ON THE ZEROS OF SUCCESSIVE DERIVATIVES OF INTEGRAL FUNCTIONS

BY

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1. The Gontcharoff polynomials

\[ G_0(z) = 1; \quad G_n(z; z_1, z_2, \ldots, z_n) = \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} dz^{(n)} \quad (n \geq 1) \]

have applications to a certain class of interpolation problem (Whittaker [7]). In this paper I obtain some formulae connected with these polynomials and use them to improve and extend a theorem due to Levinson [3, 4], and to shorten the proof of and extend a theorem due to Schoenberg [6].

**Levinson's Theorem.** If \( f(z) \) is an integral function satisfying

\[ \limsup_{r \to \infty} \frac{\log M(r)}{r} < 0.7199, \]

and if \( f(z) \) and each of its derivatives have at least one zero in or on the unit circle, then \( f(z) = 0 \).

The constant 0.7199 is not the "best possible" but cannot be replaced [5] by a number as great as 0.7378.

The "best possible" value of this constant is known as the Whittaker constant \( W \). Among new results in this paper, I prove that \( W \) cannot be less than 0.7259.

**Schoenberg's Theorem.** If \( f(z) \) is an integral function satisfying

\[ \limsup_{r \to \infty} \frac{\log M(r)}{r} < \frac{\pi}{4}, \]

and if \( f(z) \) and each of its derivatives have at least one zero in the segment \(-1 \leq x \leq 1\) of the real axis, then \( f(z) = 0 \).

The constant \( \pi/4 \) is the "best possible" as shown by the example \( \cos(\pi z/4) + \sin(\pi z/4) \).

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(*) Numbers in brackets refer to the references cited at the end of the paper.

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2. Following the notation used by Levinson [3], let
\[ H_0(z) = 1; \quad H_n(z_1, z_2, \ldots, z_n) = G_n(0; z_1, z_2, \ldots, z_n) \quad (n \geq 1), \]
\[ M_n = \max \left| G_n(z_0; z_1, z_2, \ldots, z_n) \right| \quad (\text{all } |z_r| \leq 1), \]
\[ L_n = \max \left| H_n(z_1, z_2, \ldots, z_n) \right| \quad (\text{all } |z_r| \leq 1). \]

We first require two inequalities (2.1) and (2.4) due to Levinson and (for the sake of completeness) give his proof. Since by definition
\[ G_n(z_0; z_1, \ldots, z_n) = H_n(z_1, z_2, \ldots, z_n) - H_n(z_0, z_2, \ldots, z_n), \]
therefore, by Taylor's Theorem
\[ G_n(z_0; z_1, z_2, \ldots, z_n) = \sum_{r=0}^{n} \frac{(z_1)^{r}}{r!} H_{n-r}(z_{r+1}, z_{r+2}, \ldots, z_n) - \sum_{r=0}^{n} \frac{(z_0)^{r}}{r!} H_{n-r}(z_{r+1}, z_{r+2}, \ldots, z_n). \]

Hence
\[ \left| G_n(z_0; z_1, \ldots, z_n) \right| \leq \sum_{r=0}^{n} \frac{|z_1 - z_0|^r}{r!} L_{n-r} \]
and, if we write \( 2\alpha = \arg z_1 - \arg z_0 \),

\[ (2.1) \quad M_n \leq \max_{0 \leq \alpha \leq \pi/2} \left\{ \sum_{r=0}^{n} \frac{2|\sin r\alpha|}{r!} L_{n-r} \right\}. \]

By Euler's formula for homogeneous functions,
\[ nG_n = \sum_{r=0}^{n} z_r \frac{\partial G_n}{\partial z_r}, \]
and since
\[ (2.2) \quad \frac{\partial G_n}{\partial z_0} = G_{n-1}(z_0; z_2, \ldots, z_n), \]
\[ (2.3) \quad \frac{\partial G_n}{\partial z_r} = -G_{r-1}(z_0; z_1, \ldots, z_{r-1}) \times G_{n-r}(z_r; z_{r+1}, \ldots, z_n) \quad (r \geq 1), \]
we have the inequality
\[ (2.4) \quad nM_n \leq M_{n-1} + \sum_{r=1}^{n} M_{r-1}M_{n-r}. \]

