THE DYNAMICS OF TRANSFORMATION GROUPS

BY

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1. **Introduction.** In the study of classical dynamical systems defined by systems of ordinary differential equations, it has proved useful to regard such a system as a one-parameter transformation group acting on a topological space. An extensive body of results concerned with such general dynamical systems has been developed, particularly notable contributions being due to G. D. Birkhoff (cf. Birkhoff [2, Chapter VII])(2).

If the restriction that the transformation group be a one-parameter group is dropped and we consider the more general setting of a topological group of transformations acting on a topological space there arises the question of the natural and desirable generalizations of such properties as almost periodicity, recurrence and regional recurrence. In recent papers, Barbachine [1], Gottschalk [3, 4, 5] and Niemytzki [6] have studied the problem of this type of generalization. Definitions of almost periodicity have been formulated by all of these authors and extensive results associated with this property have been obtained by Gottschalk.

Recurrence (stability in the sense of Poisson) has been defined for transformation groups by Barbachine and Niemytzki, who announce several theorems which are analogous to the classical ones of Birkhoff. In formulating such definitions, Barbachine assumes that the group is partially ordered and the classification of orbits is dependent on that partial ordering. Niemytzki essentially makes use of a partial ordering obtained by assuming that the group is separable and locally compact, and exhausting it by an expanding sequence of compact sets. In consequence, his definitions do not make use of the algebraic structure of the group.

It appears to the authors that it would be desirable to formulate definitions and develop properties which depend on the topological structures of the group and space, on the algebraic structure of the group, and on these structures only. In the present paper such a program is initiated. The concepts of recurrent point and regional recurrence are so defined that if the group is the real axis, these definitions reduce to the corresponding classical ones. It is shown that these properties are hereditary in the sense that possession of the property by the original group implies possession of the property by any relatively dense subgroup. A topological analogue of the Poincaré Recurrence Theorem is derived, as well as a measure-theoretic generalization of that

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theorem. Several sufficiency conditions are given under which pointwise recurrence implies pointwise almost periodicity. Finally, it is shown how pointwise periodicity and pointwise recurrence can be characterized by incompressibility properties.

2. **Definitions.** Let X be a topological space and let T be an abelian multiplicative topological group with identity e. Let T act as a transformation group on X. That is to say, suppose that to $x\in X$ and $t\in T$ is assigned a point, denoted xt, of X such that: (1) $xe=x$ $(x\in X)$; (2) $(xt)s=x(ts)$ $(x\in X;$ $t,$ $s\in T)$; (3) The function xt defines a continuous transformation of $X\times T$ into X.

A set S in T is said to be a *semigroup* provided that $SS\subset S$. A semigroup S in T is said to be *replete* provided that S contains some translate of each compact set in T. A set A in T is said to be *extensive* provided that A intersects every replete semigroup in T. If $x\in X$, then T is said to be *recurrent at* x (or x is said to be *recurrent under* T) provided that to each neighborhood U of x there corresponds an extensive set A in T such that $xA\subset U$.

Let $\mathfrak{J}$ (or $\mathfrak{R}$) denote the additive group of integers (or reals) provided with its natural topology. If $T=\mathfrak{J}$ or $\mathfrak{R}$, then it is easily seen that: (1) A semigroup S in T is replete if and only if S contains a "ray"; and (2) A set A in T is extensive if and only if A contains a sequence marching to $+\infty$ and a sequence marching to $-\infty$. Thus the present notion of recurrent point under a transformation group generalizes the usual notion of recurrent point under a flow ($T=\mathfrak{J}$ or $\mathfrak{R}$). We also point out that the present notion of recurrence is intrinsic in the sense that it involves the topological structures of X and T, the algebraic structure of T, and only these structures.

Suppose S is a semigroup in T. It is easily verified that if S is replete then $T=S^{-1}S$. The converse is also true in case T is discrete. However, the converse may fail if T is not discrete. This is shown by the following example. Take $T=\mathfrak{R}$ and take S to be a semigroup in T maximal with respect to the property of containing only positive non-integral numbers. Thus the notion of a replete semigroup depends on the topology of the group.

A set A in T is said to be *relatively dense* provided that $T=AK$ for some compact set K in T.

We assume throughout that T is not only abelian, but that T is generated by some compact neighborhood of e and that G is a relatively dense closed subgroup of T. The hypotheses on T ensure the existence of "sufficiently many" replete semigroups in T. The generality of such topological groups T is indicated by the structure theorem (cf. A. Weil [7, p. 110]) that $T=K\times\mathfrak{R}^m\times\mathfrak{J}^n$ where K is a compact abelian group.

