ON THE DIFFERENTIAL EQUATION $u_{xx}+u_{yy}+N(x)u=0$

BY

MARTIN M. E. EICHLER

1. **Introduction.** It is a well known fact that the theory of linear partial differential equations of elliptic type and with analytic coefficients resembles in many essential features the theory of the Laplace equation. S. Bergman, in a long series of papers during the last years(1), has put this resemblance in a new light by showing that the general solutions of such equations can always be represented by the real parts of the following expressions:

$$u(x, y) = K(x, y)f(z) + \int E(x, y, t)f(n(z, t))dt \tag{1}$$

where $f(z)$ is an analytic function of the complex variable $z=x+iy$, and $K(x, y)$, $E(x, y, t)$, $n(z, t)$ as well as the limits of the integral in (1) are real or complex functions which have to satisfy certain conditions depending on the differential equation. Infinitely many such representations are possible [4]. Each of them yields a one-to-one correspondence between the integrals (1) of the differential equation and the analytic functions $f(z)$. Bergman and others have studied this correspondence in detail and have developed a function-theory of solutions of these equations, based upon the theory of functions of a complex variable. The author of the present paper has arrived independently at many of these results [8]; he used, however, only one special operator (1) establishing the said correspondence.

The present paper deals with some aspects of the theory of the differential equation

$$\Delta u + N(x)u = 0 \tag{2}$$

in two variables x, y where $N(x)$ is a function representable by a series

$$N(x) = x^{-2}(c_0 + c_1x^\rho + c_2x^{2\rho} + \cdots), \tag{3}$$

ρ being a real positive number. Many equations occurring in applied theories can be brought into this form.

A special form of the integral operator (1), used by Bergman [4] and by the author [8], is

$$u(x, y) = f(z) - \int_{\zeta_0}^{z} S(x, y; \zeta)f(\zeta)d\zeta, \tag{4}$$

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(1) In the bibliography at the end we quote only the most important of them. References to this bibliography will be made in the usual way by numbers in brackets.

where ζ_0 is an arbitrary constant. We shall call $S(x, y; \zeta)$ a *generating solution* of (2). It has the following properties:

$$S_{xx} + S_{yy} + N(x)S = 0, \tag{5}$$

$$\frac{\partial}{\partial x} S(x, y; z) + i \frac{\partial}{\partial y} S(x, y; z) = \frac{1}{2} N(x). \tag{6}$$

There exist always generating solutions which are functions in two variables only [8, p. 188]:

$$S(x, y; \zeta) = G(x, z - \zeta) \tag{7}$$

in which case (6) becomes:

$$G(x, 0) = \frac{1}{2} \int_{x_0}^{x} N(x) dx + \gamma_0. \tag{8}$$

In many cases a generating solution can be expressed by known functions. It is both interesting and useful, though perhaps a bit old-fashioned from a purely mathematical point of view, to seek for a variety of such cases. This is done in §3. In §4 we explain two ways of deriving generating solutions of (2) from those of certain other differential equations of this type, but with a more general function N.

One arrives at the form (4) of the general integral of (2) when representing it formally by the series

$$u(x, y) = f(z) - p_1(x) \int_{\zeta_0}^{z} f(z) dz + p_2(x) \int_{\zeta_0}^{z} \int_{\zeta_0}^{z} f(\zeta) d\zeta dz - + \cdots \tag{9}$$

which can also be written as

$$\begin{aligned} u(x, y) = f(z) - \int_{\zeta_0}^{z} \Big(& p_1(x) - p_2(x)(z - \zeta) \\ & + \frac{1}{2} p_3(x)(z - \zeta)^2 - + \cdots \Big) f(\zeta) d\zeta. \end{aligned}$$

We shall call (9) the *ascending series*. If questions of convergence are left aside for the moment the conditions that (9) is an integral of (2) are

$$\begin{aligned} p_1(x) &= \frac{1}{2} \int_{x_0}^{x} N(x) dx + \gamma_1, \\ p_2(x) &= \frac{1}{2} \int_{x_0}^{x} (p_1''(x) + N(x) p_1(x)) dx + \gamma_2, \\ & \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{10}$$

Since the generating solution

$$(11)\qquad G(x, z - \zeta) = p_1(x) - p_2(x)(z - \zeta) + \frac{1}{2} p_3(x)(z - \zeta)^2 - + \cdots$$

involves infinitely many derivatives of $N(x)$ it makes sense only if $N(x)$ is an analytic function of x, and the same is true for the ascending series. If $N(x)$ has singular points one could try to represent integrals of (2) by the *descending series*

$$(12)\qquad u(x, y) = q_0(x)f(z) + q_1(x) \frac{d}{dz} f(z) + q_2(x) \frac{d^2}{dz^2} f(z) + \cdots ,$$

where the $q_n(x)$ satisfy the recurrence formulae

$$(13)\qquad \begin{aligned} q_0'' + Nq_0 &= 0, \\ q_1'' + Nq_1 &= -2q_0', \\ \cdots\cdots&\cdots\cdots . \end{aligned}$$

In §§5 and 6 the convergence of the ascending and descending series is studied, especially in the neighborhood of $x=0$ where $N(x)$ is supposed to be singular. As we shall see the descending series allows us to construct integrals of (2) which have singularities on $x=0$ apart from those which result automatically from the singularity of $N(x)$ here. §7 mentions a connection between the ascending and the descending series.

The second paragraph is dedicated to special generating solutions which we shall call canonical. They seem to be of importance in connection with the question of analytical continuation of solutions of (2), see Theorem 2.

In §8 an extension to equations in $n>2$ variables is given.

2. **Canonical generating solutions.** It will be noticed that a generating solution is not uniquely determined by the properties (5) and (6), even if it is supposed to be of the special form (7). A generating solution of this form shall be called *canonical with respect to the point* $x=x_0$ on the x-axis if $G(x, z+\bar{z}-2x_0)=0$ ($\bar{z}$ is the complex conjugate to z). For a canonical generating solution the integration constant γ_0 in (8) must be 0. It is obviously no loss of generality if we consider only the case $x_0=0$; when referring to canonical generating solutions we then may leave out the words "with respect to the point $x=0$."

