A THEORY OF TRANSFINITE CONVERGENCE

BY

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The classical theory of convergence, resting as it does on the properties of the real numbers, is intimately connected with the notion of denumerability. As examples of what is meant we recall that a sequence is a function defined on a denumerable set and that category is a property relative to a sequence of sets each having a certain topological property. An early abstract formulation of this fact was made by Hausdorff in his two denumerability axioms. The general problem of extending the methods of analysis beyond this cardinal number restriction has been studied on the topological side in terms of the concept of uniform space [1, 2](1) and on the algebraic side in terms of non-archimedean number fields [3, 4, 5].

In this paper we consider a class of ordinal numbers ξ^* and corresponding uniform topologies in terms of which certain fundamental theorems in classical analysis find natural extensions. The theory contains the theorem that a complete space is of the second category, both concepts being defined in a manner appropriate to the ordinal ξ^*. It also contains extensions of the covering theorems of Lindelöf and Borel-Lebesgue [6]. The relation of the theory to non-archimedean order fields is indicated by an example. It is shown that there is a related class of uniform topologies which are complete and of the first category.

The ordinal ξ^* and the space S. The topology of the space S to which the theory applies is defined in terms of an ordinal number ξ^* having the properties of ω, the first transfinite ordinal, which play a role in convergence theory. By ξ^* we mean a limiting ordinal such that if $\eta^* < \xi^*$ and ξ_η is a single-valued function on $\eta < \eta^*$ to $\xi < \xi^*$ then the least upper bound of ξ_η is less than ξ^*:

$$(*) \qquad \sup\,[\xi_\eta \mid \eta < \eta^*] < \xi^*.$$

We note that ξ^* is the initial ordinal of its cardinal Ξ^*. Otherwise, there is an ordinal $\eta^* < \xi^*$ which is the initial ordinal of Ξ^*. Let $C = [c]$ be a class of cardinal Ξ^*. Then C can be well-ordered as c_ξ, $\xi < \xi^*$, and as c_η, $\eta < \eta^*$. Since each $c \in C$ occurs once and only once in each well-ordering, there is a 1-1 mapping $\xi = \xi(\eta)$ of $\eta < \eta^*$ onto $\xi < \xi^*$. This contradicts (*). If ω_α is a regular initial ordinal [7], then ω_α satisfies (*). For: ω_α, being regular, is not cofinal with any $\eta^* < \omega_\alpha$. Hence any single-valued ξ_η on $\eta < \eta^*$ to $\xi < \omega_\alpha$ satisfies (*). Every transfinite ordinal ω_α whose index is not a limit number is a number ξ^*.

The space S is a set of elements $x, y, z, \cdots$ in which a set $U = [U_\xi(x)]$,

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$\xi<\xi^*$ and $x\in S$ of subsets of S, is defined so that the following axioms are satisfied:

1. $\bigcap_{\xi<\xi^*} U_\xi(x)=[x]$, the set consisting of x alone.

2. If $\xi_1<\xi_2<\xi^*$ then $U_{\xi_1}(x)\supset U_{\xi_2}(x)$.

3. If $\eta<\xi^*$ there is a $\xi(\eta)$ such that $\eta\leqq\xi(\eta)<\xi^*$, and $U_{\xi(\eta)}(y)\cap U_{\xi(\eta)}(x)\neq 0$ implies $U_{\xi(\eta)}(y)\subset U_\eta(x)$.

4. If $\eta^*<\xi^*$ and $U_{\xi_\eta}(x_\eta)$, $\eta<\eta^*$, are such that $\eta_1<\eta_2<\eta^*$ implies $U_{\xi_{\eta_1}}(x_{\eta_1})\supset U_{\xi_{\eta_2}}(x_{\eta_2})$ then $\bigcap_{\eta<\eta^*} U_{\xi_\eta}(x_\eta)$ is a non-empty open set.

A set $G\subset S$ is open if for each $x\in G$ there is a $U_\xi(x)\subset G$. It is convenient in applying axiom 3 to refer to $\xi(\eta)$ as "the ordinal of axiom 3." It is clear that metric spaces are space S for which $\xi^*=\omega$. We shall give examples of spaces S for every $\xi^*>\omega$.

We state two simple properties of spaces S.

THEOREM 1. *S is a regular hausdorff space.*

Proof. That S is a hausdorff space in terms of the neighborhoods $U_\xi(x)$ is an immediate consequence of axioms 1, 2, 4 and the fact that ξ^* is a limit number. From axiom 3 it follows that the closure(2) $\overline{U}_{\xi(\eta)}(x)\subset U_\eta(x)$ for all $\eta<\xi^*$ and $x\in S$. Hence S is regular.

THEOREM 2. *If G_η, $\eta<\eta^*<\xi^*$, is an open set in S then $\bigcap_{\eta<\eta^*} G_\eta$ is open*(3).

Proof. Suppose $x\in\bigcap_{\eta<\eta^*}G_\eta$. For every $\eta<\eta^*$ there is a ξ_η such that $U_{\xi_\eta}(x)\subset G_\eta$. For $\eta_0<\eta^*$ let $\zeta(\eta_0)=\sup\,[\xi_\eta\,|\,\eta<\eta_0]$. From property (*) it follows that $\zeta(\eta_0)<\xi^*$. By axiom 2, $U_{\zeta(\eta_0)}(x)\subset U_{\xi_{\eta_0}}(x)\subset G_{\eta_0}$. If $\eta_1<\eta_2<\eta^*$ then $\zeta(\eta_1)\leqq\zeta(\eta_2)$ and, again by axiom 2, $U_{\zeta(\eta_1)}(x)\supset U_{\zeta(\eta_2)}(x)$. The family of neighborhoods $U_{\zeta(\eta)}(x)$ satisfies the condition of axiom 4 and so there is a $U_\xi(x)\subset\bigcap_{\eta<\eta^*} U_{\zeta(\eta)}(x)\subset\bigcap_{\eta<\eta^*} G_\eta$. Thus $\bigcap_{\eta<\eta^*} G_\eta$ is open.

