ON ORDERED DIVISION RINGS

BY

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TABLE OF CONTENTS

PART I. EMBEDDING AN ORDERED GROUP IN A DIVISION RING

1. **Introduction.** The investigation of geometries with certain incidence and order properties but lacking others(1) leads to the study of (fully) ordered division rings. The best-known example of such a division ring is that due to Hilbert [6, §33](2). The problem of constructing more general types of ordered division rings was attacked by Moufang [8], who embeds the group algebra of the free metabelian group of two generators in a division ring and shows this can be ordered; and then constructs a variety of related division rings. Moufang also raises the question whether the group algebra of the free group of two generators can be embedded in a division ring(3). We shall obtain, as a by-product of the first part of this paper, an affirmative answer to this question.

The semigroup algebra of the semigroup G over the ring(4) P is an algebra

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(1) Satisfying "Desargues" but not "Pappus."

(2) Numbers in brackets refer to the bibliography at the end of the paper.

(3) Ordered or not [8, pp. 203, 208].

(4) Usually the ring P of coefficients is assumed to be a division ring or even a field; and G is commonly assumed as a group.

over P whose base elements are (in 1-1 correspondence to, or identified with) the elements of G in such a way that their multiplication in the algebra coincides with their multiplication in the semigroup. The elements of the algebra can then be written as *finite* sums

$$\sum_i g_i\rho_i \qquad\qquad g_i \in G,\ \rho_i \in \mathrm{P},$$

with addition defined in the obvious manner, and multiplication by([5])

$$\Big(\sum_i g_i\rho_i\Big)\Big(\sum_j h_j\sigma_j\Big) = \sum_{i,j} k\Big(\sum_{g_ih_j=k} \rho_i\sigma_j\Big).$$

Here the coefficient of any one k is a finite sum in the coefficient ring because there are only a finite number of g_i and h_j, thus a fortiori only a finite number of pairs of them, for which $g_ih_j=k$.

Hence if we wish to embed the semigroup algebra in a ring in which certain infinite sums $\sum g_i\rho_i$ are also admitted, we have to take care that the equation $g_ih_j=k$ has always only a finite number of solutions g_i, h_j. This can be done, provided the semigroup is (fully) ordered, by means of an idea due to Hahn [4]: one admits only such infinite sums $\sum g_i\rho_i$ in which the g_i, taken in their semigroup order, are well-ordered([6]). In this way one obtains an extension of the semigroup algebra which is called a ring of formal power series.

If P is a field and G an abelian (ordered) group, then this ring is itself a field (Hahn, loc. cit.); and if moreover P is ordered, then the formal power series field can also be ordered. The case of a noncommutative ordered group G has received some attention recently. Schilling [11] sketches a proof that the formal power series ring is then a division ring; but this sketch omits to establish the existence of an inverse to every nonzero element([7]). It has been remarked that Hahn's original proof can be adapted to the noncommutative case; but nobody seems to have done it yet, and it is a formidable proof even in the commutative case.

[*Added in proof*, July 1949.] Dr. Daniel Zelinsky has, however, kindly communicated to me a formal proof along the lines of Hahn's proof, using neither commutativity nor associativity.

In the first part of the present paper we give, therefore, an independent proof that the formal power series ring of an ordered group over a division ring is itself a division ring; this can be ordered if the coefficient division

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([5]) More generally one can introduce factor sets, and so on, into this multiplication.

([6]) Following Hahn [4] one can restrict the cardinal of these sums to be less than some arbitrarily given infinite cardinal.

([7]) Cf. [11, p. 302]: "Observe that each sequence . . . has a limit. . . . " The essence of this observation occupies §§2, 3 of the present paper. Cf., however, also footnote 8.

ring is ordered. The central fact is the existence of an inverse to every nonzero element; and it suffices to establish the existence of an inverse to every formal power series beginning with $e \cdot 1$ (e the group unit, 1 the coefficient unit([8])), and thus involving after this first element only a sum of $g_i\rho_i$ with $g_i > e$ in G. The elements $g > e$ in G form an ordered semigroup G^+, and it is with this rather than with G that we operate in the first three sections. There is a certain gain in generality, because we do not at first assume G^+ embedded or even embeddable in a group; but all the same connect a group with it, corresponding to that of the formal power series beginning with $e \cdot 1$.

The proof then that every ordered group can be embedded in an ordered division ring presents no further difficulty. Now it has been shown elsewhere([9]) that the free group (of any number of generators) can be ordered, thus Moufang's question (cf. above) finds its answer.

The division ring in which we thus embed a free group of, let us say, two generators is a formal power series ring and therefore of the cardinal of the continuum even if the coefficient field is denumerable([10]); it is, in a sense, wastefully large. The group ring of the free group over the field of rationals, say, is denumerable; and so then is the division ring *generated* by this group ring, in any division ring in which it may be embedded. It is, therefore, natural to ask if a different construction would not embed the group ring, less wastefully, in the division ring it generates. It is also natural to look to the known sufficient criteria which tell one what rings possess a division ring of (left or right) quotients (Ore [10], Dubreil [3, chap. 5]).

But these criteria fail in our case; and we end the first part of this paper by showing that there are elements in the group ring of a free group which have no common left multiple. This provides an illustration of the fact([11]) that Ore's sufficient criterion is not necessary([12]).

2. **Archimedean classification in an ordered semigroup.** Let G^+ be a semigroup, that is, a system which is closed with respect to an associative multiplication. Elements of G^+ will be denoted throughout by r, s, t, u, with affixes where required. Let G^+ be fully ordered, so that for any r, s either $r < s$ or $r = s$ or $r > s$, and no two of these take place simultaneously. We further postulate:

(2.1) For all r, s, t, if $r < s$, then $rt < st$ and $tr < ts$.

([8]) I should like to emphasize that the argument in §§2–5 of the present paper follows closely the pattern of Schilling's proof sketch referred to above.

([9]) [9] and the reference to G. Birkhoff and A. Tarski there.

([10]) The last sentence of [6, §33] is erroneous. A similar error occurs in [12, top of p. 36].

([11]) Probably well known.

([12]) I learn from the referee that the results of this part have been "batted around in conversation on this side of the Atlantic." I wish, therefore, to disclaim any originality in this part. [*Added in proof*, July 1949.] A recent paper in Russian [17] appears to anticipate results and methods of this part. Also [18], [19] contain the result for which [9] is here quoted. Other references in [20].

(2.2) For all r, $r<r^2$.

The first of these entails the cancellation laws: if $rt=st$, then $r=s$; if $tr=ts$, then $r=s$. The second condition excludes idempotent elements from G^+; in fact it does more: it ensures that a product always exceeds each factor.

(2.3) LEMMA. *For all r, s,*

$$r < rs \quad \textit{and} \quad s < rs.$$

Proof. If $r \geqq rs$ then $rs \geqq rs^2$ by (2.1); but $s<s^2$ by (2.2) and $rs<rs^2$ by (2.1). Hence $r \geqq rs$ is impossible. Similarly, if $s \geqq rs$ then $rs \geqq r^2s$; but $r<r^2$ and $rs<r^2s$. Hence $s \geqq rs$ is also impossible, and the lemma follows.

We now classify the elements of G^+ according to their "Archimedean" character. If $r^n<s$ for $n=1, 2, \cdots$, we call r infinitely smaller than s and write $r \ll s$. We also call s infinitely greater than r.

(2.4) LEMMA. (i) *If $r \ll s$ then $r<s$, and $r \neq s$.*
(ii) *If $r \ll s$ and $s \ll t$ then $r \ll t$.*
(iii) *If $r \ll s$ but not $t \ll s$, then $r \ll t$.*

The proof is obvious and omitted.

If r is neither infinitely smaller nor infinitely greater than s, we call r and s "relative Archimedean." This is an equivalence relation. The set of all elements relative Archimedean to r is called the Archimedean class of r and denoted by $\eta(r)$.

(2.5) LEMMA. (i) *The statements $s \in \eta(r)$, $r \in \eta(s)$, $\eta(r)=\eta(s)$ are equivalent.*
(ii) *If $r<s<t$ and $\eta(r)=\eta(t)$ then $\eta(r)=\eta(s)$.*
(iii) *$\eta(r^n)=\eta(r)$ for all r, and $n=1, 2, \cdots$.*

The proof is obvious and omitted.

The Archimedean classes can themselves be fully ordered. We put $\eta(r)<\eta(s)$ if and only if $r \ll s$. The usual order properties are confirmed without difficulty.

(2.6) LEMMA. *The Archimedean class of a product is the class of the greatest factor*:

$$\eta(rs) = \max(\eta(r), \eta(s)) = \eta(\max(r, s)); \tag{2.61}$$

$$\eta(\prod r_\nu) = \max_{(\nu)} \eta(r_\nu) = \eta(\max_{(\nu)} r_\nu). \tag{2.62}$$

Proof. The lemma is obviously true for any finite number of factors if it is true for two factors. Now if $r \leqq s$ then $s<rs \leqq s^2$ by (2.3) and (2.1), hence $\eta(rs)=\eta(s)$ by (2.5). Similarly if $r>s$ then $r<rs<r^2$ and $\eta(rs)=\eta(r)$. In any case the lemma is established.

We may therefore look upon the Archimedean classification as a valuation of the semigroup.

3. **Well-ordered sequences in an ordered semigroup.** We consider sets of elements of G^+ which in the order of G^+ are well-ordered; such sets we shall call WO-series.

DEFINITION. A set $S \leqq G^+$ is called a WO-series if every non-null subset of S contains a least element.

If G^+ is nondenumerable one can modify this definition by admitting only subsets whose cardinal does not exceed a given $\aleph_\alpha$. The null-set N is a WO-series, and all results we prove for WO-series apply in particular to N. But to avoid trivial distinctions we shall tacitly assume non-null series where convenient.

(3.1) LEMMA. (i) *S is a* WO-*series if and only if every sequence $s_1, s_2, \cdots$ of elements of S contains a nondecreasing subsequence $s_{\nu(1)} \leqq s_{\nu(2)} \leqq \cdots$.*

(ii) *S is a* WO-*series if and only if every nonincreasing sequence $s_1 \geqq s_2 \geqq \cdots$ of elements of S is ultimately constant, that is, for some n, $s_n = s_{n+1} = \cdots$.*

The proof is obvious and omitted.

(3.2) LEMMA. *If S, T are* WO-*series then $U = ST$ is a* WO-*series.*

Proof. Let

$$u_1 = s_1 t_1, \; u_2 = s_2 t_2, \cdots \qquad (s_\nu \in S; \; t_\nu \in T)$$

be an arbitrary sequence of elements of U. There is a nondecreasing subsequence of $s_1, s_2, \cdots$, let us say $s_{\mu(1)} \leqq s_{\mu(2)} \leqq \cdots$. The corresponding sequence $t_{\mu(1)}, t_{\mu(2)}, \cdots$ in T also contains a nondecreasing subsequence, let us say $t_{\mu(\nu(1))} \leqq t_{\mu(\nu(2))} \leqq \cdots$. Then $u_{\mu(\nu(1))} \leqq u_{\mu(\nu(2))} \leqq \cdots$ is a nondecreasing subsequence of the sequence $u_1, u_2, \cdots$. Hence, by 3.1, U is a WO-series.

(3.21) COROLLARY. *If S, T are* WO-*series then for any fixed $u \in ST$ the equation $u = st$ has only a finite number of solutions, $s \in S$, $t \in T$.*

(3.22) COROLLARY. If $S_1, S_2, \cdots, S_n$ *are* WO-*series then $S_1 S_2 \cdots S_n$ is a* WO-*series.*

(3.23) COROLLARY. *If S is a* WO-*series then so is S^n for any n.*

(3.3) LEMMA. *The union of a finite set of* WO-*series is a* WO-*series. Any subset of a* WO-*series is a* WO-*series.*

The proof is obvious and omitted. Note that the union of an infinite set of WO-series is not necessarily again a WO-series.

(3.4) THEOREM. *Let S be a* WO-*series and denote by S^ω the semigroup generated by S, that is, the union of all S^n, $n = 1, 2, \cdots$. Then S^ω is a* WO-*series.*

Proof. Assume S^ω is not a WO-series. Then:

(3.41) There is a properly decreasing infinite sequence $u_1 > u_2 > \cdots$ of elements of S^ω.

Let

$$u_1 = s_{11}s_{12}\cdot\cdots\cdot s_{1\lambda(1)},$$
$$u_2 = s_{21}s_{22}\cdot\cdots\cdot s_{2\lambda(2)},$$
$$\cdots\cdots\cdots,$$
$$u_\mu = s_{\mu 1}s_{\mu 2}\cdot\cdots\cdot s_{\mu\lambda(\mu)},$$
$$\cdots\cdots\cdots,$$

where all $s_{\mu\nu}\in S$. Denote by s_μ^* the greatest $s_{\mu\nu}$, $\nu=1,\cdots,\lambda(\mu)$:

$$s_\mu^* = \max_{(\nu)} s_{\mu\nu}.$$

Then by (2.6)

$$\eta(u_1) = \eta(s_1^*),$$
$$\eta(u_2) = \eta(s_2^*),$$
$$\cdots\cdots,$$
$$\eta(u_\mu) = \eta(s_\mu^*),$$
$$\cdots\cdots.$$

Amongst $s_1^*, s_2^*, \cdots$ there is a smallest; hence amongst $\eta(s_1^*), \eta(s_2^*), \cdots$ there is a smallest. But as $u_1>u_2>\cdots$, we have $\eta(u_1)\geqq\eta(u_2)\geqq\cdots$. Hence from a certain term on all $\eta(u_\nu)$ are equal; let us say

$$\eta(u_n) = \eta(u_{n+1}) = \cdots = \frown\eta.$$

If we consider different decreasing sequences

$$u_1' > u_2' > \cdots;\ u_1'' > u_2'' > \cdots$$

we are led to possibly different classes $\frown\eta', \frown\eta'', \cdots$. But amongst all these there must be a smallest; for $\frown\eta, \frown\eta', \frown\eta'', \cdots$ are Archimedean classes belonging to certain elements of S, amongst which there must be a least. We may therefore assume, without loss of generality, that:

(3.42) The sequence $u_1, u_2, \cdots$ in (3.41) is chosen so that $\frown\eta$ is as small as possible.

Possibly omitting a finite number of terms from the infinite sequence, we may also assume that already $u_1\in\frown\eta$, and therefore

$$\eta(u_1) = \eta(u_2) = \cdots = \frown\eta.$$

There are elements of S in the class $\frown\eta$, for example $s_1^*, s_2^*, \cdots$; denote by $\frown s$ the least element of S in $\frown\eta$. Then

$$u_1 \geqq s_1^* \geqq \frown s.$$

But as $\eta(u_1)=\eta(\frown s)$, there is a natural number p such that

$$u_1 \leqq \frown s^p,$$

and as $u_\mu < u_1$ for all μ,

$$u_\mu < \frown s^p.$$

We may assume that:

(3.43) The sequence $u_1, u_2, \cdots$ in (3.41) is chosen so that, subject to (3.42), p is as small as possible. Then

$$\frown s^{p-1} < u_\mu < \frown s^p, \qquad \mu = 1, 2, \cdots \tag{3.44}$$

(as $u_\mu \geqq s_\mu^* \geqq \frown s$, $p \geqq 2$: hence $\frown s^{p-1}$ in (3.44) has a meaning in G^+). We now represent each u_μ in one of four forms, namely,

$$u_\mu = s_\mu^*, \tag{3.45}$$

$$u_\mu = u_\mu' s_\mu^*, \qquad u_\mu' \in S^\omega, \tag{3.46}$$

$$u_\mu = s_\mu^* u_\mu'', \qquad u_\mu'' \in S^\omega, \tag{3.47}$$

$$u_\mu = u_\mu' s_\mu^* u_\mu'', \qquad u_\mu', u_\mu'' \in S^\omega. \tag{3.48}$$

Only a finite number of u_μ can be of the first form, for the s_μ^* lie in S and can not form a properly descending infinite sequence. There is then an infinite sequence of u_μ of one of the other three types; let us assume they are all of the form (3.48). Then amongst the u_μ', or amongst the u_μ'', there must be a properly decreasing infinite subsequence. But from (3.44) and $s_\mu^* \geqq \frown s$ we see that

$$u_\mu' < \frown s^{p-1}, \qquad u_\mu'' < \frown s^{p-1}.$$

Thus this new decreasing sequence has the same $\frown\eta$ but a smaller p in (3.44), contrary to (3.43); or a smaller $\frown\eta$, contrary to (3.42). In any case (3.41) leads to a contradiction, and the theorem follows.

(3.5) THEOREM. *With the notation of* (3.4), *any element of* S^ω *lies in only a finite number of the sets* $S, S^2, S^3, \cdots$.

Proof. Let U denote the set of all elements $u \in S^\omega$ which lie in an infinite number of sets $S, S^2, S^3, \cdots$. If U is not the null-set it has (as subset of the WO-series S^ω) a least element $\frown u$, say. $\frown u$ has an infinity of product representations, of increasing lengths:

$$\begin{aligned} \frown u &= s_{11}s_{12}\cdot \cdots \cdot s_{1\lambda(1)} \\ &= s_{21}s_{22}\cdot \cdots \cdot s_{2\lambda(2)} \\ &= \cdots\cdots\cdots \\ &= s_{\mu 1}s_{\mu 2}\cdot \cdots \cdot s_{\mu\lambda(\mu)} \\ &= \cdots\cdots\cdots \end{aligned}$$

with $\lambda(1)<\lambda(2)<\cdots$ and all $s_{\mu\nu}\in S$. Put

$$\frown u = s_{11}u_1 = s_{21}u_2 = \cdots = s_{\mu 1}u_\mu = \cdots \tag{3.51}$$

where

$$u_\mu = s_{\mu 2}\cdot\cdots\cdot s_{\mu\lambda(\mu)} \in S^{\lambda(\mu)-1}. \tag{3.52}$$

The sequence $s_{11}, s_{21}, \cdots$ contains a nondecreasing subsequence $s_{\mu(1),1} \leqq s_{\mu(2),1} \leqq \cdots$ (by 3.1), and the corresponding subsequence $u_{\mu(1)}, u_{\mu(2)}, \cdots$ must be nonincreasing. But as all its terms belong to the WO-series S^ω, this subsequence is ultimately constant. There is, therefore, an element

$$u' = u_{\mu(n)} = u_{\mu(n+1)} = \cdots .$$

From (3.52) we see that u' lies in an infinite number of sets S^n, and thus in U; but (3.51) shows that $u'<\frown u$ contrary to the assumption that $\frown u$ is the smallest element of U. Hence U must be the null-set, and the theorem is established.