3. **Recurrence is hereditary.**

LEMMA 1. *If K is a compact set in T, then there exists a compact set H in G such that $K^n\cap G\subset H^n$ for all integers n.*

Proof. We may suppose that $T=GK$, $e\in K$ and $K=K^{-1}$. Define $H=K^3\cap G$. Now H is a compact set in G and to prove the lemma it is enough to show that $K^n\cap G\subset H^n$ for all positive integers n. Let n be a positive integer and let $k_1, \cdots, k_n$ be elements of K such that $k_1 \cdots k_n\in G$. If $t\in T$, then there exists $g\in G$ such that $t\in gK^{-1}$ whence $g\in tK$. Thus for each integer i $(1\leqq i<n)$ there exists $g_i\in G$ such that $g_i\in k_1 \cdots k_iK$. Define $g_n=k_1 \cdots k_n$. Clearly $g_n\in G$ and $g_n\in k_1 \cdots k_nK$. Now $g_i^{-1}g_{i+1}\in Kk_{i+1}K\subset K^3$ and $g_i^{-1}g_{i+1}\in G$ $(1\leqq i<n)$ whence $g_i^{-1}g_{i+1}\in H$ and $g_{i+1}\in g_iH$ $(1\leqq i<n)$. Also $g_1\in k_1K\subset K^3$ and $g_1\in G$ whence $g_1\in H$. We conclude that $k_1 \cdots k_n=g_n\in g_{n-1}H\subset g_{n-2}H^2\subset \cdots \subset g_1H^{n-1}\subset H^n$. The proof is completed.

Lemma 2. *If R is a replete semigroup in G, then there exists a replete semigroup S in T such that $S\cap G\subset R$.*

Proof. Let K be a symmetric compact neighborhood of e whose interior generates T. By Lemma 1 there exists a compact set H in G such that $K^n\cap G\subset H^n$ for all integers n. For some $g\in G$ we have $gH\subset R$. Define $S=\bigcup_{n=1}^{+\infty} g^nK^n$. Now S is a replete semigroup in T and $S\cap G\subset\bigcup_{n=1}^{+\infty} (g^nK^n\cap G)\subset\bigcup_{n=1}^{+\infty} g^nH^n\subset R$. The proof is completed.

Lemma 3. *If S is a replete semigroup in T, then $S\cap G$ is a replete semigroup in G.*

Proof. Clearly $S\cap G$ is a semigroup. Let H be a compact set in G and let K be a compact set in T for which $T=GK$. Since $K^{-1}H$ is compact, there exists $t\in T$ such that $tK^{-1}H\subset S$. Choose $g\in G$ and $k\in K$ so that $t=gk$. Now $gH=tk^{-1}H\subset S$ and $gH\subset G$. Hence $gH\subset S\cap G$ and the proof is completed.

Lemma 4. *If S is a replete semigroup in T and if K is a compact set in T such that $e\in K$, then $\bigcap_{k\in K} kS$ is a replete semigroup in T.*

Proof. Since $R=\bigcap_{k\in K} kS=\bigcap_{k\in K} (S\cap kS)$ and $S\cap kS$ is a semigroup for each $k\in K$, it follows that R is a semigroup. Let C be a compact set in T and define $D=C\cup K^{-1}C$. There exists $t\in T$ such that $tD\subset S$. Now $k\in K$ implies $C\subset D\cap kD$ whence $tC\subset tD\cap ktD\subset S\cap kS$. Thus $tC\subset R$ and the proof is completed.

Theorem 1. *If $x\in X$, then G is recurrent at x if and only if T is recurrent at x.*

Proof. To prove that the recurrence of T at x implies the recurrence of G at x it is enough by [5] to show that: (1) If A is a subset of G which is extensive in T, then A is extensive in G; and (2) If $A, B, C\subset T$ such that A is extensive in T, C is compact and $A\subset BC$, then B is extensive in T. Statement (1) follows easily from Lemma 2. Now assume the hypotheses of statement (2). Let S be a replete semigroup in T. There exists $t\in T$ such that $tC^{-1}\subset S$. Since $S\cap tS$ is a replete semigroup in T by Lemma 4, $A\cap tS\neq\emptyset$. Hence

$BC\cap tS\neq\emptyset$ and $\emptyset\neq B\cap tC^{-1}S\subset B\cap S$. Thus B is extensive in T.

To prove that the recurrence of G at x implies the recurrence of T at x it is enough to show that every extensive set in G is also extensive in T. However, this is an easy consequence of Lemma 3. The proof is completed.

The transformation group T is said to be *pointwise recurrent* provided that T is recurrent at each point of X.

