ON THE ORDER OF $\zeta(1/2+it)$

BY

SZU-HOA MIN[1]

Introduction. The problem of finding an upper bound for θ such that

$$\zeta(1/2 + it) = O(t^{\theta})$$

has been attacked by van der Corput and Koksma[2], Walfisz[3], Titchmarsh[4], Phillips[5], and Titchmarsh[6]. Their results obtained are, neglecting a factor involving log t,

$$\theta \leqq \frac{1}{6}, \quad \frac{163}{988}, \quad \frac{27}{164}, \quad \frac{229}{1392}, \quad \text{and} \quad \frac{19}{116}$$

respectively. The object of the present paper is to prove that

$$\zeta(1/2 + it) = O(t^{15/92+\epsilon}) \qquad (\epsilon > 0).$$

In this paper there are two main difficulties. The first is the vanishing of the Hessian $H(x, y)$ (see (6.7) below) along certain lines. This is solved by a suitable division of the domain of summation and by making use of a geometrical lemma (Lemma 10). The second difficulty is that if we use the straightforward way of choosing $\lambda' = \lambda'' = \lambda^2$ (see §9) we shall get, instead of (9.8), a result containing a negative power of a which will spoil the main idea. The fact that (9.8) contains no a indicates clearly that our method is a limiting case and we can get no more benefits by merely using more summations.

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1. **Lemmas quoted.**

LEMMA 1[7]. *Let $f(x)$ be a real function with continuous derivatives $f'(x)$,*

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(1) Scholar of the Sino-British Cultural and Educational Endowment Fund.

(2) *Sur l'ordre de grandeur de la fonction $\zeta(s)$ de Riemann dans le bande critique*, Annales de Toulouse (3) vol. 22 (1930) pp. 1–39.

(3) *Zur Abschätzung von $\zeta(1/2+it)$*, Göttingen Nachrichten (1924) pp. 155–158.

(4) *On van der Corput's method and the zeta-function of Riemann* (II), Quart. J. Math. Oxford Ser. vol. 2 (1931) pp. 313–320. This will be referred to as (II).

(5) *The zeta-function of Riemann; further developments of van der Corput's method*, Quart. J. Math. Oxford Ser. vol. 4 (1933) pp. 209–225. This will be referred to as P.

(6) *On the order of $\zeta(1/2+it)$*, Quart. J. Math. Oxford Ser. vol. 13 (1942) pp. 11–17. This will be referred to as loc. cit.

(7) (II), Theorem 4, p. 315.

$f''(x)$ and $f'''(x)$. Let $f'(x)$ be steadily decreasing, $f'(b)=\alpha$, $f'(a)=\beta$ and[8]

$$\lambda_2 \leqq |f''(x)| < A\lambda_2, \qquad |f'''(x)| < A\lambda_3$$

for $a \leqq x < b$. Let n_ν be such that

$$f'(n_\nu) = \nu \qquad (\alpha \leqq \nu \leqq \beta).$$

Then

$$\sum_{a\leqq n\leqq b} e^{2\pi i f(n)} = e^{-\pi i/4} \sum_{\alpha\leqq\nu\leqq\beta} \frac{e^{2\pi i[f(n_\nu)-\nu n_\nu]}}{|f''(n_\nu)|^{1/2}} + O(\lambda_2^{-1/2})$$
$$+ O[\log\{2 + (b-a)\lambda_\alpha\}] + O[(b-a)\lambda_2^{1/5}\lambda_3^{1/5}].$$

LEMMA 2[9]. *If $F(n)$ is a real function, ρ, a and b are integers and $0<\rho<b-a$, then*

$$\left|\sum_{n=a}^{b} e^{2\pi i F(n)}\right| \leqq \frac{1}{\rho}\left\{4(b-a)^2\rho + 2(b-a)\left|\sum_{r=1}^{\rho-1}(\rho - r)\sum_{m=a}^{b-r} e^{2\pi i\Phi(r,m)}\right|\right\}^{1/2}$$

where

$$\Phi(r, m) = F(m+r) - F(m) = \int_0^1 \frac{\partial}{\partial t}F(m+rt)dt.$$

LEMMA 3. *Let $a_{\mu\nu}$ be any numbers, real or complex, such that if $S_{m,n} = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} a_{\mu\nu}$ then $|S_{m,n}| \leqq G$ $(1\leqq m\leqq M;\ 1\leqq n\leqq N)$. Let $b_{m,n}$ denote real numbers, $0\leqq b_{m,n}\leqq H$ and let each of the expressions*

$$b_{m,n} - b_{m,n+1}, \qquad b_{m,n} - b_{m+1,n}, \qquad b_{m,n} - b_{m+1,n} - b_{m,n+1} + b_{m+1,n+1}$$

be of constant sign for values of m and n in question. Then

$$\left|\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n}b_{m,n}\right| \leqq 5GH.$$

LEMMA 4. *Let $f(x, y)$ be a real function of x and y, and*

$$S = \sum\sum e^{2\pi i f(m,n)},$$

the sum being taken over the lattice points of a region D included in the rectangle $a\leqq x\leqq b$, $\alpha\leqq y\leqq\beta$. Let

$$S' = \sum\sum e^{2\pi i\phi_1(m,n)}, \qquad S'' = \sum\sum e^{2\pi i\phi_2(m,n)}$$

where

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(8) Throughout this paper we use A to denote a positive constant, not necessarily the same at each occurrence.

(9) Titchmarsh, *On van der Corput's method and the zeta-function of Riemann*, Quart. J. Math. Oxford Ser. vol. 2 (1931) p. 166.

$$\phi_1(m, n) = f(m + \mu, n + \nu) - f(m, n) = \int_0^1 \frac{\partial}{\partial t} f(m + \mu t, n + \nu t) dt,$$

$$\phi_2(m, n) = f(m + \mu, n - \nu) - f(m, n) = \int_0^1 \frac{\partial}{\partial t} f(m + \mu t, n - \nu t) dt,$$

μ and ν are integers, and S' is taken over values of m and n such that both (m, n) and $(m+\mu, n+\nu)$ belong to D; and similarly for S''. Let ρ be a positive integer not greater than $b-a$, and let ρ' be a positive integer not greater than $\beta-\alpha$. Then

$$S = O\left\{\frac{(b-a)(\beta-\alpha)}{(\rho\rho')^{1/2}}\right\} + O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'} \sum_{\mu=1}^{\rho-1} \sum_{\nu=0}^{\rho'-1} |S'|\right\}^{1/2}\right]$$
$$+ O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'} \sum_{\mu=0}^{\rho-1} \sum_{\nu=0}^{\rho'-1} |S''|\right\}^{1/2}\right].$$

This lemma (as well as the next lemma) evidently remains true when ρ is not an integer but greater than 1. In that case $\sum_{\mu=1}^{\rho-1} \phi(\mu)$ is to be interpreted as $\sum_{1 \leqq \mu \leqq \rho-1} \phi(\mu)$, and so on. A similar interpretation should be made when ρ' is not an integer but greater than 1.

LEMMA 5. *If $0 < \rho \leqq b-a$, then*

$$S = O\left\{\frac{(b-a)(\beta-\alpha)}{\rho^{1/2}}\right\} + O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho} \sum_{\mu=1}^{\rho-1} |S'''|\right\}^{1/2}\right]$$

where

$$S''' = \sum\sum e^{2\pi i \phi(m,n)}$$

with

$$\phi(m, n) = f(m + \mu, n) - f(m, n) = \int_0^1 \frac{\partial}{\partial t} f(m + \mu t, n) dt$$

the sum being taken over values of (m, n) such that (m, n) and $(m+\mu, n)$ belong to D.

LEMMA 6. *Let $f(x, y)$ be a real differentiable function of x and y. Let $f_x(x, y)$ be a monotone function of x for each value of y considered, and $f_y(x, y)$ be a monotone function of y for each value of x considered. Let $|f_x| \leqq 3/4$, $|f_y| \leqq 3/4$, for $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$ where $b-a \leqq l$, $\beta-\alpha \leqq l$ $(l \geqq 1)$. Let D be the rectangle $(a, b; \alpha, \beta)$ or part of the rectangle cut off by a continuous monotone curve. Then*

$$\sum_D\sum e^{2\pi i f(m,n)} = \iint_D e^{2\pi i f(x,y)} dx dy + O(l).$$

Lemmas 3, 4, 5, 6 are either quotations or simple modifications of Lemmas α, β, γ, δ of a paper by Titchmarsh.

2. **Lemmas concerning double exponential integrals.** In this section we give a refinement of a theorem due to Titchmarsh(10).

