ABEL'S THEOREM AND A GENERALIZATION OF ONE-PARAMETER GROUPS

BY

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1. Let a be a given point on a Riemann surface S of positive genus p and $b_1, \cdots, b_{p+1}$ an arbitrary set of $p+1$ points of S. By Abel's theorem, there exists on S a set of p points, $c_1, \cdots, c_p$, dependent on the b, such that, given any integral of the first kind on S, with a for lower limit, the sum of the $p+1$ values of the integral with the b as upper limits equals the sum of the p values with the c as upper limits.

The set of c may be regarded as a type of product of the $p+1$ points b. As the reduction of any number $q>p$ of integrals to p integrals leads generally to a unique result, the product operation has an associative quality. For $p=1$, the operation makes S a one-parameter Lie group in which b_1 and b_2 have c_1 for product.

We study a question of associative analytic combinations which is exemplified by the case of a surface of arbitrary postitive genus(1).

We use p symmetric functions $f_i\ (x_1, \cdots, x_{p+1})$, $i=1, \cdots, p$ of $p+1$ variables, analytic when the variables are all zero.

Let $P_i\ (u_1, \cdots, u_p)$ represent, for $i=1, \cdots, p$, the elementary symmetric function (e. s. f.) of degree i of $u_1, \cdots, u_p$. We assume that when an x_j vanishes, f_i reduces to P_i of the x other than x_j.

Now let there be given p relations

$$P_i(y_1, \cdots, y_p) = f_i(x_1, \cdots, x_{p+1}), \qquad i = 1, \cdots, p. \tag{1}$$

When the $|x|$ are all small, (1) determines a set of y of small modulus. This set of p numbers y will be called the *product* of $x_1, \cdots, x_{p+1}$. We write

$$y_1, \cdots, y_p = \{x_1, \cdots, x_{p+1}\}.$$

We shall study the conditions under which this product operation is associative. The condition which we put is that

$$\{\{x_1, \cdots, x_{p+1}\}, x_{p+2}\} = \{x_1, \{x_2, \cdots, x_{p+2}\}\}. \tag{2}$$

When this condition is satisfied, we describe the system (1) as *associative.*

We find that, *for* (1) *to be an associative system, it is necessary and sufficient that* (1) *be equivalent*(2) *to a system of relations*

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(1) The case of $p=1$ is essentially the problem of Abel's paper on indefinitely symmetric functions. See Abel, *Oeuvres complètes*, Christiania, 1881, vol. 1, p. 61.

(2) In saying that (1) and (3) are equivalent, we mean that, for small $|x|$, (3) is satisfied by the set of y which (1) determines and by no other set.

$$\sum_{j=1}^{p} \Phi_i(y_j) = \sum_{j=1}^{p+1} \Phi_i(x_j), \qquad i = 1, \cdots, p, \tag{3}$$

in which each $\Phi_i(z)$ *is analytic for* $z=0$ *and has an expansion about* $z=0$ *which starts effectively with a term in* z^i (3).

Any system (3), as just described, is equivalent to an associative system (1).

In §14, we treat relations (1) in which the f involve any number $q>p$ of variables x.

Partial differential equations

2. We let Y_i denote, for each i, the first member of (1). Let γ_i represent $P_i(x_2, \cdots, x_{p+1})$. We shall show that (1) can be solved for the γ in terms of x_1 and the Y.

By §1, f_i reduces to γ_i when $x_1=0$. Hence the expansion of f_i in powers of x_1 begins with γ_i. The coefficients of the positive powers of x_1, which are symmetric in the other x, can be transformed into power series in the γ. On this basis we write (1)

$$Y_i = \gamma_i + g_i(x_1, \gamma_1, \cdots, \gamma_p). \tag{4}$$

The jacobian of the second member of (4) with respect to the γ equals unity when x_1 and the γ are zero. Thus each γ is analytic in x_1 and the Y when these quantities are of small modulus. Each γ vanishes when x_1 and the Y are zero.

We have then

$$\frac{\partial Y_i}{\partial x_1} = \alpha_i(x_1, Y_1, \cdots, Y_p), \qquad i = 1, \cdots, p, \tag{5}$$

with each α analytic when its arguments are zero.

