ABEL'S THEOREM AND A GENERALIZATION OF ONE-PARAMETER GROUPS

BY

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1. Let \( a \) be a given point on a Riemann surface \( S \) of positive genus \( p \) and \( b_1, \ldots, b_{p+1} \) an arbitrary set of \( p+1 \) points of \( S \). By Abel's theorem, there exists on \( S \) a set of \( p \) points, \( c_1, \ldots, c_p \), dependent on the \( b_i \), such that, given any integral of the first kind on \( S \), with \( a \) for lower limit, the sum of the \( p+1 \) values of the integral with the \( b_i \) as upper limits equals the sum of the \( p \) values with the \( c_i \) as upper limits.

The set of \( c_i \) may be regarded as a type of product of the \( p+1 \) points \( b_i \). As the reduction of any number \( q>p \) of integrals to \( p \) integrals leads generally to a unique result, the product operation has an associative quality. For \( p=1 \), the operation makes \( S \) a one-parameter Lie group in which \( b_1 \) and \( b_2 \) have \( c_1 \) for product.

We study a question of associative analytic combinations which is exemplified by the case of a surface of arbitrary positive genus\(^{(1)}\).

We use \( p \) symmetric functions \( f_i(x_1, \ldots, x_{p+1}), i=1, \ldots, p \) of \( p+1 \) variables, analytic when the variables are all zero.

Let \( P_i(u_1, \ldots, u_p) \) represent, for \( i=1, \ldots, p \), the elementary symmetric function (e. s. f.) of degree \( i \) of \( u_1, \ldots, u_p \). We assume that when an \( x_i \) vanishes, \( f_i \) reduces to \( P_i \) of the \( x \) other than \( x_i \).

Now let there be given \( p \) relations

\[
(1) \quad P_i(y_1, \ldots, y_p) = f_i(x_1, \ldots, x_{p+1}), \quad i = 1, \ldots, p.
\]

When the \( |x| \) are all small, (1) determines a set of \( y \) of small modulus. This set of \( p \) numbers \( y \) will be called the product of \( x_1, \ldots, x_{p+1} \). We write

\[
y_1, \ldots, y_p = \{x_1, \ldots, x_{p+1}\}.
\]

We shall study the conditions under which this product operation is associative. The condition which we put is that

\[
(2) \quad \{\{x_1, \ldots, x_{p+1}\}, x_{p+2}\} = \{x_1, \{x_2, \ldots, x_{p+2}\}\}.
\]

When this condition is satisfied, we describe the system (1) as associative.

We find that, for (1) to be an associative system, it is necessary and sufficient that (1) be equivalent\(^{(2)}\) to a system of relations

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\(^{(1)}\) The case of \( p=1 \) is essentially the problem of Abel's paper on indefinitely symmetric functions. See Abel, Oeuvres complètes, Christiania, 1881, vol. 1, p. 61.

\(^{(2)}\) In saying that (1) and (3) are equivalent, we mean that, for small \( |x| \), (3) is satisfied by the set of \( y \) which (1) determines and by no other set.

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\[ \sum_{j=1}^{p} \Phi_i(y_j) = \sum_{j=1}^{p+1} \Phi_i(x_j), \quad i = 1, \ldots, p, \]

in which each \( \Phi_i(z) \) is analytic for \( z = 0 \) and has an expansion about \( z = 0 \) which starts effectively with a term in \( z^{(3)} \).

Any system \((3)\), as just described, is equivalent to an associative system \((1)\).

In §14, we treat relations \((1)\) in which the \( f \) involve any number \( q > p \) of variables \( x \).

**Partial differential equations**

2. We let \( Y_i \) denote, for each \( i \), the first member of \((1)\). Let \( \gamma_i \) represent \( P_i(x_2, \ldots, x_{p+1}) \). We shall show that \((1)\) can be solved for the \( \gamma \) in terms of \( x_1 \) and the \( Y \).

