ON THE ZEROS OF SUCCESSIVE DERIVATIVES OF INTEGRAL FUNCTIONS

BY

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1. The Gontcharoff polynomials

$$G_0(z) = 1; \quad G_n(z; z_1, z_2, \cdots, z_n) = \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} dz^{(n)} \qquad (n \geqq 1)$$

have applications to a certain class of interpolation problem (Whittaker [7])(1). In this paper I obtain some formulae connected with these polynomials and use them to improve and extend a theorem due to Levinson [3, 4], and to shorten the proof of and extend a theorem due to Schoenberg [6].

LEVINSON'S THEOREM. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} < .7199,$$

and if $f(z)$ and each of its derivatives have at least one zero in or on the unit circle, then $f(z) \equiv 0$.

The constant .7199 is not the "best possible" but cannot be replaced [5] by a number as great as .7378.

The "best possible" value of this constant is known as the Whittaker constant W. Among new results in this paper, I prove that W cannot be less than .7259.

SCHOENBERG'S THEOREM. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} < \frac{\pi}{4},$$

and if $f(z)$ and each of its derivatives have at least one zero in the segment $-1 \leqq x \leqq 1$ of the real axis, then $f(z) \equiv 0$.

The constant $\pi/4$ is the "best possible" as shown by the example $\cos(\pi z/4) + \sin(\pi z/4)$.

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(1) Numbers in brackets refer to the references cited at the end of the paper.

2. Following the notation used by Levinson [3], let

$$H_0(z) = 1;\ H_n(z_1, z_2, \cdots, z_n) = G_n(0; z_1, z_2, \cdots, z_n) \qquad (n \geqq 1),$$
$$M_n = \max\ |G_n(z_0; z_1, z_2, \cdots, z_n)| \qquad (\text{all } |z_r| \leqq 1),$$
$$L_n = \max\ |H_n(z_1, z_2, \cdots, z_n)| \qquad (\text{all } |z_r| \leqq 1).$$

We first require two inequalities (2.1) and (2.4) due to Levinson and (for the sake of completeness) give his proof. Since by definition

$$G_n(z_0; z_1, \cdots, z_n) = H_n(z_1, z_2, \cdots, z_n) - H_n(z_0, z_2, \cdots, z_n),$$

therefore, by Taylor's Theorem

$$G_n(z_0; z_1, z_2, \cdots, z_n) = \sum_{r=1}^{n} \frac{(z_1)^r}{r!} H_{n-r}(z_{r+1}, z_{r+2}, \cdots, z_n) - \sum_{r=1}^{n} \frac{(z_0)^r}{r!} H_{n-r}(z_{r+1}, z_{r+2}, \cdots, z_n).$$

Hence

$$|G_n(z_0; z_1, \cdots, z_n)| \leqq \sum_{r=1}^{n} \frac{|z_1^r - z_0^r|}{r!} L_{n-r}$$

and, if we write $2\alpha = \arg z_1 - \arg z_0$,

$$M_n \leqq \max_{0 \leqq \alpha \leqq \pi/2} \left\{ \sum_{r=1}^{n} \frac{2|\sin r\alpha|}{r!} L_{n-r} \right\}. \tag{2.1}$$

By Euler's formula for homogeneous functions,

$$nG_n = \sum_{r=0}^{n} z_r \frac{\partial G_n}{\partial z_r},$$

and since

$$\frac{\partial G_n}{\partial z_0} = G_{n-1}(z_0; z_2, \cdots, z_n), \tag{2.2}$$

$$\frac{\partial G_n}{\partial z_r} = -G_{r-1}(z_0; z_1, \cdots, z_{r-1}) \times G_{n-r}(z_r; z_{r+1}, \cdots, z_n) \qquad (r \geqq 1), \tag{2.3}$$

we have the inequality

$$nM_n \leqq M_{n-1} + \sum_{r=1}^{n} M_{r-1} M_{n-r}. \tag{2.4}$$

It is obvious, as Levinson points out, that $L_1 = 1$, $L_2 = 3/2$, $M_1 = 2$, and hence from (2.1) he obtains $M_2 \leqq (3/2)3^{1/2} < 2.5981$, $M_3 < 3.6379$. By special