LEMMA 7. *Let D be the rectangle $(a, b; \alpha, \beta)$ and U be its longer side. Let $f(x, y)$ be a real algebraic function satisfying the following conditions in D*(11).

$$B \leqq |f_{xx}| < AB, \qquad r^2B^{-1} \leqq |f_{yy}| < AB, \qquad |f_{xy}| < AB, \tag{1}$$

$$|f_{xx}f_{yy} - f_{xy}^2| > r^2, \qquad 0 < r \leqq B, \tag{2}$$

$$|f_{xxx}| < AC, \qquad C < AB^{3/2}, \qquad CUr < AB^2, \tag{3}$$

$$|f_{xx}^2 f_{xyy} - 2f_{xx}f_{xy}f_{xxy} + f_{xy}^2 f_{xxx}| < C_1B^2, \qquad B^{1/2}C_1 \leqq r^2/2, \tag{4}$$

and, for a positive integer k,

$$B^{k-1}C_1^{2k-1}U = O(r^{4k-3}) \quad or \quad B^{1/2-1/2k}C_1^{1-1/2k}U^{1/2k} = O(r^{2-3/2k}). \tag{5}$$

Then

$$\int_a^b dx \int_\alpha^\beta e^{2\pi i f(x,y)} dy = O\left(\frac{1}{r}\right).$$

Proof. We divide D into three regions, namely

D_1: $\qquad f_x \geqq B^{1/2},$

D_2: $\qquad 0 \leqq f_x < B^{1/2},$

D_3: $\qquad f_x < 0.$

Sometimes we want to redivide D_1 into subregions. We denote by D_{11} the part of D_1 lying between the curves

$$f_y - f_x \frac{f_{xy}}{f_{xx}} = \pm \frac{r}{B^{1/2}},$$

and by D_{12} the remainder of D_1. Similarly we may divide D_2 into D_{21} and D_{22}.

(1) Consider, first, D_1. Integration by parts gives

$$\iint_{D_1} e^{2\pi i f(x,y)} dx dy = \int \left[\frac{e^{2\pi i f(x,y)}}{2\pi i f_x}\right]_{\chi(y)}^{\omega(y)} dy + \frac{1}{2\pi i}\iint_{D_1} \frac{f_{xx}}{f_x^2} e^{2\pi i f(x,y)} dx dy = I_1 + I_2, \tag{2.1}$$

say, where $x = \omega(y)$ and $x = \chi(y)$ are boundaries of D_1.

(1.1). To estimate I_1, we consider, for example,

(10) Proc. London Math. Soc. (2) vol. 38 (1935) pp. 96–115.

(11) The letters B, r, C and C_1 are used to denote positive constants.

$$\int \frac{e^{2\pi i f(\chi(y),y)}}{2\pi i f_x(\chi(y),y)}\,dy.$$

The function $\chi(y)$ is either the solution of $f_x=B^{1/2}$ or it is a constant.

In the former case we have

$$\frac{d}{dy}f(\chi(y),y)=f_y-f_x\frac{f_{xy}}{f_{xx}}=v,$$

say. Hence

$$\int_{|v|\geqq r/B^{1/2}} e^{2\pi i f(\chi(y),y)}dy=\int_{|v|\geqq r/B^{1/2}}\frac{e^{2\pi i u}du}{v}=O\left(\frac{B^{1/2}}{r}\right)$$

and

$$\int_{|v|\geqq r/B^{1/2}}\frac{e^{2\pi i f(\chi(y),y)}dy}{2\pi i f_x(\chi(y),y)}=O\left(\frac{1}{B^{1/2}}\frac{B^{1/2}}{r}\right)=O\left(\frac{1}{r}\right).$$

On the other hand,

$$\int_{|v|<r/B^{1/2}}\frac{e^{2\pi i f(\chi(y),y)}}{2\pi i f_x(\chi(y),y)}\,dy=O\left(\frac{1}{B^{1/2}}\int_{-r/B^{1/2}}^{r/B^{1/2}}\left|\frac{dy}{dv}\right|dv\right).$$

Here $f_x=B^{1/2}$, so

$$\frac{dv}{dy}=\frac{1}{f_{xx}}\left[f_{xx}f_{yy}-f_{xy}^2-f_x\frac{f_{xx}^2f_{xyy}-2f_{xx}f_{xy}f_{xxy}+f_{xy}^2f_{xxx}}{f_{xx}^2}\right]$$

and, by (1), (2) and (4)

$$\left|\frac{dv}{dy}\right|>A\frac{r^2-B^{1/2}\cdot C_1}{B}>A\frac{r^2}{B}\cdot \tag{2.2}$$

Hence

$$\int_{|v|<r/B^{1/2}}\frac{e^{2\pi i f(\chi(y),y)}}{2\pi i f_x(\chi(y),y)}\,dy=O\left(\frac{1}{B^{1/2}}\frac{r}{B^{1/2}}\frac{B}{r^2}\right)=O\left(\frac{1}{r}\right).$$

Secondly, if $\chi(y)=a$, a constant,

$$\int\frac{e^{2\pi i f(\chi(y),y)}}{2\pi i f_x(\chi(y),y)}\,dy=\int\frac{e^{2\pi i f(a,y)}}{2\pi i f_x(\chi(y),y)}\,dy.$$

By (1) and a well known formula concerning exponential integrals ([12]), this is

([12]) If $f(x)$ is a real differentiable function with $|f''(x)|>\lambda$ in (c,d) then $\int_c^d e^{2\pi i f(x)}\,dx=O(1/\lambda^{1/2})$.

$$O\left(\frac{1}{B^{1/2}}\cdot\frac{1}{(r^2B^{-1})^{1/2}}\right) = O\left(\frac{1}{r}\right).$$

(1.2). Now consider I_2. We have

$$(2\pi i)^2 I_2 = \int \left[\frac{f_{xx}}{f_x^3} e^{2\pi i f(x,y)}\right]_{\chi(y)}^{\omega(y)} dy - \int dy \int \frac{f_{xxx}}{f_x^3} e^{2\pi i f(x,y)} dx$$
$$+ 3\int\int \frac{f_{xx}^2}{f_x^4} e^{2\pi i f(x,y)} dxdy = I_1' + I_2' + I_3',$$

say. The first integral can be treated as I_1. So

$$I_1' = O(1/r).$$

We have

$$I_2' = \int\int_{D_{12}} \frac{f_{xxx}}{f_x^3} e^{2\pi i f(x,y)} dxdy + \int\int_{D_{11}} \frac{f_{xxx}}{f_x^3} e^{2\pi i f(x,y)} dxdy$$
$$= I_{22}' + I_{21}',$$

say. Let $x=\phi(y)=\phi(y, u)$ be the solution of $f_x=u$. Then

$$\frac{\partial}{\partial y} f(\phi(y), y) = f_y - f_x \frac{f_{xy}}{f_{xx}}.$$

In D_{12}, the absolute value of this expression is not less than $r/B^{1/2}$. Hence

$$I_{22}' = \int dy \int \frac{f_{xxx}}{f_x^3} e^{2\pi i f(x,y)} dx = \int dy \int \frac{f_{xxx}}{f_{xx}} e^{2\pi i f(x,y)} \frac{du}{u^3}$$
$$= \int \frac{du}{u^3} \int \frac{f_{xxx}}{f_{xx}} e^{2\pi i f(\phi(y,u),y)} dy = \int_{B^{1/2}} \frac{C}{B} O\left(\frac{B^{1/2}}{r}\right) \frac{du}{u^3}$$
$$= O\left(\frac{C}{rB^{3/2}}\right) = O\left(\frac{1}{r}\right),$$

by (3).

To estimate I_{21}', we put $u=f_x$ and $v=f_y-f_xf_{xy}/f_{xx}$. Then

$$\frac{\partial(u, v)}{\partial(x, y)} = f_{xx}f_{yy} - f_{xy}^2 - \frac{f_x}{f_{xx}^2}(f_{xx}^2 f_{xyy} - 2f_{xx}f_{xy}f_{xxy} + f_{xy}^2 f_{xxx}). \tag{2.3}$$

The absolute value of this expression is greater than Ar^2 if

$$|f_x| < r^2 C_1^{-1}/2.$$

Denote by D_{11}' the part of D_{11} in which the inequality holds and by D_{11}'' the remainder of D_{11}. Then

$$\left|\iint_{D_{11}'} \frac{f_{xxx}}{f_x^3} e^{2\pi i f(x,y)} dxdy\right|$$
$$< AC\iint \frac{dxdy}{f_x^3} = AC\int_{B^{1/2}} \frac{du}{u^3}\int^{r/B^{1/2}} \frac{dv}{|\partial(u,v)/\partial(x,y)|}$$
$$\leqq AC\frac{1}{(B^{1/2})^2}\cdot\frac{r}{B^{1/2}}\cdot\frac{1}{r^2} = \frac{AC}{B^{3/2}}\cdot\frac{1}{r} = O\left(\frac{1}{r}\right),$$

by (3). Also

$$\left|\iint_{D_{11}''} \frac{f_{xxx}}{f_x^3} e^{2\pi i f(x,y)} dxdy\right| < AC\iint \frac{dxdy}{f_x^3} = AC\int dy\int \frac{du}{|f_{xx}|\, u^3}$$
$$= O\left(\frac{C}{B}\cdot\frac{U}{(r^2C_1^{-1})^2}\right) = O\left(\frac{1}{r}\cdot\frac{CUr}{B^2}\cdot\frac{BC_1^2}{r^4}\right) = O\left(\frac{1}{r}\right),$$

by (3) and (4). Hence I_{21}' is also $O(1/r)$. Thus $I_2' = O(1/r)$. It follows that

$$(2\pi i)^2 I_2 = O\left(\frac{1}{r}\right) + 3\iint_{D_1} \frac{f_{xx}^2}{f_x^4} e^{2\pi i f(x,y)} dxdy.$$

Repeating this argument we find

$$I_2 = O\left(\frac{1}{r}\right) + O\left(\iint_{D_1} \frac{f_{xx}^k}{f_x^{2k}} e^{2\pi i f(x,y)} dxdy\right).$$

Denote the last double integral by J, then

$$J = \iint_{D_{12}} + \iint_{D_{11}} = J_2 + J_1,$$

say. We have, as before[13],

$$J_2 = \int_{B^{1/2}} \frac{du}{u^{2k}}\int f_{xx}^{k-1} e^{2\pi i f(x,y)}\, dy = \int_{B^{1/2}} O\left(\frac{B^{1/2}}{r}\right)\cdot B^{k-1}\frac{du}{u^{2k}}$$
$$= O\left(\frac{1}{r}\,\frac{B^{k-1}B^{1/2}}{B^{(2k-1)/2}}\right) = O\left(\frac{1}{r}\right).$$

To estimate J_1, we write

$$J_1 = \iint_{D_{11}'} + \iint_{D_{11}''} = J_1' + J_1''.$$

As before [14], by (1) and (2.3),

[13] See the estimation of I_{22}' above (2.3).