THEOREM 1. *If $N(x)$ is regular in the neighborhood of $x=0$ then there exists one and only one canonical generating solution with respect to this point.*

The proof can be given in two ways. One way was already indicated in [8, p. 188]: write

$$G(x, z - \zeta) = H(x, \xi), \qquad \xi = \zeta - iy.$$

Then H satisfies the differential equation

$$H_{xx} - H_{\xi\xi} + N(x)H = 0,$$

which is of hyperbolic type. The initial conditions are

$$H(x, x) = \frac{1}{2}\int_0^x N(x)dx, \qquad H(x, -x) = 0.$$

It is well known that there exists one and only one solution of this problem. Because the initial conditions are analytic there exists always an analytic continuation for certain complex values of ξ. This proof shows the analogy of a generating solution with Riemann's function.

Another proof proceeds as follows: If $G(x, z-\zeta)$ is a generating solution of the form (7) which is not yet canonical, then, with an arbitrary analytic function $f(z)$,

$$G_1(x, z - \zeta) = G(x, z - \zeta) - f(z - \zeta) + \int_0^{z-\zeta} G(x, z - \zeta - t)f(t)dt \tag{14}$$

is another one, as can immediately be verified. Especially

$$G_1(x, 0) = G(x, 0) - f(0).$$

G_1 is canonical if

$$G(x, 2x) = f(2x) - \int_0^{2x} G(x, 2x - t)f(t)dt.$$

This is an integral equation of Volterra type. Since its kernel is analytic in both arguments there exists always one and only one solution in a suitable neighborhood of $x=0$.

Canonical generating solutions allow us to represent integrals of (2) in the following way:

$$u(x, y) = f(z) - \int_{-\bar{z}}^{z} G(x, z - \zeta)f(\zeta)d\zeta \tag{15}$$

which is somewhat different from (4). That (15) indeed satisfies (2) can very easily be verified. Especially let us insert $f(z) = \exp(\lambda z)$, then

$$u = \exp(\lambda z) - \int_{-\bar{z}}^{z} G(x, z - \zeta) \exp(\lambda\zeta)d\zeta = h_\lambda(x) \exp(i\lambda y)$$

with

$$h_\lambda(x) = \exp(\lambda x) - \int_{-x}^{x} G(x, x - t) \exp(\lambda t)dt.$$

Clearly

$$h_\lambda'' + (N - \lambda^2)h_\lambda = 0, \tag{16}$$

and for $x=0$:

$$h_\lambda(0) = 1, \qquad h_\lambda'(0) = \lambda. \tag{17}$$

The functions $h_\lambda(x) \exp(i\lambda y)$ are perhaps the most easily accessible solutions of (2). For them the following theorem holds:

THEOREM 2. *Suppose $N(x)$ to be regularly analytic in the interval $a \leqq x \leqq b$ which contains the point $x=0$ and $f(z)$ an analytic function defined by the series*

$$f(z) = \sum_\lambda a_\lambda \exp(\lambda z) \tag{18}$$

where λ ranges over a set of real numbers. Let this series be absolutely and uniformly convergent for $x \leqq c$, $c>0$, and let $f(z)$ be regular in a domain D contained in $a \leqq x \leqq b$ with the exception of a finite or infinite number of isolated singularities. Finally let $h_\lambda(x)$ be the solutions of (16) *with the initial values* (17). *Then the series*

$$u = \sum_\lambda a_\lambda h_\lambda(x) \exp(i\lambda y) \tag{19}$$

with the same coefficients a_λ as (18) *has the following properties*: (1) *It converges absolutely and uniformly in $|x| \leqq c$.* (2) *It is analytically continuable into the intersection of D and the domain $\overline{D}$ symmetrical to D with respect to the y-axis.* (3) *It is regular where both $f(z)$ and $f(-\bar{z})$ are regular.* (4) *In a singular point of $f(z)$*

$$u = f(z) - p_1(x) \int_{\zeta_0}^{z} f(\zeta)d\zeta + \cdots \tag{20}$$

where the dots denote further integrals over $f(z)$ in the sense of the series (9) *and a function which depends analytically on both real variables x, y.*

The nature of the singularity of u in a point where $f(-\bar{z})$ is singular depends on the order of vanishing of the canonical generating solution when $\zeta \to -\bar{z}$.

Let $G(x, z-\zeta)$ be the canonical generating solution. The function $G(x, z) = H(x, y)$ is an analytic function of both real arguments x, y in a horizontal stripe S whose width may vary with x. Certainly S contains a rectangle $a \leqq x \leqq b$, $-\eta \leqq y \leqq \eta$ with a suitable η. In order to prove the theorem we can at first simplify our task by restricting y to the interval $|y| \leqq \eta$. After having completed the proof under this restriction we can apply translations $y \to y+y_0$ with arbitrary y_0 which leave the assumptions and statements of Theorem 2 unchanged since the λ in (18) are supposed to be real.

Now put

$$(21)\quad u = f(z) - \int_{-\bar{z}}^{z} G(x, z-\zeta)f(\zeta)d\zeta = f(z) - \int_{-x}^{x} G(x, x-t)f(t+iy)dt.$$

This function has the properties indicated in Theorem 2. In $|x| \leqq c$ we can insert the series (18) and carry out the integration over t for the single terms. This shows that (21) is identical with (19) and that (19) converges absolutely and uniformly in $|x| \leqq c$.