At this point we remark that Theorem 2 does not require the full force of axiom 4. The result follows if, in 4, one puts $x_\eta=x$ for $\eta<\eta^*$. However in this case the classical relation between completeness and category is lost. The existence of complete spaces of the first category which satisfy axioms 1, 2, 3, and 4 modified in the manner just described is established in the section of this paper devoted to examples.

Convergence, completeness, category. A sequence is a single-valued function x_ξ on $\xi<\xi^*$ to S. A sequence x_ξ, $\xi<\xi^*$, is called fundamental if for each $\eta<\xi^*$ there are $y_\eta\in S$ and $\zeta(\eta)<\xi^*$ such that if $\zeta(\eta)\leqq\xi<\xi^*$ then $x_\xi\in U_\eta(y_\eta)$. A sequence x_ξ has a limit x, we write $\lim_\xi x_\xi=x$, if for each $\eta<\xi^*$ there is a $\zeta(\eta)<\xi^*$ such that if $\zeta(\eta)\leqq\xi<\xi^*$ then $x_\xi\in U_\eta(x)$. We state three elementary theorems on convergence.

(2) An element $x\in S$ is a limit point of a set $E\subset S$ if every $E\cap U_\xi(x)$ contains a $y\in S$, $y\neq x$. The closure $\overline{E}$ of E is the union of E and the set of its limit points.

(3) From this theorem, axiom 2, and property (*) it follows that x is a limit point of E if and only if the cardinal number of $E\cap U_\xi(x)$ is at least Ξ^* for every $\xi<\xi^*$.

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THEOREM 3. *If x_ξ has a limit, it is unique.*

Proof. Suppose x and y are limits of the sequence x_ξ. For $\eta<\xi^*$ let $\xi(\eta)$ be the ordinal of axiom 3. There are $\xi', \xi''<\xi^*$ such that $x_\xi\in U_{\xi(\eta)}(x)$ if $\xi'\leqq\xi<\xi^*$ and $x_\xi\in U_{\xi(\eta)}(y)$ if $\xi''\leqq\xi<\xi^*$. If $\xi_0=\max[\xi', \xi'']$, then $\xi_0<\xi^*$ and $x_{\xi_0}\in U_{\xi(\eta)}(x)\cap U_{\xi(\eta)}(y)$. By axioms 1, 3, $y\in U_{\xi(\eta)}(y)\subset U_\eta(x)$. This being the case for all $\eta<\xi^*$, we have $y=x$ by axiom 1.

THEOREM 4. *x_ξ is a fundamental sequence if and only if for each $\eta<\xi^*$ there is a $\zeta(\eta)$ such that $\eta\leqq\zeta(\eta)<\xi^*$ and if $\zeta(\eta)\leqq\xi<\xi^*$ then $x_\xi\in U_\eta(x_{\zeta(\eta)})$.*

Proof. The sufficiency of the condition is evident. Suppose now that x_ξ is a fundamental sequence and that $\eta<\xi^*$ is given. Let $\xi_0=\xi(\eta)$ be the ordinal of axiom 3. Since x_ξ is a fundamental sequence there are a $\xi_1(\xi_0)<\xi^*$ and a $y_0\in S$ such that if $\xi_1(\xi_0)\leqq\xi<\xi^*$ then $x_\xi\in U_{\xi_0}(y_0)$. Clearly we may choose $\xi_1(\xi_0)\geqq\eta$. Since $x_{\xi_1(\xi_0)}\in U_{\xi_0}(y_0)\cap U_{\xi_0}(x_{\xi_1(\xi_0)})$ we have, from axiom 3, $U_{\xi_0}(y_0)\subset U_\eta(x_{\xi_1(\xi_0)})$. Hence $x_\xi\in U_\eta(x_{\xi_1(\xi_0)})$ for $\xi_1(\xi_0)\leqq\xi<\xi^*$. This establishes the necessity of the condition since $\xi_1(\xi_0)=\xi_1(\xi(\eta))=\zeta(\eta)\geqq\eta$.

THEOREM 5. *If* $\lim_\xi x_\xi=x$ *then x_ξ is a fundamental sequence.*

Proof. If we put $x=y_\eta$ for each $\eta<\xi^*$, it is an immediate consequence of the definitions that if x_ξ has x as the limit it is a fundamental sequence.

We come now to the concepts of completeness and category. The space S is called ξ^*-complete if every fundamental sequence has a limit in S. The space S is said to be of the first ξ^*-category if it is the union of a sequence of sets N_ξ, $\xi<\xi^*$, each nowhere dense in S. If the space S is not of the first ξ^*-category it is said to be of the second ξ^*-category. In order to establish the theorem that if S is ξ^*-complete then S is of the second ξ^*-category, we find it convenient to formulate the concept of a well-pinned sequence of sets. A sequence of sets $E_\xi, \xi<\xi^*$, is said to be well-pinned if for every $\eta<\xi^*, \bigcap_{\xi<\eta}E_\xi\neq 0$.