[14] See the estimation of I_{21}'.

$$| J_1' | < AB^k \int_{B^{1/2}} \frac{du}{u^{2k}} \int^{r/B^{1/2}} \frac{dv}{|\partial(u, v)/\partial(x, y)|}$$

$$= O\left(B^k \frac{1}{B^{(2k-1)/2}} \cdot \frac{r}{B^{1/2}} \cdot \frac{1}{r^2}\right) = O\left(\frac{1}{r}\right),$$

$$| J_1'' | < AB^{k-1} \iint_{D_{11}''} \frac{|f_{xx}|\, dxdy}{f_x^{2k}} < AB^{k-1} \iint \frac{dudy}{u^{2k}}$$

$$= O\left(\frac{B^{k-1} U}{(r^2C^{-1})^{2k-1}}\right) = O\left(\frac{1}{r} \frac{B^{k-1} C_1^{2k-1} U}{r^{4k-3}}\right) = O\left(\frac{1}{r}\right),$$

by (5). Combining these results we find that J is $O(1/r)$. Hence I_2 is $O(1/r)$.

(2). Now consider the integral over D_2. Putting $f_x = u$, we have

$$\iint_{D_{22}} e^{2\pi i f(x,y)} dxdy = \int_0^{B^{1/2}} du \int \frac{e^{2\pi i f(x,y)}}{f_{xx}} dy.$$

As in (1.2), we have $\partial f(\phi(y, u), y)/\partial y \geqq r/B^{1/2}$. Hence the inner integral is $O((1/B)\cdot(B^{1/2}/r)) = O(1/rB^{1/2})$. The result follows for this part.

Finally, by (2) and (4) we have, using (3) and the fact that $|f_x| < B^{1/2}$,

$$\left|\iint_{D_{21}} e^{2\pi i f(x,y)} dxdy\right| \leqq \iint_{D_{21}} dxdy = \int_0^{B^{1/2}} \int_0^{r/B^{1/2}} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

$$= \int_0^{B^{1/2}} \int_0^{r/B^{1/2}} O\left(\frac{1}{r^2 - (B^{1/2}/B^2)\cdot C_1 B^2}\right) dudv$$

$$= O\left(B^{1/2} \cdot \frac{r}{B^{1/2}} \cdot \frac{1}{r^2}\right) = O\left(\frac{1}{r}\right).$$

(3). We have established the stated for $D_1 + D_2$, that is, the region $f_x \geqq 0$. A similar proof can be applied to D_3.

LEMMA 8. *Let D' be the part of D cut off by a curve (or several curves) whose equation is of the form $x = g(y)$ where $g(y)$ is an algebraic function satisfying*

(6) $$| U f_{xx} g''(y) | < Kr$$

where K is a sufficiently small constant. Then if we replace the condition (1) *in Lemma* 7 *by*

(1′) $$B \leqq |f_{xx}| < AB, \quad |f_{xy}| < AB, \quad |f_{yy}| < AB, \quad f_{xx} f_{xy} > 0$$

we have

$$\iint_{D'} e^{2\pi i f(x,y)} dxdy = O\left(\frac{1 + |\log B| + |\log U|}{r}\right).$$

In particular, the curve may be a straight line $x=py+q$.

Proof. If $|f_{yy}| \geqq B/2$, the condition (1) holds if we replace B by $B/2$. Now suppose that $|f_{yy}| < B/2$. We put $x=\xi+\eta$, $y=\eta$. Then

$$\left|\frac{\partial^2}{\partial\xi^2} f(x, y)\right| = |f_{xx}| \geqq B,$$

$$\left|\frac{\partial^2}{\partial\eta^2} f(x, y)\right| = |f_{xx} + 2f_{xy} + f_{yy}| > \frac{B}{2}\cdot$$

Thus the condition (1) is restored. Conditions (2) and (3) remain true. So do (4) and (5) since the expression on the left-hand side of (4) is an invariant under our transformation. We may therefore assume that all these conditions are satisfied. We need only to consider integrals of the form

$$\int \frac{e^{2\pi i f(g(y), y)}}{f_x(g(y), y)} dy \qquad (|f_x| > B^{1/2}).$$

We divide the interval of integration into three parts:

$$(1) \qquad |f_x g'(y) + f_y| \geqq r/B^{1/2},$$

$$(2) \qquad |f_x g'(y) + f_y| < r/B^{1/2}, \qquad |f_{xx} g'(y) + f_{xy}| \geqq r/2,$$

$$(3) \qquad |f_x g'(y) + f_y| < r/B^{1/2}, \qquad |f_{xx} g'(y) + f_{xy}| < r/2.$$

In the first part,

$$\int \frac{e^{2\pi i f(g(y), y)}}{f_x(g(y), y)} dy = \int \frac{e^{2\pi i \xi} d\xi}{f_x(f_x g' + f_y)} = O\left(\frac{1}{r}\right).$$

In the second part,

$$\left|\int \frac{e^{2\pi i f(g(y), y)}}{f_x(g(y), y)} dy\right| \leqq \left|\int \frac{dy}{f_x}\right| \leqq \frac{2}{r}\left|\int \frac{f_{xx} g'(y) + f_{xy}}{f_x(g(y), y)} dy\right|$$

$$= \frac{2}{r}\left| [\log f_x(g(y), y)] \right| = O\left(\frac{1 + |\log B| + |\log U|}{r}\right).$$

In the third part, we put $u = f_x g' + f_y$, then

$$\frac{du}{dy} = f_{xx} g'^2 + 2 f_{xy} g' + f_{yy} + f_x g''$$

$$= f_{xx}^{-1}[(g' f_{xx} + f_{xy})^2 + f_{xx} f_{yy} - f_{xy}^2 + f_{xx} f_x g''].$$

If $|f_x| > Ur$, the theorem is true. If otherwise,

$$\left|\frac{du}{dy}\right| > A\frac{r^2}{B}\cdot$$

Hence

$$\int \frac{e^{2\pi i f(g(y),y)}}{f_x(g(y),\,y)}\,dy = O\left(\frac{1}{B^{1/2}}\int^{r/B^{1/2}}\left|\frac{dy}{du}\right|du\right) = O\left(\frac{r}{(B^{1/2})^2}\cdot\frac{B}{r^2}\right) = O\left(\frac{1}{r}\right).$$

The lemma follows.

3. **Lemmas concerning network.** Suppose there is a network of which each cell is a rectangle S_0 of area U and with sides of lengths l and m. Suppose S is a rectangle with sides parallel to lines in the network and of lengths a and b respectively. Suppose L_1 and L_2 are parallel lines which bound with side of S a strip of area A. Let L be either of them and let the area of S under L be A_L.

LEMMA 9. *The number of rectangles S_0 lying partially or entirely within S and entirely under L is*

$$N_L = \frac{A_L}{U} + O\left(\frac{a}{l}+\frac{b}{m}+1\right).$$

The number of rectangles S_0 lying partially or entirely within S and partially or entirely under L is

$$N_L' = \frac{A_L}{U} + O\left(\frac{a}{l}+\frac{b}{m}+1\right).$$

Proof. (1) Without loss of generality, we may assume that the sides of S coincide with lines belonging to the network. For otherwise we may replace S by one with this kind of sides so that the variations of A_L and N_L are respectively

$$O\left[\left(\frac{a}{l}+\frac{b}{m}+1\right)U\right] \quad \text{and} \quad O\left[\frac{a}{l}+\frac{b}{m}+1\right].$$

Without loss of generality we may assume that S_0 is a unit square so that $U=l=m=1$. For, only the ratios of areas and lengths really matter. Without loss of generality we may also assume that L is of positive slope.

Now consider all the vertical lines of the network which are not entirely outside S. Let the line nearest the left-hand side of S be l_1 and the next l_2, and so on. Let the first of them which meets L inside S be l_k. Let the points of intersection of L with $l_k, l_{k+1} \cdots$ be $P_k, P_{k+1}, \cdots$.

We draw from P_i $(i=k, k+1, \cdots)$ a horizontal line toward the right until it reaches l_{i+1}. We denote the part of A_L which is below these horizontal line-segments by A_L^B and the remaining part by A_L^A. Then the first part of the lemma follows from the fact that $A_L^A=O(b)$, $0\leqq A_L^B-N_L=O(a)$.

(2) The second part of the lemma can be proved by drawing horizontal lines toward the left instead of the right.