COROLLARY 1. *G is pointwise recurrent if and only if T is pointwise recurrent.*

4. Regional recurrence is hereditary.

LEMMA 5. *If R is a replete semigroup in G and if K is a compact set in T such that $e\in K$, then there exists a replete semigroup S in T such that $SK\cap G\subset R$.*

Proof. By Lemma 2 there exists a replete semigroup Q in T such that $Q\cap G\subset R$. Define $S=\bigcap_{k\in K} Qk^{-1}$. By Lemma 4, S is a replete semigroup in T. Now $SK\subset Q$ whence $SK\cap G\subset Q\cap G\subset R$. The proof is completed.

LEMMA 6. *If V is a neighborhood of e and if $t\in T$, then there exists a positive integer n such that $t^n\in GV$.*

Proof. Let K be a compact set in T for which $T=GK$. Choose a neighborhood W of e so that $WW^{-1}\subset V$. Let F be a finite collection of translates of W which covers K. To each positive integer n there correspond $g_n\in G$ and $k_n\in K$ such that $t^n=g_nk_n$. Select positive integers p, q so that $p>q$ and k_p, $k_q\in W_0$ for some $W_0\in F$. Then $t^{p-q}=g_pg_q^{-1}k_pk_q^{-1}\in Gk_pk_q^{-1}\subset GW_0W_0^{-1}\subset GWW^{-1}\subset GV$ and the proof is completed.

LEMMA 7. *If V is a neighborhood of e, if K is a compact set in T and if $k_1, k_2, \cdots$ is a sequence of elements of K, then there exist finitely many positive integers $i_1, \cdots, i_n$ $(n\geqq 1)$ such that $i_1<\cdots<i_n$ and $k_{i_1}\cdots k_{i_n}\in GV$.*

Proof. It follows readily from Lemma 6 that there exist finitely many open sets $V_1, \cdots, V_m$ in T and positive integers $p_1, \cdots, p_m$ such that $K\subset\bigcup_{j=1}^m V_j$ and $V_j^{p_j}\subset GV$ $(j=1, \cdots, m)$. There exists an integer j $(1\leqq j\leqq m)$ such that $k_i\in V_j$ for infinitely many positive integers i. Define $n=p_j$. Choose positive integers $i_1, \cdots, i_n$ such that $i_1<\cdots<i_n$ and $k_{i_1}, \cdots, k_{i_n}\in V_j$. Hence $k_{i_1}\cdots k_{i_n}\in V_j^n\subset GV$ and the proof is completed.

The transformation group T is said to be *regionally recurrent* provided that to each open set U and X there corresponds an extensive set A in T such that $a\in A$ implies $U\cap Ua\neq\emptyset$.

THEOREM 2. *G is regionally recurrent if and only if T is regionally recurrent.*

Proof. It follows easily from Lemma 3 that the regional recurrence of G implies the regional recurrence of T.

Now assume that T is regionally recurrent. Let $x_0\in X$, let U be an open neighborhood of x_0 and let R be a replete semigroup in G. There exists an open neighborhood U_0 of x_0 and a compact symmetric neighborhood V of e in T such that $U_0V\subset U$. By Lemma 5 we can find a replete semigroup S in T such that $SV\cap G\subset R$.

There exists $s_1\in S$ for which $U_0\cap U_0s_1\neq\emptyset$. Choose $x_1\in U_0$ so that $x_1s_1\in U_0$ and an open neighborhood U_1 of x_1 so that $U_1\subset U_0$ and $U_1s_1\subset U_0$. There exists $s_2\in S$ for which $U_1\cap U_1s_2\neq\emptyset$. Choose $x_2\in U_1$ so that $x_2s_2\in U_1$ and an open neighborhood U_2 of x_2 so that $U_2\subset U_1$ and $U_2s_2\subset U_1$. Continuing this process we obtain a sequence $\{s_i\}$ of elements of S and a sequence $\{U_i\}$ of nonvacuous open sets such that $U_i\subset U_{i-1}$ and $U_is_i\subset U_{i-1}$ $(i=1, 2, \cdots)$.