It is interesting to compare Theorem 2 with a similar theorem of Bergman [2, p. 141]. Both are immediate consequences of special generating solutions; in our case this is the canonical one, in Bergman's case it is that which he called "of the first kind." Both the canonical generating solution and that of the first kind are distinguished with respect to a certain point, and the correspondence between analytic functions (18) and certain solutions (19) of (2) is defined with respect to this point. If we use the point $z = -\infty$ to define generating solutions of the first kind Bergman's theorem can be stated in our particular case as follows:

THEOREM 3. *Suppose* $N(x)$ *to be a series*

$$N(x) = \sum_{n=0}^{\infty} a_n \exp(nx),$$

absolutely and uniformly convergent for $-\infty \leqq x \leqq b$. *The functions* $h_\lambda(x)$ *may again be defined as solutions of* (16) *with the initial values*

$$(22)\qquad h_\lambda(x) \exp(-\lambda x) = 1, \qquad h_\lambda'(x) \exp(-\lambda x) = \lambda \qquad \text{for } x = -\infty$$

instead of (17). *Let* $f(z)$ *have the properties given in Theorem* 2, *but with* $\lambda = 0, 1, 2, \cdots$. *Then the same statements concerning the series* (19) *are true with the exception that* u *is singular only where* $f(z)$ *is singular.*

Theorems 2 and 3 can be used to obtain asymptotic formulae for the solutions h_λ of (16) with the initial conditions (17) or (22). For example, take

$$f(z) = \sum_{n=0}^{\infty} \exp(n(x-\xi))$$

where ξ is a real number with $0 < \xi \leqq b$. Then (20) yields immediately in the case of (17):

$$h_n(\xi) \to \exp(n\xi).$$

However, we shall not follow this line any further.

3. **Examples.** Most of the following examples are obtained by seeking for such functions $N(x)$ for which the process (10) can be carried out explicitly.

So $N(x)=\lambda x^{-2}$ shows that $p_n(x)=a_n x^{-n}$ with certain constants a_n. However, after having summed the series (11) in closed form, one may check (5) and (6) directly. For a fuller treatment see [8].

A large family of equations having comparatively simple generating solutions are obtained if the series of functions (10) breaks up or consists of the repetition of a finite number of them:

$$p_m(x) = \lambda p_1(x)$$

with a constant λ. Now $p_{m_0+nm}=\lambda^n p_{m_0}$. With regard to (8) we thus obtain a differential equation of order $2(m-1)$ for $N(x)$. Solutions of this differential equation depend on $2(m-1)$ arbitrary constants.

EXAMPLE 1.

$$N(x) = \lambda, \qquad G(x, z - \zeta) = \left(\frac{\lambda(\bar{z} + \zeta)}{4(z - \zeta)}\right)^{1/2} J_1((\lambda(z - \zeta)(\bar{z} + \zeta))^{1/2})$$

where J_1 is the Bessel function of first kind and order 1. Another generating solution is obtained by replacing $\bar{z}+\zeta$ by $\bar{z}$.

EXAMPLE 2.

$$N(x) = \lambda x^{-2}, \qquad G(x, z - \zeta) = x^{-1}h(v), \qquad v = \frac{z - \zeta}{2x},$$

where $h(v)$ is $-\lambda/2$ times the hypergeometric function with the parameters

$$\alpha = \frac{3}{2} + \left(\frac{1}{4} - \lambda\right)^{1/2}, \qquad \beta = \frac{3}{2} - \left(\frac{1}{4} - \lambda\right)^{1/2}, \qquad \gamma = 2.$$

EXAMPLE 3.

$$N(x) = \lambda x^{-2} + \mu.$$

Put

$$G(x, z - \zeta) = \sum_{n=0}^{\infty} x^{2n-1} h_n(v).$$

In order that G satisfies (5) the following recurrence formulae must hold:

$$v(1 - v)h_n'' - 2(n - 1)(1 - 2v)h_n' - ((2n - 1)(2n - 2) + \lambda)h_n = \mu h_{n-1}.$$

For

$$h_n = (v(v - 1))^{n-1/2} g_n$$

they become

$$v(v - 1)[v(v - 1)g_n'' + (2v - 1)g_n' + \lambda g_n] = (n - 1/2)^2 g_n + \mu g_{n-1}.$$

A solution is

$$g_n(v) = \rho_n g_0(v) + \sigma_n g_1(v)$$

with $\rho_0=1$, $\sigma_0=0$, $\rho_1=0$, $\sigma_1=1$,

$$\sigma_n = \frac{-\mu}{n(n-1)}\sigma_{n-1}, \qquad \rho_n = \frac{-\mu}{n(n-1)}(\rho_{n-1} - \sigma_n).$$

Hence

$$G(x, z - \zeta) = x^{-1}h_0(v)\sum_{n=0}^{\infty}\rho_n[(z-\zeta)(\bar{z}+\zeta)]^n$$
$$+ xh_1(v)\sum_{n=1}^{\infty}\sigma_n[(z-\zeta)(\bar{z}+\zeta)]^{n-1}.$$

Especially

$$G(x, 0) = x^{-1}h_0(0) + xh_1(0);$$

if therefore

$$h_0(0) = -\frac{\lambda}{2}, \qquad h_1(0) = \frac{\mu}{2},$$

(6) is satisfied.

EXAMPLE 4.

$$N(x) = \frac{\lambda}{4}(\cosh x)^{-2} \text{ resp. } \frac{\lambda}{4}(\sinh x)^{-2}.$$

In this case it would be better to replace z by $\log z$ and so arrive at the differential equation

$$\Delta u + \frac{\lambda}{(x^2 + y^2 \pm 1)}u = 0.$$

A generating solution is

$$S(x, y; \zeta) = \frac{\bar{z}}{z\bar{z} \pm 1}H(V), \qquad V = \frac{\bar{z}(z-\zeta)}{z\bar{z} \pm 1},$$

where H satisfies the hypergeometric differential equation

$$V(1-V)H'' + 2(1-2V)H' - (2 \mp \lambda/4)H = 0$$

and the initial condition $H(0) = \pm\lambda/4$.