LEMMA 10. *The number of rectangles S_0 lying partially or entirely between L_1 and L_2 and partially or entirely within S is*

$$N = \frac{A}{U} + O\left(\frac{a}{l} + \frac{b}{m} + 1\right).$$

Proof. We have $N = N_{L_1} - N'_{L_2}$ and the lemma follows from Lemma 9.

4. We have to consider sums of the form

$$S_1 = \sum_{n=a}^{b} n^{-it} = \sum_{n=a}^{b} e^{-it\log n}, \qquad a < b \leqq 2a. \tag{4.1}$$

By Lemma 2,

$$|S_1| \leqq \frac{1}{\rho}\left\{4(b-a)^2\rho + 2(b-a)\left|\sum_{r=1}^{\rho-1}(\rho - r)\sum_{m=a}^{b-r} e^{-it\log(m+r)/m}\right|\right\}^{1/2} \tag{4.2}$$

provided that

$$0 < \rho < b - a. \tag{C$_1$}$$

Let

$$S_2 = \sum_{r=1}^{\rho-1}(\rho - r)\sum_{m=a}^{b-r} e^{-it\log(m+r)/m}; \tag{4.3}$$

then, by Lemma 1,

$$\begin{aligned} S_2 = e^{-\pi i/4}\sum_{r=1}^{\rho-1}(\rho - r)\sum_{\alpha\leqq\nu\leqq\beta}\frac{e^{2\pi i\phi(r,\nu)}}{|f''(m_\nu)|^{1/2}} + O(a^{3/2}t^{-1/2}\rho^{3/2}) \\ + O(\rho^2\log t) + O(a^{-2/5}t^{2/5}\rho^{12/5}) \end{aligned} \tag{4.4}$$

where

$$\begin{gathered} f(y) = f(r, y) = -\frac{t}{2\pi}\log\frac{y+r}{y}, \\ f'(m_\nu) = \nu, \qquad \phi(\nu) = f(m_\nu) - \nu m_\nu, \\ \alpha = f'(b-r), \qquad \beta = f'(a), \qquad b \leqq 2a. \end{gathered} \tag{4.5}$$

Let

$$b = O(t^{1/2}); \tag{C$_2$}$$

then

$$\nu = f'(m_\nu) = \frac{tr}{2\pi m_\nu(m_\nu + r)} > \frac{Atr}{m_\nu^2} > Ar$$

and

$$(4.6)\qquad \rho = O(\beta).$$

Let

$$S_3 = \sum_{x=R+1}^{R'} \sum_{y=N+1}^{N'} e^{2\pi i\phi(x,y)}, \qquad R < R' \leqq 2R < \rho, \qquad N < N' \leqq 2N \leqq \beta.$$

Applying Lemma 5 twice and Lemma 4 once, we have

$$S_3 = O\left(\frac{RN}{\lambda^{1/2}}\right) + O\left(\frac{(RN)^{7/8}}{\lambda^{3/2}}\left\{\sum_{y_1=1}^{\lambda-1}\left[\sum_{y_2=1}^{\lambda^2-1}\left(\sum_{x_3=1}^{\lambda'^2-1}\sum_{y_3=0}^{\lambda''^2-1}|S_4|\right)^{1/2}\right]^{1/2}\right\}^{1/2}\right)$$

$$(4.7)\qquad + O\left(\frac{(RN)^{7/8}}{\lambda^{3/2}}\left\{\sum_{y_1=1}^{\lambda-1}\left[\sum_{y_2=1}^{\lambda^2-1}\left(\sum_{x_3=1}^{\lambda'^2-1}\sum_{y_3=1}^{\lambda''^2-1}|S_4'|\right)^{1/2}\right]^{1/2}\right\}^{1/2}\right)$$

where

$$(4.8)\qquad S_4 = \sum_{x=R+1}^{R''}\sum_{y=N+1}^{N''} e^{2\pi i\psi(x,y)}, \quad R'' = R' - x_3, \quad N'' = N' - y_1 - y_2 - y_3$$

with

$$(4.9)\qquad \psi(x, y) = \int\int\int_0^1 \frac{\partial^3}{\partial t_1\partial t_2\partial t_3}\phi(x + x_3t_3, y + y_1t_1 + y_2t_2 + y_3t_3)dt_1dt_2dt_3$$

and S_4' is a similar sum. Here we assumed that

$$(C_3)\qquad 1 \leqq \lambda'^2 \leqq R, \qquad \lambda'^2 \leqq \lambda''^2 \leqq N, \qquad \lambda'\lambda'' = \lambda^2.$$

Since S_4' can be estimated as S_4, we consider the latter only.

5. In this section we shall reduce $\psi(x, y)$ to a convenient form. We have

$$(5.1)\qquad \psi(x, y) = y_1y_2\int\int\int_0^1 (x_3\overset{*}{\phi}_{xy^2} + y_3\overset{*}{\phi}_{y^3})dt_1dt_2dt_3$$

where

$$(5.2)\qquad \begin{aligned} \overset{*}{\phi}_{xy^2} &= \frac{\partial^3}{\partial x^*\partial y^{*2}}\phi(x^*, y^*), \qquad & \overset{*}{\phi}_{y^3} &= \frac{\partial^3}{\partial y^{*3}}\phi(x^*, y^*), \\ x^* &= x + x_3t_3, & y^* &= y + y_1t_1 + y_2t_2 + y_3t_3. \end{aligned}$$

We have

$$f_x(x, y) = -\frac{t}{2\pi}\frac{1}{x+y}, \qquad f_y(x, y) = -\frac{t}{2\pi}\left(\frac{1}{x+y} - \frac{1}{y}\right).$$

From $f_y(x, m_y(x)) = y$ we find, by choosing the proper sign,

$$(5.3)\qquad m_y(x) = -\frac{x}{2} + \frac{x}{2}\left(1 + \frac{2t}{\pi xy}\right)^{1/2}.$$

Since

$$\phi_x(x, y) = f_x(x, m_y(x)) + f_y(x, m_y(x)) \frac{\partial}{\partial x} m_y(x) - y \frac{\partial}{\partial x} m_y(x) = f_x(x, m_y(x)),$$

$$\phi_y(x, y) = f_y(x, m_y(x)) \frac{\partial}{\partial y} m_y(x) - y \frac{\partial}{\partial y} m_y(x) - m_y(x) = - m_y(x),$$

we have, by (5.3),

$$\begin{aligned} \phi_x(x, y) &= y\left(\frac{1}{2} - \left(1 + \frac{2t}{\pi xy}\right)^{1/2}\right) \\ &= \frac{y}{2} - \left(\frac{2t}{\pi}\right)^{1/2} \frac{y^{1/2}}{x^{1/2}}\left[1 + \frac{1}{2}\frac{\pi xy}{2t} - \frac{1}{8}\left(\frac{\pi xy}{2t}\right)^2 + \cdots\right], \\ \phi_y(x, y) &= x\left(\frac{1}{2} - \left(1 + \frac{2t}{\pi xy}\right)^{1/2}\right) \\ &= \frac{x}{2} - \left(\frac{2t}{\pi}\right)^{1/2} \frac{x^{1/2}}{y^{1/2}}\left[1 + \frac{1}{2}\frac{\pi xy}{2t} - \frac{1}{8}\left(\frac{\pi xy}{2t}\right)^2 + \cdots\right]. \end{aligned} \tag{5.4}$$

Differentiation gives

$$\begin{aligned} \phi_{xyy}(x, y) &= \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} x^{-1/2}y^{-3/2}\left[1 - \frac{3}{2}\frac{\pi xy}{2t} + \frac{15}{8}\left(\frac{\pi xy}{2t}\right)^2 + \cdots\right], \\ \phi_{yyy}(x, y) &= -\frac{3}{4}\left(\frac{2t}{\pi}\right)^{1/2} x^{1/2}y^{-5/2}\left[1 - \frac{1}{6}\frac{\pi xy}{2t} - \frac{1}{8}\left(\frac{\pi xy}{2t}\right)^2 + \cdots\right]. \end{aligned} \tag{5.5}$$

Hence

$$\begin{aligned} \psi(x, y) = {} & \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1y_2 \int\int\int_0^1 x^{*-1/2}y^{*-5/2}(x_3y^* - 3y_3x^*)dt_1dt_2dt_3 \\ & -\frac{1}{8}\left(\frac{2t}{\pi}\right)^{1/2} y_1y_2 \\ & \cdot\int\int\int_0^1 \frac{\pi x^*y^*}{t} x^{*-1/2}y^{*-5/2}(3x_3y^* - y_3x^*)dt_1dt_2dt_3 + \cdots. \end{aligned} \tag{5.6}$$

6. In this section we consider the Hessian of $\psi(x, y)$, that is,

$$H(x, y) = \psi_{xx}\psi_{yy} - \psi_{xy}^2.$$

We denote the first term on the right-hand side of (5.6) by $\psi^0(x, y)$ and write $\Phi(x, y) = x^{-1/2}y^{-5/2}(x_3y - 3y_3x)$. Then

$$\Phi_{xx}(x, y) = 3x^{-5/2}y^{-5/2}(x_3y + y_3x)/4,$$
$$\Phi_{xy}(x, y) = 3x^{-3/2}y^{-7/2}(x_3y + 5y_3x)/4, \tag{6.1}$$
$$\Phi_{yy}(x, y) = 3x^{-1/2}y^{-9/2}(5x_3y - 35y_3x)/4.$$

From this it is obvious that, for $R+1 \leqq x < 2R$, $N+1 \leqq y < 2N$,

$$\Phi_{xx} = O(R^{-5/2}N^{-5/2}Q), \quad \Phi_{xy} = O(R^{-3/2}N^{-7/2}Q), \quad \Phi_{yy} = O(R^{-1/2}N^{-9/2}Q) \tag{6.2}$$

where

$$Q = x_3N + (y_3 + 1)R. \tag{6.3}$$

Hence

$$\Phi_{x^4} = O(R^{-9/2}N^{-5/2}Q), \quad \Phi_{x^3y} = O(R^{-7/2}N^{-7/2}Q), \quad \Phi_{x^2y^2} = O(R^{-5/2}N^{-9/2}Q), \tag{6.4}$$

and so on.