4. **Formal power series over a ring.** Now let P be a ring with a set Ω of operators. Elements of P are denoted by ρ, σ, τ, with affixes where required; the zero of P is denoted by 0; the unit element, if P has a unit element, by 1. The operators are denoted generically by ω, the identical operator by ϵ; operators are written as exponents.

DEFINITION. A function ϕ defined on G^+ with values in P will be called a FP-series (formal power series) if there is a WO-series $S=S(\phi)$ such that $\phi(s)\neq 0$ implies $s\in S(\phi)$. The set of all FP-series will be denoted by Π.

In Π we define addition and (right) scalar multiplication in the obvious manner: if $\phi\in\Pi$, $\chi\in\Pi$, $\rho\in$P, then

$$\psi = \phi + \chi, \qquad \psi' = \phi\rho,$$

where

$$\psi(s) = \phi(s) + \chi(s), \tag{4.11}$$

$$\psi'(s) = \phi(s)\rho. \tag{4.12}$$

(4.2) LEMMA. *Π is closed with respect to this addition and scalar multiplication. Π is a* P*-module with respect to this addition and scalar multiplication.*

The proof is obvious and omitted.

In order to define a multiplication in Π we first introduce factor sets.

(4.3) DEFINITION. A pair of functions γ, ω is called a factor set on $\{G^+, \text{P}, \Omega\}$ if γ is defined on G^+, G^+ with values in P and ω is defined on G^+ with values in Ω, and if they satisfy

$$\gamma(r, s)\rho^{\omega(r)\omega(s)} = \rho^{\omega(rs)}\gamma(r, s), \tag{4.31}$$

$$\gamma(rs, t)\gamma(r, s)^{\omega(t)} = \gamma(r, st)\gamma(s, t) \tag{4.32}$$

identically in r, s, t ($\in G^+$) and ρ ($\in$P)([13]). If $\gamma \equiv 0$ we call the factor set degenerate.

We now choose a factor set([14]), and define a multiplication in Π.

(4.4) DEFINITION. If $\phi \in \Pi$, $\chi \in \Pi$, we define the product $\psi = \phi\chi$ by

$$\psi(t) = \sum_{rs=t} \gamma(r, s)\phi(r)^{\omega(s)}\chi(s). \tag{4.41}$$

The usual power notation is used for iterated multiplication([15]):

$$\phi\phi = \phi^2, \qquad \phi\phi^{n-1} = \phi^n.$$

Observe that if $S(\phi) = R$, $S(\chi) = S$, $RS = T$, then $\psi(t) = 0$ unless $t \in T$; and when $t \in T$, then the right-hand side of (4.41) has only a finite number of nonzero terms (cf. 3.21). Hence we have the following lemmas.

(4.5) LEMMA. Π *is closed with respect to the multiplication* (4.4).

(4.6) LEMMA. Π *is a ring with respect to addition and multiplication as defined.*

Associativity of multiplication follows as usual from the factor set identities (4.31)–(4.32). Distributivity is obvious. The zero element of Π is the FP-series which vanishes identically on G^+. We denote it by 0. There is no unit element in Π.

(4.7) THEOREM. *If* $\phi \in \Pi$ *and* $\rho_n \in$ P, $n = 1, 2, \cdots$, *then the power series*

$$\phi^* = \sum_1^\infty \phi^n \rho_n \tag{4.71}$$

is meaningful and $\in \Pi$. *The power series*

$$\bar{\phi} = \sum_1^\infty \pm \phi^n \tag{4.72}$$

(with an arbitrary sequence of signs) is also meaningful and $\in \Pi$([16]).

Proof. If $S(\phi) = S$, then clearly $\phi^*(t) = 0$ unless $t \in S^\omega$. By (3.5), if $t \in S^\omega$, then $\phi^n(t) = 0$ except for a finite number of values of n. Hence for any $t \in G^+$ the right-hand side of (4.71) is a finite sum of nonzero terms only; thus ϕ^* has an obvious meaning. By (3.4), S^ω is a WO-series; hence ϕ^* is a FP-series;

([13]) Note that three different multiplications enter (4.31), (4.32): those in G^+, P, and Ω. They are all denoted by juxtaposition.

([14]) We shall later assume it nondegenerate. If P contains the unit element 1, and if Ω contains the identical operator ϵ, then there certainly is the trivial (nondegenerate) factor set $\gamma \equiv 1$, $\omega \equiv \epsilon$.

([15]) Functional iteration does not apply to elements of Π, as they have different domain and range.

([16]) (4.72) is a special case of (4.71) if P contains a unit element.

similarly for $\bar{\phi}$.

(4.8) DEFINITION. With each element $\phi \in \Pi$ we associate a symbol

$$1 + \phi.$$

The set of all $1+\phi$, for $\phi \in \Pi$, is denoted by Γ. In Γ we define a multiplication by

$$(1 + \phi)(1 + \chi) = 1 + (\phi + \chi + \phi\chi) \tag{4.81}$$

(where addition and multiplication in the brackets on the right-hand side refer to the ring operations in Π).

(4.9) THEOREM. *Γ is a group.*

Proof. Γ is obviously closed with respect to multiplication. One confirms easily that multiplication is associative. The neutral element of multiplication is $1+0$ which we shall abbreviate to 1. If

$$\bar{\phi} = \sum_{1}^{\infty} (-\phi)^n$$

then $1+\bar{\phi}$ is the inverse of $1+\phi$, and by (4.7) is contained in Γ.

(4.91) COROLLARY. *If* P *contains (a subring isomorphic to the field of) all rationals, and if the factor set γ, ω is trivial*[17], *then Γ is a complete group, that is, every element of Γ is an nth power in Γ for every positive integer n.*

This follows from (4.7), using the binomial expansion for roots.

5. **Formal power series division rings.** We now have all the tools to carry out the construction of the formal power series division ring.

Let G be a fully ordered group with order relation $<$ and unit element 1. Denote by G^+ the semigroup of all elements r with $1 < r$. Then G^+ satisfies the assumptions of §2. Let P now be a division ring. The operators ω are then automorphisms of P[18].

The definitions of WO-series, factor sets[19], FP-series, addition and multiplication of FP-series can be extended without difficulty to apply to G instead of G^+. We now assume that the factor set entering the definition of multiplication is nondegenerate, and that for all $r \in G$,

$$\gamma(1, r) = \gamma(r, 1) = 1, \qquad \omega(1) = \epsilon[20]. \tag{5.1}$$

The analogues of (4.2), (4.5), (4.6) are again proved without difficulty.

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[17] That is, $\gamma \equiv 1$, $\omega \equiv \epsilon$.

[18] We exclude the degenerate operator which maps all P on 0.

[19] Not every factor set on $\{G^+, \mathrm{P}, \Omega\}$ can be extended to a factor set on $\{G, \mathrm{P}, \Omega\}$. Thus, for example, if γ vanishes for any arguments in G it must vanish identically.

[20] This entails no loss of generality, as one can show without difficulty; and it cuts out some unnecessary complications.

We denote by π_r the special FP-series defined by

$$\pi_r(r) = 1, \qquad \pi_r(s) = 0 \qquad \text{when } s \neq r. \tag{5.2}$$

Their multiplication is described by

$$\pi_r\pi_s = \pi_{rs}\gamma(r, s). \tag{5.3}$$

π_1 is the unit element of multiplication, and the scalar multiples of π_1 form a division ring isomorphic to P([21]). If ϕ is a FP-series we further introduce ϕ^s by

$$\phi\pi_s = \pi_s\phi^s. \tag{5.4}$$

One can easily calculate ϕ^s explicitly:

$$\phi^s(t) = \gamma(s, t)^{-1}\gamma(sts^{-1}, s)\phi(sts^{-1})^{\omega(s)}. \tag{5.5}$$

Now any nonzero FP-series on G can be related to the FP-series on G^+, or rather to the elements of the group Γ, by representing it in the form

$$\Phi = \pi_r\rho(1 + \phi), \qquad r \in G,\ \rho \in \mathrm{P},\ 1 + \phi \in \Gamma. \tag{5.61}$$

If similarly

$$\Psi = \pi_s\sigma(1 + \psi),$$

then their product is found to be

$$\Phi\Psi = \pi_{rs}\gamma(r, s)\rho^{\omega(s)}\sigma(1 + \sigma^{-1}\phi^s\sigma + \psi + \sigma^{-1}\phi^s\sigma\psi). \tag{5.62}$$

By choosing $s = r^{-1}$, $\sigma = (\gamma(r, s)\rho^{\omega(s)})^{-1}$, and $1+\psi = (1+\sigma^{-1}\phi^s\sigma)^{-1}$ in G, P, Γ respectively, we obtain the inverse of Φ. Hence we have the following theorem.

(5.7) THEOREM. *The FP-series on G with coefficients in* P *form a division ring* Σ.

It may be remarked that the mapping

$$\Phi \to r$$

is a valuation([22]) of Σ. We now assume further that P is an ordered division ring, with order relation $<$; that the operators ω apply to the order of P as well as to the algebraic operations; and that the factor set is positive valued:

$$\gamma(r, s) > 0 \qquad \text{for all } r, s \in G.$$

Then we can order the FP-series by putting

(5.81) $\Phi > 0$ if and only if $\rho > 0$.

One easily satisfies oneself that in this way Σ becomes on ordered division

([21]) If $\pi_1\rho$ is identified with ρ then $\rho\pi_r = \pi_r\rho^{\omega(r)}$; in conjunction with (5.3) this determines the multiplication completely. In (5.62) (below) we write $\sigma^{-1}\phi^s\sigma$ instead of $(\pi_1\sigma)^{-1}(\phi^s\sigma)$, and so on.

([22]) Noncommutative if G is noncommutative. Cf. Schilling [11].

ring. It is also seen without difficulty that

(5.82) $\pi_r < \pi_s$ in Σ if and only if $r > s$ in G.

Finally we observe that if the trivial factor set $\gamma \equiv 1$, $\omega \equiv \epsilon$ is chosen, then the mapping

$$r \leftrightarrow \pi_r$$

is an isomorphism (cf. 5.3). Thus we have the following theorem.

(5.9) THEOREM. *Every ordered group can be embedded in an ordered division ring*([23]).

6. **Appendix. A theorem on the group ring of a free group.** We have as a corollary of our results that the group ring of a free group can be embedded in an ordered division ring. We now show that the Ore criterion (Ore [10]) does not apply to the group ring of a free group of, let us say, two generators.

Let F then be the free group generated by two elements a, b; for the purpose of this section it need not be ordered. We use the "length" of an element of F (and thus a partial order): if

$$g = \prod_{\nu=0}^{n} a^{\alpha_\nu} b^{\beta_\nu} \qquad (\alpha_\nu \neq 0,\ \nu = 1, \cdots, n;\ \beta_\nu \neq 0,\ \nu = 0, \cdots, n-1)$$

then

$$\lambda(g) = \sum |\alpha_\nu| + \sum |\beta_\nu|$$

is called the length of g. The unit element has zero length. We denote it here by e. We denote complexes (that is, subsets of F) by X, Y, and so on. $X+Y$ denotes the set theoretical sum of X and Y; XY the complex formed of all products xy, $x \in X$, $y \in Y$; and the element x and the complex whose only element is x will not here be distinguished.

(6.1) LEMMA. *Let X, Y be complexes in F, not both empty, and let*

$$X + Xa - X \cap Xa \leqq Y + Yb, \tag{6.11}$$

$$Y + Yb - Y \cap Yb \leqq X + Xa. \tag{6.12}$$

Then X or Y is infinite([24]).

Proof. We may assume X finite but not empty. Then $X+Xa$ is also finite and not empty. Let g be an element in $X+Xa$ of maximal length, and consider the elements

([23]) We have actually reversed the order of the group in the embedding process: but that is obviously immaterial.

([24]) In other words: if every element in X or in Xa but not in both is also in Y or in Yb, possibly in both; and if every element in Y or in Yb but not in both is also in X or in Xa, possibly in both; then both X and Y are empty, or else at least one of them has infinitely many elements.

(6.13) $$g_1 = ga, \qquad g_2 = ga^{-1}.$$

As g lies in X or in Xa, g_1 lies in Xa, or g_2 lies in X: in any case one of the two lies in $X+Xa$. Hence

$$\lambda(g_1) \leqq \lambda(g) \quad \text{or} \quad \lambda(g_2) \leqq \lambda(g).$$

However, the definition (6.13) shows that $\lambda(g_i) = \lambda(g) \pm 1$, the negative sign applying only when the last generator of g is cancelled by the $a^{\pm 1}$. Hence g must end in a (positive or negative) power of a.

Also either

$$\lambda(g_1) = \lambda(g) + 1 \quad \text{or} \quad \lambda(g_2) = \lambda(g) + 1,$$

so that either g_1 or g_2 does not lie in $X+Xa$. Hence g cannot lie in both X and Xa, and we see that

$$g \in X + Xa - X \cap Xa;$$

thus by (6.11) also

$$g \in Y + Yb.$$

Therefore any element of maximal length in $X+Xa$ ends in $a^{\pm 1}$ and also lies in $Y+Yb$. The symmetry of the assumptions then shows that such an element cannot be of maximal length in $Y+Yb$, because it does not end in $b^{\pm 1}$; and no longer element can be of maximal length in $Y+Yb$ because it could not lie in $X+Xa$; $Y+Yb$ has no element of maximal length, but is not empty either. Therefore $Y+Yb$ is infinite, and so then is Y.

(6.2) THEOREM. *Let* P *be a field, F the free group of two generators a, b, and* P_F *the group ring of F over* P([25]). *Then the elements*

(6.21) $$\alpha = e \cdot 1 + a \cdot 1 \quad \textit{and} \quad \beta = e \cdot 1 + b \cdot 1$$

have no common (nontrivial) left multiple in P_F.

Proof. Assume that

(6.22) $$\xi\alpha = \eta\beta = \zeta,$$

where

$$\xi = g_1\rho_1 + \cdots + g_m\rho_m, \qquad \eta = h_1\sigma_1 + \cdots + h_n\sigma_n$$

are elements of the group ring, and so is

$$\zeta = k_1\tau_1 + \cdots + k_p\tau_p.$$

We may here assume all $\rho, \sigma, \tau \neq 0$. Denote the sets of components of ξ, η, ζ by

([25]) The theorem can be considerably generalised by weakening the assumptions.

$$X = (g_1, \cdots, g_m), \qquad Y = (h_1, \cdots, h_n), \qquad Z = (k_1, \cdots, k_p).$$

Then clearly

$$X + Xa - X \cap Xa \leqq Z \leqq X + Xa$$

and also

$$Y + Yb - Y \cap Yb \leqq Z \leqq Y + Yb.$$

Hence Lemma (6.1) applies; and as X and Y are finite, they must be empty. Then $\xi = \eta = 0$, and (6.22) can only be trivially satisfied. This proves the theorem, and shows that the Ore criterion does not apply to the group ring (over any coefficient ring) of the free group.

Thus we see that if we introduce quotients into the group ring P_F[26], the resulting system will be closed with respect to multiplication and division (by elements $\neq 0$), but not with respect to addition; if we now introduce (finite) sums, we presumably lose again the multiplicative group property. Thus we can alternately close the system with respect to multiplication and (nonzero) division, and with respect to addition[27]. In this way we get a "tower" of systems, all in any division ring which contains the group ring. Adaptation of the classical Steinitz argument (Steinitz [13]) shows that the union of the tower systems is the division ring generated by P_F. It is still an open question whether the tower is finite or not; but it seems reasonable to conjecture that the tower must be infinite.

Part II. Embedding the real numbers in an ordered division ring

7. **Introduction.** The second part is independent of the first, and deals with a different problem. It is well known that every ordered field can be embedded in an ordered field enjoying certain completeness or continuity properties[28]; in particular every ordered field can be so extended as to contain (a subfield order-isomorphic to) the field of all real numbers[29]. The main result we prove here is that this remains true for ordered division rings. The methods required to establish this are, however, much more elaborate than in the commutative case.

Following the classical Steinitz procedure we adjoin elements singly. The elements we adjoin are characterised by their algebraic relationships to "real

(26) This is certainly legitimate, Ore criterion or no, as the group ring can be embedded in a division ring.

(27) Subtraction looks after itself, because -1 is already present in P.

(28) Cf. Hahn [4] for a discussion of Hilbert completeness and Veronese continuity; Artin and Schreier [2] for the algebraical theory of ordered fields.

(29) This latter fact is *not* well known. Completion of an ordered field (for example, by means of Cauchy filter bases) does not in general complete a subfield with respect to the order topology of the subfield. This fact has been repeatedly overlooked. The result quoted in the text follows, for example, from results of MacLane [16]; his method is not available in the non-commutative case.

numbers" already present in the division ring, and by their order relations with respect to them([30]). Accordingly we have to prove two things: Firstly, that the adjunction of a certain element is algebraically possible; secondly, that then the resulting extension can be ordered so as to continue the order of the given division ring and at the same time allocate to the newly adjoined element just the "right" order relations with respect to the real numbers already present.

After some preliminaries (§8) we prove (§9) that the adjunction of every element required in the process is algebraically possible. The adjunction of elements which are transcendent over the field P of real numbers already present is almost trivial. Not so the adjunction of algebraic elements over P; for it is well known that in the case of division *algebras* of finite degree over their centre certain algebraic extensions of the centre are impossible([31]).

In §10 we show that a simple transcendent extension of an ordered division ring can be ordered so as to place the adjoined transcendent in its "proper" place relative to the real numbers already present. This can in general be done in many different ways; only the "macroscopic order"—placing rational functions of the transcendent in relation to real numbers—is uniquely prescribed.

Ordering a simple algebraic extension requires very elaborate preparation; at any rate if the element θ to be adjoined is going to be "infinitely near" to elements ζ already present in the division ring([32]). The fundamental idea—very roughly speaking and under certain restrictions—is to show that a polynomial which does not vanish at θ does not change sign in a sufficiently small neighbourhood of θ; so that its sign at θ can be defined as its sign near θ. But "a sufficiently small neighbourhood of θ" has no obvious meaning before θ has been placed relative to the elements ζ; and the idea here sketched will be hardly recognisable when the details are filled in (§§11–16, especially §15).

Once all transcendent elements and all algebraic elements infinitely near to elements already present have been adjoined, and the resulting extensions ordered, there remain only algebraic elements which are not going to be infinitely near to any elements already present in the division ring. They pre-

([30]) What we mean by the real numbers already present in the given division ring will be described more precisely in §8. Amongst them are in particular the rational numbers, that is, the elements of the prime field of the given division ring; and all real numbers can of course be distinguished from each other (though not necessarily from other elements of the division ring) by their order relations relative to the rational numbers.

([31]) That is, introduce proper zero divisors; cf. Jacobson [7]. Of course such division algebras can not be ordered (cf. Wagner's Theorem 18.7). This then is the point where the fact that our division rings are *ordered* has *algebraic* consequences.