It is obvious, as Levinson points out, that \( L_1 = 1, \quad L_2 = 3/2, \quad M_1 = 2, \) and hence from (2.1) he obtains \( M_2 \leq (3/2)3^{1/2} < 2.5981, \quad M_3 < 3.6379. \) By special
choice of the $z_r$ he shows that these values are “accurate” and that in fact $M_2 = (3/2)^{3/2}$ and $M_3 > 3.6378$. It can also be proved [4] that $L_3 = 2^{-1} [2(5)^{1/2} + 3]^{1/2} + 6^{-1} [6(5)^{1/2} - 2]^{1/2} < 1.9299$, and again, by use of (2.1) he obtains [4] $M_4 < 4.8414$. He then uses (2.4) to find upper bounds for $M_6$, $M_8$, $M_7$, $M_8$, $M_8$, and (by induction) $M_n$. In fact $M_n \leq r^{n+1} (n>1)$ where $r < 1.389$. He remarks that this method would presumably yield a better value of $r$ if accurate values of some further members of the sequence $M_n$ were worked out before resorting to the use of formula (2.4). However the problem of determining $L_4$ or $M_8$ exactly is not simple and for higher $L_n$, $M_n$, this does not seem a very promising line of approach.

3. It is, however, possible to obtain upper bounds for $L_4$, and so on, by using another iteration formula involving both sequences $L_n$ and $M_n$. For Euler’s formula gives

$$H_n = \sum_{r=1}^{n} z_r \frac{\partial H_n}{\partial z_r}$$

and since

$$\frac{\partial H_n}{\partial z_r} = - H_{r-1}(z_1, z_2, \ldots, z_{r-1}) \times G_{n-r}(z_r, z_{r+1}, \ldots, z_n),$$

we have the inequality

$$(3.1) \quad nL_n \leq \sum_{r=1}^{n} L_{r-1} M_{n-r}.$$ 

In particular, when $n = 4$, $4L_4 \leq L_0 M_4 + L_1 M_2 + L_2 M_1 + L_3 M_0$, yielding $L_4 < 2.7915$, and (2.1) gives $M_5 \leq \max_{0 \leq \alpha \leq \pi/2} \phi_5(\alpha)$ where

$$\phi_5(\alpha) = 5.5830 |\sin \alpha| + 1.9299 |\sin 2\alpha| + (1/2) |\sin 3\alpha| + (1/12) |\sin 4\alpha| + (1/60) |\sin 5\alpha|.$$ 

The maximum on this curve lies between 70°27′ and 70°28′ and shows that $M_5 < 6.8223$.

Proceeding in this way by alternate use of (2.1) and (3.1), we find upper bounds for $L_6$, $L_8$, $L_9$, $L_9$, $L_{10}$; $M_6$, $M_7$, $M_8$, $M_8$, and $M_{10}$ (see appendix). The curves whose maxima have to be determined may be taken as

$$\phi_6(\alpha) = 7.6112 |\sin \alpha| + 2.7915 |\sin 2\alpha| + 0.6433 |\sin 3\alpha| + (1/8) |\sin 4\alpha| + (1/60) |\sin 5\alpha| + (1/360) |\sin 6\alpha|$$

(maximum between 69°31′ and 69°32′),

$$\phi_7(\alpha) = 10.5078 |\sin \alpha| + 3.8056 |\sin 2\alpha| + 0.9305 |\sin 3\alpha| + 0.1609 |\sin 4\alpha| + (1/40) |\sin 5\alpha| + (1/360) |\sin 6\alpha| + 2/7!$$