The present example is important in connection with potential theory: let

$$\phi(x_1, x_2, x_3) = x_3^n\psi(y_1, y_2), \qquad y_1 = \frac{x_1}{x_3}, \qquad y_2 = \frac{x_2}{x_3}$$

be a homogeneous harmonic function of degree n. Then, with

$$R = (y_1^2 + y_2^2)^{1/2}, \qquad \theta = \text{arc tan}\,\frac{y_2}{y_1} = \text{arc tan}\,\frac{x_2}{x_1},$$

$$r = \frac{(1+R^2)^{1/2}-1}{R}, \qquad x + iy = r\exp(i\theta):$$

$$\psi = \left(\frac{1+r^2}{1-r^2}\right)^n u$$

and

$$u_{xx} + u_{yy} + \frac{4n(n+1)}{(x^2+y^2+1)^2}\,u = 0.$$

For integral $n \geqq 0$, $H(V)$ can be derived from the Legendre polynomial of degree n as follows:

$$H(V) = \frac{d}{dV}P_n(2V-1), \qquad P_n(t) = -\frac{1}{2^n n!}\,\frac{d^n}{dt^n}(1-t^2)^n.$$

4. **Connections between different equations (2).** At first let N be an arbitrary function of x and y, and v a particular solution of (2) with this N. Then, for each function u satisfying (2), the integral

$$U = v^{-1}\int_{(x_0,y_0)}^{(x,y)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds \tag{23}$$

(where $\partial/\partial n$ is the derivative in normal direction and ds the tangential element for the path of integration) is independent of the path of integration, and U satisfies

$$\Delta U + N_{(v)}U = 0 \tag{24}$$

with

$$N_{(v)} = -N - 2\,\frac{v_x^2 + v_y^2}{v^2}\,. \tag{25}$$

Because this transformation of the differential equation is based upon Green's integral theorem, we have proposed to call it Green's transformation; it was studied in detail in [8].

Let $S(x, y; \zeta)$ be a generating solution of (2) and

$$\tilde{S}(x, y; \zeta) = 1 - \int^{z} S(x, y; t)dt, \tag{26}$$

then

$$g(\zeta) = \frac{1}{2}\left[u(\xi, \eta) - i\int_{(\xi_0,\eta_0)}^{(\xi,\eta)} \left(S(x, y; \zeta)\frac{\partial u(x, y)}{\partial n} - u(x, y)\frac{\partial S(x, y; \zeta)}{\partial n}\right)ds\right] = \tilde{S}[u] \tag{27}$$

is an analytic function of $\zeta = \xi + i\eta$ if u is a solution of (2) (the integration and differentiation in normal direction have to be formed with respect to the variables x, y). (27) does not depend on the path of integration. One can now express a generating solution $S_{(v)}(x, y; \zeta)$ for (24) by means of $S(x, y; \zeta)$ and $v(x, y)$ as follows:

$$S_{(v)}(x, y; \zeta) = \frac{1}{v(x, y)}\left[2\frac{d}{d\zeta}\tilde{S}[v] + \int_{(x_0,y_0)}^{(x,y)} \left(S(x, y; \zeta)\frac{\partial v(x, y)}{\partial n} - v(x, y)\frac{\partial S(x, y; \zeta)}{\partial n}\right)ds\right]. \tag{28}$$

Another transformation connects 3 differential equations with $N = N_1(x)$, $N_2(y)$, and $N_1(x) + N_2(y)$ and their generating solutions: Let

$$H_1(z - \zeta, \bar{z} + \zeta) = G_1(x, z - \zeta), \qquad H_2(z - \zeta, \bar{z} - \zeta) = G_2(y, z - \zeta)$$

be generating solutions for

$$\Delta u + N_1(x)u = 0, \qquad \Delta u + N_2(y)u = 0. \tag{29}$$

It is clear that a generating solution for the second equation (29) can be written in this form: $H_2(z, \bar{z})$ has only to satisfy

$$\frac{\partial^2}{\partial z \partial \bar{z}} H_2 + \frac{1}{4} N_2\left(\frac{z - \bar{z}}{2i}\right) H_2 = 0, \qquad \frac{\partial}{\partial \bar{z}} H_2(0, \bar{z} - z) = \frac{1}{4} N_2\left(\frac{z - \bar{z}}{2i}\right).$$

Then, with an arbitrary analytic function $f(z)$,

$$u(z, \bar{z}) = f(z) - \int_{\zeta_0}^{z} H_2(z - \zeta, \bar{z} - \zeta) f(\zeta) d\zeta \tag{30}$$

satisfies the second equation (29) and

$$\frac{\partial u(\zeta_0, \bar{z})}{\partial \bar{z}} = 0. \tag{31}$$

Consider now the function

$$U(z, \bar{z}) = u(z, \bar{z}) - \int_{\zeta_0}^{z} H_1(z - \zeta, \bar{z} + \zeta) u(\zeta, \bar{z} - z + \zeta) d\zeta. \tag{32}$$

Because H_1 is a generating solution for the first equation (29)

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + \frac{1}{4} N_1(x) U = \frac{\partial^2 u}{\partial z \partial \bar{z}} - H_1(0, 2x) \frac{\partial u}{\partial \bar{z}}$$
$$+ \int_{\zeta_0}^{z} \left(\frac{\partial H_1(z - \zeta, \bar{z} + \zeta)}{\partial \zeta} \frac{\partial u(\zeta, \bar{z} - z + \zeta)}{\partial(\bar{z} - z + \zeta)} \right.$$
$$\left. + H_1(z - \zeta, \bar{z} + \zeta) \frac{\partial^2 u(\zeta, \bar{z} - z + \zeta)}{\partial(\bar{z} - z + \zeta)^2} \right) d\zeta.$$

Integrating by parts and observing (31) we get

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + \frac{1}{4} N_1(x) U = \frac{\partial^2 u}{\partial z \partial \bar{z}} - \int_{\zeta_0}^{z} H_1(z - \zeta, \bar{z} + \zeta) \frac{\partial^2 u(\zeta, \bar{z} - z + \zeta)}{\partial \zeta \partial(\bar{z} - z + \zeta)} d\zeta$$

and because u satisfies the second equation (29):

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + \frac{1}{4} (N_1(x) + N_2(y)) U = \frac{1}{4} [\Delta U + (N_1(x) + N_2(y)) U] = 0. \tag{33}$$