choice of the z_r he shows that these values are "accurate" and that in fact $M_2=(3/2)3^{1/2}$ and $M_3>3.6378$. It can also be proved [4] that $L_3=2^{-1}[2(5)^{1/2}+3]^{1/2}+6^{-1}[6(5)^{1/2}-2]^{1/2}<1.9299$, and again, by use of (2.1) he obtains [4] $M_4<4.8414$. He then uses (2.4) to find upper bounds for M_5, M_6, M_7, M_8, M_9, and (by induction) M_n. In fact $M_n \leqq r^{n+1}$ $(n>1)$ where $r<1.389$. He remarks that this method would presumably yield a better value of r if accurate values of some further members of the sequence M_n were worked out before resorting to the use of formula (2.4). However the problem of determining L_4 or M_5 exactly is not simple and for higher L_n, M_n, this does not seem a very promising line of approach.

3. It is, however, possible to obtain upper bounds for L_4, and so on, by using another interation formula involving both sequences L_n and M_n. For Euler's formula gives

$$nH_n = \sum_{r=1}^{n} z_r \frac{\partial H_n}{\partial z_r}$$

and since

$$\partial H_n/\partial z_r = -H_{r-1}(z_1; z_2, \cdots, z_{r-1}) \times G_{n-r}(z_r; z_{r+1}, \cdots, z_n),$$

we have the inequality

$$nL_n \leqq \sum_{r=1}^{n} L_{r-1}M_{n-r}. \tag{3.1}$$

In particular, when $n=4$, $4L_4 \leqq L_0M_3+L_1M_2+L_2M_1+L_3M_0$, yielding $L_4<2.7915$, and (2.1) gives $M_5 \leqq \max_{0\leqq\alpha\leqq\pi/2} \phi_5(\alpha)$ where

$$\begin{aligned}\phi_5(\alpha) = {} & 5.5830\,|\sin\alpha| + 1.9299\,|\sin 2\alpha| + (1/2)\,|\sin 3\alpha| \\ & + (1/12)\,|\sin 4\alpha| + (1/60)\,|\sin 5\alpha|.\end{aligned}$$

The maximum on this curve lies between 70°27′ and 70°28′ and shows that $M_5<6.8223$.

Proceeding in this way by alternate use of (2.1) and (3.1), we find upper bounds for L_5, L_6, L_7, L_8, L_9, L_{10}; M_6, M_7, M_8, M_9, and M_{10} (see appendix). The curves whose maxima have to be determined may be taken as

$$\begin{aligned}\phi_6(\alpha) = {} & 7.6112\,|\sin\alpha| + 2.7915\,|\sin 2\alpha| + 0.6433\,|\sin 3\alpha| \\ & + (1/8)\,|\sin 4\alpha| + (1/60)\,|\sin 5\alpha| + (1/360)\,|\sin 6\alpha| \\ & \qquad\qquad \text{(maximum between 69°31′ and 69°32′)},\end{aligned}$$

$$\begin{aligned}\phi_7(\alpha) = {} & 10.5078\,|\sin\alpha| + 3.8056\,|\sin 2\alpha| + 0.9305\,|\sin 3\alpha| \\ & + 0.1609\,|\sin 4\alpha| + (1/40)\,|\sin 5\alpha| + (1/360)\,|\sin 6\alpha| + 2/7! \\ & \qquad\qquad \text{(maximum between 69°54′ and 69°55)},\end{aligned}$$

$$\begin{aligned}\phi_8(\alpha) = {} & 14.4630\,|\sin\alpha| + 5.2539\,|\sin 2\alpha| + 1.2686\,|\sin 3\alpha| \\ & + 0.2327\,|\sin 4\alpha| + 0.0322\,|\sin 5\alpha| + (1/240)\,|\sin 6\alpha| \\ & + 2/7! + 2/8! \qquad \text{(maximum between } 69°49' \text{ and } 69°51'),\end{aligned}$$

$$\begin{aligned}\phi_9(\alpha) = {} & 19.926924\,|\sin\alpha| + 7.2320\,|\sin 2\alpha| + 1.7513\,|\sin 3\alpha| \\ & + 0.31714\,|\sin 4\alpha| + 0.04653\,|\sin 5\alpha| + 0.00537\,|\sin 6\alpha| \\ & + 3/7! + 2/8! + 2/9! \qquad \text{(maximum between } 69°49' \text{ and } 69°51'),\end{aligned}$$