By §1, \( f_i \) reduces to \( \gamma_i \) when \( x_1 = 0 \). Hence the expansion of \( f_i \) in powers of \( x_1 \) begins with \( \gamma_i \). The coefficients of the positive powers of \( x_1 \), which are symmetric in the other \( x \), can be transformed into power series in the \( \gamma \). On this basis we write \((1)\)

\[ Y_i = \gamma_i + g_i(x_1, \gamma_1, \ldots, \gamma_p). \]

The jacobian of the second member of \((4)\) with respect to the \( \gamma \) equals unity when \( x_1 \) and the \( \gamma \) are zero. Thus each \( \gamma \) is analytic in \( x_1 \) and the \( Y \) when these quantities are of small modulus. Each \( \gamma \) vanishes when \( x_1 \) and the \( Y \) are zero.

We have then

\[ \frac{\partial Y_i}{\partial x_1} = \alpha_i(x_1, Y_1, \ldots, Y_p), \quad i = 1, \ldots, p, \]

with each \( \alpha \) analytic when its arguments are zero.

3. Let \( z_1, \ldots, z_p \) be the product of the \( y \) and \( x_{p+2} \). The \( z \) are given by either member of \((2)\). Let \( w_1, \ldots, w_p = \{x_2, \ldots, x_{p+2}\} \). Letting \( \gamma'_i = P_i(w_1, \ldots, w_p) \) and \( z_i = P_i(z_1, \ldots, z_p) \), we have, using the second member of \((2)\) and considering \((4)\),

\[ Z_i = \gamma'_i + g_i(x_1, \gamma'_1, \ldots, \gamma'_p). \]

The \( \gamma' \) are analytic in \( x_2, \ldots, x_{p+2} \). Thus the \( Z \) are analytic in \( x_1, \ldots, x_{p+2} \) when those quantities are zero. We have from \((6)\)

\[ \frac{\partial Z_i}{\partial x_1} = \alpha_i(x_1, Z_1, \ldots, Z_p) \]

where the \( \alpha \) are as in \((5)\).
4. Using the first member of (2), we have

\[ Z_i = Y_i + g_i(x_{p+2}, Y_1, \cdots, Y_p). \]

We examine the coefficient of the first power of \( x_{p+2} \) in \( g_i \). When any one of the arguments of

\[ f_i(y_1, \cdots, y_p, x_{p+2}) \]

is replaced by 0, (9) reduces to the sum of the remaining arguments. Then the terms of the first degree in the expansion of (9) make up \( Y_1 + x_{p+2} \). Let us consider the terms of any degree \( r > 1 \). They vanish for \( x_{p+2} = 0 \). Hence they amount to a polynomial in \( x_{p+2} \), with no term free of \( x_{p+2} \), the coefficients being polynomials in the \( Y \). As the coefficients vanish for \( y_1 = 0 \), they are of positive degree in the \( Y \).

On this basis, we see that, when \( x_{p+2} \) and the \( Y \) are 0, \( \partial g_i / \partial x_{p+2} \) equals unity. Thus we can express \( x_{p+2} \) in terms of \( Z_i \) and the \( Y \). We may then write, by (8),

\[ \frac{\partial Z_i}{\partial Y_j} = \beta_{ij}(Y_1, \cdots, Y_p, Z_i), \quad i, j = 1, \cdots, p, \]

where the \( \beta \) are analytic for zero values of their arguments.

A REDUCTION

5. From (7), (10), and (5), we have for every \( i \)

\[ \alpha_i(x_1, Z_1, \cdots, Z_p) = \sum_{j=1}^{p} \beta_{ij}(Y_1, \cdots, Y_p, Z_i) \alpha_j(x_1, Y_1, \cdots, Y_p), \]

the equations holding for any \( x_1, \cdots, x_{p+2} \) and for the \( Y \) corresponding to \( x_1, \cdots, x_{p+1} \). We think of the \( Z \) as obtained from \( x_{p+2} \) and the \( Y \).