There exists a compact set K in T for which $T=GK$. Hence for each positive integer i there exist $g_i\in G$ and $k_i\in K$ such that $s_i=g_ik_i$. By Lemma 7 we can find positive integers $i_1, \cdots, i_n$ $(n\geqq 1)$ for which $i_1<\cdots<i_n$ and $k_{i_1}\cdots k_{i_n}\in GV$. Since $U_{i_j}s_{i_j}\subset U_{i_{j-1}}$ $(j=2, \cdots, n)$ and $U_{i_1}s_1\subset U_0$, we conclude that $U_{i_n}s_{i_n}\cdots s_{i_1}\subset U_0$. Also $U_{i_n}\subset U_0$ whence $U_0\cap U_0s_{i_n}\cdots s_{i_1}\neq\emptyset$. Now $s_{i_n}\cdots s_{i_1}=gk_{i_n}\cdots k_{i_1}$ where $g=g_{i_n}\cdots g_{i_1}\in G$. Choose $g_0\in G$ and $v\in V$ so that $k_{i_n}\cdots k_{i_1}=g_0v$. Thus $s_{i_n}\cdots s_{i_1}=gg_0v$. Define $r=s_{i_n}\cdots s_{i_1}v^{-1}$. We observe that $r=gg_0$, $r\in SV$, $gg_0\in G$ and $SV\cap G\subset R$. Hence $r\in R$. Now $U_0v^{-1}\cap U_0s_{i_n}\cdots s_{i_1}v^{-1}\neq\emptyset$, $U_0\subset U$ and $U_0v^{-1}\subset U$. Thus $U\cap Ur\neq\emptyset$ and the proof is completed.

5. Topological and measure-theoretic recurrence theorems.

It is readily proved that if the set of recurrent points is dense in X, then T is regionally recurrent. We now point out certain conditions under which the converse is true.

Suppose that there is distinguished in T a certain class of sets called "admissible." If $x\in X$, then T is said to be *recursive at* x (or x is said to be *recursive under* T) provided that to each neighborhood U of x there corresponds an admissible set A such that $xA\subset U$. The transformation group T is said to be *regionally recursive* provided that to each open set U in X there corresponds an admissible set A such that $a\in A$ implies $U\cap Ua\neq\emptyset$. Clearly if the class of admissible sets is taken to be the class of extensive sets, then "recursive" means "recurrent."

LEMMA 8. *Let $S_1, S_2, \cdots$ be a sequence of sets in T, let a set A in T be called admissible if and only if $A\cap S_n\neq\emptyset$ for every positive integer n, and let R denote the set of recursive points in X. If X is metrizable then R is a G_δ set in X. If X is metrizable and if T is regionally recursive, then R is residual in X.*

Proof. Let ρ be a metric in X compatible with the topology in X. For positive integers n and m, let $E(n, m)$ denote the set of all points x of X such that $\rho(x, xs)\geqq 1/m$ for all $s\in S_n$. It is clear that $X-R=\bigcup_{n,m=1}^{+\infty} E(n, m)$. For fixed positive integers n and m, the set $E(n, m)$ is closed in X. Thus $X-R$ is an F_σ set and R is a G_δ set.

Now assume that T is regionally recursive. Let n and m be fixed positive integers. Suppose int $E(n, m)\neq\emptyset$. Then there exists an open set U in X such that $U \subset E(n, m)$ and $\rho(x, y) < 1/m$ $(x, y \in U)$. Since $U \cap Us \neq \emptyset$ for some $s \in S_n$, we can find $x \in U$ so that $xs \in U$ whence $\rho(x, xs) < 1/m$. This contradicts the definition of $E(n, m)$ and therefore int $E(n, m) = \emptyset$. Thus $X - R$ is of the first category in X and the proof is completed.

LEMMA 9([3]). *There exists a sequence $S_1, S_2, \cdots$ of replete semigroups in T such that each replete semigroup in T contains S_n for some positive integer n.*

Proof. According to Weil's structure theorem (§2), T contains a compact subgroup K and closed subgroup $G = \mathcal{R}^m \times \mathfrak{J}^n$ such that $T = KG$. It is evident that G is separable and thus there exists a sequence, $g_1, g_2, \cdots$, which is dense in G. Let U be a symmetric open neighborhood of e such that U generates T and $\overline{U}$ is compact. Define $S_n = \bigcup_{i=1}^{+\infty} g_n^i U^i$ $(n = 1, 2, \cdots)$. Clearly S_n $(n = 1, 2, \cdots)$ is a replete semigroup in T. Let S be a replete semigroup in T. There exist elements $t \in T$, $k \in K$, $g \in G$ such that $t = kg$ and $tK\overline{U} \subset$ int S. Choose a neighborhood V of e so that $VtK\overline{U} \subset$ int S. Then for some positive integer n, $g_n \in Vg$. Therefore $g_n U \subset VgU = Vtk^{-1}U \subset$ int $S \subset S$. Thus $S_n \subset S$ and the proof is completed.

The following theorem is a topological analogue of Theorem 4, a generalized form of the Poincaré recurrence theorem.

THEOREM 3. *If X is metrizable and if T is regionally recurrent, then the set of recurrent points is a G_δ set residual in X.*

Proof. By Lemma 9 there exists a sequence $S_1, S_2, \cdots$ of replete semigroups in T which is a "base" for the replete semigroups in T. Clearly a set A in T is extensive in T if and only if $A \cap S_n \neq \emptyset$ for every positive integer n. The desired conclusion now follows immediately from Lemma 8.