3. Let $z_1, \cdots, z_p$ be the product of the y and x_{p+2}. The z are given by either member of (2). Let $w_1, \cdots, w_p = \{x_2, \cdots, x_{p+2}\}$. Letting $\gamma_i' = P_i(w_1, \cdots, w_p)$ and $Z_i = P_i(z_1, \cdots, z_p)$, we have, using the second member of (2) and considering (4),

$$Z_i = \gamma_i' + g_i(x_1, \gamma_1', \cdots, \gamma_p'). \tag{6}$$

The γ' are analytic in $x_2, \cdots, x_{p+2}$. Thus the Z are analytic in $x_1, \cdots, x_{p+2}$ when those quantities are zero. We have from (6)

$$\frac{\partial Z_i}{\partial x_1} = \alpha_i(x_1, Z_1, \cdots, Z_p) \tag{7}$$

where the α are as in (5).

(3) By an iterative process applied to (1), the problem can be attached to the theory of abelian Lie groups of several parameters. A treatment based on this principle would be at most slightly briefer than that given here, which presupposes no knowledge of the Lie theory.

4. Using the first member of (2), we have

$$Z_i = Y_i + g_i(x_{p+2}, Y_1, \cdots, Y_p). \tag{8}$$

We examine the coefficient of the first power of x_{p+2} in g_1. When any one of the arguments of

$$f_1(y_1, \cdots, y_p, x_{p+2}) \tag{9}$$

is replaced by 0, (9) reduces to the sum of the remaining arguments. Then the terms of the first degree in the expansion of (9) make up Y_1+x_{p+2}. Let us consider the terms of any degree $r>1$. They vanish for $x_{p+2}=0$. Hence they amount to a polynomial in x_{p+2}, with no term free of x_{p+2}, the coefficients being polynomials in the Y. As the coefficients vanish for $y_1=0$, they are of positive degree in the Y.

On this basis, we see that, when x_{p+2} and the Y are 0, $\partial g_1/\partial x_{p+2}$ equals unity. Thus we can express x_{p+2} in terms of Z_1 and the Y. We may then write, by (8),

$$\frac{\partial Z_i}{\partial Y_j} = \beta_{ij}(Y_1, \cdots, Y_p, Z_1), \qquad i, j = 1, \cdots, p, \tag{10}$$

where the β are analytic for zero values of their arguments.

A REDUCTION

5. From (7), (10), and (5), we have for every i

$$\alpha_i(x_1, Z_1, \cdots, Z_p) = \sum_{j=1}^{p} \beta_{ij}(Y_1, \cdots, Y_p, Z_1)\alpha_j(x_1, Y_1, \cdots, Y_p), \tag{11}$$

the equations holding for any $x_1, \cdots, x_{p+2}$ and for the y corresponding to $x_1, \cdots, x_{p+1}$. We think of the Z as obtained from x_{p+2} and the y.

By §2, we can assign arbitrary values to x_1 and the y and then determine a set $x_2, \cdots, x_{p+1}$. We let $y_p=0$ and keep the other y arbitrary. Then Y_p becomes zero and every other Y_i becomes Y_i', the e. s. f. (§1) of degree i of $y_1, \cdots, y_{p-1}$. By §1, the corresponding values Z' of the Z are the symmetric functions of $y_1, \cdots, y_{p-1}, x_{p+2}$. The equations (11) become

$$\alpha_i(x_1, Z') = \sum \beta_{ij}(Y_1', \cdots, Y_{p-1}', 0, Z_1')\alpha_j(x_1, Y_1', \cdots, Y_{p-1}', 0). \tag{12}$$

We study (12), regarding the basic variables as $x_1, y_1, \cdots, y_{p-1}, x_{p+2}$; the Z' and Y' have definite meanings as symmetric functions of subsets of these variables. The variables $x_2, \cdots, x_{p+1}$ are suppressed in the present discussion.

First we replace the letter x_{p+2} in (12) by the letter y_p. Then we may write Y_i for Z_i', thinking of the Y as symmetric functions of $y_1, \cdots, y_p$ and as unrelated to earlier parts of our discussion. We may write (12)

$$(13)\qquad \alpha_i(x_1, Y_1, \cdots, Y_p) = \sum \delta_{ij}(y_1, \cdots, y_p)\alpha_j(x_1, Y_1', \cdots, Y_{p-1}', 0)$$

where the δ are analytic for small $|y|$ and are not necessarily symmetric.