THEOREM 6. *S is ξ^*-complete if and only if every well-pinned sequence of neighborhoods $U_{\xi_\eta}(x_\eta)$, $\eta\leqq\xi_\eta<\xi^*$, has the property $\bigcap_{\eta<\xi^*}\overline{U}_{\xi_\eta}(x_\eta)\neq 0$.*

Proof. Suppose that the condition is satisfied and let x_ξ be a fundamental sequence in S. By Theorem 4, for each $\eta<\xi^*$ there is a $\zeta(\eta)$ such that $\eta\leqq\zeta(\eta)<\xi^*$ and $x_\xi\in U_\eta(x_{\zeta(\eta)})$ if $\zeta(\eta)\leqq\xi<\xi^*$. Now if $\zeta<\xi^*$, $\zeta_0=\sup[\zeta(\eta)\mid\eta<\zeta]<\xi^*$ by (*) and $x_{\zeta_0}\in\bigcap_{\eta<\zeta}U_\eta(x_{\zeta(\eta)})$ since $\zeta(\eta)\leqq\zeta_0<\xi^*$ for $\eta<\zeta$. Therefore $U_\eta(x_{\zeta(\eta)})$ is a well-pinned sequence of neighborhoods with $\xi_\eta=\eta$. Hence there is an $x\in\bigcap_{\eta<\xi^*}\overline{U}_\eta(x_{\zeta(\eta)})$. Now for each $\eta<\xi^*$ let $\xi(\eta)$ be the ordinal of axiom 3. Then $U_{\xi(\eta)}(x)\cap U_{\xi(\eta)}(x_{\zeta(\xi(\eta))})\neq 0$ and so $x_\xi\in U_{\xi(\eta)}(x_{\zeta(\xi(\eta))})\subset U_\eta(x)$, if $\zeta(\xi(\eta))\leqq\xi<\xi^*$, by axiom 3. Hence $\lim_\xi x_\xi=x$ and the sufficiency of the condition is established.

To establish the necessity of the condition let $U_{\xi_\eta}(x_\eta)$, $\eta\leqq\xi_\eta<\xi^*$, be a well-pinned sequence of neighborhoods. We show that x_η, $\eta<\xi^*$, is a fundamental

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sequence. For $\eta_0 < \xi^*$, consider ξ_{η_0} and let $\eta_1 = \xi(\xi_{\eta_0})$ be the ordinal of axiom 3. Since $U_{\xi_\eta}(x_\eta)$ is well-pinned, $\bigcap_{\eta<\zeta+1} U_{\xi_\eta}(x_\eta) \neq 0$ for each ζ such that $\eta_1 \leqq \zeta < \xi^*$. Hence $U_{\xi_{\eta_1}}(x_{\eta_1}) \cap U_{\xi_\zeta}(x_\zeta) \neq 0$ if $\eta_1 \leqq \zeta < \xi^*$. Now since $\eta \leqq \xi_\eta$ for every $\eta < \xi^*$ we have, by axiom 2, $U_{\xi_{\eta_1}}(x_{\eta_1}) \cap U_{\xi_\zeta}(x_\zeta) \subset U_{\eta_1}(x_{\eta_1}) \cap U_\zeta(x_\zeta) \subset U_{\eta_1}(x_{\eta_1}) \cap U_{\eta_1}(x_\zeta) \neq 0$ for $\eta_1 \leqq \zeta < \xi^*$. By axioms 3, 2 we have $U_{\eta_1}(x_\zeta) \subset U_{\xi_{\eta_0}}(x_{\eta_1}) \subset U_{\eta_0}(x_{\eta_1})$. Hence for $\eta_1 \leqq \zeta < \xi^*$ we have $x_\zeta \in U_{\eta_0}(x_{\eta_1})$. Since η_0 is any ordinal less than ξ^* it follows from theorem 4 that x_η, $\eta < \xi^*$, is a fundamental sequence. By hypothesis S is ξ^*-complete and so $\lim_\eta x_\eta = x \in S$.

We show that $x \in \bigcap_{\eta<\xi^*} \overline{U}_{\xi_\eta}(x_\eta)$. Consider any $\eta_0, \xi_0 < \xi^*$. Let $\xi_1 = \xi(\xi_0)$ be the ordinal of axiom 3. Since x is the limit of x_η there is an $\eta_1 = \zeta(\xi_1) < \xi^*$ such that if $\eta_1 \leqq \eta < \xi^*$, $x_\eta \in U_{\xi_1}(x)$. Let $\zeta = \max\ [\xi_{\eta_0}, \xi_1, \eta_1]$. Then $\zeta + 1 < \xi^*$ and since the $U_{\xi_\eta}(x_\eta)$ are well-pinned, $\bigcap_{\eta<\zeta+1} U_{\xi_\eta}(x_\eta) \neq 0$. Hence there is a $y_0 \in U_{\xi_{\eta_0}}(x_{\eta_0}) \cap U_{\xi_\zeta}(x_\zeta)$ since $\eta_0 \leqq \xi_{\eta_0} \leqq \zeta$. Since $\xi_1 \leqq \zeta \leqq \xi_\zeta$, $U_{\xi_\zeta}(x_\zeta) \subset U_{\xi_1}(x_\zeta)$ by axiom 2 and so $y_0 \in U_{\xi_1}(x_\zeta)$. Since $\eta_1 \leqq \zeta$, $x_\zeta \in U_{\xi_1}(x)$ and so $U_{\xi_1}(x) \cap U_{\xi_1}(x_\zeta) \neq 0$. By axiom 3 $U_{\xi_1}(x_\zeta) \subset U_{\xi_0}(x)$. Hence $y_0 \in U_{\xi_{\eta_0}}(x_{\eta_0}) \cap U_{\xi_0}(x)$. Since ξ_0 is any ordinal less than ξ^*, $x \in \overline{U}_{\xi_{\eta_0}}(x_{\eta_0})$. Since η_0 is any ordinal less than ξ^*, $x \in \bigcap_{\eta<\xi^*} \overline{U}_{\xi_\eta}(x_\eta)$. This established the necessity of the condition.