By §2, we can assign arbitrary values to \( x_1 \) and the \( y \) and then determine a set \( x_2, \cdots, x_{p+1} \). We let \( y_p = 0 \) and keep the other \( y \) arbitrary. Then \( Y_p \) becomes zero and every other \( Y_i \) becomes \( Y_i' \), the e. s. f. (§1) of degree \( i \) of \( y_1, \cdots, y_{p-1} \). By §1, the corresponding values \( Z' \) of the \( Z \) are the symmetric functions of \( y_1, \cdots, y_{p-1}, x_{p+2} \). The equations (11) become

\[ \alpha_i(x_1, Z') = \sum \beta_{ij}(Y_1', \cdots, Y_{p-1}', 0, Z_i') \alpha_j(x_1, Y_1', \cdots, Y_{p-1}', 0). \]

We study (12), regarding the basic variables as \( x_1, y_1, \cdots, y_{p-1}, x_{p+2} \); the \( Z' \) and \( Y' \) have definite meanings as symmetric functions of subsets of these variables. The variables \( x_2, \cdots, x_{p+1} \) are suppressed in the present discussion.

First we replace the letter \( x_{p+2} \) in (12) by the letter \( y_p \). Then we may write \( Y_i \) for \( Z_i' \), thinking of the \( Y \) as symmetric functions of \( y_1, \cdots, y_p \) and as unrelated to earlier parts of our discussion. We may write (12)
(13) \( \alpha_i(x_1, Y_1, \cdots, Y_p) = \sum \delta_{ij}(y_1, \cdots, y_p) \alpha_j(x_1, Y'_1, \cdots, Y'_{p-1}, 0) \)

where the \( \delta \) are analytic for small \( |y| \) and are not necessarily symmetric.

In (13) we let \( y_{p-1} = 0 \) and then replace \( y_p \) by \( y_{p-1} \). Then \( Y_p \) becomes zero and every other \( Y_i \) becomes \( Y'_i \). Also \( Y'_{p-1} \) becomes zero and \( Y'_i \) with \( i < p-1 \) becomes \( Y''_i \), the e. s. f. of degree \( i \) of \( y_1, \cdots, y_{p-2} \). We have thus, letting

\[
\mu_i = \alpha_i(x_1, Y'_1, \cdots, Y''_{p-1}, 0, 0),
\]

(14) \( \alpha_i(x_1, Y'_1, \cdots, Y'_{p-1}, 0) = \sum \delta_{ij}(y_1, \cdots, y_{p-2}, 0, y_{p-1}) \mu_j \).

We put \( y_{p-2} = 0 \) and replace \( y_{p-1} \) by \( y_{p-2} \). Then

(15) \( \mu_i = \sum \delta_{ij}(y_1, \cdots, y_{p-3}, 0, 0, y_{p-2}) \alpha_j(x_1, Y''_1, \cdots, Y''_{p-3}, 0, 0, 0) \)

where the \( Y''' \) are combinations of \( y_1, \cdots, y_{p-3} \). We reach finally

(16) \( \alpha_i(x, y_1, 0, \cdots, 0) = \sum \delta_{ij}(0, \cdots, 0, y_1) \alpha_j(x_1, 0, \cdots, 0) \).

The first members of (14) appear in the second members of (13). We replace the former in (13) by their expressions in (14). Continuing, we obtain finally, putting \( \alpha_j(x_1, 0, \cdots, 0) = \xi_j(x_1) \), relations

(17) \( \alpha_i(x_1, Y'_1, \cdots, Y'_{p}) = \sum_{j=1}^{p} \rho_{ij}(y_1, \cdots, y_{p-1}) \xi_j(x_1) \).

We permute the \( y \) in all possible ways in (17), add the resulting set of \( p! \) equations, and divide by \( p! \). Then

(18) \( \alpha_i(x_1, Y_1, \cdots, Y_p) = \sum_{j=1}^{p} \sigma_{ij}(Y_1, \cdots, Y_p) \xi_j(x_1) \).

