the direct sum decompositions there given being replaced (roughly speaking) by representations as direct products of measure spaces. It is hoped that details may appear elsewhere.

Proof of theorem. In the present case, \((E, \sim \varphi)\) can easily be seen to satisfy postulate III, so the preceding theory applies to it. We suppose for simplicity that \(e = e'\) (cf. 10.2). Then (11.7, with a slight change in notation) \(e = c_n\), where the elements \(c_n\) are disjoint, nonzero, and homogeneous over \(U\). It is easily seen that each \(c_n\) is now invariant. Further, in view of 20.2(1), \((E(c_n), \sim \varphi)\) is isomorphic to the direct product \(A_n \otimes B_n\), where \(B_n = U(c_n)\) and \(A_n = (A_n, \mu_n)\) is a sub-algebra of \((E(c_n), \sim \varphi)\) which is also a numerical measure algebra (isometric to \(I^m_n\)). We define the measure \(\nu_n\) on \(B_n\), in the present case, to coincide with \(m\), which can easily be seen to be \(\sigma\)-finite on \(B_n\). Thus we need only verify that \(m(xz) = \mu_n(x)m(z)\) for \(x \in A_n, z \in B_n\), and it will suffice to establish this in the special case \(\mu_n(x) = 1/k\), \(k\) a positive integer. But then there exist \(k\) disjoint elements \(x_1, x_2, \ldots, x_k\) in \(A_n\) such that \(x_1 = x, \forall x_i = c_n,\) and \(\mu_n(x_i) = 1/k\). Hence \(x_i \sim \varphi x\), so that \(z\) being invariant \(x_i \sim \varphi xz,\) and therefore \(m(xz) = m(xz)/k\). Thus \(m(xz) = m(x)/k\), completing the proof.

The ideal \(E(e')\) (10.2) is disposed of similarly, using 10.6; here the measure algebras \((A_n, \mu_n)\) will be of the type described in (iv b) of the theorem. Finally \(E(e)\) (9.6) provides the measure algebras of types (iv c) and (iv d).

(3) The Banach-Tarski theorem. Apply the preceding theorem taking

\(22\) In view of the fact that in [7] and [13] we have the separable case, in which the algebras \(A_n, B_n\) can always be realised as sub-fields of the given field of measurable sets.
(E, m) to be the algebra of (Lebesgue) measurable sets modulo null sets in Euclidean space, and Φ to be the group of all rigid motions (or even of all translations). The only invariant elements are 0 and 1, and the algebra E is separable. Hence we here have c₁ = e and E = A₁. Property (iii) of the theorem (see remark (a)) then gives: m(x) = m(y) if and only if x ~_φ y (m denoting Lebesgue measure)—a well known theorem of Banach and Tarski [1, p. 277].

One can go on to obtain such generalizations as the following: For any measurable subset X of the plane, and any real number a, let mₓ(a) denote the (linear) measure of the intersection of X with the line x = a. Then if mₓ(x) = mᵧ(x) for almost all x, we can write X = Z ∪ ∪ Xₙ, Y = T ∪ ∪ Yₙ, where Z and T are of (plane) measure zero, each of the sequences {Xₙ}, {Yₙ} is disjoint, the sets Xₙ and Yₙ are all measurable, and where for each n the sets Xₙ, Yₙ correspond under some translation parallel to the y-axis. (For let E be the algebra of plane measurable sets modulo null sets, and let Φ be the group of translations parallel to the y-axis; it is not hard to see that the abstract measure function M₁[X], in the representation of (E, ~_φ) given by the theorem of 19.5, is in effect simply the function mₓ(x).)

**Bibliography**


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