$$\begin{aligned}\phi_{10}(\alpha) = {} & 27.4424\,|\sin\alpha| + 9.9635\,|\sin 2\alpha| + 2.4105\,|\sin 6\alpha| \\ & + 0.437825\,|\sin 4\alpha| + 0.0634267\,|\sin 5\alpha| + 0.0077542\,|\sin 6\alpha| \\ & + 3.8598/7! + 3/8! + 2/9! + 2/10! \\ & \qquad \text{(maximum between } 69°49' \text{ and } 69°51').\end{aligned}$$

It can be verified by direct computation that

$$M_k < 2(1.3775)^{k+1} \qquad (k = 1, 2, 3), \tag{3.2}$$

$$M_k < (1.3775)^{k+1} \qquad (k = 4, 5, 6, 7, 8, 9, 10), \tag{3.3}$$

$$L_k < (1.3775)^k \qquad (k = 1, 2, 3, 4), \tag{3.4}$$

$$L_k < 0.7692(1.3775)^k \qquad (k = 5, 6, 7, 8, 9, 10). \tag{3.5}$$

From (3.1) we have

$$\begin{aligned} nL_n < {} & M_{n-1} + M_{n-2} + 1.5M_{n-3} + 1.9299M_{n-4} + 2.7915M_{n-5} \\ & + \sum_{r=6}^{n-5} L_{r-1}M_{n-r} + 4.8414L_{n-5} + 3.6379L_{n-4} + 2.5981L_{n-3} \\ & + 2L_{n-2} + L_{n-1}. \end{aligned} \tag{3.6}$$

If we assume (3.3) and (3.5) are satisfied also for $11 \leqq k \leqq n-1$, then (3.6) gives, if we write $\gamma = 1.3775$, $\mu = 0.7692$,

$$\begin{aligned} nL_n < {} & \gamma^n + \gamma^{n-1} + 1.5\gamma^{n-2} + 1.9299\gamma^{n-3} + 2.7915\gamma^{n-4} \\ & + \mu[(n-10)\gamma^n + 4.8414\gamma^{n-5} + 3.6379\gamma^{n-4} + 2.5981\gamma^{n-3} \\ & + 2\gamma^{n-2} + \gamma^{n-1}] < n\mu\gamma^n - 0.0005\gamma^{n-5}. \end{aligned}$$

Hence $L_n < \mu\gamma^n$.

This proves (3.5) is true for all $k \geqq 11$, by induction.

From (2.1) for $n \geqq 11$,

$$\begin{aligned} M_n & \leqq \max_{0 \leqq \alpha \leqq \pi/2} \left\{ \sum_{r=1}^{6} \frac{2\,|\sin r\alpha|}{r!} L_{n-r} \right\} + \sum_{r=7}^{n} \frac{2}{r!} L_{n-r} \\ & < \mu\gamma^{n-7} \max_{0 \leqq \alpha \leqq \pi/2} \Phi(\alpha) + \sum_{r=7}^{n} \frac{2}{r!} \gamma^{7-r} \end{aligned}$$

where

$$\Phi(\alpha) = \sum_{r=1}^{6} \frac{2\,|\sin r\alpha|}{r!}\gamma^{7-r},$$

which has its maximum between 69°49′ and 69°51′, giving

$$\max_{0\leqq\alpha\leqq\pi/2} \Phi(\alpha) < 16.8520.$$

Hence

$$\begin{aligned} M_n &< 16.8520\mu\gamma^{n-7} + \gamma^{n-7}\left[\frac{2}{7!} + \frac{2}{8!}\,\frac{1}{\gamma} + \frac{2}{9!}\,\frac{1}{\gamma^2} + \cdots\right] \\ &< 16.8520\mu\gamma^{n-7} + \gamma^{n-7}\frac{2}{7!}\left[1 + \frac{1}{8\gamma} + \frac{1}{(8\gamma)^2} + \cdots\right] \\ &= 16.8520\mu\gamma^{n-7} + \frac{2\gamma^{n-7}}{7!(1 - 1/8\gamma)} \\ &< \gamma^{n-7}[12.9626 + 0.0006] \\ &< \gamma^{n+1}. \end{aligned}$$

This proves (3.3) for all $k \geqq 11$, by induction.