(maximum between 69°54′ and 69°55′).
\[ \phi_8(\alpha) = 14.4630 | \sin \alpha | + 5.2539 | \sin 2\alpha | + 1.2686 | \sin 3\alpha | \\
+ 0.2327 | \sin 4\alpha | + 0.0322 | \sin 5\alpha | + (1/240) | \sin 6\alpha | \\
+ 2/7! + 2/8! \quad (\text{maximum between } 69^\circ 49' \text{ and } 69^\circ 51'), \]

\[ \phi_9(\alpha) = 19.926924 | \sin \alpha | + 7.2320 | \sin 2\alpha | + 1.7513 | \sin 3\alpha | \\
+ 0.31714 | \sin 4\alpha | + 0.04653 | \sin 5\alpha | + 0.00537 | \sin 6\alpha | \\
+ 3/7! + 2/8! + 2/9! \quad (\text{maximum between } 69^\circ 49' \text{ and } 69^\circ 51'), \]

\[ \phi_{10}(\alpha) = 27.4424 | \sin \alpha | + 9.9635 | \sin 2\alpha | + 2.4105 | \sin 3\alpha | \\
+ 0.437825 | \sin 4\alpha | + 0.0634267 | \sin 5\alpha | + 0.0077542 | \sin 6\alpha | \\
+ 3.8598/7! + 3/8! + 2/9! + 2/10! \quad (\text{maximum between } 69^\circ 49' \text{ and } 69^\circ 51'). \]

It can be verified by direct computation that

\[ (3.2) \quad M_k < 2(1.3775)^{k+1} \quad (k = 1, 2, 3), \]

\[ (3.3) \quad M_k < (1.3775)^{k+1} \quad (k = 4, 5, 6, 7, 8, 9, 10), \]

\[ (3.4) \quad L_k < (1.3775)^k \quad (k = 1, 2, 3, 4), \]

\[ (3.5) \quad L_k < 0.7692(1.3775)^k \quad (k = 5, 6, 7, 8, 9, 10). \]

From (3.1) we have

\[ n L_n < M_{n-1} + M_{n-2} + 1.5 M_{n-3} + 1.9299 M_{n-4} + 2.7915 M_{n-5} \]

\[ + \sum_{r=6}^{n-5} L_{r-1} M_{n-r} + 4.8414 L_{n-5} + 3.6379 L_{n-4} + 2.5981 L_{n-3} \]

\[ + 2 L_{n-2} + L_{n-1}. \]

If we assume (3.3) and (3.5) are satisfied also for \( 11 \leq k \leq n-1 \), then (3.6) gives, if we write \( \gamma = 1.3775, \mu = 0.7692, \)

\[ n L_n < \gamma^n + \gamma^{n-1} + 1.5 \gamma^{n-2} + 1.9299 \gamma^{n-3} + 2.7915 \gamma^{n-4} \]

\[ + \mu [ (n - 10) \gamma^n + 4.8414 \gamma^{n-3} + 3.6379 \gamma^{n-4} + 2.5981 \gamma^{n-3} \]

\[ + 2 \gamma^{n-2} + \gamma^{n-1} ] < n \mu \gamma^n + 0.0005 \gamma^{n-5}. \]

Hence \( L_n < \mu \gamma^n \).

This proves (3.5) is true for all \( k \geq 11 \), by induction.