Inserting (30) into (32) we find the following representation of the general integral of (33) by means of single and double integrals:

$$\begin{aligned} U = f(z) &- \int_{\zeta_0}^{z} (G_1(x, z - \zeta) + G_2(y, z - \zeta)) f(\zeta) d\zeta \\ &+ \int_{\zeta_0}^{z} G_1(x, z - t) \int_{\zeta_0}^{t} G_2(y, t - \zeta) f(\zeta) d\zeta dt. \end{aligned} \tag{34}$$

But this integral operator can even be brought into the form (4): Suppose

$$G_1(x, z - \zeta) = \sum_{n=0}^{\infty} \frac{(\zeta - z)^n}{n!} p_{n+1}(x), \qquad G_2(y, z - \zeta) = \sum_{n=0}^{\infty} \frac{(\zeta - z)^n}{n!} q_{n+1}(y),$$

then (34) is identical with (4) if

$$\begin{aligned} S(x, y; \zeta) &= \sum_{n=0}^{\infty} \frac{(\zeta - z)^n}{n!} r_{n+1}(x, y), \\ r_n(x, y) &= \sum_{\mu+\nu=n+1} p_\mu(x) q_\nu(y), \quad p_0(x) = q_0(y) = r_0(x, y) = 1. \end{aligned} \tag{35}$$

5. **Convergence of the ascending series.** The convergence of the ascending series has been proved by Bergman [1] and by the author [8] in the most general case of a linear differential equation of second order and elliptic type, in the neighborhood of a point where the coefficients of the differential equaion are regular. In order to study its behavior near a singularity of $N(x)$ we have to repeat the proof. The following considerations may be restricted to an interval $0<x\leqq 2x_0$ in which $N(x)$ is regular. We put

$$N(x) = \sum_{n=0}^{\infty} a_n(x_0 - x)^n$$

and introduce for the moment the variable $x' = x_0 - x$; then the formulae (10) are also true for $-x'$ instead of x, the lower limit of the integrals becoming $x' = 0$. Now the functions $(-1)^n p_n(x')$ are of the form

$$(-1)^n p_n(x') = \sum_{\nu=0}^{\infty} a_{0,n\nu} x'^{\nu} + \gamma_1 \sum_{\nu=0}^{\infty} a_{1,n\nu} x'^{\nu} + \gamma_2 \sum_{\nu=0}^{\infty} a_{2,n\nu} x'^{\nu} + \cdots$$

where the $a_{i,n\nu}$ are polynomials in $a_0, a_1, \cdots$ with non-negative coefficients. In order to investigate the convergence of the series (11) for positive values of x' we may therefore replace the numbers a_n by positive numbers a_n^* of larger absolute values than the a_n. It is clear that for every $\epsilon > 0$, $\epsilon < x_0$ a number A exists such that

$$|a_n| < (n + 1)A/(x_0 - \epsilon)^n = a_n^*.$$

Accordingly we may replace $N(x)$ by

$$N^*(x) = \sum_{n=0}^{\infty} a_n^* x'^n = \frac{A(x_0 - \epsilon)^2}{(x - \epsilon)^2}.$$

A generating solution formed with respect to $N^*(x)$ is (see example 2):

$$G^*(x, z - \zeta) = \frac{1}{x - \epsilon} h\left(\frac{z - \epsilon - \zeta}{2(x - \epsilon)}\right)$$

where h is a hypergeometric function. It can be developed into a series of the form (11). A more general generating solution which is a function of two variables x and $z - \zeta$ is

$$\begin{aligned} G^*(x, z - \zeta) = {} & \frac{1}{x - \epsilon} h\left(\frac{z - \epsilon - \zeta}{2(x - \epsilon)}\right) + g(z - \zeta) \\ & - \frac{1}{x - \epsilon}\int_0^{z-\zeta} h\left(\frac{z - \epsilon - \zeta - t}{2(x - \epsilon)}\right) g(t)dt \end{aligned} \tag{36}$$

where $g(z - \zeta)$ is a power series. It can also be developed into a series (11) with functions $p_n(x)$ defined by (10), but with $N^*(x)$ instead of $N(x)$. Now

$$\begin{aligned} G^*(x_0, z - \zeta) = \sum_{n=0}^{\infty} \gamma_{n+1} \frac{(\zeta - z)^n}{n!} = {} & \frac{1}{x_0 - \epsilon} h\left(\frac{z - \epsilon - \zeta}{2(x_0 - \epsilon)}\right) + g(z - \zeta) \\ & - \frac{1}{x_0 - \epsilon}\int_0^{z-\zeta} h\left(\frac{z - \epsilon - \zeta - t}{2(x_0 - \epsilon)}\right) g(t)dt. \end{aligned} \tag{37}$$

Conversely, if $g(z - \zeta)$ is an analytic function satisfying (37), then (36) can be developed into a series (11) in which the integration constants of the $p_n(x)$ are γ_n.

Now (37) is an integral equation of Volterra type for $g(t)$. Since $h(v)$ is analytic for $|v|<1$, there exists always a solution $g(t)$, analytic for $|t|<2x_0-3\epsilon$, if

$$|\gamma_n| < \gamma(n-1)!(2x_0)^{-n} \tag{38}$$

with an arbitrary constant γ. If now $|z-\epsilon-\zeta|<2|x-\epsilon|$, the series development (11) of the function (36) is absolutely convergent and, passing to the limit $\epsilon\rightarrow 0$, we see that the same is true for

$$x \neq 0, \qquad |z-\zeta| < 2|x|. \tag{39}$$

Since the development of (36) was seen to be a majorant for the original series (11) in the sense that in both the terms can be replaced by their absolute values if only the constants γ_n are chosen according to (38), we have the following theorem.