Using the expansion

$$\begin{aligned}\Phi(x^*, y^*) = {} & \Phi(x, y) + x_3t_3\Phi_x(x, y) \\ & + (y_1t_1 + y_2t_2 + y_3t_3)\Phi_y(x, y) \\ & + 2^{-1}[x_3^2t_3^2\Phi_{xx}(x, y) + 2x_3t_3(y_1t_1 + y_2t_2 + y_3t_3)\Phi_{xy}(x, y) \\ & + (y_1t_1 + y_2t_2 + y_3t_3)^2\Phi_{yy}(x, y)] + \cdots\end{aligned}$$

we find that

$$\begin{aligned}\psi^0(x, y) = {} & \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1y_2\Bigg[\Phi(x, y) + \frac{x_3}{2}\Phi_x(x, y) + \frac{y_1 + y_2 + y_3}{2}\Phi_y(x, y) \\ & + \frac{1}{2}\left\{\frac{x_3^2}{3}\Phi_{xx}(x, y) + 2x_3\left(\frac{y_1 + y_2}{4} + \frac{y_3}{3}\right)\Phi_{xy}(x, y)\right. \\ & \left. + \left(\frac{y_1^2 + y_2^2 + y_3^2}{3} + \frac{y_1y_2 + y_2y_3 + y_3y_1}{2}\right)\Phi_{yy}(x, y)\right\} + \cdots\Bigg] \\ = {} & \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1y_2\Bigg[\Phi(x', y') + \frac{1}{2}\left\{\frac{x_3^2}{12}\Phi_{xx}(x, y)\right. \\ & \left. + \frac{x_3y_3}{6}\Phi_{xy}(x, y) + \frac{y_1^2 + y_2^2 + y_3^2}{12}\Phi_{yy}(x, y)\right\} + \cdots\Bigg]\end{aligned}$$

where $x' = x + x_3/2$, $y' = y + (y_1 + y_2 + y_3)/2$.

Hence, by (6.4) and (C_3),

$$\overset{0}{\psi}_{xx} = \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1 y_2\left[\Phi_{xx}(x', y') + O\left(R^{-5/2}N^{-5/2}Q\left(\frac{Q_0}{RN}\right)^2\right)\right],$$

$$\overset{0}{\psi}_{xy} = \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1 y_2\left[\Phi_{xy}(x', y') + O\left(R^{-3/2}N^{-7/2}Q\left(\frac{Q_0}{RN}\right)^2\right)\right],$$

$$\overset{0}{\psi}_{yy} = \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1 y_2\left[\Phi_{yy}(x', y') + O\left(R^{-1/2}N^{-9/2}Q\left(\frac{Q_0}{RN}\right)^2\right)\right],$$

since $\lambda'^2/R+\lambda''^2/N=O(Q_0/RN)$ where $Q_0=\lambda'^2N+\lambda''^2R$.

Hence, by (5.6),

$$\psi_{xx} = \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1 y_2\Phi_{xx}(x', y') + O\left(t^{1/2}y_1y_2R^{-5/2}N^{-5/2}Q\left(\frac{Q_0}{RN}\right)^2\right) + O(t^{-1/2}y_1y_2R^{-3/2}N^{-3/2}Q),$$

(6.5) $$\psi_{xy} = \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1 y_2\Phi_{xy}(x', y') + O\left(t^{1/2}y_1y_2R^{-3/2}N^{-7/2}Q\left(\frac{Q_0}{RN}\right)^2\right) + O(t^{-1/2}y_1y_2R^{-1/2}N^{-5/2}Q),$$

$$\psi_{yy} = \frac{1}{4}\left(\frac{2t}{\pi}\right)^{1/2} y_1 y_2\Phi_{yy}(x', y') + O\left(t^{1/2}y_1y_2R^{-1/2}N^{-9/2}Q\left(\frac{Q_0}{RN}\right)^2\right) + O(t^{-1/2}y_1y_2R^{1/2}N^{-7/2}Q).$$

We may omit the second error term from each of these relations provided that

(C_4) $$R^3N = O(t).$$

Hence, by (6.2) and (C_3),

(6.6) $$\psi_{xx} = O(t^{1/2}y_1y_2R^{-5/2}N^{-5/2}Q), \qquad \psi_{xy} = O(t^{1/2}y_1y_2R^{-3/2}N^{-7/2}Q),$$
$$\psi_{yy} = O(t^{1/2}y_1y_2R^{-1/2}N^{-9/2}Q).$$

Further, by (6.1),

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 = \frac{9t}{32\pi} y_1^2y_2^2x'^{-3}y'^{-7}(x_3^2y'^2 - 10x_3y_3x'y' - 15y_3^2x'^2) + O\left(ty_1^2y_2^2R^{-3}N^{-7}Q^2\left(\frac{Q_0}{RN}\right)^2\right)$$

or

(6.7) $$H(x, y) = (9t/32\pi)\, y_1^2y_2^2x'^{-3}y'^{-7}[x_3y' + (2(10)^{1/2} - 5)x_3y'] \cdot [x_3y' - (2(10)^{1/2} + 5)y_3x'] + O\left(ty_1^2y_2^2R^{-3}N^{-7}Q^2\left(\frac{Q_0}{RN}\right)^2\right).$$

REMARKS. The inequalities (6.5) to (6.7) obviously remain true if we replace $\psi(x, y)$ by a partial sum containing only the first n ($\geqq 1$) terms on the right-hand side of (5.6). Further, the general term is of the form

$$t^{1/2}y_1y_2\int\int\int_0^1\left(\frac{x^*y^*}{t}\right)^n x^{*-1/2}y^{*-5/2}(c_1x_3y^* + c_2y_3x^*)dt_1dt_2dt_3$$

$$= t^{1/2}y_1y_2\int\int\int_0^1\left(\frac{xy}{t}\right)^n x^{-1/2}y^{-5/2}\left(1+\frac{x_3t}{x}\right)^{n-1/2}$$

$$\times\left(1+\frac{y_1t_1+y_2t_2+y_3t_3}{y}\right)^{n-5/2}$$

$$\times\left[c_1x_3y\left(1+\frac{y_1t_1+y_2t_2+y_3t_3}{y}\right)+c_2y_3x\left(1+\frac{x_3t_3}{x}\right)\right]dt_1dt_2dt_3$$

$$= t^{1/2}y_1y_2x^{1/2}y^{-3/2}\left[P(x^{-1}, y^{-1})+O\left\{\left(\frac{x_3}{x}\right)^h\right\}\right.$$

$$\left.+O\left\{\left(\frac{y_1+y_2+y_3}{y}\right)^h\right\}\right]$$

where $P(x^{-1}, y^{-1})$ is a polynomial in x^{-1}, y^{-1} (depending on y_1, y_2, y_3, x_3 and h). Now suppose that

$$(C_5)\qquad \lambda'^2 < Rt^{-\epsilon},\qquad \lambda''^2 < Nt^{-\epsilon}\qquad (\epsilon > 0)\text{(15)}.$$

When h is large enough, the inequalities (6.5) to (6.7) remain true if we neglect the terms which are

$$O\left[\left(\frac{x_3}{x}\right)^h\right]+O\left[\left(\frac{y_1+y_2+y_3}{y}\right)^h\right]$$

from each term on the right-hand side of (5.6). So we can write $\psi(x,y) = \psi_1(x, y)+\psi_2(x, y)$ where $\psi_1(x, y)$ is an algebraic function satisfying (6.5) to (6.7) and

$$\psi_2(x, y) = t^{1/2}y_1y_2x^{1/2}y^{-3/2}\left[O\left\{\left(\frac{x_3}{x}\right)^h\right\}+O\left\{\left(\frac{y_1+y_2+y_3}{y}\right)^h\right\}\right]$$

which can be made as small as we please by taking h sufficiently large. In fact, we can choose h so that, for a given positive δ, $\psi_2(x, y) = O(t^{-\epsilon})$.

7. Now return to the sum S_4. Let

$$(7.1)\qquad l_1 = c\frac{R^{5/2}N^{5/2}}{t^{1/2}y_1y_2Q},\qquad l_2 = c\frac{R^{3/2}N^{7/2}}{t^{1/2}y_1y_2Q},$$

(15) We use ϵ to denote a small positive number, which, like the symbol A, may or may not keep the same value.

where c is some positive constant.