From (2.1) for \( n \geq 11 \),

\[ M_n \leq \max_{0 \leq \alpha \leq \pi/2} \left\{ \sum_{r=1}^{6} \frac{2 | \sin r \alpha |}{r!} L_{n-r} \right\} + \sum_{r=7}^{n} \frac{2}{r!} L_{n-r} \]

\[ < \mu \gamma^{n-7} \max_{0 \leq \alpha \leq \pi/2} \Phi(\alpha) + \sum_{r=7}^{n} \frac{2}{r!} \gamma^{7-r} \]
where
\[ \Phi(\alpha) = \sum_{r=1}^{8} \frac{2 | \sin r\alpha |}{r!} \gamma^{r-r}, \]
which has its maximum between 69°49' and 69°51', giving
\[ \max_{0 \leq \alpha \leq \pi/2} \Phi(\alpha) < 16.8520. \]

Hence
\[ M_n < 16.8520 \mu \gamma^{n-7} + \gamma^{n-7} \left[ \frac{2}{7!} + \frac{2}{8!} \frac{1}{\gamma} + \frac{2}{9!} \frac{1}{\gamma^2} + \cdots \right] \]
\[ < 16.8520 \mu \gamma^{n-7} + \gamma^{n-7} \left[ \frac{1}{7!} \frac{1}{8\gamma} + \frac{1}{(8\gamma)^2} + \cdots \right] \]
\[ = 16.8520 \mu \gamma^{n-7} + \frac{2\gamma^{n-7}}{7!(1 - 1/8\gamma)} \]
\[ < \gamma^{n-7} [12.9626 + 0.0006] \]
\[ < \gamma^{n+1}. \]

This proves (3.3) for all \( k \geq 11 \), by induction.

Since \( G_n \) is analytic in the \( z_r \) it follows that its maximum modulus is assumed when each \( z_r \) is on the circumference of the unit circle. Thus we have the following theorem.

**Theorem I.** If \( z_r \) is a sequence of points in the unit circle, then
\[ M_n = \max |G_n(z_0; z_1, z_2, \ldots, z_n)| < (1.3775)^{n+1} \quad (n \geq 4). \]

4. Now consider the Gontcharoff polynomials for the case discussed by Schoenberg, namely \( G_n(x; x_1, x_2, \ldots, x_n) \) where
\[-1 \leq x_r \leq +1 \quad (1 \leq r \leq n).\]

Consider any one of the \( 2^{n-r} \) polynomials
\[ G_n(x; x_1, x_2, \ldots, x_r, \pm 1, \pm 1, \ldots, \pm 1) \quad (1 \leq r \leq n), \]
\[ \frac{\partial G_n}{\partial x_r} = -G_{n-1}(x_1, x_2, \cdots, x_{r-1}) \times G_{n-r}(x_r, \pm 1, \pm 1, \cdots). \]

As \( x_r \) varies between \(-1\) and \(+1\), keeping \( x_1, x_2, \cdots, x_{r-1} \) fixed, \( \partial G_n/\partial x_r \) is of constant sign, that is, \( G_n(x; x_1, \cdots, x_r, \pm 1, \pm 1, \cdots, \pm 1) \) increases or decreases steadily. Hence \( |G_n(x; x_1, \cdots, x_r, \pm 1, \pm 1, \cdots, \pm 1)| \) attains its maximum when \( x_r \) is an end point.

If we take \( r = 1, 2, \cdots, n \), it follows that \( |G_n(x; x_1, \cdots, x_n)| \) \((-1 \leq x_r \leq +1, \cdots, \leq +1)\)
attains its maximum for any given value of \( x \) \((-1 \leq x \leq 1)\) when \( x_r = \pm 1 \) \((1 \leq r \leq n)\).

So, in order to find an upper bound for \( |G_n(x; x_1, x_2, \ldots, x_n)| \) \((-1 \leq x_r \leq 1)\), it is sufficient to consider the \(2^n\) polynomials \( |G_n(x; \pm 1, \pm 1, \ldots, \pm 1)| \) \((-1 \leq x \leq 1)\).