Since G_n is analytic in the z_r it follows that its maximum modulus is assumed when each z_r is on the circumference of the unit circle. Thus we have the following theorem.

THEOREM I. *If z_r is a sequence of points in the unit circle, then*

$$M_n = \max\,|G_n(z_0; z_1, z_2, \cdots, z_n)| < (1.3775)^{n+1} \qquad (n \geqq 4).$$

4. Now consider the Gontcharoff polynomials for the case discussed by Schoenberg, namely $G_n(x; x_1, x_2, \cdots, x_n)$ where

$$-1 \leqq x_r \leqq +1 \quad (1 \leqq r \leqq n).$$

Consider any one of the 2^{n-r} polynomials

$$G_n(x; x_1, x_2, \cdots, x_r, \pm 1, \pm 1, \cdots, \pm 1) \qquad (1 \leqq r \leqq n),$$

$$\frac{\partial G_n}{\partial x_r} = -G_{r-1}(x; x_1, x_2, \cdots, x_{r-1}) \times G_{n-r}(x_r, \pm 1, \pm 1, \cdots).$$

As x_r varies between -1 and $+1$, keeping $x_1, x_2, \cdots, x_{r-1}$ fixed, $\partial G_n/\partial x_r$ is of constant sign, that is, $G_n(x; x_1, \cdots, x_r, \pm 1, \pm 1, \cdots, \pm 1)$ increases or decreases steadily. Hence $|G_n(x; x_1, \cdots, x_r, \pm 1, \pm 1, \cdots, \pm 1)|$ attains its maximum when x_r is an end point.

If we take $r = 1, 2, \cdots, n$, it follows that $|G_n(x; x_1, \cdots, x_n)|$ $(-1 \leqq x_r$

$\leqq 1$) attains its maximum for any given value of x $(-1 \leqq x \leqq 1)$ when $x_r = \pm 1$ $(1 \leqq r \leqq n)$.

So, in order to find an upper bound for $|G_n(x; x_1, x_2, \cdots; x_n)|$ $(-1 \leqq x_r \leqq 1)$, it is sufficient to consider the 2^n polynomials $|G_n(x; \pm 1, \pm 1, \cdots, \pm 1)|$ $(-1 \leqq x \leqq 1)$.

Clearly if $0 \leqq x \leqq 1$ and $x_r = \pm 1$,

$$|G_n(x; +1, x_2, \cdots, x_n)| = |G_n(-x; -1, -x_2, \cdots, -x_n)| \tag{4.1}$$

$$\leqq |G_n(0; -1, -x_2, \cdots, -x_n)|$$

$$\leqq |G_n(x; -1, -x_2, \cdots, -x_n)|. \tag{4.2}$$

I shall prove that if $0 \leqq x \leqq 1$ and $x_r = \pm 1$ $(1 \leqq r \leqq n)$ for all n,

$$|G_n(x; x_1, x_2, \cdots, x_n)| \leqq 2\left(\frac{4}{\pi}\right)^{n-1} \sin \frac{\pi}{4}(x+1). \tag{4.3}$$

By (4.2), it is sufficient to prove (4.3) for the case $x_1 = -1$, that is, it is sufficient to prove

$$|G_n(x; -1, +1, x_3, \cdots, x_n)| \leqq 2\left(\frac{4}{\pi}\right)^{n-1} \sin \frac{\pi}{4}(x+1) \tag{4.4}$$

and

$$|G_n(x; -1, -1, x_3, \cdots, x_n)| \leqq 2\left(\frac{4}{\pi}\right)^{n-1} \sin \frac{\pi}{4}(x+1). \tag{4.5}$$

Proof of (4.4).