THEOREM 4. *Let x_0 be a positive constant such that $N(x)$ is regular in $0<x\leqq 2x_0$. If the integration constants γ_n in* (10) *satisfy* (38) *with an arbitrary γ the series* (11) *is absolutely convergent under the assumption* (39). *The ascending series* (9) *is absolutely convergent for every regularly analytic function $f(z)$ if $x\neq 0$, $|z-\zeta_0|<2|x|$ hold.*

The statement on the convergence of the ascending series is an immediate consequence of the first part of Theorem 4.

6. **The descending series.** In order to study all functions (12) it is sufficient to consider only those which are formed by two particular sets of solutions of (13) whose initial functions $q_0(x)$ are linearly independent. Indeed a change of the $q_{1n}(x)$, $q_{2n}(x)$ in

$$u = q_{10}(x)f_1(z) + q_{11}(x)f_1'(z) + \cdots + q_{20}(x)f_2(z) + q_{21}(x)f_2'(z) + \cdots$$

amounts to a suitable change of $f_1(z), f_2(z)$. We shall show that two such particular sets exist, so that (12) becomes convergent for every analytic function $f(z)$.

Particular solutions of (2) result by inserting polynomials for $f(z)$. The real parts of them form a complete set of real functions on the boundary of an arbitrary region in the x-y-plane where $q_0(x)$ is regular and equal to 0, and therefore they form a complete set of solutions of (2) in the interior. We could base the proof for the existence of convergent descending series on this fact; however, we shall apply a more constructive method.

Let α_1 and α_2 with $|\alpha_1|\leqq|\alpha_2|$ be the roots of the quadratic equation

$$\alpha(\alpha-1) = -c_0. \tag{40}$$

If β_1 and β_2 are their real parts,

$$\beta_1 + \beta_2 = 1. \tag{41}$$

Three different cases have to be considered:

(1) $N(x)$ is regular at $x=0$.

(2) $(\alpha_2-\alpha_1)/\rho$ is not an integer, $N(x)$ is not regular at $x=0$.

(3) $(\alpha_2-\alpha_1)/\rho$ is an integer, $N(x)$ is not regular at $x=0$.

In the third case α_1 and α_2 are real, and we may suppose $\alpha_1 \leqq \alpha_2$. Let $q_{10}(x)$ and $q_{20}(x)$ be two linearly independent solutions of the first equation (13), their Wronskian, being a constant, may be assumed to be 1. Both functions are power series in x in the first case or series of the form

$$(42) \qquad x^{\alpha_1}\sum_{n=0}^{\infty} a_n x^{\rho n} + x^{\alpha_2}\sum_{n=0}^{\infty} b_n x^{\rho n}, \qquad x^{\alpha_1}\sum_{n=0}^{\infty} a_n x^{\rho n} + x^{\alpha_2}\sum_{n=0}^{\infty} (a_n' \log x + b_n) x^{\rho n}$$

in the second and third cases respectively. Without loss of generality we can assume $b_n=0$ for $q_{10}(x)$ and $a_n(=a_n')=0$ for $q_{20}(x)$. In a suitable neighborhood of $x=0$ the following inequalities will hold:

$$(43) \qquad \begin{array}{ll} (1) & \begin{array}{ll} |q_{10}(x)| < C, & |q_{10}'(x)| < C, \\ |q_{20}(x)| < C, & |q_{20}'(x)| < C, \end{array} \\ (2) & \begin{array}{ll} |q_{10}(x)| < C|x|^{\beta_1}, & |q_{10}'(x)| < C|x|^{\beta_1-1}, \\ |q_{20}(x)| < C|x|^{\beta_2}, & |q_{20}'(x)| < C|x|^{\beta_2-1}, \end{array} \\ (3) & \begin{array}{ll} |q_{10}(x)| < C|x|^{\beta_1-\epsilon} & |q_{10}'(x)| < C|x|^{\beta_1-\epsilon-1}, \\ |q_{20}(x)| < C|x|^{\beta_2}, & |q_{20}'(x)| < C|x|^{\beta_2-1}, \end{array} \end{array}$$

where C is a suitable positive constant and ϵ an arbitrarily small one. (Even in the third case one could take $\epsilon=0$ except when $\alpha_1=\alpha_2=1/2$.)

Now the $q_{1n}(x)$ can be defined by the recurrence formulae

$$(44) \qquad q_{1,n+1}(x) = -2\left[q_{10}(x)\int_0^x q_{20}(\xi)q_{1n}'(\xi)d\xi - q_{20}(x)\int_0^x q_{10}(\xi)q_{1n}'(\xi)d\xi\right],$$

then

$$(45) \qquad q_{1,n+1}'(x) = -2\left[q_{10}'(x)\int_0^x q_{20}(\xi)q_{1n}'(\xi)d\xi - q_{20}'(x)\int_0^x q_{10}(\xi)q_{1n}'(\xi)d\xi\right].$$

The integrals (44) and (45) make proper sense only if the integrands are greater than $O(x^{-1})$ at $x=0$. As we shall see this is indeed the case for sufficiently large n. Otherwise we want them to be understood in the way that

$$\int_0^x x^{-1}dx = \log x, \qquad \int_0^x x^{-1}\log x dx = \frac{1}{2}(\log x)^2, \cdots$$

$$\int_0^x x^{\alpha}dx = \frac{1}{\alpha+1}x^{\alpha+1},$$

$$\int_0^x x^\alpha \log x dx = \frac{1}{\alpha+1}\left(\log x - \frac{1}{\alpha+1}\right)x^{\alpha+1}, \cdots$$

for arbitrary values of $\alpha \neq -1$. The majorization of such integrals by means of a majorization of their integrands is possible and follows the same rules as that of ordinary integrals(2).