Clearly if \( 0 \leq x \leq 1 \) and \( x_r = \pm 1 \),

\begin{equation}
|G_n(x; 1, x_2, \ldots, x_n)| = |G_n(-x; -1, -x_2, \ldots, -x_n)| \leq |G_n(0; -1, -x_2, \ldots, -x_n)| \leq |G_n(x; -1, -x_2, \ldots, -x_n)|.
\end{equation}

I shall prove that if \( 0 \leq x \leq 1 \) and \( x_r = \pm 1 \) \((1 \leq r \leq n)\) for all \( n \),

\begin{equation}
|G_n(x; x_1, x_2, \ldots, x_n)| \leq 2 \left( \frac{4}{\pi} \right)^{n-1} \sin \frac{\pi}{4} (x + 1).
\end{equation}

By (4.2), it is sufficient to prove (4.3) for the case \( x_1 = -1 \), that is, it is sufficient to prove

\begin{equation}
|G_n(x; -1, +1, x_3, \ldots, x_n)| \leq 2 \left( \frac{4}{\pi} \right)^{n-1} \sin \frac{\pi}{4} (x + 1)
\end{equation}

and

\begin{equation}
|G_n(x; -1, -1, x_3, \ldots, x_n)| \leq 2 \left( \frac{4}{\pi} \right)^{n-1} \sin \frac{\pi}{4} (x + 1).
\end{equation}

**Proof of (4.4).**

\[ |G_{n+1}(x; -1, +1, x_3, \ldots, x_{n+1})| = \int_{-1}^{x} |G_n(x'; +1, x_3, \ldots, x_{n+1})| \, dx' \]

where

\[ I_1 = \int_{-1}^{0} |G_n(x'; +1, x_3, \ldots, x_{n+1})| \, dx', \]

\[ I_2 = \int_{0}^{x} |G_n(x'; +1, x_3, \ldots, x_{n+1})| \, dx'. \]

If we use (4.1),

\[ I_1 = \int_{-1}^{0} |G_n(-x'; -1, -x_2, \ldots, -x_{n+1})| \, dx'. \]

If we substitute \( x = -x' \),
\[ I_1 = \int_0^1 |G_n(x; -1, -x_3, \ldots, -x_{n+1})| \, dx. \]

Now if we assume that (4.3) is true if \( n \) is replaced by any number \( m \leq n \),

\[
\begin{align*}
I_1 &\leq 2 \left( \frac{4}{\pi} \right)^{n-1} \int_0^1 \sin \frac{\pi}{4} (x + 1) \, dx = 2^{1/2} \left( \frac{4}{\pi} \right)^n, \\
I_2 &= \int_0^x dx' \int_{x'}^1 \left| G_{n-1}(x''; x_3, \ldots, x_{n+1}) \right| \, dx'' \\
&\leq 2 \left( \frac{4}{\pi} \right)^{n-2} \int_0^x dx' \int_{x'}^1 \sin \frac{\pi}{4} (x'' + 1) \, dx'' \\
&= 2 \left( \frac{4}{\pi} \right)^n \sin \frac{\pi}{4} (x + 1) - 2^{1/2} \left( \frac{4}{\pi} \right)^n.
\end{align*}
\]

Therefore \( I_1 + I_2 \leq 2(4/\pi)^n \sin (\pi/4)(x+1) \).

But (4.4) is true when \( n = 0, 1 \).

Hence (4.4) is true for all \( n \) by induction.

**Proof of (4.5).**

\[ G_{n+1}(x; -1, -1, x_3, \ldots, x_n) = \int_{-1}^{x} |G_n(x'; -1, x_3, \ldots, x_n)| \, dx' = I_3 + I_4 \]

where

\[
\begin{align*}
I_3 &= \int_{-1}^{0} |G_n(x'; -1, x_3, \ldots, x_n)| \, dx', \\
I_4 &= \int_{0}^{x} |G_n(x'; -1, x_3, \ldots, x_n)| \, dx'.
\end{align*}
\]

If we use (4.1),

\[ I_3 = \int_{-1}^{0} |G_n(-x'; +1, -x_3, \ldots, -x_n)| \, dx'. \]