The \( \sigma \) are analytic for small \( |Y| \), the \( \xi \) for small \( |x_1| \).

6. We shall effect transformations which will bring us to relations of type (18) in which the expansion of \( \xi_j \) about \( x_1 = 0 \) starts with a term \( a_{ij} x_1^{r-1} \) with \( a_{ij} \neq 0 \).

We use (5), the \( \alpha \) being given by (18). Of course, the \( Y \) are now the second members of (1). As \( f_1 \) contains \( x_1 \) as a term, we see from (5) with \( i = 1 \) that not all \( \xi \) vanish for \( x_1 = 0 \). Rearranging subscripts if necessary, we assume that \( \xi_1(0) \) is not zero. We now arrange so that \( \xi_j(0) = 0 \) for \( j > 1 \). For this we subtract a suitable multiple of \( \xi_1 \) from each \( \xi_j \) with \( j > 1 \) and add to each \( \sigma_{1j} \) a suitable linear combination(*) of the other \( \sigma_{ij} \).

Now, \( r \) being any positive integer with \( 1 \leq r < p \), let us suppose that

(a) each \( \xi_j \) with \( j \leq r \) starts with \( a_{ij} x_1^{r-1} \) where \( a_{ij} \neq 0 \), and

(b) each \( \xi_j \) with \( j > r \) contains no term of degree less than \( r \). This situa-

(*) The same for all \( i \).
tion has been realized for $r = 1$. We shall arrange so that (a) and (b) hold with $r + 1$ replacing $r$.

7. Let us examine any $f_i$ in (1). It contains no terms of degree less than $i$. If such a term existed it would be possible to replace some $x$ by 0 and let the term survive; this would contradict our stipulation of §1. For the same reason, a power product of degree $i$ in $f_i$ must involve $i$ distinct $x$. We see now that the terms of degree $i$ are an e. s. f. of $x_1, \ldots, x_{p+1}$.

Thus the first member of (5) with $i = r + 1$ starts with terms which make up $\gamma_r$ of §2.

8. We now examine $\alpha_{r+1}$ as given by (18). Let $\sigma_j$ stand for $\sigma_{r+1,j}$. Let each $\sigma_j$ with $j \leq r$ be regarded as a power series in $x_1, \ldots, x_{p+1}$. We shall show that no such $\sigma_j$ contains terms of degree less than $r - j + 1$.

Let this be false. Of all integers $s \leq r$ such that some $\sigma_j$ with $j \leq s$ contains terms of degree less than $s - j + 1$, let $t$ be the least. For $j \leq t$, $\sigma_j$ has no terms of degree less than $t - j$; for $j < t$ this follows from the minimal character of $t$ and for $j = t$ it is trivial. Let $A_j, j = 1, \ldots, t$, be the sum of the terms of degree $t - j$ in $\sigma_j$. Some of the $A$ may be zero but our assumption with regard to $t$ implies that not all are.

The terms of degree $t - 1$ in the second member of (5) amount to

$$a_1 A_1 + a_2 x_1 A_2 + \cdots + a_{t-1} x_1^{t-1} A_t.$$  

As $t - 1 < r$ and as $\partial Y_{r+1}/\partial x_1$ starts with $\gamma_r$, (19) must be zero. The $A$ are symmetric in $x_1, \ldots, x_{p+1}$. Hence if we replace $x_1$ in (19) by any of $x_2, \ldots, x_{p+1}$, we get zero. Thus the equation

$$a_1 A_1 + a_2 A_2 w + \cdots + a_t A_t w^{t-1} = 0,$$

which is of degree at most $t - 1$ in $w$ and which is not an identity, has at least $p + 1$ distinct solutions for $w$. As $t < p$ we have a contradiction which proves the absence of terms of degree less than $r - j + 1$ from $\sigma_j, j = 1, \ldots, r$.