THEOREM 7. *If S is ξ^*-complete then S is of the second ξ^*-category.*

Proof. Let N_ξ, $\xi < \xi^*$, be a sequence of nowhere dense subsets of S and let $T = \bigcup_{\xi<\xi^*} N_\xi$. We show that T is a proper subset of S. For any $U_1(x_1)$ there is a $U_{\xi_1}(y_1) \subset U_1(x_1)$ such that $\overline{U}_{\xi_1}(y_1) \cap N_1 = 0$ since N_1 is nowhere dense and S is regular. Suppose that for $\eta^* < \xi^*$ there are neighborhoods $U_{\xi_\eta}(y_\eta)$, $\eta < \eta^*$, such that

(a) $\eta \leqq \xi_\eta < \xi^*$ if $\eta < \xi^*$,

(b) $\eta_1 < \eta_2 < \eta^*$ implies $U_{\xi_{\eta_1}}(y_{\eta_1}) \supset U_{\xi_{\eta_2}}(y_{\eta_2})$,

(c) $\eta < \eta^*$ implies $(\bigcup_{\zeta \leqq \eta} N_\zeta) \cap \overline{U}_{\xi_\eta}(y_\eta) = 0$.

The $U_{\xi_\eta}(y_\eta)$ satisfy the condition of axiom 4. Hence there is a $U_\xi(y) \subset \bigcap_{\eta<\eta^*} U_{\xi_\eta}(y_\eta)$. Since N_{η^*} is nowhere dense in the regular space S there is a $U_\zeta(y_{\eta^*})$ with $\overline{U}_\zeta(y_{\eta^*}) \subset U_\xi(y)$, such that $\overline{U}_\zeta(y_{\eta^*}) \cap N_{\eta^*} = 0$. Since $\eta^* < \xi^*$

$$\xi_{\eta^*} = \max\ (\zeta, \eta^*, \sup\ [\xi_\eta \mid \eta < \eta^*]) < \xi^*.$$

By axiom 2, $U_{\xi_{\eta^*}}(y_{\eta^*}) \subset U_\zeta(y_{\eta^*}) \subset \bigcap_{\eta<\eta^*} U_{\xi_\eta}(y_\eta)$. Now the family $U_{\xi_\eta}(y_\eta)$, $\eta \leqq \eta^*$, satisfies (a) if $\eta \leqq \eta_2^*$, (b) if $\eta_1 < \eta \leqq \eta^*$ and (c) if $\eta \leqq \eta^*$. Only (c) warrants an explicit argument: $\overline{U}_{\xi_{\eta^*}}(y_{\eta^*}) \cap N_{\eta^*} \subset \overline{U}_\zeta(y_{\eta^*}) \cap N_{\eta^*} = 0$ and $\overline{U}_{\xi_{\eta^*}}(y_{\eta^*}) \cap N_\zeta \subset U_{\xi_\eta}(y_\eta) \cap N_\zeta = 0$ for $\zeta < \eta^*$ yield the result. By transfinite induction it follows that there exists a sequence $U_{\xi_\eta}(y_\eta)$, $\eta < \xi^*$, of neighborhoods such that

(A) $\eta \leqq \xi_\eta < \xi^*$ for $\eta < \xi^*$,

(B) $\eta_1 < \eta_2 < \xi^*$ implies $U_{\xi_{\eta_1}}(y_{\eta_1}) \supset U_{\xi_{\eta_2}}(y_{\eta_2})$,

(C) $\eta < \xi^*$ implies $(\bigcup_{\zeta \leqq \eta} N_\zeta) \cap \overline{U}_{\xi_\eta}(y_\eta) = 0$.

From (B) and axiom 4 it follows that $U_{\xi_\eta}(y_\eta)$, $\eta < \xi^*$, is a well-pinned sequence of sets. Since S is ξ^*-complete and (A) holds it follows from Theorem 6 that $\bigcap_{\eta<\xi^*} \overline{U}_{\xi_\eta}(y_\eta)$ contains an $x \in S$. From (C) it follows that x is not in any N_ξ,

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$\xi<\xi^*$. Hence T is a proper subset of S and so S is of the second ξ^*-category.

Separability and compactness. In the system of real numbers the concepts of compactness and separability are related by the theorem that every nondenumerable set has a limit point, the proof resting on the density of the denumerable set of rationals. This leads us to formulate the axiom 5.

5. There is a set $D=[x_\xi \mid \xi<\xi^*]$ of distinct points $x_\xi \in S$ such that $\overline{D}=S$.

It might be useful to call S Ξ^*-separable if it satisfies axiom 5. A set $E\subset S$ is called Ξ^*-compact if every subset $M\subset E$ of cardinal Ξ^* has a limit point in S. In this section we shall state covering theorems analogous to those of Lindelöf and of Heine, Borel, Lebesgue. The omitted proofs may be supplied by the reader.

THEOREM 8. *The set* $\mathcal{D}=[U_\xi(x_\eta)]$, $\xi<\xi^*$ *and* $x_\eta\in D$, *is equivalent to the set of all neighborhoods of* S (*Lindelöf*).

LEMMA. *If* $\mathcal{G}=[G_\lambda]$, $\lambda<\lambda^*$, *is a family of open subsets of* S, *then there is a family* $\mathcal{H}=[H_\mu]\subset\mathcal{G}$, $\mu<\mu^*$, *of cardinal number not exceeding* Ξ^* *such that* $\bigcup_{\mu<\mu^*}H_\mu=\bigcup_{\lambda<\lambda^*}G_\lambda$.

LEMMA. *If* F_ξ, $\xi<\xi^*$, *is a sequence of decreasing, closed, non-empty sets and* F_1 *is* Ξ^*-*compact, then* $\bigcap_{\xi<\xi^*}F_\xi\neq 0$.

THEOREM 9. *A set* $E\subset S$ *is closed and* Ξ^*-*compact if and only if for every* $\mathcal{G}=[G_\lambda]$, $\lambda<\lambda^*$, *whose union contains E there is a subset* $\mathcal{H}\subset\mathcal{G}$ *of cardinal less than* Ξ^* *whose union contains E* (*Borel-Lebesgue*).