If we substitute \( x = -x' \),

\[ I_3 = \int_{0}^{1} |G_n(x; +1, -x_3, \ldots, -x_n)| \, dx = \int_{0}^{1} \int_{x'}^{1} |G_{n-1}(x'; -x_3, \ldots, -x_n)| \, dx'. \]

Hence, if we assume (4.3) is true if \( n \) is replaced by any number \( m \leq n \),
\[ I_3 \leq 2 \left( \frac{4}{\pi} \right)^{n-2} \int_0^1 dx \int_x^1 \sin \frac{\pi}{4} (x' + 1) dx' \]
\[ = \left( \frac{4}{\pi} \right)^n (2 - 2^{1/2}). \]
\[ I_4 \leq 2 \left( \frac{4}{\pi} \right)^{n-1} \int_0^x \sin \frac{\pi}{4} (x' + 1) dx' \]
\[ = \left( \frac{4}{\pi} \right)^n \left( -2 \cos \frac{\pi}{4} (x + 1) + 2^{1/2} \right). \]

Now \( 1 - \cos \left( \frac{\pi}{4} (x + 1) \right) \leq \sin \left( \frac{\pi}{4} (x + 1) \right), \quad 0 \leq x \leq 1. \) Hence \( I_3 + I_4 \leq 2 \left( \frac{4}{\pi} \right)^n \sin \left( \frac{\pi}{4} (x + 1) \right). \)

But (4.5) is true when \( n = 0, 1. \) Hence (4.5) is true for all \( n \) by induction.

Since (4.4) and (4.5) are true, we have proved (4.3). It follows by substituting \(-x\) for \(x\), that for \(-1 \leq x \leq 0\) and \(-1 \leq x_r \leq 1 \quad (1 \leq r \leq n),\)

\[ \left| G_n(x; x_1, x_2, \cdots, x_n) \right| \leq 2 \left( \frac{4}{\pi} \right)^{n-1} \cos \frac{\pi}{4} (x + 1), \]

and we have the following theorem.

**Theorem II.** If \( z_r \) is a sequence of points on the real axis, satisfying \(-1 \leq z_r \leq 1,\) then

\[ \left| G_n(z_0; z_1, z_2, \cdots, z_n) \right| \leq 2 \left( \frac{4}{\pi} \right)^{n-1}. \]

5. I shall now discuss extensions of Theorems I and II in which some of the points of the sequence \( z_r \) lie outside the unit circle, and the segment \(-1 \leq x \leq 1\) respectively.

Let \( z_r = x_r + y_r, \) where both \( x_r \) and \( y_r \) may be complex, then since \( G_n(z_0; z_1, \cdots, z_n) \) is a polynomial in each \( z_r (0 \leq r \leq n), \) we may apply Taylor's series and write

\[ G_n(z_0; z_1, \cdots, z_n) = \exp \left( \sum_{r=0}^n y_r \frac{\partial}{\partial x_r} \right) G_n(x_0; x_1, \cdots, x_n). \]

Now, writing \( G_n(x_0; x_1, x_2, \cdots, x_n) = G_n, \) using (2.2) and (2.3), we note that \( \partial G_n/\partial x_r \quad (0 \leq r \leq n) \) is either one multiple integral or the product of two such integrals, in each case the total multiplicity being \( n - 1. \) Similarly \( \partial^k G_n/\partial x_r \partial x_s \cdots \partial x_t, \) where \( r, s, \cdots, t \) may all take any values between 0 and \( n \) inclusive, is either zero (for example, \( \partial^k G_n/\partial x_0 \partial x_1 \)) or the product of not more than \( k + 1 \) multiple integrals, the total multiplicity being \( n - k. \)

Now suppose that positive constants \( A, \gamma \) can be found such that

\[ \left| G_n \right| < A \gamma^{n+1} \]

provided that the sequence \( \{ x_r \} \) belongs to a given set of points \( S \) which in-
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eludes \( z = 0 \). Such a set exists by Theorem I.