COROLLARY 2. *If X is a complete metric space and if T is regionally recurrent, then the set of recurrent points is dense in X.*

COROLLARY 3. *If X is a complete metric space, then the closure of the set of recurrent points in X is exactly the maximal invariant set in X on which T is regionally recurrent.*

The following theorem is a generalization of the Poincaré recurrence theorem.

THEOREM 4. *If X is separable and metrizable, if T is separable, if μ is a non-negative countably additive measure function in X defined exactly for the Borel sets, if μ is invariant under T and if μX is finite, then almost all points of X are recurrent.*

[3] In the original proof of Lemma 9, it was assumed that T was separable. The authors are indebted to Professor J. C. Oxtoby for the modified proof presented here which permits the omission of the hypothesis of separability and consequently the omission of that hypothesis in Theorem 3, Corollaries 2 and 3, and Theorem 9.

Proof. Let $t_1, t_2, \cdots$ be a sequence of elements of T which is dense in T. For each positive integer n define R_n to be the set of recurrent points under the discrete flow generated by powers of t_n. Define R to be the set of recurrent points under T. It is readily proved that $\bigcap_{n=1}^{+\infty} R_n \subset R$. The Poincaré recurrence theorem asserts that each R_n $(n=1, 2, \cdots)$ is a G_δ set with $\mu R_n = \mu X$. By Lemmas 8 and 9, R is a G_δ set. Hence $\mu R = \mu X$ and the proof is completed.

6. **When does pointwise recurrence imply pointwise almost periodicity?**

LEMMA 10. *If A is a relatively dense set in T, then A is extensive in T.*

Proof. Let S be a replete semigroup in T. There exists a compact set K in T such that $T = AK$. For some $t \in T$, $tK^{-1} \subset S$. There exists $a \in A$ such that $t \in aK$. Hence $a \in tK^{-1} \subset S$ and $A \cap S \neq \emptyset$. The proof is completed.

If $x \in X$, then T is said to be *almost periodic* at x (or x is said to be *almost periodic under* T) provided that to each neighborhood U of x there corresponds a relatively dense set A in T such that $xA \subset U$. The transformation group T is said to be *pointwise almost periodic* provided that T is almost periodic at each point of X.

It follows from Lemma 10 that an almost periodic point is a recurrent point and that a pointwise almost periodic transformation group is pointwise recurrent. The converse of the latter statement is not generally true. (See [3, p. 764].) We now indicate several sufficiency conditions under which the converse does hold.

LEMMA 11. *If Y is a subset of X such that every replete semigroup in T contains a compact set Q for which $Y \subset YQ$, then there exists a compact set C in T for which $YT = YC$.*

Proof. Let U be a symmetric open neighborhood of e such that U generates T and $\overline{U}$ is compact. Define $H = \overline{U}^2$ and $K = \overline{U}^3 = \overline{U}H$.

We first show that there exists a positive integer n such that if $k \in K$, then $Y \subset \bigcup_{i=1}^{n} Yk(kH)^i$. To show this it is enough to prove that if $k_0 \in K$, then there exists a positive integer m and a neighborhood V of e such that $k \in k_0 V$ implies $Y \subset \bigcup_{i=1}^{m} Yk(kH)^i$. Now suppose $k_0 \in K$. Define $S = \bigcup_{i=1}^{+\infty} k_0(k_0 U)^i$. The set S is an open replete semigroup. Hence S contains a compact set Q such that $Y \subset YQ$. Choose a symmetric compact neighborhood V of e for which $V \subset U$ and $QV \subset S$. Since QV is compact, there exists a positive integer m such that $QV \subset \bigcup_{i=1}^{m} k_0(k_0 U)^i$ and hence $YV \subset YQV \subset \bigcup_{i=1}^{m} Yk_0(k_0 U)^i$. Let $k \in k_0 V$ and $y \in Y$. Choose $v \in V$ so that $k_0 = kv$. Then $yv \in \bigcup_{i=1}^{m} Yk_0(k_0 U)^i$ $= \bigcup_{i=1}^{m} Ykv(kvU)^1 \subset \bigcup_{i=1}^{m} Ykv(kH)^i$ and $y \in \bigcup_{i=1}^{m} Yk(kH)^i$. This completes the proof that there exists a positive integer n such that $Y \in \bigcup_{i=1}^{n} Yk(kH)^i$ if $k \in K$. Let n denote such an integer.