9. We shall now prove that at least one $x_j$ with $j > r$ starts with a term in $x_1$. Let this be false. Then, for $j > r$, $\xi_j$ has no terms of degree $r$ or less.

Let $B_j$ be the sum of the terms of degree $r - j + 1$ in $\sigma_j, j = 1, \ldots, r$. By §§7, 8,

$$\gamma_r = a_1 B_1 + a_2 x_1 B_2 + \cdots + a_r x_1^{r-1} B_r.$$  

Then $\gamma_r$ can be obtained from $a_1 B_1$ by replacing $x_1$ by 0. Thus $\gamma_r$ gives those terms of $a_1 B_1$ which are free of $x_1$. As $B_1$ is of degree $r < p$, each term of $B_1$ lacks some $x$. By the symmetry of $B_1$, the terms of $a_1 B_1$ which lack $x_j$ with $j > 1$ are obtained by replacing $x_j$ in $\gamma_r$ by $x_1$. Thus $a_1 B_1$ is the e. s. f. of degree $r$ of $x_1, \ldots, x_{p+1}$. This furnishes a contradiction when $r = 1$, since the second member of (20) is then merely $a_1 B_1$.

(1) If $r = p - 1$, (b) is suppressed when we consider $r + 1$. 
Let $r > 1$. We have then $a_1B_1 = \gamma_r + x_1\gamma_{r-1}$ and (20) becomes

$$-\gamma_{r-1} = a_2B_2 + \cdots + a_r x_1^{r-2} B_r.$$  

We see as above that $a_2B_2 = -\gamma_{r-1} - x_1\gamma_{r-2}$, which is contradictory if $r = 2$.

Continuing, we find a contradiction for every possible $r$.

10. Thus some $\xi_j$ with $j > r$ starts with a term in $x'_1$. Let this be the case for $\xi_{r+1}$. By suitable subtractions and by an adjustment of the $\sigma_{r+1}$, we arrange so that such $\xi_j$ with $j > r + 1$ as may exist contain no term of degree $r$. This completes the induction commenced in §6.

We thus assume that, in (18), $\xi_j$ starts with $a_j x_1^{j-1}$ where $a_j \neq 0$.

**Completion of proof**

11. Let $\Phi_j(x_1)$ be that integral of $\xi_j(x_1)$ which vanishes for $x_1 = 0$. Then $\Phi_j$ starts with a term in $x'_1$. We consider the functions $\phi_i$ of $x_1, \cdots, x_{p+1}$ given by

$$(21) \quad \phi_i = \sum_{j=1}^{p+1} \Phi_i(x_j), \quad i = 1, \cdots, p.$$  

Because of the symmetry of the $Y$, (5) holds if $x_1$ is replaced by any of $x_2, \cdots, x_{p+1}$. From elementary considerations of linear dependence, we see, using (18), that the jacobian of any $\phi_i$ and the $p$ functions $Y_j$ vanishes identically in $x_1, \cdots, x_{p+1}$.

Given any $\phi_i$, we shall show that it can be expressed as a function of the $Y$, analytic when the $|Y|$ are small. The $Y$ are expressed by (4) as functions of $x_1$ and the $\gamma$. As $\phi_i$ is symmetric in $x_2, \cdots, x_{p+1}$, it is a function of $x_1$ and the $\gamma$, analytic for small $|x_1|$ and $|\gamma|$. We have

$$(22) \quad \frac{\partial(\phi_i, Y_1, \cdots, Y_p)}{\partial(x_1, \cdots, x_{p+1})} = J \frac{\partial(\phi_i, Y_1, \cdots, Y_p)}{\partial(x_1, \gamma_1, \cdots, \gamma_p)},$$  

where $J$ is the jacobian of $x_1$ and the $\gamma$ with respect to $x_1, \cdots, x_{p+1}$. As there is no dependence among $x_1$ and the $\gamma$, $J$ does not vanish identically. Hence the multiplier of $J$ in (22) vanishes. By §2, the jacobian of the $Y$ with respect to the $\gamma$ is unity when $x_1$ and the $\gamma$ are zero. The theory of functional dependence tells us that $\phi_i$ is analytic in the $Y$ for small $|Y|$.