With the aid of (43) and (45) we can prove inequalities

$$|q_{1n}(x)| < C_n |x|^{\rho_n}. \tag{46}$$

Indeed the first one is true with $C_0 = C$ and $\rho_0 = 0$, $\beta_1 - 1$, $\beta_1 - \epsilon - 1$ in the three cases. Inserting (46) into (45) we get because of (41)

$$|q'_{1,n+1}(x)| < C_n \frac{4C^2}{\rho_n+1}|x|^{\rho_n+1} \text{ resp. } < C_n \frac{4C^2}{\rho_n+1}|x|^{\rho_n+1-\epsilon},$$

consequently (46) is true with

$$\begin{aligned}
&(1)\quad \rho_n = n, && C_n = \frac{2^{2n}C^{2n+1}}{n!},\\
&(2)\quad \rho_n = \beta_1 + n - 1, && C_n = \frac{2^{2n}C^{2n+1}}{(n-1)!},\\
&(3)\quad \rho_n = \beta_1 - \epsilon - 1 + n(1-\epsilon), && C_n = \frac{2^{2n}C^{2n+1}}{(1-\epsilon)(n-1)!}.
\end{aligned} \tag{47}$$

From (44) and (46) we have

$$|q_{1,n+1}(x)| < C_n \frac{4C^2}{\rho_n+1}|x|^{\rho_n+2}$$

and this, together with (47), shows that the descending series taken with the solutions (44) of (13), can be majorized in the following way:

$$\sum_{n=0}^{\infty}\left|q_{1n}(x)\frac{d^n}{dz^n}f(z)\right| < \begin{cases} C|x|\sum_{n=0}^{\infty}\frac{(4C^2|x|)^n}{n!}\left|\frac{d^n}{dz^n}f(z)\right| \text{ in case 1,}\\ C|x|^{\beta_1}\sum_{n=0}^{\infty}\frac{(4C^2|x|)^n}{(n-1)!}\left|\frac{d^n}{dz^n}f(z)\right| \text{ in case 2,}\\ C|x|^{\beta_1-\epsilon}\sum_{n=0}^{\infty}\frac{((4/(1-\epsilon))C^2|x|^{1-\epsilon})^n}{(n-1)!}\left|\frac{d^n}{dz^n}f(z)\right| \\ \qquad\qquad \text{in case 3.}\end{cases}$$

(2) In other words, we take the "finite part" of these integrals. This concept was introduced by J. Hadamard, *Lectures on Cauchy's problem*, New Haven and London, 1923.

The series on the right-hand side are absolutely convergent for

$$|x| \leqq \frac{|z - z_0|}{4C^2} - \eta(\epsilon) \quad \text{with } \eta(\epsilon) \to 0 \text{ when } \epsilon \to 0$$

where z_0 is the nearest singular point of $f(z)$. Because of (43) and the assumption that the Wronskian

$$q_{10}(x)q'_{20}(x) - q'_{10}(x)q_{20}(x) = 1$$

we have

$$2C^2 \geqq 1, \qquad \frac{1}{4C^2} \leqq \frac{1}{2},$$

hence the descending series is absolutely convergent for

$$0 < |x| < \frac{1}{2}|z - z_0|.$$

Another set of solutions of (13) is obtained when $q_{10}(x)$ and $q_{20}(x)$ are interchanged. All considerations remain unaltered, only β_1 resp. $\beta_1 - \epsilon$ have to be replaced by β_2 and β_2. So we have proved the following theorem:

THEOREM 5. *There exist two sets* $q_{10}(x)$, $q_{20}(x)$ *of solutions of* (13) *with the property*

$$\frac{q_{1,n+1}(x)}{q_{1n}(x)} = x^{1-\epsilon}O(x), \qquad \frac{q_{2,n+1}(x)}{q_{2n}(x)} = x^{1-\epsilon}O(x) \qquad \text{for } x \to 0.$$

The descending series (12) *formed with either of them is convergent in a sufficiently small neighborhood of* $x = 0$ *and for*

$$0 < |x| < |z - z_0|/2 \tag{48}$$

where z_0 *is the nearest singular point of* $f(z)$.

The line $x = 0$ itself must naturally be excluded since some of the $q_{\nu n}(x)$ may be infinite here.

The behavior near $x = 0$ of a solution of (2) which is represented by a descending series with a regular function $f(z)$ is now quite clear according to Theorem 5. One could call such a solution regular in $x = 0$. But the descending series can even be formed with a function $f(z)$ which is singular in a point z_0 on $x = 0$, and Theorem 5 says that it converges at least in a part of the neighborhood of its singular point, whereas the representation of such a function by the ascending series fails.

7. A connection between the ascending and the descending series. It is interesting to notice that the domains of convergence of the ascending and the descending series as given by Theorems 4 and 5 are complementary to each other if the constants ξ_0 and z_0 are equal. This fact finds an easy explanation in the case of example 2:

Consider the integrals

$$u_n = (z - z_0)^n - \frac{1}{x}\int_{z_0}^{z} h\left(\frac{z-\zeta}{2x}\right)(\zeta - z_0)^n d\zeta \qquad (n = 0, 1, \cdots).$$

After introduction of

$$\frac{z - z_0}{2x} = w, \qquad \frac{\zeta - z_0}{2x} = \theta$$

they become

$$u_n = x^n 2^n\left[w^n - 2\int_0^w h(w-\theta)\theta^n d\theta\right] = x^n g_n(w).$$

Because u_n satisfies (2) with $N(x) = \lambda x^{-2}$,

$$w(1 - w)g_n'' - (n-1)(1-2w)g_n' - (n(n-1)+\lambda)g_n = 0.$$

u_n is regular in $w=0$, and so $g_n(w)$ is to within a certain constant factor a hypergeometric function with the parameters

$$\alpha_n = \frac{1}{2} + \left(\frac{1}{4} - \lambda\right)^{1/2} - n, \quad \beta_n = \frac{1}{2} - \left(\frac{1}{4} - \lambda\right)^{1/2} - n, \quad \gamma_n = 1 - n.$$

Developing $g_n(w)$ into a series in w we obtain

$$u_n = \sum_{\nu=0}^{\infty} a_{\nu,n}(z - z_0)^{n+\nu}x^{-\nu}$$

which must be identical with the ascending series for $f(x) = (z - z_0)^n \cdot \text{const}$. Of course, this result is trivial since we started from the ascending series.