Proof. Suppose that E is closed, Ξ^*-compact and that $E\subset\bigcap_{\lambda<\lambda^*}G_\lambda$. There is $\mathcal{H}=[H_\mu]\subset\mathcal{G}$, $\mu<\mu^*$, of cardinal not greater than Ξ^* whose union contains E. Suppose that the cardinal of $\mathcal{H}$ is Ξ^*. We may take $\mu^*=\xi^*$ since ξ^* is the initial ordinal of Ξ^*. Let $O_\nu=\bigcup_{\mu\leqq\nu}H_\mu$, $\nu<\xi^*$, and let $\tilde{O}_\nu$ be the complement of O_ν. Then $F_\nu=E\cap\tilde{O}_\nu$ is a sequence of decreasing, closed, Ξ^*-compact sets. If no F_ν is empty then there is an $x\in\bigcap_{\nu<\xi^*}F_\nu$. But now $x\in E$, $x\notin\bigcup_{\nu<\xi^*}O_\nu=\bigcup_{\mu<\xi^*}H_\mu$, which contradicts $E\subset\bigcup_{\mu<\xi^*}H_\mu$. Hence some $F_\nu=0$ and so $E\subset\bigcup_{\mu<\nu}H_\mu$ where $\nu<\xi^*$. Since ξ^* is the initial ordinal of Ξ^*, the cardinal of $\mathcal{H}^*=[H_\mu]$, $\mu<\nu$, is less than Ξ^*.

Now let E be a set for which the condition of the converse holds and which is not both closed and Ξ^*-compact. Then there is a set $M=[x_\eta \mid \eta<\xi^*]\subset E$ in which the x_η are distinct and without a limit point in E. It follows that for each $y\in E$ there is a $U_{\xi(y)}(y)$ such that $M\cap U_{\xi(y)}(y)=0$ or $[y]$, the set consisting of y alone. Any subset $\mathcal{H}$ of $\mathcal{G}=[U_{\xi(y)}(y) \mid y\in E]$, whose union covers E, contains all $U_{\xi(x_\eta)}(x_\eta)$, $\eta<\xi^*$. Since ξ^* is the initial ordinal of Ξ^*, the cardinal of $\mathcal{H}$ is at least Ξ^*. This contradicts the hypothesis and the converse is proved.

Examples, relation to non-archimedean fields. As an example of a space S satisfying axioms 1–4 we construct the order closure $\overline{K}$ of a non-archimedean

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field K as follows. Let u_ξ, $\xi<\xi^*$, be the basis of a set Γ of forms

$$\alpha = \sum_1^n r_k u_{\xi_k}$$

where the r_k are real numbers and $\xi_1>\xi_2> \cdots >\xi_n$. We define $\alpha=\alpha_1\pm\alpha_2$ where

$$\alpha_1 = \sum_1^m r_i u_{\xi_i}, \qquad \alpha_2 = \sum_1^n s_j u_{\eta_j}$$

by

$$\alpha = \sum_1^p t_k u_{\zeta_k}$$

where the union $[\zeta_k]=[\xi_i]\cup[\eta_j]$ is ordered as decreasing function of k and $t_k=r_i$, $\pm s_j$, $r_i\pm s_j$ according as $\zeta_k=\xi_i\notin[\eta_j]$, $\zeta_k=\eta_j\notin[\xi_i]$, $\zeta_k=\xi_i=\eta_j$. In this way Γ is an abelian group whose identity is the form with all $r_k=0$. Order in Γ is defined by: $\alpha>0$ if $r_1>0$; $\alpha_1>\alpha_2$ if $\alpha_1-\alpha_2>0$.

The elements of the field K are the forms

$$x = \sum_{\rho<\sigma} a_\rho t^{\alpha_\rho}$$

where σ is an ordinal number, a_ρ is real, $\alpha_\rho\in\Gamma$ and $\alpha_{\rho_1}>\alpha_{\rho_2}$ if $\rho_1<\rho_2<\sigma$. The field operations in K are defined by the rules for the addition and multiplication of formal power series [4, 5]. Order in K is defined by: $x>0$ if for some $\rho_0<\sigma$, $a_{\rho_0}>0$ and $a_\rho=0$ for $\rho<\rho_0$; $x_1>x_2$ if $x_1-x_2>0$.

We shall use the following lemma.

LEMMA 1. *If* $x>0$ *then for some* $\xi<\xi^*$, $x>t^{-u_\xi}$.

Proof. $x=a_{\rho_0}t^{\alpha_{\rho_0}}+ \cdots$, $a_{\rho_0}>0$. If $\alpha_{\rho_0}\geqq 0$, then $\alpha_{\rho_0}>-u_\xi$ for all $\xi<\xi^*$. If $\alpha_{\rho_0}<0$, then $u_\xi+\alpha_{\rho_0}>0$ for all ξ such that $\xi_1<\xi<\xi^*$ where $\alpha_{\rho_0}=r_1u_{\xi_1}+ \cdots$, $r_1<0$. Since ξ^* is a limit number, there is a $\xi<\xi^*$ such that $\alpha_{\rho_0}>-u_\xi$. Hence $x-t^{-u_\xi}=a_{\rho_0}t^{\alpha_{\rho_0}}+ \cdots +at^{-u_\xi}+ \cdots >0$ and so $x>t^{-u_\xi}$.

We now construct the order closure of K. This is the set $\overline{K}$ of sets $X\subset K$ such that [8]:

1. $X\neq 0, K$.
2. If $x\in X$ and $y<x$ then $y\in X$.
3. If $x\in X$ there is $y\in X$ such that $x<y$.

In order to prepare for the introduction of neighborhoods in $\overline{K}$, we shall derive some of the needed properties of $\overline{K}$.