([32]) By this we mean that there are (of necessity infinitely many) elements ζ in the division ring which have precisely the same order relations to the real numbers as θ will have to have.

sent no further problem, macroscopic order being all that is required in their case (§17).

8. **Preliminaries and notation.** We denote by $\hat{}\mathbf{P}^*$ the field of real numbers in their natural order; by $\hat{}\mathbf{P}_0$ the prime field of rational numbers contained in it. Subfields of $\hat{}\mathbf{P}^*$ and elements of $\hat{}\mathbf{P}^*$ (that is, real numbers) will be marked by the same accent $\hat{}$, where they are used for purposes of comparison. Thus $\hat{}0$, $\hat{}1$ are the zero and unit element of $\hat{}\mathbf{P}^*$; but suffixes, exponents, and so on, are written without accents.

Let Σ be an ordered division ring, Z its centre, $\mathbf{P}_0$ the prime field of Σ. Then $\mathbf{P}_0 \leqq \mathbf{Z}$, and $\mathbf{P}_0$ is order-isomorphic([33]) to $\hat{}\mathbf{P}_0$. The aim of this part is to show that Σ can be extended to an ordered division ring $\Sigma^* \geqq \Sigma$ which contains in its centre $\mathbf{Z}^*$ a subfield $\mathbf{P}^*$ order-isomorphic to $\hat{}\mathbf{P}^*$.

Following the method of Steinitz [13] we adjoin new elements one at a time, first the transcendent elements, then the algebraic elements. In fact we describe only simple extensions: the usual well-order and "tower" arguments are omitted.

The order-isomorphism([34]) between $\mathbf{P}_0$ and $\hat{}\mathbf{P}_0$ sets up a "natural" mapping

$$\sigma \rightarrow \hat{}\sigma$$

of Σ into the compact system obtained from $\hat{}\mathbf{P}^*$ by adding two elements $+\hat{}\infty$ and $-\hat{}\infty$ with the usual algebraic and order conventions. Thus if $\sigma \in \Sigma$ and if $\sigma > \rho_0$ for all $\rho_0 \in \mathbf{P}_0$, then we put $\hat{}\sigma = +\hat{}\infty$; if $\sigma < \rho_0$ for all $\rho_0 \in \mathbf{P}_0$, then we put $\hat{}\sigma = -\hat{}\infty$. If neither of these two cases obtains, then those $\rho_0 \in \mathbf{P}_0$ which are greater than σ on the one hand, and those which are smaller than σ on the other hand, define a Dedekind section in $\mathbf{P}_0$, and thus a Dedekind section in $\hat{}\mathbf{P}_0$. This in its turn defines a unique real number $\hat{}\sigma \in \hat{}\mathbf{P}^*$. This mapping will be used throughout.

(8.1) LEMMA. *If* $\sigma > \tau$ *then* $\hat{}\sigma \geqq \hat{}\tau$.

(8.2) LEMMA. *If* $\sigma_1 \in \Sigma$, $\sigma_2 \in \Sigma$, $\hat{}\sigma_1 \in \hat{}\mathbf{P}^*$, $\hat{}\sigma_2 \in \hat{}\mathbf{P}^*$, *and* $\sigma_1 + \sigma_2 = \tau_1$, $\sigma_1\sigma_2 = \tau_2$, *then*

$$\hat{}\tau_1 \in \hat{}\mathbf{P}^* \quad \textit{and} \quad \hat{}\tau_1 = \hat{}\sigma_1 + \hat{}\sigma_2;$$
$$\hat{}\tau_2 \in \hat{}\mathbf{P}^* \quad \textit{and} \quad \hat{}\tau_2 = \hat{}\sigma_1\hat{}\sigma_2.$$

The proofs are obvious and omitted.

It follows that those elements of Σ which are mapped onto elements of $\hat{}\mathbf{P}^*$ (that is, the elements not infinitely large compared with the unit element of Σ) form a ring, which we denote by $\mathfrak{R}$. The mapping of $\mathfrak{R}$ into $\hat{}\mathbf{P}^*$ is a homomorphism. The kernel of this homomorphism consists of the ele-

([33]) We distinguish between "isomorphism," relating to the algebraic operations, and "order-isomorphism," relating to the order as well.

([34]) There is only one.

ments mapped onto the zero $\frown 0$ of $\frown P^*$ (that is, the elements infinitely small compared with the unit element of Σ); it is a (two-sided) maximal prime ideal in $\mathfrak{N}$ which will be denoted by $\mathfrak{P}$. We shall sometimes refer to the elements of $\Sigma - \mathfrak{N}$, $\mathfrak{N}$, $\mathfrak{P}$ simply as "infinite," "finite," "infinitely small" elements.

(8.3) LEMMA. *The map* $\frown\mathfrak{N}$ *of* $\mathfrak{N}$ *in* $\frown P^*$ *is a field.*

The proof is obvious and omitted. $\frown\mathfrak{N}$ may, of course, coincide with $\frown P^*$ or with $\frown P_0$.

We denote by P an arbitrary subfield of $Z \cap \mathfrak{N}$. That there are subfields of $Z \cap \mathfrak{N}$ is obvious: P_0 is one of them. The mapping $\mathfrak{N} \to \frown\mathfrak{N}$ maps P onto a subfield $\frown P$ of $\frown P^*$. This mapping, being a homomorphism between two fields, must be an isomorphism; we shall repeatedly use the fact that the step from elements of $\frown P$ back to elements of P is unique.

Amongst the subfields P of $Z \cap \mathfrak{N}$ there are maximal subfields; for the union of an increasing chain of such fields is again a field, and again in the centre Z, and again in $\mathfrak{N}$. The maximal subfields of $Z \cap \mathfrak{N}$ are not in general uniquely determined; they need not even be isomorphic. We denote one of these maximal subfields by P_{max}.

We may of course assume that $\frown P_{max} \neq \frown P^*$—otherwise no extension would be required. It is then possible to adjoin to $\frown P_{max}$ an element $\frown\theta \in \frown P^* - \frown P_{max}$; the corresponding adjunction of an element θ to P_{max}, and thus to Σ, forms the subject of the following paragraphs. We have to distinguish various cases, for example, according as $\frown\theta$ is transcendent or algebraic over $\frown P_{max}$. In much of what follows the maximality of P_{max} is not essential, and we operate with an arbitrary P.

We shall make extensive use of the polynomial domains $\Sigma[x]$, $\mathfrak{N}[x]$, $\mathfrak{P}[x]$, $P[x]$, $\frown P[\frown x]$. The variable will throughout be assumed commutative with the coefficients. Following Albert [1] we call a polynomial "monic" if its highest coefficient is 1.

Finally we mention two simple facts which will be found useful:

(8.4) LEMMA. *If* $\rho \in \mathfrak{N}$, $\sigma \in \Sigma$, $\rho' = \sigma^{-1}\rho\sigma$, *then* $\frown\rho' = \frown\rho$.

Proof. The mapping $\rho \to \rho'$ is an order-automorphism of Σ. Elements of the centre, and in particular elements of P_0, are left invariant by it. Hence ρ and ρ' define the same Dedekind section in P_0, $\frown\rho$ and $\frown\rho'$ define the same Dedekind section in $\frown P_0$, and are therefore the same real number.

(8.5) COROLLARY. *If* $p \in \mathfrak{N}[x]$, $\sigma \in \Sigma$, $p' = \sigma^{-1}p\sigma$, *then* $\frown p' = \frown p$.

9. **Algebraic adjunction of an element.** The process of adjoining an element θ to Σ naturally divides into two parts: Adjoining θ algebraically, that is, forming a division ring $\Sigma(\theta)$, not yet ordered, which contains, and is generated by, Σ and θ; and then ordering $\Sigma(\theta)$. In this paragraph we deal with the first step.

We say that "θ can be adjoined to Σ" or that the adjunction of θ to Σ is "possible" if the result of such an adjunction is again a *division* ring. If the division ring of real quaternions is extended by the algebraic adjunction of $(-1)^{1/2}$ to the centre, the extension which results (namely, quaternions with complex coefficients) is no longer a division ring; we should, therefore, say that $(-1)^{1/2}$ cannot be adjoined to the real quaternions (so as to lie in the centre).

Let $\frown\theta$ be chosen in $\frown$P* but not in $\frown$P([25]). If $\frown\theta$ is transcendent over $\frown$P, we have to adjoin a transcendent θ to P: we do this by adjoining a transcendent to Σ.

(9.1) LEMMA. *A (commutative) transcendent θ can be adjoined to any([36]) division ring Σ.*

Proof. We first form the polynomial ring $\Sigma[\theta]$ of a variable θ, commutative with all coefficients, and then the left-quotient division ring $\Sigma(\theta)$. That this exists follows directly from results of Ore [10]([37]).

Now let $\frown\theta$ be algebraic over $\frown$P, and let

$$(9.21) \qquad \frown f(\frown x) = \frown x^n + \frown\alpha_{n-1}\frown x^{n-1} + \cdots + \frown\alpha_0, \qquad \frown\alpha_\nu \in \frown \mathrm{P},$$

be the irreducible monic polynomial with coefficients in $\frown$P of which $\frown\theta$ is a root:

$$(9.22) \qquad \frown f(\frown\theta) = \frown 0.$$

Then correspondingly

$$(9.23) \qquad f(x) = x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0, \qquad \alpha_\nu \in \mathrm{P},$$

is the irreducible monic polynomial with coefficients in P a root of which we have to adjoin to P([38]). Note that (9.23) is uniquely determined by the isomorphism between P and $\frown$P([39]).

(9.3) LEMMA. *If $f(x)$ is irreducible over Σ, then a root θ of $f=0$ can be adjoined to Σ.*

Proof. In the polynomial ring $\Sigma[x]$ (of the *commutative* variable x), $f(x)$ generates a two-sided principal ideal without right-ideal divisors; the residue class ring with respect to this ideal is the division ring $\Sigma(\theta)$([40]). Note incidentally that θ lies in the centre of $\Sigma(\theta)$.

([35]) Not assumed maximal.

([36]) Not necessarily ordered.

([37]) Alternatively one can form the power series division ring by means of the free abelian group of one generator θ in the obvious order (cf. first part of this paper), then pick out the division ring generated by θ. Cf. also Dubreil [3, chap. 5] and Jacobson [7, chap 3].

([38]) Such a root may or may not be present in Σ already.

([39]) The notation (9.2), especially (9.23), will remain in use throughout.

([40]) Cf., for example, Jacobson [7, chap. 3]. Σ need not be ordered for this.

(9.4) LEMMA. *If $f(x)$ is reducible over Σ*

$$f(x) = g(x) \cdot h(x) \tag{9.41}$$

and if g and h are monic, then all coefficients of g and h lie in $\mathfrak{R}$[41].

Proof. Let

$$g(x) = x^k + \beta_{k-1}x^{k-1} + \cdots + \beta_0, \tag{9.42}$$

$$h(x) = x^l + \gamma_{l-1}x^{l-1} + \cdots + \gamma_0, \tag{9.43}$$

and assume that not all $\beta_\pi \in \mathfrak{R}$. Let κ be the greatest suffix for which β_κ is not in $\mathfrak{R}$ (that is, infinite). Consider the coefficient $\alpha_{\kappa+l}$ of $x^{\kappa+l}$ in $f(x)$

$$\alpha_{\kappa+l} = \beta_\kappa + \beta_{\kappa+1}\gamma_{l-1} + \beta_{\kappa+2}\gamma_{l-2} + \cdots .$$

As $\alpha_{\kappa+l}$ is finite, and so are $\beta_{\kappa+1}, \beta_{\kappa+2}, \cdots$, at least one of the γ_λ must be infinite.

Now let

$$\beta = \max(|\beta_0|, |\beta_1|, \cdots, |\beta_{k-1}|), \qquad \gamma = \max(|\gamma_0|, |\gamma_1|, \cdots, |\gamma_{l-1}|),$$

and put

$$g' = \gamma^{-1}\beta^{-1}g\gamma, \qquad h' = \gamma^{-1}h.$$

Then the coefficients of g' and h' all lie between -1 and $+1$, and therefore certainly in $\mathfrak{R}$. Also both g' and h' have at least one coefficient equal to ± 1, so neither of them lies in $\mathfrak{P}[x]$. Hence their product $g'h'$ does not lie in $\mathfrak{P}[x]$ either[42]. But then, as β and γ are infinite,

$$f(x) = \beta\gamma g'(x)h'(x)$$

has at least one infinite coefficient, which is absurd. Hence g can have no infinite coefficient; similarly for h, and the lemma is proved.

(9.44) COROLLARY. *If $\theta \in \Sigma$ is algebraic over a field $\mathbf{P} \leqq \mathbf{Z} \cap \mathfrak{R}$ then $\theta \in \mathfrak{R}$.*

For $f(x)$ (given by (9.23)) then has $x - \theta$ as left factor in $\Sigma[x]$ (cf. Jacobson [7, chap. 3]):

$$f(x) = (x - \theta)h(x),$$

and the lemma applies.

(9.5) LEMMA. *If $f(x)$ is factorised in two ways into monic factors*

$$f = gh = g'h' \tag{9.51}$$

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[41] This is analogous to Gauss' lemma. The absence of commutativity is seen to make little difference to the proof.

[42] Mapping $\mathfrak{R}[x]$ onto $\widehat{\mathfrak{R}}[\widehat{x}]$ we have $\widehat{g'} \neq \widehat{0}$, $\widehat{h'} \neq \widehat{0}$, hence $\widehat{(g'h')} \neq \widehat{0}$. Or else: as $\mathfrak{P}$ is a prime ideal in $\mathfrak{R}$, $\mathfrak{P}[x]$ is a prime ideal in $\mathfrak{R}[x]$.

and if g and g' differ infinitely little:

$$g \equiv g' \pmod{\mathfrak{P}[x]} \tag{9.52}$$

then they are identical:

$$g = g', \qquad h = h'. \tag{9.53}$$

Proof. Let g, h be given by (9.42), (9.43) and

$$g'(x) = x^k + \beta'_{k-1}x^{k-1} + \cdots + \beta'_0, \tag{9.54}$$

$$h'(x) = x^l + \gamma'_{l-1}x^{l-1} + \cdots + \gamma'_0. \tag{9.55}$$

By (9.4) all coefficients β_κ, γ_λ, β'_κ, γ'_λ are in $\mathfrak{R}$. Now by (9.51)

$$gh - g'h' = (g - g')h + g'(h - h') = 0, \tag{9.56}$$

identically in x. Here $g-g'$ is a polynomial of degree at most $k-1$. Assume now that it is not identically 0, and let

$$\delta = \max\left(|\beta_0 - \beta'_0|, \cdots, |\beta_{k-1} - \beta'_{k-1}|\right) \neq 0.$$

Put

$$g'' = (g - g')\delta^{-1}, \qquad h'' = \delta h\delta^{-1}, \qquad h''' = (h - h')\delta^{-1}.$$

Then (9.56) leads to

$$g''h'' + g'h''' = 0. \tag{9.57}$$

From its definition, g'' has coefficients between -1 and $+1$; at least one of them is equal to ± 1; hence

$$\frown g'' \neq \frown 0.$$

Also h'' is a transform of h; hence (by 8.5)

$$\frown h'' = \frown h.$$

Moreover, from (9.52)

$$\frown g' = \frown g.$$

Finally, as g'', h'', g' all have finite coefficients and g' is monic, one sees from (9.57) that h''' also has finite coefficients, and $\frown h'''$ is meaningful in $\frown\mathrm{P}^*[\frown x]$. Thus, mapping (9.57) into $\frown\mathrm{P}^*[\frown x]$, we obtain

$$\frown g''\frown h + \frown g\frown h''' = \frown 0. \tag{9.58}$$

But the degree of $\frown g''$ is at most $k-1$, that is, less than the degree k of $\frown g$. Hence $\frown g$ and $\frown h$ must have a nonconstant common factor, and $\frown f = \frown g\frown h$ has a multiple factor. But this it cannot have, as it is irreducible over a (trivially) separable field. Thus the assumption that $g-g'$ is not zero leads

to a contradiction, and the lemma is proved.

(9.6) THEOREM. *If f factorises in $\Sigma[x]$ into monic factors, then these factors lie in $\mathrm{Z}[x]$.*

Proof. Let $f=gh$, g and h monic, and let $\sigma\in\Sigma$ be arbitrary. Then

$$f = \sigma^{-1}f\sigma = \sigma^{-1}g\sigma\cdot\sigma^{-1}h\sigma.$$

Putting $\sigma^{-1}g\sigma=g'$, $\sigma^{-1}h\sigma=h'$, we see from (9.3) that g, h, g', h' all lie in $\mathfrak{R}[x]$, then that g' and h' are also monic, from (8.5) that g' differs infinitely little from g, from (9.5) that g and g' are identical. Hence $\sigma g=g\sigma$, that is, g is a centre polynomial. So then is h, and the theorem is proved.

(9.71) COROLLARY. *If $\theta\in\Sigma$ is algebraic over a field $\mathrm{P}\leqq\mathrm{Z}\cap\mathfrak{R}$, then $\theta\in\mathrm{Z}$.*

For $f(x)$ (given by (9.23)) then has $x-\theta$ as left factor in $\Sigma[x]$[43]:

$$f(x) = (x-\theta)h(x),$$

and the theorem applies. This fact should be contrasted with the situation in division *algebras* (of finite degree) over a real number field; it illustrates the (well known) transcendental character of ordered properly noncommutative division rings. One naturally expects that the restriction to a field in $\mathrm{Z}\cap\mathfrak{R}$ is too stringent, and that in fact every algebraic element over the centre lies in the centre. This is indeed true, and can be proved very simply and elegantly[44] by an argument used by Albert [15] to prove a theorem of Wagner [14].

(9.72) COROLLARY. *If $\theta\in\Sigma$ is algebraic over a field $\mathrm{P}\leqq\mathrm{Z}\cap\mathfrak{R}$, then $\mathrm{P}(\theta)\leqq\mathrm{Z}\cap\mathfrak{R}$.*

For every element of $\mathrm{P}(\theta)$ is algebraic over P; so (9.44) and (9.71) apply.

(9.73) COROLLARY. *If $\mathrm{P}_{\max}$ is maximal in $\mathrm{Z}\cap\mathfrak{R}$, then Σ contains no algebraic elements over $\mathrm{P}_{\max}$ outside $\mathrm{P}_{\max}$.*

(9.8) THEOREM. *If $f\in\mathrm{P}_{\max}[x]$ is irreducible there then f is irreducible in $\Sigma[x]$.*

Proof. If f factorises in $\Sigma[x]$ then by (9.6) it factorises in $\mathrm{Z}[x]$; hence commutative algebra applies. In particular the coefficients of any factor of f in $\mathrm{Z}[x]$ are algebraic over $\mathrm{P}_{\max}$, hence lie in $\mathrm{P}_{\max}$ by (9.73). But then f factorises already in $\mathrm{P}_{\max}[x]$. Thus the theorem is established.