In (13) we let $y_{p-1}=0$ and then replace y_p by y_{p-1}. Then Y_p becomes zero and every other Y_i becomes Y_i'. Also Y_{p-1}' becomes zero and Y_i' with $i<p-1$ becomes Y_i'', the e. s. f. of degree i of $y_1, \cdots, y_{p-2}$. We have thus, letting

$$\mu_j = \alpha_j(x_1, Y_1'', \cdots, Y_{p-2}'', 0, 0),$$

$$(14)\qquad \alpha_i(x_1, Y_1', \cdots, Y_{p-1}', 0) = \sum \delta_{ij}(y_1, \cdots, y_{p-2}, 0, y_{p-1})\mu_j.$$

We put $y_{p-2}=0$ and replace y_{p-1} by y_{p-2}. Then

$$(15)\qquad \mu_i = \sum \delta_{ij}(y_1, \cdots, y_{p-3}, 0, 0, y_{p-2})\alpha_j(x_1, Y_1''', \cdots, Y_{p-3}''', 0, 0, 0)$$

where the Y''' are combinations of $y_1, \cdots, y_{p-3}$. We reach finally

$$(16)\qquad \alpha_i(x, y_1, 0, \cdots, 0) = \sum \delta_{ij}(0, \cdots, 0, y_1)\alpha_j(x_1, 0, \cdots, 0).$$

The first members of (14) appear in the second members of (13). We replace the former in (13) by their expressions in (14). Continuing, we obtain finally, putting $\alpha_j(x_1, 0, \cdots, 0)=\xi_j(x_1)$, relations

$$(17)\qquad \alpha_i(x_1, Y_1, \cdots, Y_p) = \sum_{j=1}^{p} \rho_{ij}(y_1, \cdots, y_p)\xi_j(x_1).$$

We permute the y in all possible ways in (17), add the resulting set of $p!$ equations, and divide by $p!$. Then

$$(18)\qquad \alpha_i(x_1, Y_1, \cdots, Y_p) = \sum_{j=1}^{p} \sigma_{ij}(Y_1, \cdots, Y_p)\xi_j(x_1).$$

The σ are analytic for small $|Y|$, the ξ for small $|x_1|$.

6. We shall effect transformations which will bring us to relations of type (18) in which the expansion of ξ_j about $x_1=0$ starts with a term $a_jx_1^{j-1}$ with $a_j\neq 0$.

We use (5), the α being given by (18). Of course, the Y are now the second members of (1). As f_1 contains x_1 as a term, we see from (5) with $i=1$ that not all ξ vanish for $x_1=0$. Rearranging subscripts if necessary, we assume that $\xi_1(0)$ is not zero. We now arrange so that $\xi_j(0)=0$ for $j>1$. For this we subtract a suitable multiple of ξ_1 from each ξ_j with $j>1$ and add to each σ_{i1} a suitable linear combination[4] of the other σ_{ij}.

Now, r being any positive integer with $1\leqq r<p$, let us suppose that

(a) each ξ_j with $j\leqq r$ starts with $a_jx_1^{j-1}$ where $a_j\neq 0$, and

(b) each ξ_j with $j>r$ contains no term of degree less than r. This situa-

[4] The same for all i.

tion has been realized for $r=1$. We shall arrange so that (a) and (b) hold with $r+1$ replacing r[5].

7. Let us examine any f_i in (1). It contains no terms of degree less than i. If such a term existed it would be possible to replace some x by 0 and let the term survive; this would contradict our stipulation of §1. For the same reason, a power product of degree i in f_i must involve i distinct x. We see now that the terms of degree i are an e. s. f. of $x_1, \cdots, x_{p+1}$.

Thus the first member of (5) with $i=r+1$ starts with terms which make up γ_r of §2.

8. We now examine α_{r+1} as given by (18). Let σ_j stand for $\sigma_{r+1,j}$. Let each σ_j with $j \leqq r$ be regarded as a power series in $x_1, \cdots, x_{p+1}$. We shall show that no such σ_j contains terms of degree less than $r-j+1$.