LEMMA 2. *If* X_1, $X_2\in\overline{K}$, *then just one of* $X_1\subset X_2$, $X_1=X_2$, $X_2\subset X_1$ *holds.*

Proof. Suppose $X_1\not\subset X_2$. There is $x_0\in X_1-X_2$. Now $x\in X_2$ implies $x<x_0$.

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Otherwise $x_0 \leqq x$ and $x_0 \in X_2$ which is false. Since $x_0 \in X_1$, $X_2 \subset X_1$.

We define $X_1 < X_2$ in $\overline{K}$ if $X_1 \subset X_2$, $X_1 \neq X_2$ in K. From Lemma 2 we get the following lemma.

Lemma 3. *$\overline{K}$ is simply ordered by the relation $X_1 < X_2$.*

By a direct application of the definitions we have the following lemma.

Lemma 4. *If $E \subset \overline{K}$ is bounded above, then*

$$\sup E = \bigcup_E X \in \overline{K}.$$

If $E \subset \overline{K}$ is bounded below and $M = \bigcap_E X$, then

$$\inf E = M - [\max M] \in \overline{K}$$

where the set $[\max M]$ is empty if M has no maximum.

For $X \in \overline{K}$ and $y \in K$ we define $X+y$ as the set of $x+y$ for all $x < X$. We have the following lemma.

Lemma 5. *$X+y \in \overline{K}$. $(X+y)+z = X+(y+z)$. $y \leqq 0$ implies $X+y \leqq X$. $X_1 \leqq X_2$ implies $X_1+y \leqq X_2+y$.*

We note that the set X_0 of $x \in K$ such that for some real $a = a(x)$, $x \in at^0$ has the properties $X_0 + t^0 = X_0 \in \overline{K}$. The set $X(x_0)$ of $x \in K$ such that $x < x_0$ has the properties $X(x_0) + y \neq X(x_0) \in \overline{K}$ if $y \neq 0$. We call X singular if for some $y \neq 0$, $X+y = X$. We call X regular if it is not singular. From Lemma 5 we have the following lemma.

Lemma 6. *If X is singular then for all $z \in K$, $X+z$ is singular.*

Lemma 7. *X is singular if and only if there is a $\xi < \xi^*$ such that $X + t^{-u_\xi} = X$.*

Proof. The sufficiency is evident. To establish the necessity, suppose X singular. Then there is a $y \neq 0$ such that $X+y = X$. Suppose $y > 0$. Then by Lemma 1 there is a $\xi < \xi^*$ such that $y > t^{-u_\xi} > 0$. Now by Lemma 5, $X = X+y \geqq X + t^{-u_\xi} \geqq X$. Hence $X = X + t^{-u_\xi}$. If $y \leqq 0$, $X = X - y$, by Lemma 5, and the lemma follows.

Lemma 8. *If $X_1 < X_2$ there are $\xi_1, \xi_2 < \xi^*$ such that $X_1 \leqq X_1 + t^{-u_{\xi_1}} < X_2 - t^{-u_{\xi_2}} \leqq X_2$.*

Proof. Since $X_1 < X_2$, there is $y_1 \in X_2 - X_1$. Since X_2 has no maximum, there are $y_2, y_3 \in X_2$ such that $y_1 < y_2 < y_3$. By Lemma 1 there are $\xi_1, \xi_2 < \xi^*$ such that $0 < t^{-u_{\xi_1}} < y_2 - y_1$, $0 < t^{-u_{\xi_2}} < y_3 - y_2$. Now for all $x \in X_1$,

$$x < x + t^{-u_{\xi_1}} < y_1 + t^{-u_{\xi_1}} < y_2 < y_3 - t^{-u_{\xi_2}} < y_3 \in X_2.$$

Hence $y_2 \in (X_2 - t^{-u_{\xi_2}}) - (X_1 + t^{-u_{\xi_1}})$ and the lemma follows. As a corollary we have the following lemma.

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Lemma 9. Sup $[X-t^{-u_\xi}|\xi<\xi^*]=X=\inf [X+t^{-u_\xi}|\xi<\xi^*]$.

We define the neighborhoods in $\overline{K}$ by:

$$U_\xi(X)=\begin{cases}[Y\mid X-t^{-u_\xi}<Y<X+t^{-u_\xi}] & \text{if } X \text{ is regular,}\\ [Y\mid X-t^{-u_\xi}\leqq Y\leqq X+t^{-u_\xi}] & \text{if } X \text{ is singular.}\end{cases}$$

Lemma 10. *If X is singular then for some $\xi<\xi^*$, $U_\xi(X)=X$.*

Proof. This follows from Lemmas 6, 7.

Theorem 10. *The space $\overline{K}$ with the neighborhoods $U_\xi(X)$ satisfies axioms* 1–4.

Proof. That axiom 1 holds follows from Lemma 9. That axiom 2 holds follows from the fact that $\xi_1<\xi_2<\xi^*$ implies $0<t^{-u_{\xi_2}}<t^{-u_{\xi_1}}$ and Lemma 5.

To verify axiom 3 we consider $U_{\eta+1}(X_1)\cap U_{\eta+1}(X_2)\neq 0$. Suppose $X_1\leqq X_2$. Then $X_2-t^{-u_{\eta+1}}\leqq X_1+t^{-u_{\eta+1}}$. Since for all real a, $at^{-u_{\eta+1}}<t^{-u_\eta}$, $X_2+t^{-u_{\eta+1}}\leqq X_1+3t^{-u_{\eta+1}}\leqq X_1+t^{-u_\eta}$. But $X_1-t^{-u_\eta}\leqq X_1-t^{-u_{\eta+1}}\leqq X_2-t^{-u_{\eta+1}}$. Hence $X_1-t^{-u_\eta}\leqq X_2-t^{-u_{\eta+1}}\leqq X_2+t^{-u_{\eta+1}}\leqq X_1+t^{-u_\eta}$. Now if $Y\in U_{\eta+1}(X_2)$ and Y is an end point of $U_\eta(X_1)$, then it is an end point of $U_{\eta+1}(X_2)$. Hence X_2 and $Y=X_2\pm t^{-u_{\eta+1}}$ are singular; but $Y=X_1\pm t^{-u_\eta}$ and so X_1 is singular by Lemma 6. Hence $Y\in U_\eta(X_1)$. A similar argument holds if $X_2<X_1$. Hence $U_{\eta+1}(X_2)\subset U_\eta(X_1)$ and $\xi(\eta)=\eta+1$ serves as the ordinal of axiom 3.