(9.9) COROLLARY. *If P is chosen as a maximal subfield $\mathrm{P}_{\max}\leqq\mathrm{Z}\cap\mathfrak{R}$, then no algebraic element over $\mathrm{P}_{\max}$ is contained in Σ; but the adjunction of any such element to Σ (and so to $\mathrm{P}_{\max}$) is possible.*

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(43) Cf. (9.44) above.

(44) The author wishes to thank the referee for pointing this out.

This follows immediately from (9.73), (9.8), (9.3).

10. **Ordering a simple transcendent extension.** We now have to show how an extension $\Sigma(\theta)$ of Σ can be ordered. We are interested only in such orderings of $\Sigma(\theta)$ which continue the given order of Σ and at the same time allot the same order relations to θ with respect to elements $\sigma\in\Sigma$ as $\widehat{\ }\theta$ has with respect to the corresponding $\widehat{\ }\sigma$; and this is implied when we say that $\Sigma(\theta)$ can be ordered. In fact the requirements upon the order relations of θ and the elements $\sigma\in\Sigma$ will evidently be satisfied as soon as θ has the "correct" order relations with respect to the rationals, that is, if

$$\widehat{\ }\rho_1 < \widehat{\ }\theta < \widehat{\ }\rho_2 \text{ implies } \rho_1 < \theta < \rho_2 \tag{10.1}$$

when $\rho_1, \rho_2\in P_0$, and $\widehat{\ }\rho_1$, $\widehat{\ }\rho_2$ are the corresponding elements in $\widehat{\ }P_0$.

In this section we deal with the case that $\widehat{\ }\theta$ is transcendent over $\widehat{\ }P$ or, what amounts to the same (§9), that θ is transcendent over Σ. The elements of $\Sigma(\theta)$ are of the form (cf. (9.1)) $Q^{-1}P$ where $P\in\Sigma[\theta]$, $Q\in\Sigma[\theta]$ are polynomials in θ. We first order $\Sigma[\theta]$; the ordering of $\Sigma(\theta)$ will then follow easily. In fact we begin by ordering only those polynomials whose coefficients are finite but not all infinitely small.

Let

$$p = \sum_0^n \rho_\nu\theta^\nu \in \mathfrak{R}[\theta] - \mathfrak{P}[\theta] \tag{10.21}$$

be a polynomial in θ, and put

$$\widehat{\ }p(\widehat{\ }x) = \sum_0^n \widehat{\ }\rho_\nu\widehat{\ }x^\nu. \tag{10.22}$$

Then $\widehat{\ }p(\widehat{\ }\theta)$ is the element of $\widehat{\ }P^*$ onto which p is to be mapped by the mapping $\theta\to\widehat{\ }\theta$[45]. Hence we define (as we have to, by (10.1))

$$p > 0 \qquad \text{when } \widehat{\ }p(\widehat{\ }\theta) > \widehat{\ }0, \tag{10.31}$$

and correspondingly

$$p < 0 \qquad \text{when } \widehat{\ }p(\widehat{\ }\theta) < \widehat{\ }0. \tag{10.32}$$

This we shall refer to as the "macroscopic ordering"[46]; it allocates a sign to p whenever $\widehat{\ }p(\widehat{\ }\theta)\neq\widehat{\ }0$.

If $\widehat{\ }\theta$ is algebraic over $\widehat{\ }\mathfrak{R}$[47], denote by

$$\widehat{\ }g(\widehat{\ }x) = \widehat{\ }x^k + \widehat{\ }\beta_{k-1}\widehat{\ }x^{k-1} + \cdots + \widehat{\ }\beta_0 \tag{10.41}$$

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[45] Observe that to any $p\in\mathfrak{R}[\theta]-\mathfrak{P}[\theta]$ there corresponds a unique *polynomial* $\widehat{\ }p(\widehat{\ }x)$ and a unique *real number* $\widehat{\ }p(\widehat{\ }\theta)$.

[46] It is (in general) a partial ordering only.

[47] $\widehat{\ }\theta$ is transcendent over $\widehat{\ }P$, but may be algebraic over $\widehat{\ }\mathfrak{R}$ ($\geqq\widehat{\ }P$). In fact $\widehat{\ }\theta\in\widehat{\ }\mathfrak{R}$ is not excluded. In this case $\widehat{\ }g$ is linear.

the monic irreducible polynomial in $\frown\mathfrak{R}[\frown x]$ of which $\frown\theta$ is a root:

$$\frown g(\frown\theta) = \frown 0. \tag{10.42}$$

Then to $\frown p(\frown x)$ (given by (10.22)) we can find a number $\lambda \geqq 0$ and another polynomial $\frown q(\frown x) \in \frown\mathfrak{R}[\frown x]$ such that

$$\frown p(\frown x) = (\frown g(\frown x))^{\lambda} \frown q(\frown x), \qquad \frown q(\frown\theta) \neq \frown 0. \tag{10.43}$$

We now define

$$p > 0 \qquad \text{if } \frown q(\frown\theta) > \frown 0, \tag{10.51}$$

$$p < 0 \qquad \text{if } \frown q(\frown\theta) < \frown 0. \tag{10.52}$$

This contains the macroscopic ordering (10.3) as a special case, namely, $\lambda = 0$. There is a certain arbitrariness in this definition. Thus, for example, if

$$g = \theta^k + \beta_{k-1}\theta^{k-1} + \cdots + \beta_0$$

is any polynomial[48] in θ whose coefficients map onto those of $\frown g(\frown x)$, then (10.5) decrees that $g > 0$. One could equally well make $g < 0$ by defining

$$p \gtrless 0 \quad \text{according as } (-\frown 1)^{\lambda} \frown q(\frown\theta) \gtrless \frown 0;$$

and other, more complicated, ordering conventions can also be devised, unless $\frown\theta$ is transcendent over $\frown\mathfrak{R}$, when the macroscopic ordering already exhausts all possibilities.

The ordering of $\Sigma[\theta]$ and $\Sigma(\theta)$ is now automatic. If $P \in \Sigma[\theta]$ is any non-zero polynomial, denote by π the greatest of the absolute values of its coefficients, and put

$$P = \pi p.$$

Then the coefficients of p are finite because they lie between -1 and $+1$, and they are not all infinitely small because at least one of them is ± 1. Hence (10.5) applies to p, and we put

$$P \gtrless 0 \qquad \text{according as } p \gtrless 0. \tag{10.61}$$

Finally if $Q^{-1}P$ is any element of $\Sigma(\theta)$, we define

$$Q^{-1}P \gtrless 0 \qquad \text{according as } QP \gtrless 0[49]. \tag{10.62}$$

We now have to show that we have in fact defined an order in $\Sigma[\theta]$. For the remainder of this section $p, p_1, \cdots, q, q_1, \cdots$ are polynomials with finite coefficients not all infinitely small[50], that is, in $\mathfrak{R}[\theta] - \mathfrak{P}[\theta]$; $P, P_1, \cdots, Q, Q_1, \cdots$ are polynomials with arbitrary coefficients, that is, in $\Sigma[\theta]$, and

[48] g is not uniquely determined by (10.22), but only modulo $\mathfrak{P}[\theta]$.

[49] It is evidently sufficient to define what is to be positive and what negative; then any two elements can be compared by the sign of their difference.

[50] Except in the proof of (10.83).

the elements of $\Sigma(\theta)$ are represented in the form $Q^{-1}P$.

(10.71) LEMMA. *If* $p_1>0$, $p_2>0$, *then* $p_1+p_2>0$ *and* $p_1p_2>0$.

Proof. Introduce (as in (10.43)) λ, μ, $\frown q_1(\frown x)$, $\frown q_2(\frown x)$ by

$$\frown p_1(\frown x) = (\frown g(\frown x))^{\lambda}\frown q_1(\frown x), \qquad \frown q_1(\frown\theta) > \frown 0,$$
$$\frown p_2(\frown x) = (\frown g(\frown x))^{\mu}\frown q_2(\frown x), \qquad \frown q_2(\frown\theta) > \frown 0,$$

and similarly ν, $\frown q(\frown x)$ by

$$\frown p_1(\frown x) + \frown p_2(\frown x) = (\frown g(\frown x))^{\nu}\frown q(\frown x), \qquad \frown q(\frown\theta) \neq \frown 0.$$

Then $\nu=\min(\lambda, \mu)$ and

$$\frown q(\frown\theta) = \frown q_1(\frown\theta) \text{ or } \frown q(\frown\theta) = \frown q_2(\frown\theta) \text{ or } \frown q(\frown\theta) = \frown q_1(\frown\theta) + \frown q_2(\frown\theta),$$

according as $\lambda<\mu$ or $\mu<\lambda$ or $\lambda=\mu$. Hence in any case $\frown q(\frown\theta)>\frown 0$, and $p_1+p_2>0$ follows. Moreover

$$\frown p_1(\frown x)\frown p_2(\frown x) =.(\frown g(\frown x))^{\lambda+\mu}\frown q_1(\frown x)\frown q_2(\frown x)$$

and

$$\frown q_1(\frown\theta)\frown q_2(\frown\theta) > \frown 0.$$

Thus also $p_1p_2>0$, and the lemma follows.

(10.72) LEMMA. *If* $p_1\equiv p_2 \pmod{\mathfrak{P}[\theta]}$, *and* $p_1>0$, *then* $p_2>0$.

Proof. As corresponding coefficients of p_1 and p_2 differ only infinitely little, the corresponding real polynomials coincide:

$$\frown p_1(\frown x) = \frown p_2(\frown x).$$

Hence also

$$\frown q_1(\frown x) = \frown q_2(\frown x),$$

and

$$\frown q_2(\frown\theta) = \frown q_1(\frown\theta) > \frown 0.$$

(10.73) COROLLARY. *If* $\sigma\in\Sigma$, *and* $p>0$, *then* $\sigma^{-1}p\sigma>0$.

This follows from (10.72) and (8.5).

(10.74) LEMMA. *If* $p\in\mathfrak{R}[\theta]-\mathfrak{P}[\theta]$ *then either* $p>0$ *and* $-p<0$, *or* $p<0$ *and* $-p>0$, *but not both.*

The proof is obvious and omitted.

(10.81) LEMMA. *If* $P\in\Sigma[\theta]$ *then either* $P=0$, *or* $P>0$ *and* $-P<0$, *or* $P<0$ *and* $-P>0$, *and these three cases are mutually exclusive.*

The proof is obvious and omitted.

(10.82) LEMMA. *If* $\sigma\in\Sigma$, *and* $\sigma>0$, *then* $\sigma P\gtrless 0$ *according as* $P\gtrless 0$.

For if the maximum of the absolute values of the coefficients of P is π, then that of σP is $\sigma\pi$.

(10.83) LEMMA. *If* $P_1>0$, $P_2>0$, *then* $P_1+P_2>0$.

Proof. Denote by π the maximum of the absolute values of all the coefficients of P_1, P_2, and P_1+P_2, and put

$$P_1 = \pi p_1, \qquad P_2 = \pi p_2.$$

Then also

$$P_1 + P_2 = \pi(p_1 + p_2).$$

Now $p_1>0$ and $p_2>0$ by (10.82), and it suffices to show that $p_1+p_2>0$. Also p_1, p_2, and p_1+p_2 all have finite coefficients, and at least two of them have coefficients not all infinitely small. If $p_1\in\mathfrak{R}[\theta]-\mathfrak{P}[\theta]$ and $p_2\in\mathfrak{R}[\theta]-\mathfrak{P}[\theta]$, then (10.71) shows that $p_1+p_2>0$; if $p_1\in\mathfrak{R}[\theta]-\mathfrak{P}[\theta]$ but $p_2\in\mathfrak{P}[\theta]$, then (10.72) applies, and the lemma follows.

(10.84) LEMMA. *If* $\sigma\in\Sigma$, *and* $P>0$, *then* $\sigma^{-1}P\sigma>0$.

Proof. Let $P=\pi p$, $\pi>0$, $p\in\mathfrak{R}[\theta]-\mathfrak{P}[\theta]$. Then $\sigma^{-1}P\sigma=\sigma^{-1}\pi\sigma\cdot\sigma^{-1}p\sigma$, $\sigma^{-1}\pi\sigma>0$ and $\sigma^{-1}p\sigma\in\mathfrak{R}[\theta]-\mathfrak{P}[\theta]$. Hence (10.73) and (10.82) apply, and the lemma follows.

(10.85) LEMMA. *If* $P_1>0$, $P_2>0$, *then* $P_1P_2>0$.

Proof. Let again

$$P_1 = \pi_1 p_1, \qquad \pi_1 > 0,\ p_1 \in \mathfrak{R}[\theta] - \mathfrak{P}[\theta],$$

and

$$P_2 = \pi_2 p_2, \qquad \pi_2 > 0,\ p_2 \in \mathfrak{R}[\theta] - \mathfrak{P}[\theta].$$

Then $P_1P_2=\pi_1\pi_2\cdot\pi_2^{-1}p_1\pi_2\cdot p_2$, $\pi_1\pi_2>0$, and $\pi_2^{-1}p_1\pi_2\cdot p_2>0$ by (10.73) and (10.71), hence $P_1P_2>0$ by (10.82).

(10.86) COROLLARY. *If* $P_1P_2\neq 0$, *then of* P_1, P_2, P_1P_2 *just one or all three are positive.*

(10.91) LEMMA. *If* $Q_1^{-1}P_1=Q_2^{-1}P_2$ *and* $Q_1^{-1}P_1>0$ *then* $Q_2^{-1}P_2>0$([51]).

Proof. As $Q_1^{-1}P_1=Q_2^{-1}P_2$, there are two polynomials R_1, $R_2\in\Sigma[\theta]$ such

([51]) The order convention (10.62) does not depend on the element of $\Sigma(\theta)$ only, but also on its representation as a left-quotient of polynomials. The lemma shows that the ordering is in fact independent of the representation.

that

$$R_1P_1 = R_2P_2, \qquad R_1Q_1 = R_2Q_2, \qquad R_1R_2 \neq 0.$$

As $P_1Q_1>0$ by assumption, P_1 and Q_1 are both >0 or both <0 by (10.86); hence R_1P_1 and R_1Q_1 are both >0 or both <0, and P_2 and Q_2 are both >0 or both <0, all by (10.86). Thus again $P_2Q_2>0$, and the lemma follows.

(10.92) LEMMA. *If* $Q^{-1}P_1>0$, $Q^{-1}P_2>0$, *then* $Q^{-1}P_1+Q^{-1}P_2>0$.

(10.93) LEMMA. *If* $Q_1^{-1}P_1>0$, $Q_2^{-1}P_2>0$, *then* $Q_1^{-1}P_1+Q_2^{-1}P_2>0$.

The proofs are obvious and omitted.

(10.94) LEMMA. *If* $Q_1^{-1}P_1>0$, $Q_2^{-1}P_2>0$, *then* $(Q_1^{-1}P_1)(Q_2^{-1}P_2)>0$.

Proof. There are polynomials R_1, R_2 such that

$$R_1P_1 = R_2Q_2 \tag{10.941}$$

and with any such pair R_1, R_2

$$(Q_1^{-1}P_1)(Q_2^{-1}P_2) = (R_1Q_1)^{-1}(R_2P_2). \tag{10.942}$$

By assumption $P_1Q_1>0$, $P_2Q_2>0$; hence P_1 and Q_1 are simultaneously $\lesseqgtr 0$, and so are P_2 and Q_2; then also R_1P_1 and R_1Q_1 are simultaneously $\lesseqgtr 0$, and so R_2P_2 and R_2Q_2. Then, because of (10.941), R_1Q_1 and R_2P_2 are simultaneously $\lesseqgtr 0$, and thus $(R_1Q_1)(R_2P_2)>0$. This combines with (10.942) and (10.62) to give the lemma.

(10.95) LEMMA. *If* $Q^{-1}P\in\Sigma(\theta)$, $Q\in\Sigma[\theta]$, $P\in\Sigma[\theta]$, *then either* $P=0$, *or* $Q^{-1}P>0$, *or* $Q^{-1}P<0$, *and these three cases are mutually exclusive.*

The proof is obvious and omitted.

Combining all these facts we have now the following theorem.

(10.10) THEOREM. *The transcendent extension* $\Sigma(\theta)$ *is ordered by the conventions* (10.5), (10.6); *its order continues that of* Σ, *and* (10.1) *is satisfied.*

We have given the proof in some detail here; in §17 we shall meet with an analogous situation and shall then refer back to this argument. We note that all lemmas (10.8) follow from (10.61) and (10.7) without further reference to (10.5), and all lemmas (10.9) follow from (10.62) and (10.8) without further reference to (10.5), (10.61), or (10.7). One easily shows in the same way:

(10.11) COROLLARY. *If a ring is ordered and if any two elements have a (nontrivial) common left multiple, then the division ring* Δ *of its left quotients*[52] *can be ordered uniquely so as to continue the order of the given ring.*

[52] An ordered ring can have no proper divisors of zero; hence the existence of Δ follows from Ore [10].

11. Relative orders of magnitude and alternators. Ordering an algebraic extension of Σ requires rather more preparation([53]). It is convenient to introduce a sort of Bachman-Landau notation([54]): If $\sigma, \tau \in \Sigma$, we write

$$\sigma = O(\tau) \quad \text{if } \sigma = \rho\tau \text{ with } \rho \in \mathfrak{R}, \tag{11.11}$$

$$\sigma = o(\tau) \quad \text{if } \sigma = \pi\tau \text{ with } \pi \in \mathfrak{P}; \tag{11.12}$$

and we use these symbols in conjunction with addition and multiplication in the customary manner. We also use as abbreviations

$$O(\tau_1 \vee \tau_2) = O(\max(|\tau_1|, |\tau_2|)), \tag{11.13}$$

$$O(\tau_1 \,\&\, \tau_2) = O(\min(|\tau_1|, |\tau_2|)), \tag{11.14}$$

and similarly $o(\tau_1 \vee \tau_2)$ and $o(\tau_1 \,\&\, \tau_2)$. Thus, for example, $\sigma = o(\tau_1 \vee \tau_2)$ means that $\sigma = o(\tau_1)$ or $\sigma = (\tau_2)$; and all the following formulae are equivalent:

$$\tau_1 - \tau_2 = o(\tau_1 \vee \tau_2), \tag{11.21}$$

$$\tau_2 = \tau_1 + o(\tau_1), \tag{11.22}$$

$$\tau_2 = \tau_1(1 + o(1)), \tag{11.23}$$

$$\tau_2 = (1 + o(1))\tau_1, \tag{11.24}$$

$$\tau_1 = \tau_2 + o(\tau_2) = \tau_2(1 + o(1)) = (1 + o(1))\tau_2, \tag{11.25}$$

$$\tau_1 - \tau_2 = o(\tau_1 \,\&\, \tau_2)([55]). \tag{11.26}$$

We also use the "alternator"([56])

$$A_\sigma\tau = \sigma\tau - \tau\sigma. \tag{11.31}$$

The following (familiar) properties of the alternator are easily confirmed.

$$A_\tau\sigma = -A_\sigma\tau, \tag{11.32}$$

$$A_\sigma(\tau_1 + \tau_2) = A_\sigma\tau_1 + A_\sigma\tau_2, \tag{11.33}$$

$$A_\sigma(\tau_1\tau_2) = (A_\sigma\tau_1)\tau_2 + \tau_1 A_\sigma\tau_2, \tag{11.34}$$

$$A_\sigma\tau^m = \sum_{\mu+\nu=m-1} \tau^\mu(A_\sigma\tau)\tau^\nu. \tag{11.35}$$

(11.4) LEMMA. *If* $\sigma = O(1)$, $\tau = O(1)$, *then*

$$A_\sigma\tau = o(\sigma \,\&\, \tau).$$

Proof. Because of the antisymmetry (11.32) it suffices to show that

([53]) If $\widehat{\ }\theta$, the algebraic element to be adjoined, does not lie in $\widehat{\ }\mathfrak{R}$, the procedure can be very considerably simplified. §§11–15 pave the way for ordering $\Sigma(\theta)$ when $\widehat{\ }\theta \in \widehat{\ }\mathfrak{R}$.