12. Replacing the $Y$ by their expressions symmetric in the $y$ of §1, we write $\phi_i = \xi_i(y_1, \cdots, y_p)$ where the $\xi$ are symmetric, and analytic for small $|y|$.

We consider (21). When $x_{p+1} = 0$, we may suppose that $y_j = x_j$, $j = 1, \cdots, p$. It follows that

$$\xi_i = \sum_{j=1}^p \Phi_i(y_j)$$  

(9) If $r = 2$, we understand here that $\gamma_0 = 1$. 


and the relations (21) become (3).

13. Let there be given \( p \) relations (3), \( \Phi_i(z) \) starting with a term in \( z \). For any \( i \leq p \), \( \sum y_j^i \) can be expressed in terms of the \( Y \). It equals \(-iY_i\) plus terms in the \( Y_j \) with \( j<i \). The latter terms have degrees exceeding unity. Hence the first members of (3), which are functions of the \( Y \), have a jacobian with respect to the \( Y \) which does not vanish when the \( Y \) do. We can thus solve (3) for the \( Y \) and obtain (1) with \( f \) which are analytic for small \( |x| \). By (3), the system (1) obtained is associative.

**AN EXTENSION**

14. Suppose that, in (1), we use any number \( q>p \) of variables \( x \). We assume that, when any \( q-p \) of the \( x \) are zero, \( f_i \) reduces to \( P_i \) of the remaining \( x \). The associativity relation asserts that, given any \( 2q-p \) quantities \( x \), we can bracket the first \( q \) or the last \( q \). We shall derive relations (3) with \( q \) replacing \( p+1 \) as the upper limit of the second sum.

The case of \( q=p+1 \) has been treated. We perform an induction from \( q=r \), where \( r>p \), to \( q=r+1 \). Using \( f \) with \( r+1 \) variables \( x \), let

\[
g_i(x_1, \ldots, x_r) = f_i(x_1, \ldots, x_r, 0).
\]

Let us see that the system \( Y_i = g_i \), which comes under the case of \( q=r \), is associative. We compare the results of bracketing the first \( r \), and then the last \( r \), of \( x_1, \ldots, x_{2r-p} \). The first of these results would be obtained, using the \( f \) system, by taking the product of \( x_1, \ldots, x_r, 0 \) and combining it with \( x_{r+1}, \ldots, x_{2r-p}, 0 \). The associativity and symmetry of the \( f \) permits us to combine \( 0, x_1, \ldots, x_{2r-p} \) and the product of \( x_{r-p+1}, \ldots, x_{2r-p}, 0 \). This gives the result of the second bracketing for the \( g \).

Thus the \( g \) system is equivalent to a system (3) with \( r \) replacing \( p+1 \) in the second sum.

Suppose that, for the \( g \) system, \( y_1, \ldots, y_p \) is the product of \( x_1, \ldots, x_r \); also that \( z_1, \ldots, z_p \) is the product of the \( y, x_{r+1} \) and \( r-p-1 \) zeros. Then

\[
\sum_{i=1}^{p} \Phi_i(z_j) = \sum_{i=1}^{r+1} \Phi_i(x_j).
\]

For the \( f \) system, the above \( y \) are the product of \( x_1, \ldots, x_r, 0 \), the \( z \) are the product of the \( y, x_{r+1} \), and \( r-p \) zeros. By the associativity and symmetry of the \( f \), we get the same \( z \) if we take the product \( w_1, \ldots, w_p \) of \( x_1, \ldots, x_{r+1} \) and then combine the \( w \) with \( r+1-p \) zeros. The latter combination produces the \( w \) again. Thus the \( w \) are the \( z \), (23) describes the \( f \), and the induction is accomplished.

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(1) Perron, *Lehrbuch der Algebra*, vol. 1, p. 157