By (4.6) we have

$$R = O(N), \qquad l_1 = O(l_2). \tag{7.2}$$

We divide the region of summation of S_4, that is,

$$R+1 \leqq x \leqq R'', \qquad N+1 \leqq y \leqq N''$$

into rectangles with sides parallel to the axes and of lengths l_1 and l_2 and parts of such rectangles. We may enumerate these subregions and denote them by Δ_p, $p=1, 2, \cdots$. If c is small enough, the variations of ψ_x and ψ_y in each Δ_p will be less than 1/2. Hence to each Δ_p correspond integers μ and ν such that if $\psi_p(x, y)=\psi(x, y)-\mu x-\nu y$ the absolute value of the first derivatives of ψ_p is not greater than 3/4. So for each Δ_p we have, by Lemma 6,

$$\sum_{\Delta_p}\sum e^{2\pi i\psi(x,y)} = \sum_{\Delta_p}\sum e^{2\pi i\psi_p(x,y)} = \iint_{\Delta_p} e^{2\pi i\psi_p(x,y)}dxdy + O(l_2) \tag{7.3}$$

provided that

$$l_2 \geqq 1. \tag{C$_6$}$$

Hence

$$S_4 = \sum_p\left\{\iint_{\Delta_p} e^{2\pi i\psi_p(x,y)}dxdy + O(l_2)\right\}.$$

The system of parallel lines

$$\left| x_3y - (2(10)^{1/2}-5)y_3x\right| = 4^m\xi, \qquad m = 0, 1, \cdots,$$

divides each Δ_p into strips. Hence

$$\begin{aligned} S_4 &= \sum_p\iint_{\Delta_p'} e^{2\pi i\psi_p(x,y)}dxdy + \sum_p\sum_{m=0}^{L-1}\iint_{\Delta_{p,m}} e^{2\pi i\psi_p(x,y)}dxdy \\ &\quad + \sum_p\iint_{\Delta_p''} e^{2\pi i\psi_p(x,y)}dxdy + \sum_p O(l_2) = J_0 + J_1 + J_2 + J_3, \end{aligned} \tag{7.4}$$

say, where

$$L = \left[\frac{\log(\xi^{-1}Q)}{\log 4}\right].$$

$\Delta_{p,m}$ denotes the part of Δ_p for which

$$4^m\xi < \left| x_3y - (2(10)^{1/2}-5)y_3x\right| < 4^{m+1}\xi \qquad (m = 0, 1, \cdots, L-1), \tag{7.5}$$

Δ_p' denotes the part for which

$$|x_3y-(2(10)^{1/2}-5)y_3x|<\xi \tag{7.6}$$

and Δ_p'' denotes the part for which

$$|x_3y-(2(10)^{1/2}-5)y_3x|>4^L\xi(>AQ). \tag{7.7}$$

Evidently

$$|J_0|\leqq\sum_p\iint_{\Delta'}dxdy=\underset{|x_3y-(2(10)^{1/2}-5)y_3x|<\xi}{\int_{R+1}^{R''}\int_{N+1}^{N''}}dxdy=O\left(\frac{R\xi}{x_3}\right). \tag{7.8}$$

In the next section we shall prove, under certain conditions, that

$$\begin{aligned}&\iint_{\Delta_{p,m}}e^{2\pi i\psi_p(x,y)}dxdy=O\left(\frac{R^{3/2}N^{7/2}\log t}{t^{1/2}y_1y_2Q^{1/2}\cdot 2^m\xi^{1/2}}\right),\\&\iint_{\Delta_p'}e^{2\pi i\psi_p(x,y)}dxdy=O\left(\frac{R^{3/2}N^{7/2}\log t}{t^{1/2}y_1y_2Q}\right).\end{aligned} \tag{C$_7$}$$

On assuming this,

$$J_1=O\left(\sum_{m=0}^{L-1}\sum_p^{(m)}\frac{R^{3/2}N^{7/2}\log t}{t^{1/2}y_1y_2Q^{1/2}\cdot 2^m\xi^{1/2}}\right)$$

where (m) denotes that the sum runs over only those p for which Δ_p lie partially or entirely in the strip (7.5). By Lemma 10, the number of such Δ_p is

$$O\left(\frac{4^m\xi R}{x_3l_1l_2}\right)+O\left(\frac{R}{l_1}+\frac{N}{l_2}+1\right)=O\left(\frac{4^m\xi t y_1^2y_2^2Q^2}{x_3R^3N^6}\right)+O\left(\frac{t^{1/2}y_1y_2Q}{R^{3/2}N^{5/2}}+1\right).$$

Therefore

$$\begin{aligned}J_1&=O\left[\log t\sum_{m=0}^{L-1}\left\{\frac{2^m\xi^{1/2}t^{1/2}y_1y_2Q^{3/2}}{x_3R^{3/2}N^{5/2}}+\frac{Q^{1/2}N}{2^m\xi^{1/2}}+\frac{R^{3/2}N^{7/2}}{t^{1/2}y_1y_2Q^{1/2}\cdot 2^m\xi^{1/2}}\right\}\right]\\&=O\left[\log t\left\{\frac{t^{1/2}y_1y_2Q^2}{x_3R^{3/2}N^{5/2}}+\frac{Q^{1/2}N}{\xi^{1/2}}+\frac{R^{3/2}N^{7/2}}{t^{1/2}y_1y_2Q^{1/2}\xi^{1/2}}\right\}\right].\end{aligned} \tag{7.9}$$

Similarly

$$\begin{aligned}J_2&=O\left[\left(\frac{R}{l_1}+1\right)\left(\frac{N}{l_2}+1\right)\frac{R^{3/2}N^{7/2}\log t}{t^{1/2}y_1y_2Q}\right]\\&=O\left[\log t\left\{\frac{t^{1/2}y_1y_2Q}{R^{3/2}N^{3/2}}+\frac{R^{3/2}N^{7/2}}{t^{1/2}y_1y_2Q}\right\}\right]\end{aligned} \tag{7.10}$$

since $R/l_1=N/l_2$ and $(x+1)^2=O(x^2+1)$. Finally,

$$(7.11)\qquad \begin{aligned} J_3 &= O\left[\left(\frac{R}{l_1}+1\right)\left(\frac{N}{l_2}+1\right)l_2\right] = O\left(\frac{RN}{l_1}+l_2\right) \\ &= O\left(\frac{t^{1/2}y_1y_2Q}{R^{3/2}N^{3/2}}\right) + O\left(\frac{R^{3/2}N^{7/2}}{t^{1/2}y_1y_2Q}\right). \end{aligned}$$

From (7.8) to (7.11)

$$S_4 = O\left(\frac{R\xi}{x_3}\right) + O\left[\log t\left\{\frac{t^{1/2}y_1y_2Q^2}{x_3R^{3/2}N^{5/2}} + \frac{Q^{1/2}N}{\xi^{1/2}} + \frac{R^{3/2}N^{7/2}}{t^{1/2}y_1y_2Q^{1/2}\xi^{1/2}}\right\}\right]$$

since $\xi < Q$ and $Q \geqq x_3N$, by (6.3).

If we put $R\xi/x_3 = (Q^{1/2}N/\xi^{1/2}) \log t$, we shall get $\xi = ((x_3NQ^{1/2}/R) \log t)^{2/3}$. But we take the bigger value

$$(7.12)\qquad \xi = A\frac{Q}{R^{2/3}}\left(\frac{\lambda^2\lambda''^2}{y_1y_2(y_3+1)}\right)^{1/2}\log^{2/3} t, \qquad A > 1.$$

The value is certainly bigger by (6.3). The reason for doing so will be seen in the following sections. We have

$$(7.13)\qquad \begin{aligned} S_4 &= O\left[\frac{R^{1/3}Q}{x_3}\left(\frac{\lambda^3\lambda''^2}{y_1y_2(y_3+1)}\right)^{1/2}\log^{2/3} t\right] + O\left[\frac{t^{1/2}y_1y_2Q^2}{x_3R^{3/2}N^{5/2}}\log t\right] \\ &\quad + O\left[\frac{R^{11/6}N^{7/2}}{t^{1/2}y_1y_2Q}\right]. \end{aligned}$$

Remarks. If (C_6) is not true, the second term is not less than $O(RN)$. Hence (7.13) remains true.