To verify axiom 4 we consider $\eta^*<\xi^*$ and neighborhoods $U_{\xi_\eta}(X_\eta)$ which decrease on $\eta<\eta^*$. Let $E=\bigcap_{\eta<\eta^*}U_{\xi_\eta}(X_\eta)$, $X'_\eta=X_\eta-t^{-u_{\xi_\eta}}$, $X''_\eta=X_\eta+t^{-u_{\xi_\eta}}$. We have for $\eta_1<\eta_2<\eta^*$, $X'_{\eta_1}\leqq X'_{\eta_2}\leqq X''_{\eta_2}\leqq X''_{\eta_1}$. Hence

$$X'=\sup [X'_\eta\mid \eta<\eta^*]\in\overline{K},\qquad X''=\inf [X''_\eta\mid\eta<\eta^*]\in\overline{K},\qquad X'\leqq X''.$$

We show that $X'\in E$ if and only if X' is singular. First suppose X' is singular. For $\eta<\eta^*$, $X'_\eta\leqq X'\leqq X''_\eta$. If either equality holds, $X_\eta\pm t^{-u_{\xi_\eta}}$, and so X_η, is singular by Lemma 6. It follows that $X'\in U_{\xi_\eta}(X_\eta)$ and so $X'\in E$. Next suppose that $X'\in E$ and that X' is regular. Then $X'\neq X_\eta-t^{-u_{\xi_\eta}}$ since if $U_{\xi_\eta}(X_\eta)$ contains an end point, X_η and $X_\eta-t^{-u_{\xi_\eta}}$ are singular. It follows that $X'_\eta<X'$ for $\eta<\eta^*$. By Lemma 8 and the regularity of X' there is $\xi_\eta<\xi^*$ such that $X'_\eta<X'-t^{-u_{\xi_\eta}}<X'$. By (*) $\zeta_0=\sup [\zeta_\eta|\eta<\eta^*]+1<\xi^*$. Since $\zeta_\eta<\zeta_0$, $t^{-u_{\zeta_0}}<t^{-u_{\zeta_\eta}}$ and $X'_\eta<X'-t^{-u_{\zeta_\eta}}<X'-t^{-u_{\zeta_0}}<X'$ for $\eta<\eta^*$. This contradicts the definition: $X'=\sup [X'_\eta\mid\eta<\eta^*]$. Hence X' is singular. A similar argument shows that $X''\in E$ if and only if X'' is singular.

Now suppose $X'=X''=X$. Then X is singular. Otherwise, X is regular. By (*) $\xi_0=\sup [\xi_\eta|\eta<\eta^*]+1<\xi^*$. There is an $\eta<\eta^*$ such that, since X is regular,

$$X-t^{-u_{\xi_0}}<X_\eta-t^{-u_{\xi_\eta}}\leqq X_\eta+t^{-u_{\xi_\eta}}<X+t^{-u_{\xi_0}}.$$

By the definition of order in $\overline{K}$ there are $x\in X$, $x_\eta\in X_\eta$ such that

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$$x - t^{-u_{\xi_0}} < x_\eta - t^{-u_{\xi_\eta}} < x_\eta + t^{-u_{\xi_\eta}} < x + t^{-u_{\xi_0}}.$$

Hence $t^{-u_{\xi_\eta}} < t^{-u_{\xi_0}}$ which is false since $\xi_\eta < \xi_0$. Therefore X is singular, $X \in E$, and X is an inner point of E by Lemma 10. Since $X' = X'' = X$, X is the only point in E. Hence $E = \bigcap_{\eta<\eta^*} U_{\xi_\eta}(x_\eta)$ is a non-empty open set. Finally suppose that $X' < X''$. Let $I = [Y | X' \leqq Y \leqq X'']$, $I_0 = [Y | X' < Y < X'']$. Then $I \supset E \supset I_0 \neq 0$. By Lemma 10, if X' or X'' belongs to E, it is an inner point of E. Hence $\bigcap_{\eta<\eta^*} U_{\xi_\eta}(X_\eta) = I \cap E$ is a non-empty open set.

THEOREM 11. *$\overline{K}$ is ξ^*-complete.*

Proof. Suppose that $U_{\xi_\eta}(X_\eta)$, $\eta = \xi_\eta$, is a well-pinned sequence of neighborhoods in $\overline{K}$. Then for $\zeta < \xi^*$ there is $Y_\zeta \in \overline{K}$ such that $X_\eta - t^{-u_{\xi_\eta}} \leqq Y_\zeta \leqq X_\eta + t^{-u_{\xi_\eta}}$ for $\eta < \zeta$. Hence for $\zeta < \xi^*$, $X_1 - t^{-u_{\xi_1}} \leqq Y_\zeta \leqq X_1 + t^{-u_{\xi_1}}$ and $[Y_\zeta]$ is a bounded set. It follows that

$$\begin{aligned} Z_\lambda &= \sup\,[Y_\zeta \,|\, \lambda \leqq \zeta < \xi^*] \in \overline{K}, \ \lambda < \xi^*, \\ Z_{\lambda_1} &\leqq Z_{\lambda_2}, \ \lambda_2 < \lambda_1, \\ X &= \inf\,[Z_\lambda \,|\, \lambda < \xi^*] \in \overline{K}. \end{aligned}$$

For all $\eta < \xi^*$, $X_\eta - t^{-u_{\xi_\eta}} \leqq X \leqq X_\eta + t^{-u_{\xi_\eta}}$. Hence $X \in \bigcap_{\eta<\xi^*} \overline{U}_{\xi_\eta}(X_\eta)$. It follows from Theorem 6 that $\overline{K}$ is ξ^*-complete.