Suppose now that $\alpha_n - \beta_n$ is not an integer. Then $g_n(w)$ can also be developed as follows:

$$g_n(w) = W^{\alpha_0 - n}\sum_{\nu=0}^{\infty} A_{\nu,n}W^{\nu} + W^{\beta_0 - n}\sum_{\nu=0}^{\infty} B_{\nu,n}W^{\nu}, \qquad W = w^{-1}.$$

Two sets of solutions of (13) are

$$q_{1\nu}(x) = A_\nu x^{\alpha_0+\nu}, \qquad q_{2\nu}(x) = B_\nu x^{\beta_0+\nu} \qquad (\nu = 0, 1, \cdots)$$

with certain constants A_ν, B_ν. Hence

$$u_n = \sum_{\nu=0}^{\infty} C_{\nu,n}q_{1\nu}(x)\frac{d^\nu}{dz^\nu}(z - z_0)^{n-\alpha_0} + \sum_{\nu=0}^{\infty} D_{\nu,n}q_{2\nu}(x)\frac{d^\nu}{dz^\nu}(z - z_0)^{n-\beta_0}.$$

The constants $C_{\nu,n}$ $D_{\nu,n}$ must be equal:

$$C_{\nu,n} = C_{0,n}, \qquad D_{\nu,n} = D_{0,n}$$

because u_n satisfies (2). So we have also expressed u_n as a sum of two descending series for $f(z)=(z-z_0)^{n-\alpha_0}$ const. and $f(z)=(z-z_0)^{n-\beta_0}$ const.

Now it is clear why the domains of convergence of the ascending and the descending series are complementary to each other: they are identical with the domains of convergence of series in w and w^{-1} representing the same hypergeometric functions in w.

This consideration can be extended to the most general case. One has to start from a representation

$$G(x, z - \zeta) = \sum_{n=0}^{\infty} x^{\rho n-1} h_{1,n}(v) + \log x \sum_{n=0}^{\infty} x^{\rho n-1} h_{2,n}(v)$$

of a generating solution. The functions $h_{\nu,n}(v)$ satisfy a system of recurrence formulae which are of the hypergeometric type. But calculations analogous to those above are rather cumbersome.

8. **An extension to differential equations in $n>2$ variables.** In this section the differential equation

$$\Delta U + N(r^2)U = 0 \tag{49}$$

with $r^2=x_1^2+x_2^2+\cdots+x_n^2$ in n variables $x_1, x_2, \cdots, x_n$ may be given. It was considered first in connection with similar questions by Bergman in the case $n=3$ [9]. Bergman's differential equation seems to be slightly more general but is reduced to the form (49) when U is replaced by $M(r^2)U$ with a suitable function $M(r^2)$. The integral

$$U(x_1, \cdots, x_n) = u(x_1, \cdots, x_n) - \int_0^1 S(r^2, t)u(tx_1, \cdots, tx_n)dt \tag{50}$$

satisfies (49) if u is a harmonic function and

$$r^2 \frac{\partial^2 S}{(\partial r^2)^2} - t\frac{\partial^2 S}{\partial t \partial r^2} + \left(\frac{n}{2} - 1\right)\frac{\partial S}{\partial r^2} + \frac{1}{4}NS = 0, \tag{51}$$

$$\left[\frac{\partial S}{\partial r^2}\right]_{t=1} = \frac{1}{4}N. \tag{52}$$

The proof follows by differentiating under the integral sign and integrating by parts.

In (51), (52) substitute

$$r^2 = \exp x,\ t = \exp\frac{y-x}{2}, \qquad S(r^2, t) = t^{n/4-1}T(x, y); \tag{53}$$

then

$$T_{xx} - T_{yy} + \frac{1}{4} \exp xN(\exp x)T = 0, \tag{54}$$

$$\frac{d}{dx} T(x, x) = \frac{1}{4} \exp xN(\exp x). \tag{55}$$

The problem to find a function T with these properties is evidently independent of n. It is a problem similar to that of finding a generating solution for (2) (see the first proof of Theorem 1 in §2). In our present case we even need a solution only for real values of the variables. It is known that solutions exist in the whole part of the x-y-plane in which $N(\exp x)$ is regular.

Now insert a homogeneous spherical harmonic $u=u_m$ of degree m into (50). Then

$$U = P_m(r)u_m r^{-m} \tag{56}$$

with

$$P_m(r) = r^m\left(1 - \int_0^1 S(r^2, t)t^m dt\right). \tag{57}$$

Because U satisfies (49)

$$P_m'' + \frac{n-1}{r} P_m' + \left(N(r^2) - \left(\frac{m}{r}\right)^2\right)P_m = 0. \tag{58}$$

Between the harmonic functions u and the solutions U of (49) a connection similar to that stated in Theorem 2 holds. Assume $N(r^2)$ to be regular in a sphere R about the origin and

$$u = \sum_{m=0}^{\infty} u_m \tag{59}$$

to be a harmonic function regular in a domain D; (59) may be absolutely and uniformly convergent for $r<r_0$. Suppose a light source to be placed in the origin and D' to be defined as that part of the intersection of R and D which does not lie in the shadow of any boundary points of D.

THEOREM 6. *The series*

$$U = \sum_{m=0}^{\infty} P_m(r)u_m r^{-m} \tag{60}$$

formed with the same functions u_m as (59) *and with $P_m(r)$ as defined by* (57) *is absolutely and uniformly convergent for $r<r_0$ and analytically continuable into the domain D' described above, U is regularly analytic in D'.*

The proof is, just as in the case of Theorem 2, almost trivial. We have only to define U as the integral (50).

Three improvements of Theorem 6 seem to be quite easy: (1) description of the $P_m(r)$ without reference to an operator; for this purpose a special operator will have to be distinguished, (2) description of the behavior of U in singular points of u, (3) it seems likely that Theorem 6 would hold with D' replaced by the intersection of R and D. However, we do not want to go into too many details without a particular problem urging us to do so. Our considerations seem sufficient to give a rough idea of the properties of functions satisfying partial differential equations which are revealed by the new operational approach.

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Mathematisches Seminar, Universität Münster,
Münster, Germany.