8. Proof of (C_7) under certain conditions. We consider, for example, the first relation in (C_7) only. By the remarks at the end of §6, we can write $\psi_p = \psi_{p,1} + \psi_{p,2}$ where $\psi_{p,1}$ satisfies (6.5) to (6.7) and $\psi_{p,2} = O(t^{-\delta})$ where δ can be made as large as we please. Hence

$$\begin{aligned} \iint_{\Delta_{p,m}} e^{2\pi i\psi_p(x,y)}dxdy &= \iint_{\Delta_{p,m}} e^{2\pi i\psi_{p,1}(x,y)}dxdy \\ &\quad + \sum_{j=1}^{\infty} O\left[\iint_{\Delta_{p,m}} \frac{|\psi_{p,2}(x,y)|^j}{j!}dxdy\right] \\ &= \iint_{\Delta_{p,m}} e^{2\pi i\psi_{p,1}(x,y)}dxdy \\ &\quad + O\left(\frac{R^{3/2}N^{7/2}}{t^{1/2}y_1y_2Q^{1/2}\cdot 2^m\xi^{1/2}}\right). \end{aligned}$$

Write $\psi_{p,1} = \psi^*$. We need only to examine the conditions of Lemma 8. Let

$$B = At^{1/2}y_1y_2R^{-5/2}N^{-5/2}Q,$$

then

$$t^{1/2}y_1y_2R^{-5/2}N^{-5/2}Q\left(\frac{Q_0}{RN}\right)^2 = O(Bt^{-2\epsilon})$$

by (C_5). By the remarks at the end of §6, ψ^* satisfies (6.5) to (6.7). Hence, by (6.1),

$$(8.1) \qquad B < \psi^*_{xx} < AB, \qquad 0 < \psi^*_{xy} < A\,\frac{BR}{N}, \qquad |\psi^*_{yy}| < A\,\frac{BR^2}{N^2}\cdot$$

Thus condition (1′) of Lemma 8 is satisfied. In condition (2), we may take, by (6.7), $r_0^2 = Aty_1^2y_2^2R^{-3}N^{-7}Q\cdot 4^m\xi$([16]), provided that $ty_1^2y_2^2R^{-3}N^{-7}Q^2(Q_0/RN)^2 < Kr_0^2$ for a sufficiently small K. By choosing the constant A in (7.12) sufficiently large, this can be achieved, provided that

$$(C_8) \qquad Q_0 = O(R^{2/3}N).$$

In condition (3), we take $C = ABR^{-1}$, $U = \min(N, l_2)$. Then we want $BR^{-1} < AB^{3/2}$ and $BR^{-1}Nr < AB^2$, that is, $R^{-2} < AB$ and $r_0N < ABR$. Since $2^m\xi^{1/2} < Q^{1/2}$, we have $r_0N < t^{1/2}y_1y_2R^{-3/2}N^{-5/2}Q < ABR$. The second condition is satisfied. Since $Q > x_3N$, we have $B > At^{1/2}R^{-5/2}N^{-3/2}$, and the first condition reduces to

$$(C_9) \qquad RN^3 = O(t).$$

By taking k sufficiently large, we can replace the conditions (4) and (5) by a stronger condition

$$(8.2) \qquad B^{1/2}C_1 = O(r_0^2t^{-\epsilon}).$$

By differentiating ψ^*_{xy} with respect to y we get an extra factor N^{-1}. Hence $\psi^*_{xyy} = O(BRN^{-1})$. Similarly $\psi^*_{xxy} = O(BN^{-1})$, $\psi^*_{xxx} = O(BR^{-1})$. Therefore

$$|\psi^{*2}_{xx}\psi^*_{xyy} - 2\psi^*_{xx}\psi^*_{xy}\psi^*_{xxy} + \psi^{*2}_{xy}\psi^*_{xxx}| < B^3RN^{-2}.$$

We now take $C_1 = BRN^{-2}$. Using (7.12) we find

$$(8.3) \qquad r_0^2 > AB^2R^2N^{-2}Q^{-1}\xi > AB^2R^2N^{-2}R^{-2/3}\left(\frac{\lambda^3\lambda''^2}{y_1y_2(y_3+1)}\right)^{1/2}.$$

The relation (8.2) becomes

$$R^{-1/3} = O\left[\left(B\,\frac{\lambda^3\lambda''^2}{y_1y_2(y_3+1)}\right)^{1/2}t^{-\epsilon}\right].$$

Since $Q > (y_3+1)R$, we have $B > At^{1/2}y_1y_2(y_3+1)R^{-3/2}N^{5/2}$. So the last condi-

([16]) Here we write r_0 for the r in Lemma 8 to avoid confusion.

tion reduces to

$$(C_{10}) \qquad R^{-1/3} = O\left[(t^{1/2}\lambda^3\lambda''^2R^{-3/2}N^{-5/2})^{1/2}t^{-\epsilon}\right].$$

9. Now consider S_3. Since S_4' can be estimated as S_4 we have, by (7.13) and (4.7),

$$S_3 = O\left(\frac{RN}{\lambda^{1/2}}\right)$$

$$(9.1) \qquad + O\left(\frac{(RN)^{7/8}}{\lambda^{3/2}}\left\{\sum_{y_1=1}^{\lambda-1}\left[\sum_{y_2=1}^{\lambda^2-1}\left(\sum_{x_3=1}^{\lambda'^2-1}\sum_{y_3=1}^{\lambda''^2-1}\left\{\frac{R^{1/3}Q}{x_3}\frac{\lambda^{3/2}\lambda''}{y_1^{1/2}y_2^{1/2}(y_3+1)^{1/2}}\log^{2/3}t + \frac{t^{1/2}y_1y_2Q^2}{x_3R^{3/2}N^{5/2}}\log t + \frac{R^{11/6}N^{7/2}}{t^{1/2}y_1y_2Q}\right\}\right)^{1/2}\right]^{1/2}\right\}^{1/2}\right).$$

We choose λ' and λ'' such that $\lambda'^2N=\lambda''^2R$, then, since $\lambda'\lambda''=\lambda^2$,

$$(9.2) \qquad \lambda'' = \left(\frac{N}{R}\right)^{1/4}\lambda, \qquad \lambda' = \left(\frac{N}{R}\right)^{-1/4}\lambda.$$

This is possible provided that

$$(C_{11}) \qquad NR^{-1} \leqq \lambda^4.$$

Thus $Q=O(\lambda'^2N)=O(\lambda''^2R)=O(\lambda^2N^{1/2}R^{1/2})$. Hence

$$(9.3) \qquad \begin{aligned} S_3 &= O\left(\frac{RN}{\lambda^{1/2}}\right) + O\left((RN)^{7/8}\left\{R^{1/2}N\log^{5/3}t + \frac{t^{1/2}\lambda^5}{RN}\log^2 t + \frac{R^{4/3}N^3}{t^{1/2}\lambda^5}\log t\right\}^{1/8}\right) \\ &= O\left(\frac{RN}{\lambda^{1/2}}\right) + O(t^{1/16}R^{3/4}N^{3/4}\lambda^{5/8}\log^{1/4}\lambda) \end{aligned}$$

provided that

$$(9.4) \qquad R^{1/3}N = O(t^{1/2}R^{-1}N^{-1}\lambda^5), \qquad t^{-1/2}\lambda^{-5}R^{4/3}N^3\log t = O(t^{1/2}R^{-1}N^{-1}\lambda^5).$$

Choose λ so that $RN\lambda^{-1/2}=t^{1/16}R^{3/4}N^{3/4}\lambda^{5/8}\log^{1/4}t$, then

$$(9.5) \qquad \lambda = \left(\frac{R^{1/4}N^{1/4}}{t^{1/16}}\right)^{8/9}\log^{-1/18} t = \frac{R^{2/9}N^{2/9}}{t^{1/18}}\log^{-1/18} t.$$

Inserting this value in (9.4), we find that the first relation is more stringent. It can be replaced by

$$(C_{12}) \qquad RN^4 = O(t^{-\epsilon}).$$

Inserting (9.5) into (9.3),

$$S_3 = O(t^{1/36}R^{8/9}N^{8/9}\log^{1/36} t) = O(t^{1/36+\epsilon}R^{8/9}N^{8/9}). \tag{9.6}$$

We now return to (4.4). We observe that the above argument applies equally well if S_3 is over part of a rectangle cut off by either or both of the curves $\nu=\alpha$ and $\nu=\beta$. In fact, the equations of the two curves are, by (4.5),

$$\nu = -\frac{t}{2\pi}\left(\frac{1}{b} - \frac{1}{b-r}\right), \qquad \nu = -\frac{t}{2\pi}\left(\frac{1}{a+r} - \frac{1}{a}\right).$$

Hence along these curves $d^2r/d\nu^2 = O(a^3/t^2)$. In Lemma 8 the condition (6) is satisfied if (see (C_7)) $|\psi_{pxx}l_2a^3t^{-2}| < Kr_0$ where ψ_p, l_2 and r_0 are given in §§7 and 8 and K is sufficiently small. By our choice of l_2, $|\psi_{pxx}l_2| < NR^{-1}$. By (6.3) and (7.12), $r_0 > At^{1/2}y_1y_2R^{-3/2}N^{-7/2}Q^{1/2}\xi^{1/2} > At^{1/2}R^{-3/2}N^{-7/2}NR^{-1/3}$. Thus the condition reduces to $Kt^{1/2}R^{-11/6}N^{-9/2} > NR^{-1}a^3t^{-2}$ for a sufficiently small K. That is, $a^3R^{5/6}N^{7/2} < Kt^{5/2}$. Using the fact $N = O(Rt/a^2)$ or $N^{3/2} = O(R^{3/2}t^{3/2}/a^3)$, we reduced it to $R^{7/3}N^2 < Kt$. This is included in (C_{12}). Hence it is legitimate to use Lemma 8 in estimating S_4. We may also use Lemma 10 to get an upper bound for the number of rectangles (or parts of rectangles) $\Delta_{p,m}$ in a strip (7.5), since the domain of summation lies entirely within a rectangle of side-lengths R and N.