$$|G_{n+1}(x; -1, +1, x_3, \cdots, x_{n+1})| = \int_{-1}^{x} |G_n(x'; +1, x_3, \cdots, x_{n+1})|\, dx'$$

$$= I_1 + I_2,$$

where

$$I_1 = \int_{-1}^{0} |G_n(x'; +1, x_3, \cdots, x_{n+1})|\, dx',$$

$$I_2 = \int_{0}^{x} |G_n(x'; +1, x_3, \cdots, x_{n+1})|\, dx'.$$

If we use (4.1),

$$I_1 = \int_{-1}^{0} |G_n(-x'; -1, -x_3, \cdots, -x_{n+1})|\, dx'.$$

If we substitute $x = -x'$,

$$I_1 = \int_0^1 \left| G_n(x; -1, -x_3, \cdots, -x_{n+1}) \right| dx.$$

Now if we assume that (4.3) is true if n is replaced by any number $m \leqq n$,

$$I_1 \leqq 2\left(\frac{4}{\pi}\right)^{n-1} \int_0^1 \sin \frac{\pi}{4}(x+1)dx = 2^{1/2}\left(\frac{4}{\pi}\right)^n,$$

$$I_2 = \int_0^x dx' \int_{x'}^1 \left| G_{n-1}(x''; x_3, \cdots, x_{n+1}) \right| dx''$$

$$\leqq 2\left(\frac{4}{\pi}\right)^{n-2} \int_0^x dx' \int_{x'}^1 \sin \frac{\pi}{4}(x''+1)dx''$$

$$= 2\left(\frac{4}{\pi}\right)^n \sin \frac{\pi}{4}(x+1) - 2^{1/2}\left(\frac{4}{\pi}\right)^n.$$

Therefore $I_1 + I_2 \leqq 2(4/\pi)^n \sin(\pi/4)(x+1)$.

But (4.4) is true when $n = 0, 1$.

Hence (4.4) is true for all n by induction.

Proof of (4.5).

$$G_{n+1}(x; -1, -1, x_3, \cdots, x_n) = \int_{-1}^x \left| G_n(x'; -1, x_3, \cdots, x_n) \right| dx'$$
$$= I_3 + I_4$$

where

$$I_3 = \int_{-1}^0 \left| G_n(x'; -1, x_3, \cdots, x_n) \right| dx',$$

$$I_4 = \int_0^x \left| G_n(x'; -1, x_3, \cdots, x_n) \right| dx'.$$

If we use (4.1),

$$I_3 = \int_{-1}^0 \left| G_n(-x'; +1, -x_3, \cdots, -x_n) \right| dx'.$$

If we substitute $x = -x'$,

$$I_3 = \int_0^1 \left| G_n(x; +1, -x_3, \cdots, -x_n) \right| dx$$

$$= \int_0^1 \int_x^1 \left| G_{n-1}(x'; -x_3, \cdots, -x_n) \right| dx'.$$

Hence, if we assume (4.3) is true if n is replaced by any number $m \leqq n$,

$$I_3 \leqq 2\left(\frac{4}{\pi}\right)^{n-2} \int_0^1 dx \int_x^1 \sin \frac{\pi}{4}(x'+1)dx'$$
$$= \left(\frac{4}{\pi}\right)^n (2 - 2^{1/2}).$$
$$I_4 \leqq 2\left(\frac{4}{\pi}\right)^{n-1} \int_0^x \sin \frac{\pi}{4}(x'+1)dx'$$
$$= \left(\frac{4}{\pi}\right)^n \left(-2 \cos \frac{\pi}{4}(x+1) + 2^{1/2}\right).$$

Now $1-\cos\ (\pi/4)(x+1) \leqq \sin\ (\pi/4)(x+1)$, $0 \leqq x \leqq 1$. Hence $I_3+I_4 \leqq 2(4/\pi)^n \sin\ (\pi/4)(x+1)$.

But (4.5) is true when $n=0, 1$. Hence (4.5) is true for all n by induction.

Since (4.4) and (4.5) are true, we have proved (4.3). It follows by substituting $-x$ for x, that for $-1 \leqq x \leqq 0$ and $-1 \leqq x_r \leqq 1$ $(1 \leqq r \leqq n)$,

$$|G_n(x; x_1, x_2, \cdots, x_n)| \leqq 2\left(\frac{4}{\pi}\right)^{n-1} \cos \frac{\pi}{4}(x+1), \tag{4.6}$$

and we have the following theorem.