([54]) Cf. Hardy [5, p. 5] for the references.

([55]) The whole symbolism of "orders of infinity" can be easily, and not unprofitably, applied to any non-Archimedean ordering. Thus, for example, one could add $\tau_1 \sim \tau_2$ as another equivalent formulation.

([56]) "Inner derivation"—Jacobson [7, p. 102]. Many of the formulae derived here and later are well known.

$A_\sigma\tau = o(\tau)$. If $\tau = 0$, then the result is trivial; so we may assume $\tau \neq 0$. As $\sigma = O(1)$, σ defines a Dedekind section of the rationals. If ρ_1, $\rho_1 \in P_2$ and $\rho_1 < \sigma < \rho_2$, then also $\rho_1 < \tau^{-1}\sigma\tau < \rho_2$; hence $\tau^{-1}\sigma\tau$ defines the same Dedekind section of the rationals as σ. Thus

$$\tau^{-1}\sigma\tau - \sigma = o(1),$$

whence $\sigma\tau - \tau\sigma = o(\tau)$, and the lemma follows.

(11.51) LEMMA. *If $\sigma = O(1)$, $\tau = O(1)$, m a positive integer, then*

$$\sigma^m - \tau^m = \sum_{\mu+\nu=m-1} \sigma^\mu\tau^\nu(\sigma - \tau) + o(\sigma - \tau).$$

Proof. One has

$$\begin{aligned}\sigma^m - \tau^m &= \sigma^m - \sigma^{m-1}\tau + \sigma^{m-1}\tau - \sigma^{m-2}\tau^2 + \sigma^{m-2}\tau^2 - + \cdots + \sigma\tau^{m-1} - \tau^m \\ &= \sigma^{m-1}(\sigma - \tau) + \sigma^{m-2}(\sigma - \tau)\tau + \cdots + (\sigma - \tau)\tau^{m-1} \\ &= (\sigma^{m-1} + \sigma^{m-2}\tau + \cdots + \sigma\tau^{m-2} + \tau^{m-1})(\sigma - \tau) \\ &\quad + \sigma^{m-2}A_{\sigma-\tau}\tau + \sigma^{m-3}A_{\sigma-\tau}\tau^2 + \cdots + A_{\sigma-\tau}\tau^{m-1}.\end{aligned}$$

But all the alternator terms are $o(\sigma - \tau)$ by (11.4), and the lemma follows.

When σ and τ are infinitely close to each other, one can say more.

(11.52) LEMMA. *If $\sigma = O(1)$, $\tau = \sigma + o(\sigma)$, then*

$$\sigma^m - \tau^m = m\sigma^{m-1}(\sigma - \tau) + o(\sigma - \tau).$$

Proof. We may assume $\sigma \neq 0$. Put

$$\pi = \tau\sigma^{-1} - 1.$$

Then $\pi = o(1)$ and $\tau = \sigma(1+\pi)$; also

$$\begin{aligned}\tau^\nu &= (\sigma(1+\pi))^\nu = \sigma^\nu \cdot \sigma^{-(\nu-1)}(1+\pi)\sigma^{\nu-1} \cdot \sigma^{-(\nu-2)}(1+\pi)\sigma^{\nu-2} \cdot \cdots \cdot (1+\pi) \\ &= \sigma^\nu \prod_{\nu=1}^{0} (1 + \sigma^{-\kappa}\pi\sigma^\kappa).\end{aligned}$$

But $\sigma^{-\kappa}\pi\sigma^\kappa = o(1)$ and $\prod (1+o(1)) = 1+o(1)$; hence $\tau^\nu = \sigma^\nu(1+o(1))$. (11.51) then gives

$$\sigma^m - \tau^m = m\sigma^{m-1}(1 + o(1))(\sigma - \tau) + o(\sigma - \tau)$$

and the desired result follows immediately([57]).

(11.53) LEMMA. *If $\sigma = O(1)$, $\tau = O(1)$, and σ and τ are permutable with each other, then*

([57]) When $\sigma = o(1)$, both (11.51) and (11.52) become trivial and useless. For then both terms on the right-hand side are $o(\sigma - \tau)$, and the second term may not be negligible compared with the first.

$$\sigma^m - \tau^m = m\sigma^{m-1}(\sigma - \tau) + O((\sigma - \tau)^2).$$

The proof is obvious and omitted. The fact is, of course, quite useless unless $\tau = \sigma + o(\sigma)$.

12. **Rate of change of a polynomial and approximations of roots.** We now study a polynomial with coefficients in $\mathfrak{R}$ for arguments in $\mathfrak{R}$; and in particular for "infinitely small changes of argument." Let

$$p(x) = \sum_0^k \rho_\kappa x^\kappa \in \mathfrak{R}[x]. \tag{12.11}$$

We then *define*, when $\xi \in \Sigma$,

$$p(\xi) = \sum_0^k \rho_\kappa \xi^\kappa. \tag{12.12}$$

This notation should not be looked upon as a trivial application of the customary convention; for in (12.11) x commutes with the coefficients: x is by definition in the centre of $\mathfrak{R}[x]$. But in (12.12), ξ need not commute with the coefficients: neither ξ nor the ρ_κ need be in the centre of Σ. Thus we do not have to distinguish between ρx and $x\rho$; they are the same element of $\mathfrak{R}[x]$. But $\rho\xi$ and $\xi\rho$ may well be different. It should be observed then that (12.12) is not the most general polynomial expression in ξ (though (12.11) is the most general polynomial expression in x)(58).

We further introduce the "derivative"

$$p'(x) = \sum_1^k \kappa\rho_\kappa x^{\kappa-1}, \tag{12.21}$$

and correspondingly

$$p'(\xi) = \sum_1^k \kappa\rho_\kappa \xi^{\kappa-1}. \tag{12.22}$$

Obviously then $\frown p'(\frown x)$ is the derivative of $\frown p(\frown x)$ in $\frown\mathfrak{R}[\frown x]$.

(12.31) LEMMA. *If* $\xi_1 = O(1)$, $\xi_2 = O(1)$, *then*

$$p(\xi_1) - p(\xi_2) = O(\xi_1 - \xi_2).$$

This is an obvious corollary of (11.51). Similarly (11.52) and (11.53) give immediately:

(12.32) LEMMA. *If* $\xi_1 = O(1)$, $\xi_2 = \xi_1 + o(\xi_1)$, *then*

$$p(\xi_1) - p(\xi_2) = p'(\xi_1)(\xi_1 - \xi_2) + o(\xi_1 - \xi_2).$$

(58) In §§9, 10 we have sometimes substituted an element θ of Σ, or of an extension $\Sigma(\theta)$ of Σ, for x in a polynomial $p(x) \in \Sigma[x]$. But then we either knew (§10) that the element θ was in the centre of $\Sigma(\theta)$, or we knew (§9) that the coefficients of $p(x)$ were in the centre of Σ.

(12.33) LEMMA. *If* $\xi_1=O(1)$, $\xi_2=O(1)$, *and* ξ_1 *and* ξ_2 *are permutable with each other, then*

$$p(\xi_1) - p(\xi_2) = p'(\xi_1)(\xi_1 - \xi_2) + O((\xi_1 - \xi_2)^2).$$

Note that we need not here assume ξ_1 or ξ_2 permutable with the coefficients of the polynomial.

(12.4) COROLLARY. *If* $\xi_1=O(1)$, $\xi_2=\xi_1+o(\xi_1)$, *and* $p'(\xi_1)\neq o(1)$, *then*

$$p(\xi_1) - p(\xi_2) = p'(\xi_1)(\xi_1 - \xi_2)(1 + o(1)), \tag{12.41}$$

and also

$$\xi_1 - \xi_2 = O(p(\xi_1) - p(\xi_2)). \tag{12.42}$$

(12.32) and (12.33) suggest some simple applications of Newton's approximation method:

(12.5) LEMMA. If $p\in\mathfrak{R}[x]$, $\xi=O(1)$

$$\xi \not\equiv o(1)\text{(59)}, \tag{12.51}$$

$$p(\xi) = o(1), \tag{12.52}$$

$$p'(\xi) \not\equiv o(1), \tag{12.53}$$

then there is a $\xi_1\in\mathfrak{R}$ *such that*

$$p(\xi_1) = o(p(\xi)), \tag{12.54}$$

$$p'(\xi_1) \not\equiv o(1). \tag{12.55}$$

Proof. We put

$$\xi_1 = \xi - p'(\xi)^{-1}p(\xi). \tag{12.56}$$

Then $\xi_1=\xi+o(\xi)$ and we can apply (12.32). Thus

$$p(\xi) - p(\xi_1) = p'(\xi)\cdot p'(\xi)^{-1}p(\xi) + o(p'(\xi)^{-1}p(\xi)) = p(\xi) + o(p(\xi)),$$

and (12.54) follows. (12.55) follows easily from (12.31) (applied to p' instead of p).

(12.6) LEMMA. *If* $p\in\mathbf{Z}[x]\cap\mathfrak{R}[x]$, *that is, if* p *is a polynomial with finite centre coefficients, and if* $\xi=O(1)$,

$$\xi \not\equiv o(1), \tag{12.61}$$

$$p(\xi) = o(1), \tag{12.62}$$

$$p'(\xi) \not\equiv o(1), \tag{12.63}$$

then there is a $\xi_1\in\mathfrak{R}$ *with* $\xi_1=\xi+o(\xi)$ *and*

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(59) One can omit the assumption $\xi\not\equiv o(1)$ if one strengthens the next one to $p(\xi)=o(\xi)$.

$$p(\xi_1) = O(p(\xi)^2), \tag{12.64}$$

$$p'(\xi_1) \neq o(1). \tag{12.65}$$

Proof. We again define ξ_1 by (12.56) and notice that it is then permutable with ξ. Hence we can apply (12.33) to give us (12.64). (12.65) follows as (12.55) before.

13. **Product polynomials and higher alternators.** We now study the connection between the value of a product of two polynomials for an argument $\xi \in \Sigma$, and the product of the values of the polynomials for the same argument.

Let $p(x)$, $q(x)$ be two polynomials in $\mathfrak{R}[x]$ and put

$$\begin{aligned} p(x) + q(x) &= r(x), \\ p(x)q(x) &= s(x). \end{aligned} \tag{13.11}$$

Then we define

$$(p + q)(\xi) = r(\xi), \tag{13.12}$$

$$(pq)(\xi) = s(\xi). \tag{13.13}$$

Obviously then

$$(p + q)(\xi) = p(\xi) + q(\xi), \tag{13.14}$$

but $(pq)(\xi)$ may be different from $p(\xi)q(\xi)$. How much (or how little) is shown by the following lemma.

(13.2) LEMMA. *If* $\xi = O(1)$ *then*

$$(pq)(\xi) = p(\xi)q(\xi) + o(q(\xi)\ \&\ \xi).$$

Proof. If $p(x) = \sum_\kappa \rho_\kappa x^\kappa$, $q(x) = \sum_\lambda \sigma_\lambda x^\lambda$, then

$$(pq)(x) = \sum_{\kappa,\lambda} \rho_\kappa \sigma_\lambda x^{\kappa+\lambda}.$$

Thus

$$\begin{aligned} p(\xi)q(\xi) - (pq)(\xi) &= \Big(\sum_\kappa \rho_\kappa \xi^\kappa\Big)\Big(\sum_\lambda \sigma_\lambda \xi^\lambda\Big) - \sum_{\kappa,\lambda} \rho_\kappa \sigma_\lambda \xi^{\kappa+\lambda} \\ &= \sum_{\kappa,\lambda} (\rho_\kappa \xi^\kappa \sigma_\lambda \xi^\lambda - \rho_\kappa \sigma_\lambda \xi^{\kappa+\lambda}) \\ &= \sum_\kappa \rho_\kappa \Big(\sum_\lambda (\xi^\kappa \sigma_\lambda \xi^\lambda - \sigma_\lambda \xi^\lambda \xi^\kappa)\Big) \\ &= \sum_\kappa \rho_\kappa \Big(\sum_\lambda A_{\xi^\kappa}(\sigma_\lambda \xi^\lambda)\Big) \\ &= \sum_\kappa \rho_\kappa A_{\xi^\kappa}\Big(\sum_\lambda \sigma_\lambda \xi^\lambda\Big) = \sum_\kappa \rho_\kappa A_{\xi^\kappa} q(\xi), \end{aligned}$$

or

$$(pq)(\xi) = p(\xi)q(\xi) - \sum_{\kappa} \rho_\kappa A_{\xi^\kappa} q(\xi), \tag{13.21}$$

and the lemma follows from (11.4).

The lack of symmetry in (13.2) is not fortuitous. It arises from the unsymmetry of the definition (12.12). It is not difficult to show by an example that $p(\xi)q(\xi) - (pq)(\xi)$ need not even be an $O(p(\xi))$. We shall, however, require a better estimate later, under more stringent assumptions. To obtain this estimate we first elaborate our alternator technique (§11) (see footnote 56).

The alternator $A_\sigma\tau$ is used as left operator, and its iterates are written as powers

$$A_\sigma^{\mu+1}\tau = A_\sigma(A_\sigma^\mu\tau). \tag{13.31}$$

Then (11.33) leads immediately to

$$A_\sigma^\mu(\tau_1 + \tau_2) = A_\sigma^\mu\tau_1 + A_\sigma^\mu\tau_2, \tag{13.32}$$

and (11.34) leads to a "Leibnitz formula" for higher alternators of products, which we shall not require. We only need that ([60])

$$A_\sigma^\mu(\sigma^\nu\tau) = \sigma^\nu A_\sigma^\mu\tau. \tag{13.33}$$

The definition (11.31) of the alternator can be rewritten as

$$\tau\sigma = (\sigma - A_\sigma)\tau. \tag{13.34}$$

Repeated application gives

$$\tau\sigma^\kappa = (\sigma - A_\sigma)^\kappa\tau. \tag{13.35}$$

The power on the right-hand side can be evaluated by the commutative binomial formula, because of (13.33). Then

$$\tau\sigma^\kappa = \left(\sigma^\kappa - \sum_{\mu=1}^{\kappa} (-1)^{\mu-1} C_{\kappa,\mu}\sigma^{\kappa-\mu}A_\sigma^\mu\right)\tau, \tag{13.36}$$

which gives (directly or on comparison with (13.34))

$$A_{\sigma^\kappa}\tau = \left(\sum_{\mu=1}^{\kappa} (-1)^{\mu-1} C_{\kappa,\mu}\sigma^{\kappa-\mu}A_\sigma^\mu\right)\tau. \tag{13.37}$$

Now let again

$$p(x) = \sum_{\kappa} \rho_\kappa x^\kappa, \qquad q(x) = \sum_{\lambda} \sigma_\lambda x^\lambda,$$

([60]) σ^ν can be replaced by any ρ which is permutable with σ; and right-multiplication by ρ also permutes with alternation.

so that (see (13.21))

$$(pq)(\xi) = p(\xi)q(\xi) - \sum_{\kappa} \rho_\kappa A_{\xi^\kappa} q(\xi).$$

Expanding the alternators by means of (13.37), this gives

$$(pq)(\xi) = p(\xi)q(\xi) - \sum_{\kappa,\mu} \rho_\kappa (-1)^{\mu-1} C_{\kappa,\mu} \xi^{\kappa-\mu} A_\xi^\mu q(\xi).$$

The coefficient of $A_\xi^\mu q(\xi)$ is easily seen to be (apart from a numerical factor) a higher derivative of p at ξ. With the usual notation, $p^{(\mu)}$ for the μth derivative, we have then the following identity.

(13.4) LEMMA([61]). *If the degree of p is k, then*

$$(pq)(\xi) = p(\xi)q(\xi) + \sum_{\mu=1}^{k} (-1)^\mu \frac{1}{\mu!} p^{(\mu)}(\xi) A_\xi^\mu q(\xi).$$

(13.41) COROLLARY. *If $p, q \in \mathfrak{R}[x]$, $\xi = O(1)$, then*

$$(pq)(\xi) = p(\xi)q(\xi) - p'(\xi)A_\xi q(\xi) + O(A_\xi^2 q(\xi)), \tag{13.42}$$

or more simply

$$(pq)(\xi) = p(\xi)q(\xi) + O(A_\xi q(\xi)). \tag{13.43}$$

We get better extensions of (13.2) by comparing with a finite centre polynomial.

(13.5) LEMMA. *If $f \in \mathbf{Z}[x] \cap \mathfrak{R}[x]$, $\xi, \tau = O(1)$, then*

$$A_{f(\xi)\tau} = f'(\xi) A_\xi \tau + O(A_\xi^2 \tau).$$

Proof. In (13.41) we assume p in the centre so that it is permutable with q

$$pq = qp. \tag{13.51}$$

Also $p(\xi)$ is then permutable with ξ, so that $A_\xi p(\xi) = 0$ and by (13.4)

$$(qp)(\xi) = q(\xi)p(\xi). \tag{13.52}$$

Combining (13.51), (13.52), and (13.42), we obtain

$$p(\xi)q(\xi) - q(\xi)p(\xi) = p'(\xi)A_\xi q(\xi) + O(A_\xi^2 q(\xi)). \tag{13.53}$$

Here we identify p with f, and specialise q to be the constant τ, and the lemma results.