Choose a positive integer p $(p \geqq n)$ so large that if $k_1, \cdots, k_{p+1} \in K$, then for some $n+1$ of the elements $k_1, \cdots, k_{p+1}$, let us say $k_1, \cdots, k_{n+1}$, we have

$k_i^{-1}k_j \in U$ $(i, j=1, \cdots, n+1)$. We now show that $YT \subset YK^p$, which will complete the proof. Assume $YT \not\subset YK^p$. Then $YK^{p+1} \not\subset YK^p$ for otherwise $YT \subset \bigcup_{i=1}^{+\infty} YK^i \subset \bigcup_{i=p}^{+\infty} YK^i \subset YK^p$. Select $y \in Y$ and $k_1, \cdots, k_{p+1} \in K$ for which $yk_1 \cdots k_{p+1} \notin YK^p$. There exist $n+1$ of the elements $k_1, \cdots, k_{p+1}$, let us say $k_1, \cdots, k_{n+1}$, such that $k_i^{-1}k_j \in U$ $(i, j=1, \cdots, n+1)$. Let r be a positive integer such that $r \leqq n$. It follows that $yk_1k_2 \cdots k_{r+1} \notin YK^r$ and $yk_1(k_1u_2) \cdots (k_1u_{r+1}) \notin YK^r$ where $u_2, \cdots, u_{r+1}$ are elements of U for which $k_2 = k_1u_2, \cdots, k_{r+1} = k_1u_{r+1}$. Thus $yk_1^{r+1} \notin YH^r$ and $y \notin Yk_1^{-1}(k_1^{-1}H)^r$. We conclude that $y \notin \bigcup_{i=1}^{n} Yk_1^{-1}(k_1^{-1}H)^i$. Since $k_1^{-1} \in K$, this contradicts the definition of n. The proof is completed.

Lemma 12. *If U is an open set in X such that $\overline{U}$ is compact and if to each compact set K in T there corresponds $x \in U$ and $t \in T$ such that $xtK \cap U = \emptyset$, then there exists a point y of $\overline{U}$ and a replete semigroup S in T such that $yS \cap U = \emptyset$.*

Proof. Assume the conclusion is false. Then for each point x of $\overline{U}$ and each replete semigroup S in T, $xS \cap U \neq \emptyset$ whence $x \in US^{-1}$. Since the inverse of a replete semigroup is a replete semigroup, it follows that for each replete semigroup S in T, $\overline{U} \subset US$; since U is open and $\overline{U}$ is compact, we can choose a finite set F in S such $\overline{U} \subset UF$. Thus each replete semigroup in T contains a finite set F for which $U \subset UF$. By Lemma 11 there exists a compact set C in T such that $UT = UC$. Hence $x \in U$ and $t \in T$ implies $xt \in UC$ and $xtC^{-1} \cap U \neq \emptyset$. This contradicts the hypothesis. The proof is completed.

Theorem 5. *If X is locally compact, if the collection of orbit-closures is a partition of X, and if T is pointwise recurrent, then T is pointwise almost periodic.*

Proof. We may suppose that X is a minimal orbit-closure. Assume some point x of X is not almost periodic. Then there exists an open neighborhood U of x such that $\overline{U}$ is compact and such that to each compact set K in T there corresponds $t \in T$ for which $xtK \cap U = \emptyset$. By Lemma 12 there exists a point y of $\overline{U}$ and a replete semigroup S in T such that $yS \cap U = \emptyset$. Hence $x \notin \overline{yS}$. Since y is recurrent, $\overline{yS} = \overline{yT}$. Therefore $x \notin \overline{yT}$. This is impossible since X is assumed to be a minimal orbit-closure. The proof is completed.

The transformation group T is said to be *locally weakly almost periodic* provided that if $x \in X$ and if U is a neighborhood of x, then there exists a neighborhood V of x and a compact set K in T such that $y \in V$ and $t \in T$ implies that $ytK \cap U \neq \emptyset$. It may be verified that if T is locally weakly almost periodic, then T is pointwise almost periodic. Simple examples show that the converse is generally false.

Theorem 6. *If X is locally compact and zero-dimensional and if T is pointwise recurrent, then T is locally weakly almost periodic.*

Proof. Suppose $x \in X$ and U is a neighborhood of x. Choose an open closed compact neighborhood V of x so that $V \subset U$. It is enough to show that there exists a compact set K in T such that $y \in V$ and $t \in T$ implies $ytK \cap U \neq \emptyset$. Assume this to be false. Then by Lemma 12 there exists $y \in V$ and a replete semigroup S in T such that $yS \cap V = \emptyset$. Hence y is not recurrent. This is a contradiction and the proof is completed.

7. **Recurrence and incompressibility.** The following theorem characterizes pointwise recurrence in terms of an "incompressibility" property.