THEOREM II. *If z_r is a sequence of points on the real axis, satisfying $-1 \leqq z_r \leqq 1$, then*

$$|G_n(z_0; z_1, z_2, \cdots, z_n)| \leqq 2\,(4/\pi)^{n-1}. \tag{4.7}$$

5. I shall now discuss extensions of Theorems I and II in which some of the points of the sequence z_r lie outside the unit circle, and the segment $-1 \leqq x \leqq 1$ respectively.

Let $z_r = x_r + y_r$, where both x_r and y_r may be complex, then since $G_n(z_0; z_1, \cdots, z_n)$ is a polynomial in each $z_r (0 \leqq r \leqq n)$, we may apply Taylor's series and write

$$G_n(z_0; z_1, \cdots, z_n) = \exp\left(\sum_{r=0}^{n} y_r \frac{\partial}{\partial x_r}\right) G_n(x_0; x_1, \cdots, x_n). \tag{5.1}$$

Now, writing $G_n(x_0; x_1, x_2, \cdots, x_n) = G_n$, using (2.2) and (2.3), we note that $\partial G_n/\partial x_r$ $(0 \leqq r \leqq n)$ is either one multiple integral or the product of two such integrals, in each case the total multiplicity being $n-1$. Similarly $\partial^k G_n/\partial x_r \partial x_s \cdots \partial x_t$, where $r, s, \cdots, t$ may all take any values between 0 and n inclusive, is either zero (for example, $\partial^2 G_n/\partial x_0 \partial x_1$) or the product of not more than $k+1$ multiple integrals, the total multiplicity being $n-k$.

Now suppose that positive constants A, γ can be found such that

$$|G_n| < A\gamma^{n+1} \tag{5.2}$$

provided that the sequence $\{x_r\}$ belongs to a given set of points S which in-

cludes $z=0$. Such a set exists by Theorem I.

Setting $n=0$, we see that $A\gamma>1$. Hence

$$\left|\frac{\partial^k G_n}{\partial x_r \partial x_s \cdots \partial x_t}\right| < A^{k+1}\gamma^{n+1}.$$

Suppose also that the values of y_r are restricted in such a way that

$$\sum_{r=1}^{n} |y_r| \leqq nh \tag{5.3}$$

for certain values of n. Then (5.1) gives, for these values of n,

$$\begin{aligned} |G_n(z_0; z_1, z_2, \cdots, z_n)| &< \sum_{k=0}^{\infty} \frac{(|y_0| + nh)^k}{k!} A^{k+1}\gamma^{n+1} \\ &= A\gamma^{n+1} \exp\{A(|y_0| + nh)\}. \end{aligned} \tag{5.4}$$

If the sequence $\{z_r\}$ is such that all its limit points belong to S, then (5.3) is satisfied for arbitrarily small h and sufficiently large n, and (5.4) gives

$$|G_n(z_0; z_1, \cdots, z_n)| < Ae^{A|y_0|}(\gamma + \epsilon)^{n+1}, \qquad n \geqq n_0(\epsilon), \tag{5.5}$$

and hence for all z in any given finite domain, and all n,

$$|G_n(z; z_1, z_2, \cdots, z_n)| < A'(\gamma + \epsilon)^{n+1}. \tag{5.6}$$

6. Suppose now that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an integral function satisfying

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} = \sigma < \frac{1}{\gamma},$$

it follows that for any $\tau > \sigma$, and sufficiently large n

$$n!\,|a_n| < \tau^n. \tag{6.1}$$

Then, if $f(z_1)=0$, $f^{(n-1)}(z_n)=0$, clearly

$$f(z) = \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} f^{(n)}(z)dz,$$

or, following Levinson [3, §1], if we replace $f^n(z)$ by its power series, we obtain

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} (n+k)!\,\frac{a_{n+k}}{k!} \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} z^k dz \\ &= \sum_{k=0}^{\infty} (n+k)!\,a_{n+k} G_{n+k}(z; z_1, z_2, \cdots, z_n, 0, 0, \cdots, 0). \end{aligned}$$

Now since the sequence $\{z_n\}$ is such that all its limit points belong to S, then for large n and for all z in any finite domain we have by (5.6) and (6.1)

$$(6.2)\qquad |f(z)| < \sum_{k=0}^{\infty} \tau^{n+k} A'(\gamma+\epsilon)^{n+k+1} = \frac{A'(\gamma+\epsilon)^{n+1}\tau^n}{1-\tau(\gamma+\epsilon)}$$

provided $\tau < 1/(\gamma+\epsilon)$. But letting $n \to \infty$ in (6.2) we have $f(z) \equiv 0$.