(13.6) COROLLARY. *If $f \in \mathbf{Z}[x] \cap \mathfrak{R}[x]$, $\xi, \tau = O(1)$, and $f'(\xi) \neq o(1)$, then*

([61]) This identity is true in any ring which contains the rationals; or in any ring whatsoever if $(1/\mu!)p^{(\mu)}$ is understood as a single, integrally defined symbol.

$$A_\xi\tau = O(A_{f(\xi)}\tau). \tag{13.61}$$

For $A_\xi^2\tau = o(A_\xi\tau)$ by (11.4).

The significance of the following refinement of 13.2 will not become apparent until later (§16).

(13.7) LEMMA. *Let* p, $q \in \mathfrak{R}[x]$, $\xi = O(1)$, *and assume the existence of a finite centre polynomial*

$$f \in \mathrm{Z}[x] \cap \mathfrak{R}[x] \tag{13.71}$$

with the properties

$$f(\xi) = O((pq)(\xi)), \tag{13.72}$$

$$f'(\xi) \neq o(1). \tag{13.73}$$

Then

$$(pq)(\xi) = p(\xi)q(\xi)(1 + o(1)). \tag{13.74}$$

Proof. We apply (13.6) with $q(\xi)$ in the place of τ to obtain first

$$A_\xi q(\xi) = O(A_{f(\xi)}q(\xi)),$$

then (11.4) to the right-hand side of this to see that

$$A_\xi q(\xi) = o(f(\xi)).$$

Then (13.72) gives

$$A_\xi q(\xi) = o((pq)(\xi)),$$

and this in conjunction with (13.43) gives

$$(pq)(\xi) = p(\xi)q(\xi) + o((pq)(\xi)),$$

from which the lemma is obvious.

14. **Degree of approximation to a root.** We have seen in §12 that if we can solve an algebraic equation (with coefficients in $\mathfrak{R}$, or in $\mathfrak{R} \cap \mathrm{Z}$) approximately (in $\mathfrak{R}$), then we can improve the approximation; in this paragraph we investigate (roughly speaking) just how near one can get to an exact solution.

Let $\frown\theta \in \frown\mathfrak{R}$ (cf. footnote 53) be algebraic over $\frown\mathrm{P}$, of degree $n > 1$. There are then elements in $\mathfrak{R}$ which map onto $\frown\theta$. We denote them by ζ, with suffixes where required[62]:

$$\frown\zeta = \frown\theta. \tag{14.11}$$

Denote by $\frown f$ the irreducible monic polynomial $\in \frown\mathrm{P}[\frown x]$ of which $\frown\theta$ is a

[62] We reserve the letter θ for the element corresponding to $\frown\theta$ and algebraic over P which we wish to adjoint to Σ; it will not come into this section at all.

zero; and by f the polynomial $\in \mathbf{P}[x]$ which corresponds to $\frown f$ (uniquely) by virtue of the order-isomorphism of $\frown\mathbf{P}$ and $\mathbf{P}$. Then f is also irreducible over $\mathbf{P}$, and monic. Translating from $\frown\mathfrak{R}$ into $\mathfrak{R}$, we see that for all ζ

$$\zeta \not\equiv o(1) \quad \text{because} \quad \frown\theta \not\equiv \frown 0, \tag{14.12}$$

$$f(\zeta) = o(1) \quad \text{because} \quad \frown f(\frown\theta) = \frown 0, \tag{14.13}$$

$$f'(\zeta) \not\equiv o(1) \quad \text{because} \quad \frown f'(\frown\theta) \not\equiv \frown 0. \tag{14.14}$$

Also:

(14.15) If $g \in \mathbf{P}[x]$ and $g(\zeta) = o(1)$, then $g = f \cdot g_1$, $g_1 \in \mathbf{P}[x]$.

For $\frown g(\frown\theta) = \frown 0$, hence $\frown f$ divides $\frown g$ in $\frown\mathbf{P}[\frown x]$, and the isomorphism between $\frown\mathbf{P}$ and $\mathbf{P}$ does the rest.

(14.2) THEOREM. *Let $\mathfrak{p}(\frown\theta)$ denote the set of those (and only those) elements $\pi \in \mathfrak{R}$ which satisfy*

$$\pi = O(g(\zeta)) \tag{14.21}$$

for all $g \in \mathbf{P}[x]$, $g \neq 0$, and all ζ (defined by (14.11)). Let $\mathfrak{p}_1(\frown\theta)$ denote the set of those (and only those) $\pi_1 \in \mathfrak{R}$ which satisfy

$$\pi_1 = O(f(\zeta)) \tag{14.22}$$

for f and all ζ defined by (14.1). Then

$$\mathfrak{p}_1(\frown\theta) = \mathfrak{p}(\frown\theta); \tag{14.23}$$

(14.24) $\mathfrak{p}(\frown\theta)$ is a two-sided prime ideal in $\mathfrak{R}$.

Proof. If $\pi \in \mathfrak{p}(\frown\theta)$ $[\pi_1 \in \mathfrak{p}_1(\frown\theta)]$ and $\pi' = O(\pi)$ $[\pi_1' = O(\pi_1)]$, then $\pi' = O(g(\zeta))$ $[\pi_1' = O(f(\zeta))]$ for all $g \neq 0$ [for f] and all ζ, hence $\pi' \in \mathfrak{p}(\frown\theta)$ $[\pi_1' \in \mathfrak{p}_1(\frown\theta)]$. Therefore $\mathfrak{p}(\frown\theta)$ $[\mathfrak{p}_1(\frown\theta)]$ is a two-sided ideal in $\mathfrak{R}$.

Next let $\rho \in \mathfrak{R}$, $\rho \not\equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$. Then there is a ζ which satisfies (14.11)-(14.14) and

$$f(\zeta) = o(\rho).$$

We apply (12.6) to f and ζ, and see that there is a ζ_1 which also satisfies (14.11)-(14.14), and

$$f(\zeta_1) = O(f(\zeta)^2) = o(\rho^2).$$

Hence also $\rho^2 \not\equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$. If now $\sigma \in \mathfrak{R}$, $\sigma \not\equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$, and $|\rho| \leqq |\sigma|$, say, then $\rho^2 \leqq |\rho\sigma|$, hence $\rho^2 = O(\rho\sigma)$, hence $\rho\sigma \not\equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$; thus $\mathfrak{p}_1(\frown\theta)$ is a prime ideal in $\mathfrak{R}$.

The elements of $\mathfrak{p}(\frown\theta)$ have to satisfy, *inter alia*, (14.22); obviously then

$$\mathfrak{p}(\frown\theta) \leqq \mathfrak{p}_1(\frown\theta). \tag{14.25}$$

Now let $\rho \in \mathfrak{R}$, $\rho \not\equiv 0 \pmod{\mathfrak{p}(\frown\theta)}$. Then there is a $g \in \mathbf{P}[x]$ and a ζ such that

$$g(\zeta) = o(\rho).$$

By (14.15) g is divisible by f. Let

$$g = f^{\lambda} \cdot h, \qquad h \in \mathrm{P}[x],$$

h not divisible by f. Then $h(\zeta) \neq o(1)$. Also[63]

$$g(\zeta) = (f(\zeta))^{\lambda} \cdot h(\zeta).$$

Hence if we put $f(\zeta) = \sigma$, then

$$\sigma^{\lambda} = o(\rho).$$

Thus $\rho \equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$ would lead to $\sigma^{\lambda} \equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$, hence $\sigma \equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$; but this is impossible, as for suitable ζ_1

$$f(\zeta_1) = o(f(\zeta)) = o(\sigma).$$

We see therefore that $\rho \not\equiv 0 \pmod{\mathfrak{p}(\frown\theta)}$ entails $\rho \not\equiv 0 \pmod{\mathfrak{p}_1(\frown\theta)}$, in other words

$$\mathfrak{p}_1(\frown\theta) \leqq \mathfrak{p}(\frown\theta). \tag{14.26}$$

This combines with (14.25) to give (14.23), which completes the proof of the theorem.

(14.31) COROLLARY. $\mathfrak{p}(\frown\theta)$ *consists of those (and only those) elements* $\pi \in \mathfrak{R}$ *which satisfy*

$$\pi = o(g(\zeta)) \tag{14.32}$$

for all $g \neq 0$, $g \in \mathrm{P}[x]$, *and all* ζ. $\mathfrak{p}(\frown\theta)$ *also consists of those (and only those) elements* $\pi \in \mathfrak{R}$ *which satisfy*

$$\pi = o(f(\zeta)) \tag{14.33}$$

for f *and all* ζ.

(14.34) COROLLARY. *If* $\sigma \in \Sigma$ *then*

$$\sigma^{-1}\mathfrak{p}(\frown\theta)\sigma = \mathfrak{p}(\frown\theta).$$

For if $\pi \in \mathfrak{R}$ then $\sigma^{-1}\pi\sigma \in \mathfrak{R}$; if $\pi = o(f(\zeta))$ then $\sigma^{-1}\pi\sigma = o(\sigma^{-1}f(\zeta)\sigma) = o(f(\sigma^{-1}\zeta\sigma))$ because the coefficients of f are in the centre. Also, when ζ ranges over all elements with $\frown\zeta = \frown\theta$, then $\sigma^{-1}\zeta\sigma$ also ranges over all these elements; hence $\sigma^{-1}\zeta\sigma = o(f(\zeta))$ for all ζ. This shows that $\sigma^{-1}\mathfrak{p}(\frown\theta)\sigma \leqq \mathfrak{p}(\frown\theta)$. Replacing σ by σ^{-1} and combining the results, one obtains (14.34).

We have defined $\mathfrak{p}(\frown\theta)$ only for elements $\frown\theta$ which lie in $\frown\mathfrak{R}$, do not lie in $\frown\mathrm{P}$, and are algebraic over $\frown\mathrm{P}$. None of these restrictions is essential. For elements outside $\frown\mathfrak{R}$ one would have to replace "all ζ such that $\frown\zeta = \frown\theta$" (of which there are then none) by "all ζ of an interval $\rho < \zeta < \sigma$ such that the

[63] (12.6) need not be applied, as here all polynomials have coefficients in P, therefore in the centre.

corresponding interval $(\hat{\rho}, \hat{\sigma})$ contains $\hat{\theta}$ but no real algebraical conjugate of $\hat{\theta}''$; but one then has to restrict g in 14.21 to polynomials such that $\hat{g}$ possesses no root in that interval except (possibly) $\hat{\theta}$ itself. For transcendent elements over $\hat{\mathrm{P}}$ and for elements in $\hat{\mathrm{P}}$ the definition (*mutatis mutandis*) also applies. One satisfies oneself without difficulty that $\mathfrak{p}(\hat{\theta}) = \mathfrak{P}$ if $\hat{\theta}$ is transcendent over $\hat{\mathrm{P}}$, or if $\hat{\theta}$ is outside $\hat{\mathfrak{R}}$; in all other cases, that is, if $\hat{\theta}$ is algebraic over $\hat{\mathrm{P}}$ and $\in \hat{\mathfrak{R}}$, $\mathfrak{p}(\hat{\theta}) < \mathfrak{P}$; and $\mathfrak{p}(\hat{\theta}) = (0)$ if (but not necessarily only if) $\hat{\theta} \in \hat{\mathrm{P}}_{\max}$([64]).

Now let $\mathfrak{p}$ be an infinitely small([65]) prime ideal in $\mathfrak{R}$, and denote the residue class ring modulo $\mathfrak{p}$ by

$$\mathfrak{R}_1 = \mathfrak{R}/\mathfrak{p}. \tag{14.41}$$

Then by forming (left or right) quotients in $\mathfrak{R}_1$ one can embed it in a division ring Σ_1. The order of $\mathfrak{R}$ induces an order in $\mathfrak{R}_1$, and this in turn leads to an order relation for Σ_1 (cf. (10.11)). Hence the results we have already derived can be applied to Σ_1.

The ring of elements which are $O(1)$ in Σ_1 is of course just $\mathfrak{R}_1$. The ideal of $o(1)$ in $\mathfrak{R}_1$ is

$$\mathfrak{P}_1 = \mathfrak{P}/\mathfrak{p}. \tag{14.42}$$

The homomorphic mapping of $\mathfrak{R}_1$ into the real field $\hat{\mathrm{P}}^*$ gives

$$\hat{\mathfrak{R}}_1 = \hat{\mathfrak{R}}. \tag{14.43}$$

If Z_1 denotes the centre of Σ_1, then $\mathrm{Z}_1 \cap \mathfrak{R}_1$ contains a field P_1 which is mapped on $\hat{\mathrm{P}}$

$$\hat{\mathrm{P}}_1 = \hat{\mathrm{P}}, \tag{14.44}$$

and therefore is order-isomorphic to P. But now $\mathrm{Z}_1 \cap \mathfrak{R}_1$ may contain bigger fields then P_1; and that even if P is a maximal field $\mathrm{P}_{\max}$ in $\mathrm{Z} \cap \mathfrak{R}$.

(14.5) THEOREM. *If $\hat{\theta} \in \hat{\mathfrak{R}}$ is algebraic over $\hat{\mathrm{P}}$, and if $\mathfrak{p} \neq (0)$, then a necessary and sufficient condition for the existence of an element $\theta_1 \in \mathfrak{R}_1$ with*

$$\hat{\theta}_1 = \hat{\theta}, \tag{14.51}$$

$$\mathrm{P}_1(\theta_1) \leqq \mathrm{Z}_1 \cap \mathfrak{R}_1, \tag{14.52}$$

is that

$$\mathfrak{p}(\hat{\theta}) < \mathfrak{p}. \tag{14.53}$$

Proof. If (14.53) is satisfied there is a $\pi \equiv 0 \pmod{\mathfrak{p}}$, $\pi \not\equiv 0 \pmod{\mathfrak{p}(\hat{\theta})}$. Hence there is a $\zeta \in \mathfrak{R}$, $\hat{\zeta} = \hat{\theta}$ for which

([64]) $\mathfrak{p}(\hat{\theta}) = (0)$ characterises those elements which can be obtained by completing Σ with respect to its order topology.

([65]) That is, one contained in $\mathfrak{P}$.

$$f(\zeta) = O(\pi) \equiv 0 \pmod{\mathfrak{p}}. \tag{14.54}$$

Denoting by θ_1 the element of $\mathfrak{R}_1$ onto which ζ is mapped when $\mathfrak{R}$ is reduced mod $\mathfrak{p}$, we see that in $\mathfrak{R}_1$

$$f_1(\theta_1) = 0. \tag{14.55}$$

Here f_1 is the (monic and irreducible) polynomial with coefficients in P_1 which corresponds to f and $\frown f$ under the order-isomorphism between P_1, P, and $\frown P$. (14.51) is obviously satisfied, and (14.52) follows from (9.72).

Conversely (14.51) and (14.52) lead to (14.55) in $\mathfrak{R}_1$. Translated back into $\mathfrak{R}$ this means that for any $\zeta \in \mathfrak{R}$ which is mapped onto θ_1 (mod $\mathfrak{p}$) we have

$$f(\zeta) \equiv 0 \pmod{\mathfrak{p}}. \tag{14.56}$$

Now either $f(\zeta) = 0$, then clearly $\mathfrak{p}(\frown\theta) = (0) \neq \mathfrak{p}$, and (14.53) is trivially satisfied; or $f(\zeta) \neq 0$, and then $f(\zeta) \not\equiv 0 \pmod{\mathfrak{p}(\frown\theta)}$ by (14.3). Hence (14.53) follows, and the theorem is proved. The theorem can be interpreted thus: The irreducible polynomial $f \in P[x]$ has a zero modulo $\mathfrak{p}$, corresponding to the root $\frown\theta$ of $\frown f = \frown 0$, if and only if $\mathfrak{p}$ properly contains $\mathfrak{p}(\frown\theta)$. Note that zeros in other positions lead to (possibly) different left-hand sides in (14.53).

(14.6) LEMMA. *If* $\frown\theta \in \frown\mathfrak{R}$ *is algebraic over* $\frown P$, *and if*

$$\frown\theta' \in \frown P(\frown\theta), \tag{14.61}$$

then[66]

$$\mathfrak{p}(\frown\theta') \leqq \mathfrak{p}(\frown\theta). \tag{14.62}$$

Proof. If $\frown\theta \in \frown P_{max}$ then also $\frown\theta' \in \frown P_{max}$, and $\mathfrak{p}(\frown\theta) = \mathfrak{p}(\frown\theta') = (0)$. If $\frown\theta \notin \frown P_{max}$ put $\mathfrak{p} = \mathfrak{p}(\frown\theta')$ and apply Theorem (14.5). Then $Z_1 \cap \mathfrak{R}_1$ contains no subfield $P_1(\theta_1')$ with $\frown\theta_1' = \frown\theta'$, hence a fortiori no subfield $P_1(\theta_1)$ with $\frown\theta_1 = \frown\theta$. Hence $\mathfrak{p}(\frown\theta) \not< \mathfrak{p}(\frown\theta')$, and the (obvious) linear order (by inclusion) of the ideals entails (14.62).

(14.63) COROLLARY. *If* $\frown\theta'$ *and* $\frown\theta$ *generate the same algebraic extension of* $\frown P$, *then*

$$\mathfrak{p}(\frown\theta') = \mathfrak{p}(\frown\theta).$$

Thus the prime ideal $\mathfrak{p}(\frown\theta)$ depends on the extension $\frown P(\frown\theta)$ of $\frown P$ rather than on $\frown\theta$ itself.

(14.7) LEMMA. *If* $\frown\theta'$, $\frown\theta''$, $\frown\theta$ *are algebraic over* $\frown P$ *and*

$$\frown P(\frown\theta', \frown\theta'') = \frown P(\frown\theta), \tag{14.71}$$

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(66) If $\frown\theta$ is not in $\frown\mathfrak{R}$, or transcendent over $\frown P$, then $\mathfrak{p}(\frown\theta) = \mathfrak{P}$, and (14.62) becomes trivial.

then

$$\mathfrak{p}(\frown\theta) = \mathfrak{p}(\frown\theta') \vee \mathfrak{p}(\frown\theta''). \tag{14.72}$$

Proof. Application of (14.6) gives immediately

$$\mathfrak{p}(\frown\theta') \vee \mathfrak{p}(\frown\theta'') \leqq \mathfrak{p}(\frown\theta). \tag{14.73}$$

On the other hand, if $\mathfrak{p}$ is a prime ideal in $\mathfrak{R}$ such that

$$\mathfrak{p}(\frown\theta') \vee \mathfrak{p}(\frown\theta'') < \mathfrak{p}, \tag{14.74}$$

then we apply (14.5) and see that (in the notation there used) $\mathbf{Z}_1 \cap \mathfrak{R}_1$ contains $\mathbf{P}_1(\theta_1')$ and $\mathbf{P}_1(\theta_1'')$, therefore also $\mathbf{P}_1(\theta_1', \theta_1'') = \mathbf{P}_1(\theta_1)$. Then

$$\mathfrak{p}(\frown\theta) < \mathfrak{p}. \tag{14.75}$$

As (14.74) implies (14.75) for any $\mathfrak{p}$, then

$$\mathfrak{p}(\frown\theta) \leqq \mathfrak{p}(\frown\theta') \vee \mathfrak{p}(\frown\theta''), \tag{14.76}$$

which together with (14.73) completes the proof of (14.72).