Setting \( n = 0 \), we see that \( A' > 1 \). Hence

\[
\left| \frac{\partial^k G_n}{\partial x_r \partial x_s \cdots \partial x_t} \right| < A^{k+1} \gamma^{n+1}.
\]

Suppose also that the values of \( y_r \) are restricted in such a way that

\[
(5.3) \quad \sum_{r=1}^{n} |y_r| \leq nh
\]

for certain values of \( n \). Then (5.1) gives, for these values of \( n \),

\[
(5.4) \quad \left| G_n(z_0; z_1, \ldots, z_n) \right| < \sum_{k=0}^{\infty} \left( \frac{|y_0| + nh}{k!} \right)^k A^{k+1} \gamma^{n+1} = A \gamma^{n+1} \exp \left\{ A (|y_0| + nh) \right\}.
\]

If the sequence \( \{z_r\} \) is such that all its limit points belong to \( S \), then (5.3) is satisfied for arbitrarily small \( h \) and sufficiently large \( n \), and (5.4) gives

\[
(5.5) \quad \left| G_n(z_0; z_1, \ldots, z_n) \right| < A \gamma^{n+1}, \quad n \geq n_0(\epsilon),
\]

and hence for all \( z \) in any given finite domain, and all \( n \),

\[
(5.6) \quad \left| G_n(z; z_1, z_2, \ldots, z_n) \right| < A' (\gamma + \epsilon)^{n+1}.
\]

6. Suppose now that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is an integral function satisfying

\[
\lim_{r \to \infty} \frac{\log M(r)}{r} = \sigma < \frac{1}{\gamma},
\]

it follows that for any \( r > \sigma \), and sufficiently large \( n \)

\[
(6.1) \quad n |a_n| < r^n.
\]

Then, if \( f(z_1) = 0 \), \( f^{(n-1)}(z_n) = 0 \), clearly

\[
f(z) = \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} f^{(n)}(z) dz,
\]

or, following Levinson [3, §1], if we replace \( f^n(z) \) by its power series, we obtain

\[
f(z) = \sum_{k=0}^{\infty} (n + k)! \frac{a_{n+k}}{k!} \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} z^k dz
\]

\[
= \sum_{k=0}^{\infty} (n + k)! a_{n+k} G_n(z; z_1, z_2, \ldots, z_n, 0, 0, \ldots, 0).
\]

Now since the sequence \( \{z_n\} \) is such that all its limit points belong to \( S \), then for large \( n \) and for all \( z \) in any finite domain we have by (5.6) and (6.1)
provided \( r < 1/(\gamma + \epsilon) \). But letting \( n \to \infty \) in (6.2) we have \( f(z) = 0 \).

In the particular case in which \( S \) is the unit circle, Theorem I shows that (5.2) is satisfied with \( \gamma = 1.3775 < 1/0.7259 \) for all values of \( n \), so we now have the following theorem.

**Theorem III.** If \( f(z) \) is an integral function satisfying

\[
\limsup_{r \to \infty} \frac{\log M(r)}{r} < 0.7259,
\]

and if \( f(z_0) = 0, f^{n-1}(z_n) = 0 \) \( (n \geq 2) \), the sequence \( \{z_r\} \) having all its limit points in the unit circle, then \( f(z) = 0 \).

In the particular case in which \( S \) is the segment \( 0 \leq x \leq 1 \), Theorem II shows that (5.2) is satisfied for all \( n \) with \( \gamma = 4/\pi \) and we have the following extension of Schoenberg’s theorem.

**Theorem IV.** If \( f(z) \) is an integral function satisfying

\[
\limsup_{r \to \infty} \frac{\log M(r)}{r} < \frac{\pi}{4},
\]

and if \( f(z_0) = 0, f^{n-1}(z_n) = 0 \) \( (n \geq 2) \), the sequence \( \{z_r\} \) having all its limit points on the segment \( -1 \leq x \leq 1 \) of the real axis, then \( f(z) = 0 \).