Let this be false. Of all integers $s \leqq r$ such that some σ_j with $j \leqq s$ contains terms of degree less than $s-j+1$, let t be the least. For $j \leqq t$, σ_j has no terms of degree less than $t-j$; for $j<t$ this follows from the minimal character of t and for $j=t$ it is trivial. Let A_j, $j=1, \cdots, t$, be the sum of the terms of degree $t-j$ in σ_j. Some of the A may be zero but our assumption with regard to t implies that not all are.

The terms of degree $t-1$ in the second member of (5) amount to

$$a_1A_1 + a_2x_1A_2 + \cdots + a_t x_1^{t-1} A_t. \tag{19}$$

As $t-1<r$ and as $\partial Y_{r+1}/\partial x_1$ starts with γ_r, (19) must be zero. The A are symmetric in $x_1, \cdots, x_{p+1}$. Hence if we replace x_1 in (19) by any of $x_2, \cdots, x_{p+1}$, we get zero. Thus the equation

$$a_1A_1 + a_2A_2w + \cdots + a_tA_tw^{t-1} = 0,$$

which is of degree at most $t-1$ in w and which is not an identity, has at least $p+1$ distinct solutions for w. As $t<p$ we have a contradiction which proves the absence of terms of degree less than $r-j+1$ from σ_j, $j=1, \cdots, r$.

9. We shall now prove that at least one ξ_j with $j>r$ starts with a term in x_1^r. Let this be false. Then, for $j>r$, ξ_j has no terms of degree r or less.

Let B_j be the sum of the terms of degree $r-j+1$ in σ_j, $j=1, \cdots, r$. By §§7, 8,

$$\gamma_r = a_1B_1 + a_2x_1B_2 + \cdots + a_r x_1^{r-1} B_r. \tag{20}$$

Then γ_r can be obtained from a_1B_1 by replacing x_1 by 0. Thus γ_r gives those terms of a_1B_1 which are free of x_1. As B_1 is of degree $r<p$, each term of B_1 lacks some x. By the symmetry of B_1, the terms of a_1B_1 which lack x_j with $j>1$ are obtained by replacing x_j in γ_r by x_1. Thus a_1B_1 is the e. s. f. of degree r of $x_1, \cdots, x_{p+1}$. This furnishes a contradiction when $r=1$, since the second member of (20) is then merely a_1B_1.

[5] If $r=p-1$, (b) is suppressed when we consider $r+1$.

Let $r>1$. We have then $a_1B_1=\gamma_r+x_1\gamma_{r-1}$ and (20) becomes

$$-\gamma_{r-1} = a_2B_2 + \cdots + a_r x_1^{r-2} B_r.$$

We see as above that $a_2B_2=-\gamma_{r-1}-x_1\gamma_{r-2}$, which is contradictory if $r=2$(6). Continuing, we find a contradiction for every possible r.

10. Thus some ξ_j with $j>r$ starts with a term in x_1^r. Let this be the case for ξ_{r+1}. By suitable subtractions and by an adjustment of the $\sigma_{i,r+1}$, we arrange so that such ξ_j with $j>r+1$ as may exist contain no term of degree r. This completes the induction commenced in §6.

We thus assume that, in (18), ξ_j starts with $a_jx_1^{j-1}$ where $a_j\neq 0$.

Completion of proof

11. Let $\Phi_j(x_1)$ be that integral of $\xi_j(x_1)$ which vanishes for $x_1=0$. Then Φ_j starts with a term in x_1^j. We consider the functions ϕ_i of $x_1, \cdots, x_{p+1}$ given by

$$\phi_i = \sum_{j=1}^{p+1} \Phi_i(x_j), \qquad i = 1, \cdots, p. \tag{21}$$

Because of the symmetry of the Y, (5) holds if x_1 is replaced by any of $x_2, \cdots, x_{p+1}$. From elementary considerations of linear dependence, we see, using (18), that the jacobian of any ϕ_i and the p functions Y_j vanishes identically in $x_1, \cdots, x_{p+1}$.