We observe that $|f_{yy}(r, y)| > Atra^{-3}$. Hence, by partial summations

$$\begin{aligned} S_2 &= O(\rho(t\rho a^{-3})^{-1/2}t^{1/36+\epsilon}\rho^{8/9}\beta^{8/9}) + O(a^{3/2}t^{-1/2}\rho^{3/2}) \\ &\quad + O(\rho^2 \log t) + O(a^{-2/5}t^{2/5}\rho^{12/5}) \\ &= O(t^{15/36+\epsilon}a^{-5/18}\rho^{41/18}) + O(a^{3/2}t^{-1/2}\rho^{3/2}) + O(\rho^2\log t) + O(a^{-2/5}t^{2/5}\rho^{12/5}) \end{aligned}$$

since $\beta = O(t\rho a^{-2})$. Therefore, by (4.2)

$$\begin{aligned} S_1 &= O(a\rho^{-1/2}) + O(t^{15/72+\epsilon}a^{13/36}\rho^{5/36}) + O(a^{5/4}t^{-1/4}\rho^{-1/4}) \\ &\quad + O(a^{1/2}\log^{1/2} t) + O(a^{3/10}t^{1/5}\rho^{1/5}). \end{aligned}$$

The first two terms are of the same order if

$$\rho = (t^{-15/72-\epsilon}a^{23/36})^{36/23} = t^{-15/46-\epsilon}a. \tag{9.7}$$

This gives, for $a = O(t^{1/2})$,

$$\begin{aligned} S_1 &= O(t^{15/92+\epsilon}a^{1/2}) + O(t^{-31/184+\epsilon}a) + O(a^{1/2}\log^{1/2} t) + O(t^{31/230}a^{1/2}) \\ &= O(t^{15/92+\epsilon}a^{1/2}). \end{aligned}$$

Hence, by partial summation

$$\sum_{n=a}^{b} \frac{1}{n^{1/2+it}} = O(t^{15/92+\epsilon}). \tag{9.8}$$

10. Let us examine the conditions we assumed. The conditions (C_1), (C_4) and (C_9) are included in (C_{12}). By (9.2), (C_3) is not stronger than (C_5). By the remarks at the end of §7 and by §8, the conditions (C_6) and (C_7) can

be deleted. The conditions (C_2) is satisfied so far as we do not consider the case $a > At^{1/2}$. It remains, therefore, to consider (C_5), (C_8), (C_{10}), (C_{11}) and (C_{12}).

Since $Q_0 = O(\lambda'^2 N)$ and $\lambda'^2 N = \lambda''^2 R$, (C_5) and (C_8) can be replaced by $\lambda'^2 = O(R^{2/3} t^{-\epsilon})$. By (9.2) and (9.5), this can be reduced to the trivial condition $R^5 N^{-1} = O(t^{2-\epsilon})$.

Using (9.2) and (9.5), (C_{10}) can be written as

$$R^{-1/3} = O\left[\left(t^{1/2}\frac{R^{10/9}N^{10/9}}{t^{5/18}}\left(\frac{N}{R}\right)^{1/2} R^{-3/2}N^{-5/2}\right)^{1/2} t^{-\epsilon}\right]$$

which is actually equivalent to (C_{12}). Since $N = O(Rt/a^2)$ (use the relation above (4.6)), (C_{12}) is equivalent to $R^5 t^{3+\epsilon} = O(a^3)$. By (9.7), this reduces to $t^{-75/46} t^{3+\epsilon} = O(a^3)$. That is,

$$\text{(C)} \qquad a > At^{21/46+\epsilon}.$$

Now consider (C_{11}). By (9.5), the condition is

$$NR^{-1} < t^{-2/9} R^{8/9} N^{8/9} \log^{-2/9} t$$

or

$$\text{(C')} \qquad t^2 \log^2 t \leqq AR^{17} N^{-1}.$$

Using $N > ARta^{-2}$, this can be reduced to

$$\text{(C}_1'\text{)} \qquad t^3 \log^2 t = O(R^{16} a^2) \quad \text{or} \quad R > A\left(\frac{t^3 \log^2 t}{a^2}\right)^{1/16}.$$

11. If both (C) and (C′) are satisfied we have nothing to justify. Now suppose that one of them is not true.

We shall not take the values for λ' and λ'' given in (9.2). We can, as did Professor Titchmarsh[17] in his paper, take $\lambda'' = \lambda^2$ and omit the x_3-summation. This amounts to using Lemma 5 three times.

We are compelled to examine the whole proof afresh, keeping to its original form as closely as possible. §3 is now useless. In §4, we omit all the x_3-summations and put $x_3 = 0$, $\lambda' = 1$ whenever they occur elsewhere. In §5, we put $x_3 = 0$. In §6, we put $x_3 = \lambda' = 0$. Then, in (6.7), the first term on the right-hand side is now "positive definite."

Now §7 can be greatly simplified, for we have no need of redividing Δ_ρ. We may take Δ_ρ as Δ_ρ'' there and put $J_0 = J = 0$. By arguing as before, we find

$$S_4 = J_2 + J_3 = O\left(\frac{RN}{l_1} + l_2\right)\log t = O\left[\frac{t^{1/2}y_1y_2y_3}{R^{1/2}N^{3/2}} + \frac{R^{1/2}N^{7/2}}{t^{1/2}y_1y_2y_3}\right]\log t.$$

(17) Loc. cit. p. 13.

Inserting this result into (4.7) we obtain

$$S_3 = O(RN/\lambda^{1/2}) + O(R^{13/16}N^{11/16}t^{1/16}\lambda^{7/8}\log^{1/8} t) + O(R^{15/16}N^{21/16}t^{-1/16}\lambda^{-7/8}\log^{1/4} t). \tag{11.1}$$

The first two terms are of the same order if

$$\lambda = \left[\left(\frac{R^3N^5}{t\log^2 t}\right)^{1/22}\right]. \tag{11.2}$$

This gives

$$S_3 = O(R^{41/44}N^{39/44}t^{1/44}\log^{1/22} t) \tag{11.3}$$

provided that the last term in (11.1) is negligible. This is true if $RN^5 \log t = O(t\lambda^{14})$. Using (11.2) and the fact that $N<AtRa^{-2}$, we reduce this to

$$t^{16}R^{10}\log^{25} t = O(a^{40}). \tag{11.4}$$

First, suppose that (C) is true and (C′) is false. Then (11.4) becomes $t^{16}(a^{-2}t^3\log^2 t)^{5/8}\log^{25} t = O(a^{40})$. This can be reduced to $a>t^{143/330}\log^{7/11} t$, a consequence of (C).

We expect that (11.3) implies (9.6). This is true of $R^{17}N^{-1}<t^{2+\epsilon}$ which is weaker than the negation of (C′). Thus (9.6) is proved for this case.

Next, suppose that (C) is untrue. Then we have, as before,

$$S_2 = O(\rho^{51/22}t^{9/22}a^{-3/11}\log^{1/22} t) + O(a^{3/2}t^{-1/2}\rho^{3/2}) + O(\rho^2\log t) + O(a^{-2/5}t^{2/5}\rho^{12/5}).$$

Hence

$$S_1 = O(a\rho^{-1/2}) + O(a^{4/11}\rho^{7/44}t^{9/44}\log^{1/44} t) + O(a^{5/4}\rho^{-1/4}t^{-1/4}) + O(a^{1/2}\log^{1/2} t) + O(a^{3/10}\rho^{1/5}t^{1/5}).$$

The first two terms are of the same order if

$$\rho = [(a^{28}t^{-9}\log^{-1} t)^{1/29}]. \tag{11.5}$$

This gives

$$\sum_{n=a}^{b} n^{-it} = O(a^{15/29}t^{9/58}\log^{1/58} t) + O(a^{117/116}t^{-5/29}\log^{1/116} t) + O(a^{1/2}\log^{1/2} t) + O(a^{143/290}t^{4/29}).$$

It can be verified that the last three terms are negligible and all conditions except (11.4) can be removed. By partial summation,

$$\sum_{n=a} \frac{1}{n^{1/2+it}} = O(a^{1/58}t^{9/58}\log^{1/58} t) = O(t^{15/92}\log^{1/58} t)$$

since (C) is untrue and $(21/46)\times 1/58+9/58=15/92$. By (11.5) we may reduce (11.4) to

$$(\mathrm{C}^*) \qquad a > t^{17/40}\log^{143/176} t.$$

Thus we have proved (9.8) completely under the sole condition (C*).

12. **Completing the proof.** We use, first, the inequality

$$\sum_{n=N}^{N'} \frac{1}{n^{1/2+it}} = O(N'^{5/82}t^{11/82}) + O(N^{-17/328}t^{61/328}) \qquad (N > t^{11/36}).$$

For $N' < t^{17/40+\epsilon}$, the first term is $O(t^{15/92+\epsilon})$, for

$$\frac{17}{40}\cdot\frac{5}{82}+\frac{11}{82}=\frac{105}{656}<\frac{15}{92}.$$

The second term is $O(t^{15/92})$ if $N \geqq t^{172/391}$.

For $N < t^{172/391}$, we use the result[18]

$$\sum_{a\leqq n\leqq b} n^{-1/2+it} = O(t^k a^{l-k-1/2})$$

where $a<b<2a<t/\pi$, and $k=97/696$, $l=480/696$. The sum is $O(t^{15/92})$ since

$$\frac{97}{696}+\left(\frac{480}{696}-\frac{97}{696}-\frac{1}{2}\right)\cdot\frac{172}{391}=\frac{97}{696}+\frac{35\times 172}{696\times 391}<\frac{15}{92}.$$

By the approximate functional equation, we have

$$\zeta(1/2+it) = O(t^{15/92+\epsilon}) \qquad (\epsilon > 0)$$

where the constant implied by O depends only on ϵ.

EXETER COLLEGE, OXFORD UNIVERSITY,
OXFORD, ENGLAND.

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