THEOREM 7. *In order that T be pointwise recurrent it is both necessary and sufficient that if M be a closed set in X and S be a replete semigroup in T such that $MS \subset M$, then $MS = M$.*

Proof. We prove the necessity. Let M be a closed set in X, let S be a replete semigroup in T and let $MS \subset M$. Choose a compact set K in T such that K contains a symmetric open neighborhood of e which generates T. Select $t \in T$ so that $H = tK \subset S$. Define $R = \bigcup_{n=1}^{+\infty} H^n$. Since R is a replete semigroup in T and every point of M is recurrent, $M \subset \overline{MR}$. Now $MH \subset MS \subset M$ whence $MH^n \subset MH$ $(n = 1, 2, \cdots)$ and $MR \subset MH$. Since M is closed and H is compact, $\overline{MH} = MH$. Thus $M \subset \overline{MR} \subset \overline{MH} = MH \subset MS$ and the necessity is proved.

We prove the sufficiency. Let $x \in X$ and let S be a replete semigroup in T. Define $M = \overline{x \cup xS}$. Now $MS \subset \overline{xS} \subset M$. From hypothesis $MS = M$ whence $M = \overline{xS}$ and $x \in \overline{xS}$. Thus x is recurrent and the proof is completed.

LEMMA 13. *If A is a nonvacuous subset of T such that every replete semigroup in T contains a compact set Q for which $A \subset AQ$, then A is relatively dense in T.*

Proof. Consider T acting as a transformation group upon itself and apply Lemma 11.

LEMMA 14. *If S is an extensive semigroup in T, then S is relatively dense in T.*

Proof. Let R be a replete semigroup in T. Choose $t \in S \cap R^{-1}$. Clearly $t^{-1} \in R$ and $S \subset St^{-1}$. The conclusion now follows from Lemma 13.

If $x \in X$, then T is said to be *periodic at* x (or x is said to be *periodic under* T) provided that there exists a relatively dense subgroup A of T such that $xA = x$. The transformation group T is said to be *pointwise periodic* provided that T is periodic at each point of x.

The following theorem characterizes pointwise periodicity in terms of an "incompressibility" property.

THEOREM 8. *In order that T be pointwise periodic it is both necessary and sufficient that if M be a set in X and S be a replete semigroup in T such that $MS \subset M$, then $MS = M$.*

Proof. We prove the necessity. Let M be a set in X and let S be a replete semigroup in T such that $MS \subset M$. Let $m \in M$. There exists a relatively dense subgroup A of T such that $mA = m$. By Lemma 10 there exists $t \in A \cap S$. Hence $m = mt \in MS$. The necessity is proved.

We prove the sufficiency. Let $x \in X$ and let A be the maximal subset of T for which $xA = x$. Now A is a subgroup of T. By Lemma 14 it is enough to show that A is extensive in T. Let S be a replete semigroup in T. Define $M = x \cup xS$. Then $MS \subset M$ whence $MS = M$ and $xs = x$ for some $s \in S$. Therefore $A \cap S \neq \emptyset$. The proof is completed.

8. An example.

LEMMA 15. *If G is a subgroup of T which is not relatively dense in T and if K is a compact set in T, then there exists a compact set C such that every translate of C contains a translate of K disjoint from G.*

Proof. We may suppose that K contains a symmetric neighborhood of e which generates T. If the lemma is not true, then corresponding to each positive integer m there exists an element $t_m \in T$ such that G intersects every translate of K in $t_m K^m$. It follows that there exists $g_m \in t_m K \cap G$. Since G is a group, every translate of K in $g_m^{-1} t_m K^m$ intersects G. Let K_0 be an arbitrary translate of K. Choose a positive integer m so that $K_0 K \subset K^m$. Now $g_m t_m^{-1} \in K$ and hence $K_0 g_m t_m^{-1} \subset K^m$ or $K_0 \subset g_m^{-1} t_m K^m$. It follows that $G \cap K_0 \neq \emptyset$ and thus G is relatively dense in T, contrary to hypothesis.

LEMMA 16. *Let $G_1, G_2, \cdots, G_m$ be subgroups of T, each of which is not relatively dense in T, let S be a replete semigroup in T and let K_0 be a compact subset of T. Then there exists a translate of K_0 contained in S and disjoint from $\bigcup_{i=1}^m G_i$.*

Proof. By Lemma 15 there exist compact sets $K_1, K_2, \cdots, K_m$ such that for each $i = 1, 2, \cdots, m$ every translate of K_i contains a translate of K_{i-1} disjoint from G_i. Since S contains some translate of K_m, the conclusion follows.

LEMMA 17. *If T is not compact, then there exists a replete semigroup not containing the identity.*

Proof. For otherwise $[e]$ would be relatively dense by Lemma 14 and thus T would be compact.