Further examples arise from function spaces in which the functions have values in a space S satisfying axioms 1–4. Let $P = [p]$ be an arbitrary set. We denote by S_P the set of functions f on P to S. We define $V_\xi(f)$ as the set of functions g such that $g(p) \in U_\xi(f(p))$ for all $p \in P$ where the U_ξ are the neighborhoods in S. It is clear that S satisfies axioms 1–4.

We now remark that axiom 4 cannot be weakened by putting $x_\eta = x$ for all $\eta < \eta^*$, without losing the theorem that the ξ^*-completeness of S implies that S is of the second ξ^*-category. To establish this we consider axiom 4′.

4′. If $\eta^* < \xi^*$ then $\bigcap_{\eta<\eta^*} U_{\xi_\eta}(x_\eta)$ is an open set.

THEOREM 12. *For every $\xi^* > \omega$ there is a space S' satisfying axioms* 1, 2, 3, 4′ *which is ξ^*-complete and of the first ξ^*-category.*

Proof. Consider a denumerable set $P_0 = [p_n | n < \omega]$ and the function space S_{P_0} with neighborhoods $V_\xi(f)$ as defined above. Let θ be a fixed element of the ξ^*-complete space S which is the value domain of the functions $f(p)$. We assume that every neighborhood $U_\xi(\theta)$ of θ contains at least two elements. For $S = \overline{K}$, θ may be taken as any regular element of $\overline{K}$. Let S' be the set of those $f \in S_{P_0}$ such that $f(p) = \theta$ except on a finite set $E(f) \subset P_0$. For each $f \in S'$ let $W_\xi(f) = V_\xi(f) \cap S'$. The space S' with neighborhoods $W_\xi(f)$ satisfies axioms 1, 2, 3, 4′. For: If $g \neq f$ in S' then for some n, $g(p_n) \neq f(p_n)$ and there is $\xi < \xi^*$ such that $g(p_n) \notin U_\xi((fp_n))$ in S. Hence $g \notin W_\xi(f) = V_\xi(f) \cap S'$. Thus axiom 1 is verified. If $\xi_1 < \xi_2 < \xi^*$, then for $f \in S'$, $n < \omega$, $U_{\xi_2}(f(p_n)) \subset U_{\xi_1}(f(p_n))$. Hence $W_{\xi_2}(f) \subset W_{\xi_1}(f)$ and axiom 2 is verified. The ordinal of axiom 3 for S serves

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as the ordinal of axiom 3 for S'. Now consider $\eta^* < \xi^*$ and $W_{\xi_\eta}(f)$ which decrease on $\eta < \eta^*$. Since S satisfies axiom 4, $\bigcap_{\eta<\eta^*} U_{\xi_\eta}(f(p_n))$ contains some $U_{\zeta_\eta}(f(p_n))$ for each n. By property (*) and the hypothesis $\omega < \xi^*$, $\zeta = \sup [\zeta_n | n < \omega] < \xi^*$. By axiom 2 for the space S, $U_\zeta(f(p_n)) \subset U_{\zeta_n}(f(p_n)) \subset \bigcap_{\eta<\eta^*} U_{\zeta_\eta}(f(p_n))$ for all n. Hence $W_\zeta(f) \subset \bigcap_{\eta<\eta^*} W_{\xi_\eta}(f)$. Hence S' satisfies axiom 4'.

We show that S' is ξ -complete. Suppose that f_ξ, $\xi < \xi^*$, is a fundamental sequence in S'. Then f_ξ is a fundamental sequence in S_{P_0}. Since S_{P_0} is ξ^*-complete, there is an $f \in S_{P_0}$ such that $\lim_\xi f_\xi = f$ in the topology of S_{P_0}. Let $E \subset P_0$ be the set on which $f(p_n) \neq \theta$. If E is not finite, $E = [p_{n_k} | k < \omega]$. For each k there is $\zeta_k < \xi^*$ such that $\theta \notin U_{\zeta_k}(f(p_{n_k}))$. Now by (*) $\zeta = \sup [\zeta_k | k < \omega] + 1 < \xi^*$. By axiom 2, $\theta \notin U_\zeta(f(p_{n_k}))$ for $k < \omega$. Since $\lim_\xi f_\xi = f$ in S_{P_0} there is an $\eta_\zeta < \xi^*$ such that, for all $k < \omega$, $f_\xi(p_{n_k})) \in U_\xi(f(p_{n_k}))$ if $\eta_\zeta \leqq \xi < \xi^*$. Hence $f_\xi(p_{n_k}) \neq \theta$ for $k < \omega$ and $\eta_\zeta \leqq \xi < \xi^*$. This is false since $f_\xi \in S'$, $\xi < \xi^*$. Hence E is finite and $f \in S'$. It follows that $\lim_\xi f_\xi = f$ in the topology of S' and so S' is ξ^*-complete.

But S' is of the first ξ^*-category. For: Let E_n be the set of $f \in S'$ such that $f(p_k) = \theta$ for $k > n$. Clearly $S' = \bigcup_n E_n$. For each n and $W_\xi(f)$ there is a $g \in W_\xi(f)$ such that $g(p_{n+1}) \neq \theta$ since every $U_\xi(\theta)$ in S is assumed to have at least two points. Hence there is $W_\eta(g) \subset W_\xi(f)$ such that $W_\eta(g) \cap E_n = 0$. It follows that each E_n is nowhere dense in S' and so S' is of the first ξ^*-category.

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