In the particular case in which S is the unit circle, Theorem I shows that (5.2) is satisfied with $\gamma = 1.3775 < 1/.7259$ for all values of n, so we now have the following theorem.

THEOREM III. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} < .7259,$$

and if $f(z_1)=0, f^{(n-1)}(z_n)=0$ $(n \geqq 2)$, the sequence $\{z_r\}$ having all its limit points in the unit circle, then $f(z) \equiv 0$.

In the particular case in which S is the segment $0 \leqq x \leqq 1$, Theorem II shows that (5.2) is satisfied for all n with $\gamma = 4/\pi$ and we have the following extension of Schoenberg's theorem.

THEOREM IV. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} < \frac{\pi}{4},$$

and if $f(z_1)=0, f^{(n-1)}(z_n)=0$ $(n \geqq 2)$, the sequence $\{z_r\}$ having all its limit points on the segment $-1 \leqq x \leqq 1$ of the real axis, then $f(z) \equiv 0$.

This result has been stated by Kamenetsky [2, Theorem VIII] but I have been unable to find a published proof. It seems unlikely from the context that his method has anything in common with the one which I have used here.

7. A further theorem follows as a consequence of inequalities (5.2) and (5.6) for the case in which the limit points of the sequence of zeros lie inside the locus of points distant h from the segment $-1 \leqq x \leqq 1$ of the real axis. We shall call the domain enclosed by this curve H. In this case, if we restrict the sequence $\{x_r\}$ to the segment $-1 \leqq x \leqq 1$ (all r) and $z_r = x_r + y_r$ (all $r \geqq 1$) where $|y_r| \leqq h$ $(r \geqq 1)$, (5.2) is satisfied with $A = \pi^2/8$, $\gamma = 4/\pi$, by Theorem II, and (5.3) is satisfied for all n since $|y_r| \leqq h (r \geqq 1)$. Hence (5.4) is satisfied for all n with these values of the constants, that is,

$$|G_n(z_0; z_1, z_2, \cdots, z_n)| \leqq \frac{\pi^2}{8}\left(\frac{4}{\pi}\right)^{n+1} \exp\left\{\frac{\pi^2}{8}(|y_0| + nh)\right\} < \bar{A}\bar{\gamma}^{n+1},$$

with $\bar{\gamma} = (4/\pi) \exp(\pi^2 h/8)$. By a second application of formulae (5.2) and (5.6), we see that, provided all the limit points of the sequence $\{z_r\}$ lie within H, (5.6) holds with $\gamma = (4/\pi) \exp(\pi^2 h/8)$, and we have the following theorem.

THEOREM V. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r\to\infty}\frac{\log M(r)}{r}<\frac{\pi}{4}\exp\left(-\frac{\pi^2 h}{8}\right),$$

and if $f(z_1)=0$, $f^{(n-1)}(z_n)=0$ $(n\geqq 2)$, where the sequence $\{z_r\}$ has all its limit points in H, then $f(z)\equiv 0$.

It is to be noted that the constant $(\pi/4)\exp(-\pi^2 h/8)$ is "better" (that is, greater) than that obtained from the circle circumscribed to H, namely, $.7259/1+h$ (which is obtained from Theorem III by the transformation $\zeta=(1+h)z$) only for small values of h. It is "better" when $h\leqq 0.23$ but not when $h=0.24$.

APPENDIX

	Upper bounds for			
n	M_{n-1}	L_n	$L_n/(1.3775)^n$	$(1.3775)^n$
1	1	1	0.7260	1.3775
2	2	1.5	0.7905	1.8975
3	2.5981	1.9299	0.7384	2.6138
4	3.6379	2.7915	0.7753	3.6005
5	4.8414	3.8056	0.7673	4.9597
6	6.8223	5.2539	0.7690	6.8320
7	9.3973	7.2315	0.7685	9.4111
8	12.9512	9.9635	0.7686	12.9638
9	17.8413	13.7212	0.7684	17.8577
10	24.5754	18.8998	0.7683	24.5989
11	33.8472			33.8850

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