15. **Comparison of polynomials near a root.** We can now introduce a property of certain algebraic elements which will play an important rôle when we define the order relations in algebraic extensions.

(15.1) DEFINITION. Let $\frown\theta \in \frown\mathfrak{R}$ be algebraic of degree greater than 1 over $\frown\mathbf{P}$, $\frown f(\frown\theta) = \frown 0$ its irreducible monic equation over $\frown\mathbf{P}$, $f \in \mathfrak{R}[x]$ the corresponding polynomial with coefficients in P. Then $\frown\theta$ will be said to possess "property (15.1)" if and only if f is irreducible modulo $\mathfrak{p}(\frown\theta)$.

Of course this property depends upon the field P. An element which has the property with respect to P may possibly lose it when some other element is adjoined to P; and elements not possessing the property will acquire it after certain extensions. We shall, however, show presently that there are always elements with property (15.1) as long as there are algebraic elements in $\frown\mathfrak{R}$.

(15.2) THEOREM. *If $\frown\mathfrak{R}$ contains an algebraic element (of degree greater than 1) over $\frown\mathbf{P}$, then it contains one which enjoys property* (15.1).

Proof. Let $\frown\theta' \in \frown\mathfrak{R}$ be algebraic over $\frown\mathbf{P}$; denote by Φ the normal extension(67) of $\frown\mathbf{P}$ generated by $\frown\theta'$ and all its algebraical conjugates over $\frown\mathbf{P}$. Put

$$\frown\Psi = \Phi \cap \frown\mathfrak{R}. \tag{15.21}$$

Let all the different subfields of $\frown\Psi$ which contain $\frown\mathbf{P}$ properly be,

$$\frown\mathbf{P}(\frown\theta), \frown\mathbf{P}(\frown\theta_1), \cdots, \frown\mathbf{P}(\frown\theta_k). \tag{15.22}$$

Clearly $\frown\Psi$ is a finite extension of $\frown\mathbf{P}$ and so are all its subfields over $\frown\mathbf{P}$; there are only a finite number of them. We may then assume they are so

(67) In $\frown\mathbf{P}^*(-1)^{1/2}$, the field of complex numbers.

arranged in (15.22) that

$$\mathfrak{p}(\frown\theta) \leqq \mathfrak{p}(\frown\theta_1) \leqq \cdots \leqq \mathfrak{p}(\frown\theta_k). \tag{15.23}$$

We shall prove the theorem by showing:

(15.3) LEMMA. *$\frown\theta$ has property* (15.1).

Proof. If $\mathfrak{p}(\frown\theta)=(0)$, then (15.3) is trivial. Assume then $\mathfrak{p}(\frown\theta)>(0)$. Let f be again the irreducible monic polynomial with coefficients in P defined by $\frown f(\frown\theta)=\frown 0$. Let

$$f \equiv g\cdot h \pmod{\mathfrak{p}(\frown\theta)}, \tag{15.31}$$

where

$$g(x) \equiv x^m + \beta_{m-1}x^{m-1} + \cdots + \beta_0 \pmod{\mathfrak{p}(\frown\theta)} \tag{15.32}$$

is irreducible modulo $\mathfrak{p}(\frown\theta)$, and $\frown g(\frown\theta)=\frown 0$. The coefficients $\frown\beta_{m-1}, \cdots, \frown\beta_0$ of $\frown g$ all lie in $\frown\mathfrak{R}$ and (being certain symmetric functions of certain algebraic conjugates of $\frown\theta$) also in Φ, hence in $\frown\Psi$. Applying (14.5) to each $\frown\beta_\mu$ in turn, with $\mathfrak{p}(\frown\theta)$ as the prime ideal modulo which $\mathfrak{R}_1$ and $\mathbf{Z}_1$ are defined, we see then that

$$\mathfrak{p}(\frown\beta_\mu) < \mathfrak{p}(\frown\theta), \qquad \mu = 0, \cdots, m-1. \tag{15.33}$$

But then no $\frown\mathrm{P}(\frown\beta_\mu)$ can occur in 15.22, that is, no $\frown\mathrm{P}(\frown\beta_\mu)$ can contain $\frown\mathrm{P}$ properly. Thus

$$\frown\beta_\mu \in \frown\mathrm{P}, \qquad \mu = 0, \cdots, m-1. \tag{15.34}$$

Then $\frown g=\frown f$, $g\equiv f \pmod{\mathfrak{p}(\frown\theta)}$, and the lemma follows. This also completes the proof of theorem (15.2).

In order to exhibit the significance of property (15.1), we now study the behaviour of a polynomial with coefficients in $\mathfrak{R}$ for arguments for which a polynomial with coefficients in P nearly vanishes. We use again the notation of the beginning of §14, especially (14.1).

(15.4) LEMMA. *Let $\mathfrak{q}(\frown\theta)$ denote the set of those (and only those) polynomials $q\in\mathfrak{R}[x]$ which satisfy*

$$q(\zeta) = O(f(\zeta)) \tag{15.41}$$

for f and all ζ defined by (14.1). *Then $\mathfrak{q}(\frown\theta)$ is a left ideal in $\mathfrak{R}[x]$.*

Proof. Let $q_1\in\mathfrak{q}(\frown\theta)$, $q_2\in\mathfrak{q}(\frown\theta)$. Then

$$(q_1 + q_2)(\zeta) = q_1(\zeta) + q_2(\zeta) = O(f(\zeta)),$$

hence $q_1+q_2\in\mathfrak{q}(\frown\theta)$. Let $q\in\mathfrak{q}(\frown\theta)$ and $p\in\mathfrak{R}[x]$. Then by (13.2)

$$(pq)(\zeta) = p(\zeta)q(\zeta) + o(q(\zeta)) = O(q(\zeta)) = O(f(\zeta)),$$

hence $pq \in \mathfrak{q}(\frown\theta)$. This proves the lemma.

(15.42) COROLLARY. *The constants in* $\mathfrak{q}(\frown\theta)$ *are*

$$\mathfrak{q}(\frown\theta) \cap \mathfrak{R} = \mathfrak{p}(\frown\theta). \tag{15.43}$$

Also

$$\mathfrak{R}[x]\mathfrak{p}(\frown\theta) \vee \mathfrak{R}[x]f \leqq \mathfrak{q}(\frown\theta). \tag{15.44}$$

Both are immediate from the definitions of $\mathfrak{p}(\frown\theta)$ and $\mathfrak{q}(\frown\theta)$, and the obvious inclusion

$$f \in \mathfrak{q}(\frown\theta).$$

The left-hand side of (15.44) is a two-sided ideal in $\mathfrak{R}[x]$, for $\mathfrak{p}(\frown\theta)$ is a two-sided ideal in $\mathfrak{R}$, and f is in the center of $\mathfrak{R}[x]$. More accurately we can describe the situation thus:

(15.5) THEOREM. *Let* $g \in \mathfrak{R}[x]$ *be the monic irreducible factor of* f *modulo* $\mathfrak{p}(\frown\theta)$ *which belongs to* $\frown\theta$, *that is,*

$$f \equiv hg \pmod{\mathfrak{p}(\frown\theta)}, \tag{15.51}$$

$$\frown g(\frown\theta) = \frown 0, \tag{15.52}$$

and g *irreducible modulo* $\mathfrak{p}(\frown\theta)$. *Then*[68]

$$\mathfrak{q}(\frown\theta) = \mathfrak{R}[x]\mathfrak{p}(\frown\theta) \vee \mathfrak{R}[x]g. \tag{15.53}$$

Proof. Let

$$f = hg + r, \tag{15.54}$$

where $r \in \mathfrak{R}[x]$ has coefficients in $\mathfrak{p}(\frown\theta)$. Then for all ζ

$$r(\zeta) = o(f(\zeta)),$$

and by (13.2)

$$f(\zeta) = h(\zeta)g(\zeta) + o(g(\zeta)) + o(f(\zeta));$$

but $h(\zeta) \neq o(1)$ because $\frown h(\frown\theta) \neq \frown 0$. Hence

$$g(\zeta) = O(f(\zeta))$$

for all ζ, that is, $g \in \mathfrak{q}(\frown\theta)$. This shows (using (15.44)) that

$$\mathfrak{q}(\frown\theta) \geqq \mathfrak{R}[x]\mathfrak{p}(\frown\theta) \vee \mathfrak{R}[x]g. \tag{15.55}$$

On the other hand let $p \in \mathfrak{R}[x]$

[68] g is not uniquely defined as an element of $\mathfrak{R}[x]$, but only modulo $\mathfrak{p}(\frown\theta)$; but the right-hand side of (15.53) is independent of the representation of g within its residue class modulo $\mathfrak{p}(\frown\theta)$.

$$p \not\equiv 0 \pmod{\mathfrak{R}[x]\mathfrak{p}(\frown\theta) \vee \mathfrak{R}[x]g}. \tag{15.56}$$

Then p and g are relatively prime modulo $\mathfrak{p}(\frown\theta)$, and there are $u \in \mathfrak{R}[x]$, $v \in \mathfrak{R}[x]$, $\rho \in \mathfrak{R}$ such that[69]

$$up + vg \equiv \rho \not\equiv 0 \pmod{\mathfrak{p}(\frown\theta)}, \tag{15.57}$$

or

$$up + vg = \rho + s, \tag{15.58}$$

where $s \in \mathfrak{R}[x]$ has coefficients in $\mathfrak{p}(\frown\theta)$. Then for all ζ

$$s(\zeta) = o(f(\zeta)) = O(f(\zeta)),$$
$$(vg)(\zeta) = v(\zeta)g(\zeta) + o(g(\zeta)) = O(g(\zeta)) = O(f(\zeta)),$$

but for suitable ζ_1,

$$\rho \not= O(f(\zeta_1)).$$

For such ζ_1 then (15.58) gives

$$(up)(\zeta_1) = u(\zeta_1)p(\zeta_1) + o(p(\zeta_1)) \not= O(f(\zeta_1)),$$

hence

$$p(\zeta_1) \not= O(f(\zeta_1)),$$

that is,

$$p \not\equiv 0 \pmod{\mathfrak{q}(\frown\theta)}.$$

This shows that

$$\mathfrak{q}(\frown\theta) \leqq \mathfrak{R}[x]\mathfrak{p}(\frown\theta) \vee \mathfrak{R}[x]g, \tag{15.59}$$

and combined with (15.55) completes the proof of the theorem.

(15.6) COROLLARY. *Necessary and sufficient condition for*

$$\mathfrak{q}(\frown\theta) = \mathfrak{R}[x]\mathfrak{p}(\frown\theta) \vee \mathfrak{R}[x]f \tag{15.61}$$

is that $\frown\theta$ *possess property* (15.1).

(15.62) COROLLARY. *If* $\frown\theta$ *possesses property* (15.1) *then* $\mathfrak{q}(\frown\theta)$ *is a two-sided prime ideal in* $\mathfrak{R}[x]$.

For $\mathfrak{R}[x]\mathfrak{p}(\frown\theta)$ is two-sided because $\mathfrak{p}(\frown\theta)$ is two-sided in $\mathfrak{R}$, and $\mathfrak{R}[x]f$ is

[69] This can be seen thus: Using the notation of (14.4) with $\mathfrak{p} = \mathfrak{p}(\frown\theta)$, we put $\mathfrak{R}_1 = \mathfrak{R}/\mathfrak{p}(\frown\theta)$, p_1 and g_1 the polynomials onto which p and g are mapped. Σ_1 is the division ring of left quotients in $\mathfrak{R}_1$; then $\Sigma_1[x]$ is a principal ideal domain. As p_1 and g_1 are relatively prime, there are U_1, V_1 in $\Sigma_1[x]$ such that in $\Sigma_1[x]$, $U_1p_1 + V_1g_1 = 1$. Let ρ_1^{-1} denote the coefficient of greatest absolute value in U_1, V_1, and put $u_1 = \rho_1 U_1$, $v_1 = \rho_1 V_1$. Then u_1, v_1 are in $\mathfrak{R}_1[x]$, and $u_1p_1 + v_1g_1 = \rho_1$ leads to (15.57).

two-sided because f is in the centre. Also f is irreducible modulo $\mathfrak{p}(\frown\theta)$, which is itself prime in $\mathfrak{R}$.

One can also show without great difficulty that $\mathfrak{q}(\frown\theta)$ is always a two-sided prime ideal, whether $\frown\theta$ enjoys property (15.1) or not; but the fact will not be needed.

(15.7) COROLLARY. *Let $\frown\theta$ have property* (15.1), *let $p\in\mathfrak{R}[x]$ have at least one coefficient not in $\mathfrak{p}(\frown\theta)$, and let p be of smaller degree than f. Then there is a $\zeta\in\mathfrak{R}$, $\frown\zeta=\frown\theta$, such that*

$$f(\zeta) = o(p(\zeta)). \tag{15.71}$$

For clearly p cannot lie in $\mathfrak{q}(\frown\theta)$.

Finally we study the behaviour of p for arguments where f is infinitely small compared with it; and also the behaviour of two such polynomials p_1, p_2 for such arguments.

(15.8) LEMMA. *If*

$$f(\zeta_1) = o(p(\zeta_1)) \tag{15.81}$$

and

$$f(\zeta_2) = O(f(\zeta_1)), \tag{15.82}$$

then also

$$f(\zeta_2) = o(p(\zeta_2)). \tag{15.83}$$

Proof. We apply (12.31) to p, and (12.42) to f:

$$\begin{aligned} p(\zeta_2) - p(\zeta_1) &= O(\zeta_2 - \zeta_1) \\ &= O(f(\zeta_2) - f(\zeta_1)) \\ &= O(f(\zeta_1)) = o(p(\zeta_1)). \end{aligned}$$

Hence

$$p(\zeta_2) = p(\zeta_1) + o(p(\zeta_1)), \tag{15.84}$$

and

$$f(\zeta_2) = O(f(\zeta_1)) = o(p(\zeta_1)) = o(p(\zeta_2)).$$

(15.91) COROLLARY. *If*

$$f(\zeta_1) = o(p(\zeta_1)), \qquad f(\zeta_2) = o(p(\zeta_2))$$

then

$$p(\zeta_2) = p(\zeta_1)(1 + o(1)).$$

This is immediate from (15.84), observing that it is symmetric in ζ_1 and ζ_2, so that we may assume (15.82) without loss of generality.

(15.92) COROLLARY. *If*

$$f(\zeta_1) = o(p_1(\zeta_1)), \qquad f(\zeta_2) = o(p_2(\zeta_2))$$

then for $\zeta = \zeta_1$ *or* $\zeta = \zeta_2$,

$$f(\zeta) = o(p_1(\zeta) \,\&\, p_2(\zeta)).$$

For if $f(\zeta_2) = O(f(\zeta_1))$, say, we choose $\zeta = \zeta_2$ and apply (15.8).

16. **Ordering a simple algebraic extension (in $\frown\Re$).** We now have all the tools for defining the order in an algebraic extension $\Sigma(\theta)$ of Σ. Throughout this section we assume that P is a maximal subfield $\mathbf{P}_{\max} \leqq \mathbf{Z} \cap \Re$, and that $\frown\theta \in \frown\Re$ is algebraic over $\frown$P and has property (15.1). We know that the algebraic adjunction of θ to Σ is possible (§9), and only have to define when an element of $\Sigma(\theta)$ shall be positive.

(16.1) DEFINITION. Let p be a polynomial with finite coefficients not all in $\mathfrak{p}(\frown\theta)$, that is, $p \in \Re[x] - \Re[x]\mathfrak{p}(\frown\theta)$, and let p be of smaller degree than f. Let $\zeta \in \Re$ be chosen such that $\frown\zeta = \frown\theta$ and

$$f(\zeta) = o(p(\zeta)). \tag{16.11}$$

(Such ζ exist by (15.7).) Then we define

$$p(\theta) \gtrless 0 \text{ in } \Sigma(\theta) \text{ according as } p(\zeta) \gtrless 0 \text{ in } \Sigma(^{70}). \tag{16.12}$$

(16.2) DEFINITION. Let $P \in \Sigma[x]$ be of smaller degree than f; and if $P \neq 0$ let $\pi \in \Sigma$, $p \in \Re[x] - \Re[x]\mathfrak{p}(\frown\theta)$ be chosen such that

$$P = \pi p. \tag{16.21}$$

Finally let ζ be chosen as in (16.1). Then we define

$$P(\theta) \lessgtr 0 \text{ in } \Sigma(\theta) \text{ according as } P(\zeta) \lessgtr 0 \text{ in } \Sigma. \tag{16.22}$$

(16.3) LEMMA. *By virtue of definitions* (16.1), (16.2) *every element of* $\Sigma(\theta)$ *is either* >0, *or* $=0$, *or* <0, *and no two of these simultaneously.*

Proof. The elements of $\Sigma(\theta)$ can be represented uniquely as the polynomials in θ, of smaller degree than f, with coefficients in Σ. If $P(\theta) \neq 0$ is such a polynomial, π a coefficient of P of greatest absolute value, and $p = \pi^{-1}P$, then the coefficients of p are all finite, but at least one of them is ± 1; hence $p \in \Re[x] - \Re[x]\mathfrak{p}(\frown\theta)$. The existence of a suitable ζ follows from (15.7). $P(\zeta) = 0$ would entail $p(\zeta) = 0$, contrary to (16.11). Hence (16.2) allocates a positive or negative sign to $P(\theta)$. This proves the first part of the lemma.

Let now $\zeta_1 \in \Re$ be chosen (differently from ζ) such that $\frown\zeta_1 = \frown\theta$ and

$$f(\zeta_1) = o(p(\zeta_1)).$$

(70) This definition contains the macroscopic order (when $p(\frown\theta) \neq \frown 0$) as a special case. It is, itself, only a special case of definition (16.2), but has been included to show more clearly the underlying principle (cf. the introductory remarks, §7) and to underline the analogy with the situation in §10 and §17.

Then by (15.91)

$$p(\zeta_1) = p(\zeta)(1 + o(1)).$$

Hence $p(\zeta_1) \gtrless 0$ according as $p(\zeta) \gtrless 0$ and (multiplying by π) also $P(\zeta_1) \gtrless 0$ according as $P(\zeta) \gtrless 0$. Hence a different choice of the ζ in (16.1) does not affect the sign allocated to an element by the definitions.