Given any ϕ_i, we shall show that it can be expressed as a function of the Y, analytic when the $|Y|$ are small. The Y are expressed by (4) as functions of x_1 and the γ. As ϕ_i is symmetric in $x_2, \cdots, x_{p+1}$, it is a function of x_1 and the γ, analytic for small $|x_1|$ and $|\gamma|$. We have

$$\frac{\partial(\phi_i, Y_1, \cdots, Y_p)}{\partial(x_1, \cdots, x_{p+1})} = J\,\frac{\partial(\phi_i, Y_1, \cdots, Y_p)}{\partial(x_1, \gamma_1, \cdots, \gamma_p)}, \tag{22}$$

where J is the jacobian of x_1 and the γ with respect to $x_1, \cdots, x_{p+1}$. As there is no dependence among x_1 and the γ, J does not vanish identically. Hence the multiplier of J in (22) vanishes. By §2, the jacobian of the Y with respect to the γ is unity when x_1 and the γ are zero. The theory of functional dependence tells us that ϕ_i is analytic in the Y for small $|Y|$.

12. Replacing the Y by their expressions symmetric in the y of §1, we write $\phi_i=\zeta_i(y_1, \cdots, y_p)$ where the ζ are symmetric, and analytic for small $|y|$.

We consider (21). When $x_{p+1}=0$, we may suppose that $y_j=x_j$, $j=1, \cdots, p$. It follows that

$$\zeta_i = \sum_{j=1}^{p} \Phi_i(y_j)$$

(6) If $r=2$, we understand here that $\gamma_0=1$.

and the relations (21) become (3).

13. Let there be given p relations (3), $\Phi_i(z)$ starting with a term in z^i. For any $i \leqq p$, $\sum y_j^i$ can be expressed in terms of the Y(7). It equals $-iY_i$ plus terms in the Y_j with $j<i$. The latter terms have degrees exceeding unity. Hence the first members of (3), which are functions of the Y, have a jacobian with respect to the Y which does not vanish when the Y do. We can thus solve (3) for the Y and obtain (1) with f which are analytic for small $|x|$. By (3), the system (1) obtained is associative.

An extension

14. Suppose that, in (1), we use any number $q>p$ of variables x. We assume that, when any $q-p$ of the x are zero, f_i reduces to P_i of the remaining x. The associativity relation asserts that, given any $2q-p$ quantities x, we can bracket the first q or the last q. We shall derive relations (3) with q replacing $p+1$ as the upper limit of the second sum.

The case of $q=p+1$ has been treated. We perform an induction from $q=r$, where $r>p$, to $q=r+1$. Using f with $r+1$ variables x, let

$$g_i(x_1, \cdots, x_r) = f_i(x_1, \cdots, x_r, 0).$$

Let us see that the system $Y_i=g_i$, which comes under the case of $q=r$, is associative. We compare the results of bracketing the first r, and then the last r, of $x_1, \cdots, x_{2r-p}$. The first of these results would be obtained, using the f system, by taking the product of $x_1, \cdots, x_r$, 0 and combining it with $x_{r+1}, \cdots, x_{2r-p}$, 0. The associativity and symmetry of the f permits us to combine 0, $x_1, \cdots, x_{r-p}$ and the product of $x_{r-p+1}, \cdots, x_{2r-p}$, 0. This gives the result of the second bracketing for the g.

Thus the g system is equivalent to a system (3) with r replacing $p+1$ in the second sum.

Suppose that, for the g system, $y_1, \cdots, y_p$ is the product of $x_1, \cdots, x_r$; also that $z_1, \cdots, z_p$ is the product of the y, x_{r+1} and $r-p-1$ zeros. Then

$$\sum_{j=1}^{p} \Phi_i(z_j) = \sum_{j=1}^{r+1} \Phi_i(x_j). \tag{23}$$

For the f system, the above y are the product of $x_1, \cdots, x_r$, 0, the z are the product of the y, x_{r+1}, and $r-p$ zeros. By the associativity and symmetry of the f, we get the same z if we take the product $w_1, \cdots, w_p$ of $x_1, \cdots, x_{r+1}$ and then combine the w with $r+1-p$ zeros. The latter combination produces the w again. Thus the w are the z, (23) describes the f, and the induction is accomplished.

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(7) Perron, *Lehrbuch der Algebra*, vol. 1, p. 157