This result has been stated by Kamenetsky [2, Theorem VIII] but I have been unable to find a published proof. It seems unlikely from the context that his method has anything in common with the one which I have used here.

7. A further theorem follows as a consequence of inequalities (5.2) and (5.6) for the case in which the limit points of the sequence of zeros lie inside the locus of points distant \( h \) from the segment \( -1 \leq x \leq 1 \) of the real axis. We shall call the domain enclosed by this curve \( H \). In this case, if we restrict the sequence \( \{x_r\} \) to the segment \( -1 \leq x \leq 1 \) (all \( r \)) and \( z_r = x_r + y_r \) \( (r \geq 1) \) where \( |y_r| \leq h \) \( (r \geq 1) \), (5.2) is satisfied with \( A = \pi^2/8, \gamma = 4/\pi \), by Theorem II, and (5.3) is satisfied for all \( n \) since \( |y_r| \leq h(r \geq 1) \). Hence (5.4) is satisfied for all \( n \) with these values of the constants, that is,

\[
|G_n(z_0; z_1, z_2, \cdots, z_n)| \leq \frac{\pi^2}{8} \left( \frac{4}{\pi} \right)^{n+1} \exp \left\{ \frac{\pi^2}{8} (|y_0| + nh) \right\} < A \tilde{\gamma}^{n+1},
\]

with \( \tilde{\gamma} = (4/\pi) \exp (\pi^2 h/8) \). By a second application of formulae (5.2) and (5.6), we see that, provided all the limit points of the sequence \( \{z_r\} \) lie within \( H \), (5.6) holds with \( \gamma = (4/\pi) \exp (\pi^2 h/8) \), and we have the following theorem.
Theorem V. If \( f(z) \) is an integral function satisfying
\[
\limsup_{r \to \infty} \frac{\log M(r)}{r} < \frac{\pi}{4} \exp \left(-\frac{\pi^2 h}{8}\right),
\]
and if \( f(z_1) = 0, f^{(n-1)}(z_n) = 0 \) \( (n \geq 2) \), where the sequence \( \{z_r\} \) has all its limit points in \( H \), then \( f(z) = 0 \).

It is to be noted that the constant \( \left(\frac{\pi}{4}\right) \exp \left(-\frac{\pi^2 h}{8}\right) \) is "better" (that is, greater) than that obtained from the circle circumscribed to \( H \), namely, 
\[
.7259/(1+h) \quad (\text{which is obtained from Theorem III by the transformation } \zeta=(1+h)z)
\]
only for small values of \( h \). It is "better" when \( h \leq 0.23 \) but not when \( h = 0.24 \).

Appendix

Upper bounds for

\[
\begin{array}{cccc}
 n & M_{n-1} & L_n & L_n/(1.3775)^n \quad (1.3775)^n \\
1 & 1 & 1 & 0.7260 \quad 1.3775 \\
2 & 2 & 1.5 & 0.7905 \quad 1.8975 \\
3 & 2.5981 & 1.9299 & 0.7384 \quad 2.6138 \\
4 & 3.6379 & 2.7915 & 0.7753 \quad 3.6005 \\
5 & 4.8414 & 3.8056 & 0.7673 \quad 4.9597 \\
6 & 6.8223 & 5.2539 & 0.7690 \quad 6.8320 \\
7 & 9.3973 & 7.2315 & 0.7685 \quad 9.4111 \\
8 & 12.9512 & 9.9635 & 0.7686 \quad 12.9638 \\
9 & 17.8413 & 13.7212 & 0.7684 \quad 17.8577 \\
10 & 24.5754 & 18.8998 & 0.7683 \quad 24.5989 \\
11 & 33.8472 & & 0.7685 \quad 33.8850 \\
\end{array}
\]

References


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