Finally let $\pi_1 \in \Sigma$, $p_1 \in \mathfrak{R}[x] - \mathfrak{R}[x]\mathfrak{p}(\hat{\ }\theta)$ be chosen (differently from π, p) such that

$$P = \pi_1 p_1$$

and let $\zeta_1 \in \mathfrak{R}$ be such that $\hat{\ }\zeta_1 = \hat{\ }\theta$ and

$$f(\zeta_1) = o(p_1(\zeta_1)).$$

Then by (15.92) we may assume without loss of generality that also

$$f(\zeta_1) = o(p(\zeta_1)).$$

Hence the sign of $P(\theta)$ according to (16.2), using π_1, p_1, ζ_1, is the same as that determined in terms of π, p, ζ_1, and therefore also the same as that defined in terms of π, p, ζ. Thus a different choice of π, p in (16.2) does not affect the sign allocated to an element by the definitions, and the proof of the lemma is completed.

(16.4) LEMMA. *The sum of two positive elements of $\Sigma(\theta)$ is positive.*

Proof. Let P_1, $P_2 \in \Sigma[x]$, $P = P_1 + P_2$, and $P_1(\theta) > 0$, $P_2(\theta) > 0$. If π_1, π_2, p_1, p_2 are chosen according to (16.2) so that

$$P_1 = \pi_1 p_1, \qquad P_2 = \pi_2 p_2$$

and then ζ according to (15.92) such that (16.11) is satisfied simultaneously for p_1 and p_2, then

$$P(\zeta) = (P_1 + P_2)(\zeta) = P_1(\zeta) + P_2(\zeta) > 0; \tag{16.41}$$

therefore P is not identically zero. Clearly also P is of smaller degree than f, hence $P(\theta) \neq 0$, and we can apply (16.2) to it. Assume now without loss of generality that $\pi_1 \geqq \pi_2 > 0$, and put

$$\pi = \pi_1, \qquad p = p_1 + \pi_1^{-1}\pi_2 p_2,$$

so that

$$P = \pi p.$$

Then $p \in \mathfrak{R}[x]$. Also $\pi_1^{-1}\pi_2 p_2(\zeta) > 0$, whence $p(\zeta) > p_1(\zeta) > 0$ and

$$f(\zeta) = o(p(\zeta)).$$

This shows that $p \notin \mathfrak{R}[x]\mathfrak{p}(\hat{\ }\theta)$, and also that $P(\zeta) > 0$ (cf. (16.41)) entails

$P(\theta)>0$. Thus the lemma is proved.

(16.42) COROLLARY. *If* $P(\theta)>0$ *then* $-P(\theta)<0$, *and vice versa.*

In order to show that the product of positive elements is again positive we first have to extend (16.2) to the case of polynomials of degree no smaller than that of f.

(16.5) LEMMA. *Let* $P, Q, R\in\Sigma[x]$, $R\neq 0$ *of smaller degree than* f, *and*

$$P = Qf + R. \tag{16.51}$$

Let also, if possible([71]), $\pi\in\Sigma$, $p\in\mathfrak{R}[x]-\mathfrak{q}(\frown\theta)$ *be chosen such that*

$$P = \pi p, \tag{16.52}$$

and $\zeta\in\mathfrak{R}$ *such that* $\frown\zeta=\frown\theta$ *and*

$$f(\zeta) = o(p(\zeta)). \tag{16.53}$$

Then

$$P(\theta) = R(\theta) \gtrless 0 \qquad \textit{according as } P(\zeta) \gtrless 0.$$

Proof. Put $\pi^{-1}Q=q$, $\pi^{-1}R=r$. Then

$$p = qf + r \tag{16.54}$$

and consequently $q\in\mathfrak{R}[x]$, $r\in\mathfrak{R}[x]$([72]). Also

$$r \equiv p \not\equiv 0 \pmod{\mathfrak{q}(\frown\theta)};$$

thus $r\in\mathfrak{R}[x]-\mathfrak{R}[x]\mathfrak{p}(\frown\theta)$. Moreover, from (16.54), (13.2), (16.53)

$$\begin{aligned} r(\zeta) &= (p - qf)(\zeta) = p(\zeta) - (qf)(\zeta) \\ &= p(\zeta) - q(\zeta)f(\zeta) + o(f(\zeta)) = p(\zeta) + O(f(\zeta)) \\ &= p(\zeta) + o(p(\zeta)) \end{aligned}$$

or

$$r(\zeta) = p(\zeta)(1 + o(1)), \tag{16.55}$$

and also

$$f(\zeta) = o(r(\zeta)).$$

Hence (16.2) can be applied to R, π, r, ζ, to show that $R(\theta)\gtrless 0$ according as $R(\zeta)\gtrless 0$. But using (16.55),

([71]) This can be done only if the coefficients of R are not too small compared with those of P.

([72]) If qf or r had an infinite coefficient, left-multiply by the reciprocal of the absolutely greatest coefficient and consider the result modulo $\mathfrak{P}$: Then $\frown q_1\frown f+\frown r_1=\frown 0$ with $\frown r_1$ of smaller degree than $\frown f$, and not both $\frown q_1$ and $\frown r_1$ vanishing, which is absurd.

$$R(\zeta) = \pi r(\zeta) = \pi p(\zeta)(1 + o(1)) = P(\zeta)(1 + o(1)).$$

Hence $P(\zeta)$ and $R(\zeta)$ have the same sign, and the lemma follows.

(16.6) LEMMA. *If $P \in \Sigma[x]$ is of smaller degree than f, and $\sigma \in \Sigma$, then $\sigma^{-1}P(\theta)\sigma$ and $P(\theta)$ have the same sign.*

Proof. $\sigma^{-1}P\sigma$ is the polynomial obtained from P by transforming all coefficients by σ. For brevity we put, when $\xi \in \Sigma(\theta)$,

$$\sigma^{-1}\xi\sigma = \xi^{\sigma},$$

and for polynomials $P \in \Sigma[x]$

$$\sigma^{-1}P\sigma = P^{\sigma}.$$

Then obviously

$$\sigma^{-1}P(\xi)\sigma = P^{\sigma}(\xi^{\sigma}).$$

Also, as f has coefficients in the centre of Σ,

$$f^{\sigma} = f,$$

and as θ lies in the centre of $\Sigma(\theta)$,

$$\theta^{\sigma} = \theta;$$

hence

$$\sigma^{-1}P(\theta)\sigma = P^{\sigma}(\theta).$$

Let now π, p, ζ be chosen according to (16.2). Then

$$P^{\sigma} = \pi^{\sigma}p^{\sigma} \tag{16.61}$$

because $P = \pi p$, and

$$p^{\sigma} \in \mathfrak{R}[x] - \mathfrak{R}[x]\mathfrak{p}(\frown\theta), \tag{16.62}$$

because $p \in \mathfrak{R}[x] - \mathfrak{R}[x]\mathfrak{p}(\frown\theta)$, using (14.34). Transforming

$$f(\zeta) = o(p(\zeta))$$

by σ, we get on the left-hand side

$$\sigma^{-1}f(\zeta)\sigma = f^{\sigma}(\zeta^{\sigma}) = f(\zeta^{\sigma}),$$

and on the right-hand side

$$\sigma^{-1}o(p(\zeta))\sigma = o(\sigma^{-1}p(\zeta)\sigma) = o(p^{\sigma}(\zeta^{\sigma})).$$

Hence

$$f(\zeta^{\sigma}) = o(p^{\sigma}(\zeta^{\sigma})). \tag{16.63}$$

Now (16.61)-(16.63) and (16.2) (applied to P^{σ}, π^{σ}, p^{σ}, ζ^{σ}) show that $\sigma^{-1}P(\theta)\sigma$

$=P^\sigma(\theta)\gtrless 0$ according as $P^\sigma(\zeta^\sigma)\gtrless 0$; but $P^\sigma(\zeta^\sigma)\gtrless 0$ according as $P(\zeta)\gtrless 0$ because they are conjugate in Σ; and $P(\zeta)\gtrless 0$ according as $P(\theta)\gtrless 0$ by (16.2) (applied to P, π, p, ζ). This establishes the lemma.

(16.7) LEMMA. *The product of two positive elements of $\Sigma(\theta)$ is positive.*

Proof. Let P_1, $P_2\in\Sigma[x]$ be of smaller degree than f, and $P_1>0$, $P_2>0$. Put $P=P_1P_2$. This need not be of smaller degree than f. Put

$$P = Qf + R,$$

where R is of smaller degree than f. Then $R\neq 0$, for $R(\theta)=(P_1P_2)(\theta)$ $=P_1(\theta)P_2(\theta)\neq 0$. Put also

$$P_1 = \pi_1 p_1, \qquad P_2 = \pi_2 p_2$$

with $\pi_i\in\Sigma$, $p_i\in\mathfrak{R}[x]-\mathfrak{R}[x]\mathfrak{p}(\frown\theta)$; finally put

$$\pi_1\pi_2 = \pi; \qquad \pi^{-1}P = p; \qquad \pi^{-1}Q = q; \qquad \pi^{-1}R = r.$$

Then

$$p = \pi_2^{-1}p_1\pi_2 \cdot p_2. \tag{16.71}$$

As

$$\pi_2^{-1}p_1\pi_2 \in \mathfrak{R}[x] - \mathfrak{q}(\frown\theta),$$
$$p_2 \in \mathfrak{R}[x] - \mathfrak{q}(\frown\theta),$$

and $\mathfrak{q}(\frown\theta)$ is a two-sided prime ideal in $\mathfrak{R}[x]$ (by (15.62)), it follows that also

$$p \in \mathfrak{R}[x] - \mathfrak{q}(\frown\theta).$$

Therefore (16.5) can be applied. Thus we have to show only that $P(\zeta)>0$ if ζ is chosen according to (16.53).

We may assume that π_1 and π_2 are positive. Then $\pi>0$, and $P(\zeta)$ has the same sign as $p(\zeta)$. We chose ζ according to (15.7) and (15.92) so that simultaneously

$$f(\zeta) = o(p(\zeta) \,\&\, \pi_2^{-1}p_1\pi_2(\zeta) \,\&\, p_2(\zeta)). \tag{16.72}$$

Then we can apply (13.7), with $\pi_2^{-1}p_1\pi_2$, p_2, ζ, f, p in the place of p, q, ξ, f, pq respectively. We get then

$$p(\zeta) = \pi_2^{-1}p_1\pi_2(\zeta)\cdot p_2(\zeta)(1 + o(1)). \tag{16.73}$$

Now $\pi_2^{-1}p_1\pi_2(\zeta)>0$ because $p_1(\theta)>0$, using (16.6). Also $p_2(\zeta)>0$ because $p_2(\theta)>0$. Hence, by (16.73), $p(\zeta)>0$, and the lemma follows.

(16.81). LEMMA. *If $\pi\in\Sigma$, then π has the same sign qua element of $\Sigma(\theta)$ as qua element of Σ.*

The proof is obvious and omitted.

(16.82) LEMMA. *If* $\frown\rho_1 < \frown\theta < \frown\rho_2$ *with* $\frown\rho_1, \frown\rho_2 \in \frown\mathrm{P}_0$, *then for the corresponding elements* ρ_1, ρ_2 *of* P_0 *also*

$$\rho_1 < \theta < \rho_2.$$

The proof is obvious and omitted. Combining all these facts we have now the following theorem.

(16.9) THEOREM. *If* $\frown\theta$ *has property* (15.1), *then the algebraic extension* $\Sigma(\theta)$ *of* Σ *is ordered by the definitions* (16.1), (16.2); *its order continues that of* Σ, *and* (10.1) *is satisfied.*

17. **Ordering a simple algebraic extension (outside** $\frown\mathfrak{R}$**).** We have so far dealt only with the cases that $\frown\theta$ is transcendent over $\frown\mathrm{P}$ (§10) or contained in $\frown\mathfrak{R}$ (§§11–16). If $\frown\theta$ is algebraic over $\frown\mathrm{P}$ but outside $\frown\mathfrak{R}$ the ordering procedure is rather simpler[73]; in particular if we may assume $\frown\mathrm{P} = \frown\mathfrak{R}$. In fact we then have only to order the extension macroscopically.

Let then $\frown\mathrm{P} = \frown\mathfrak{R}$. Then P is obviously a maximal subfield $\mathrm{P}_{\max} \leqq \mathrm{Z} \cap \mathfrak{R}$. Hence if $f \in \mathrm{P}[x]$ is irreducible, then f is irreducible in $\Sigma[x]$ (cf. (9.8)). We assume $\frown\theta$ algebraic (of degree >1) over $\frown\mathrm{P}$, f again the monic irreducible polynomial $\in \mathrm{P}[x]$ defined by $\frown f(\frown\theta) = \frown 0$. The elements of $\Sigma(\theta)$ are the polynomials in θ of smaller degree than f, with coefficients in Σ.

(17.1) DEFINITION. Let $p \in \mathfrak{R}[x] - \mathfrak{P}[x]$ be a polynomial with finite coefficients not all infinitely small. Then we define

$$p(\theta) \gtrless 0 \qquad \text{according as } \frown p(\frown\theta) \gtrless \frown 0. \tag{17.11}$$

(17.2) DEFINITION. Let $P \in \Sigma[x]$ be a polynomial, and let $\pi \in \Sigma$, $p \in \mathfrak{R}[x] - \mathfrak{P}[x]$ be such that $\pi > 0$ and

$$P = \pi p. \tag{17.21}$$

Then we define

$$P(\theta) \gtrless 0 \qquad \text{according as } p(\theta) \gtrless 0. \tag{17.22}$$

Note that we do not here restrict the degree of p or P; but that (17.11) does not cover the case that $\frown p(\frown\theta) = \frown 0$.

(17.3) LEMMA. *By virtue of definitions* (17.1), (17.2) *every element of* $\Sigma(\theta)$ *is either* >0, *or* $=0$, *or* <0, *and no two of these simultaneously.*

Proof. Any element of $\Sigma(\theta)$ can be represented in the form $P(\theta)$, where $P \in \Sigma[x]$ has smaller degree than f. If $P \neq 0$, let π be the greatest of the absolute values of the coefficients of P, and $p = \pi^{-1}P$. Then $p \in \mathfrak{R}[x] - \mathfrak{P}[x]$, $p \neq 0$, and the degree of p is smaller than that of f. Hence $\frown p(\frown\theta) \neq \frown 0$, and

[73] With a few appropriate changes in the formulation of the definitions (in particular $\mathfrak{p}(\frown\theta)$, $\mathfrak{q}(\frown\theta)$, and property (15.1)) this case could have been subsumed under the algebraic extensions already dealt with. But any gain in conciseness would have been only slight.

(17.1) allocates a sign to $p(\theta)$, (17.2) the same sign to $P(\theta)$. This proves the first part of the lemma.

If $\pi_1 \in \Sigma$, $p_1 \in \mathfrak{R}[x] - \mathfrak{P}[x]$, $\pi_1 > 0$, and $P = \pi_1 p_1$ is a different representation (17.21) of P, we put $\pi_1^{-1}\pi = \rho$. Then $p_1 = \rho p$, and $p \in \mathfrak{R} - \mathfrak{P}$; hence $\hat{}\rho \neq \hat{}0$. In fact $\rho > 0$, and thus $\hat{}\rho > \hat{}0$. Therefore $\hat{}p_1(\hat{}\theta)$ and $\hat{}p(\hat{}\theta)$ are both positive or both negative, and (17.2) allocates the same sign to $P(\theta)$, whatever the representation (17.21).

Finally let $P_1(\theta) = P(\theta)$ be two different representations of the same element of $\Sigma(\theta)$, with P (but of course not P_1) of smaller degree than f. Then $P_1 = Qf + P$. Let also $\pi_1 \in \Sigma$, $p_1 \in \mathfrak{R}[x] - \mathfrak{P}[x]$, $\pi_1 > 0$, and $P_1 = \pi_1 p_1$, and put

$$\pi_1^{-1} Q = q, \qquad \pi_1^{-1} P = r.$$

Then

$$p_1 = qf + r. \tag{17.31}$$

Clearly $r \in \mathfrak{R}[x]$. If $r \in \mathfrak{R}[x] - \mathfrak{P}[x]$, then (17.1) and (17.2) (applied to P, π_1, r) imply that

$$P(\theta) \gtrless 0 \qquad \text{according as } \hat{}r(\hat{}\theta) \gtrless \hat{}0,$$

and $\hat{}r(\hat{}\theta) = \hat{}p_1(\hat{}\theta)$ by (17.31), and then (17.1) and (17.2) (applied to P_1, π_1, p_1) give

$$P_1(\theta) \gtrless 0 \qquad \text{according as } \hat{}p_1(\hat{}\theta) \gtrless 0.$$

Hence in this case $P(\theta)$ and $P_1(\theta)$ are allocated the same sign by the definitions. If, on the other hand, $r \in \mathfrak{P}[x]$, then (17.31) shows that $\hat{}p_1(\hat{}\theta) = \hat{}0$; hence in this case (17.1) and (17.2) (applied to P_1, π_1, p_1) are vacuous, and no other sign is allocated to $P_1(\theta)$ than to $P(\theta)$. This completes the proof of the lemma.

(17.4) THEOREM. *If* $\hat{}\mathrm{P} = \hat{}\mathfrak{R}$ *and* $\hat{}\theta$ *is algebraic over* $\hat{}\mathrm{P}$, *then the algebraic extension* $\Sigma(\theta)$ *of* Σ *is ordered by the definitions* (17.1), (17.2); *its order continues that of* Σ, *and* (10.1) *is satisfied.*

The proof follows, with the appropriate (and obvious) simplifications, the line of argument in §10 (or §16), and is omitted.

Finally we can now establish the main result of this part.

(17.5) THEOREM. *An ordered division ring* Σ *can be extended to an ordered division ring* Σ^* (*containing* Σ *and continuing its order, and*) *containing in its centre a field* $^*\mathrm{P}$ *order-isomorphic to the field* $\hat{}\mathrm{P}^*$ *of all real numbers.*

Proof ([74]). The algebraic adjunction of the required elements has been described in §9. Following the Steinitz [13] procedure we first adjoin, step by

([74]) We present a proof sketch rather than a formal proof. Note that well-order is implicitly and extensively used.

step, all transcendents, and order the resulting extensions as described in §10. Next we adjoin (again step by step) algebraic elements θ for which $\widehat{\ }\theta$ has property (15.1); and we order the resulting extensions as described in §16. By theorem (15.2) this can be carried on as long as there are algebraic elements in $\widehat{\ }\mathfrak{R}-\widehat{\ }P$ left; that is—as the transcendent elements have been adjoined already—until $\widehat{\ }P$ coincides with $\widehat{\ }\mathfrak{R}$. Then finally we adjoin (again step by step) the algebraic elements outside $\widehat{\ }\mathfrak{R}=\widehat{\ }P$, and order the resulting extensions as described in this paragraph. This process comes to an end only when $\widehat{\ }P^*$ is exhausted; the final division ring thus obtained is the desired Σ^*.

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