LEMMA 18. *If T is not compact, if S is a replete semigroup and if K is a compact set in T, then there exists a replete semigroup S^* such that $S^* \subset S$ and $S^* \cap K = \emptyset$.*

Proof. By Lemma 17 we may suppose that $S \neq T$. It is enough to show that $sS \cap K = \emptyset$ for some $s \in S$ and then to take $S^* = sS$. Assume that $s \in S$ implies $sS \cap K \neq \emptyset$ whence $s \in KS^{-1}$. Thus $S \subset KS^{-1}$, $T = SS^{-1} \subset KS^{-1}S^{-1}$

$\subset KS^{-1}$ and $T=SK^{-1}$. Choosing $t\in T$ so that $K^{-1}t\subset S$, we have $T=Tt=SK^{-1}t \subset SS\subset S$. This is a contradiction and the proof is completed.

Let $\tilde{X}$ be the collection of continuous functions on T to the unit interval and let $\tilde{X}$ be topologized by the compact-open topology. Let T act as a transformation group on $\tilde{X}$ by translation of the functions.

THEOREM 9. *If $G_1, G_2, \cdots$ is a sequence of discrete infinite cyclic subgroups in T, if each G_n $(n=1, 2, \cdots)$ is not relatively dense in T and if $G_i\cap G_j=e$ $(i\neq j; i, j=1, 2, \cdots)$, then there exists $\tilde{x}\in\tilde{X}$ such that T is recurrent at $\tilde{x}$ and each G_n $(n=1, 2, \cdots)$ is not recurrent at $\tilde{x}$.*

Proof. Let K be a compact symmetric set in T such that K contains a neighborhood of e which generates T. According to Lemmas 9 and 18 there exists a sequence $S_1, S_2, \cdots$ of replete semigroups in T such that each replete semigroup in T contains S_n for certain arbitrarily large positive integers n. Define $\tilde{x}=0$ on K. According to Lemmas 16 and 18 there exists a translate K_1 of K such that $K_1\subset S_1$ and $K_1\cap(K\cup G_1)=\emptyset$. Define $\tilde{x}$ on K_1 by translation of $\tilde{x}$ on K and define $\tilde{x}=1$ on $G_1-(K\cup K_1)$. Then $\tilde{x}$ is defined and continuous on $H_1=K\cup K_1\cup G_1$. Extend $\tilde{x}$ continuously over K^2 (Tietze's Extension Theorem). According to Lemmas 16 and 18 there exists a translate K_2 of K^2 such that $K_2\subset S_2$ and $K_2\cap(K^2\cup H_1\cup G_2)=\emptyset$. Define $\tilde{x}$ on K_2 by translation of $\tilde{x}$ on K^2 and define $\tilde{x}=1$ on $G_2-(K^2\cup K_2\cup H_1)$. Then $\tilde{x}$ is defined and continuous on $H_2=K^2\cup K_2\cup H_1\cup G_2$. Extend $\tilde{x}$ continuously over K^3. According to Lemmas 16 and 18 there exists a translate K_3 of K^3 such that $K_3\subset S_3$ and $K_3\cap(K^3\cup H_2\cup G_3)=\emptyset$. Define $\tilde{x}$ on K_3 by translation of $\tilde{x}$ on K^3 and define $\tilde{x}=1$ on $G_3-(K^3\cup K_3\cup H_2)$. Thus $\tilde{x}$ is defined and continuous on $H_3=K^3\cup K_3\cup H_2\cup G_3$. By continuing this process we define $\tilde{x}$ so that it is continuous on K^n for all positive integers n and thus $\tilde{x}$ is defined and continuous on T.

If C is any compact set in T and S is any replete semigroup in T, there exists a positive integer m such that $C\subset K^n$ provided $n\geqq m$. But there exists an integer $p\geqq m$ such that $S_p\subset S$ and thus there exists a translate C^* of C such that $C^*\subset S$ and $\tilde{x}$ is defined on C^* by translation of $\tilde{x}$ on C. It follows that if U is any neighborhood of $\tilde{x}$ in $\tilde{X}$, there exists an extensive set A in T such that $\tilde{x}A\subset U$ and thus T is recurrent at $\tilde{x}$. But by construction, $\tilde{x}=0$ at $e\in G_n$ and $\tilde{x}=1$ at all except a finite number of points of G_n. It follows that for all positive integers n, G_n is not recurrent at $\tilde{x}$. The proof of the theorem is completed.

COROLLARY 4. *It is possible for $\mathfrak{Z}^n$ $(n>1)$ to act as a transformation group on some topological space X in such a manner that $\mathfrak{Z}^n$ is recurrent at some $x\in X$ but each infinite cyclic subgroup of $\mathfrak{Z}^n$ is not recurrent at x.*

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