PARALLELISABILITY OF PRINCIPAL FIBRE BUNDLES

BY
SHIING-SHEN CHERN AND SZE-TSEN HU

1. **Introduction.** One of the natural problems concerning fibre bundles is to determine whether or not a given fibre bundle $\mathfrak{F}=\{F, G; X, B; \psi, \phi_U\}$ is equivalent with the product bundle $B\times F$. It is proved in [1, §1](1) that $\mathfrak{F}$ is equivalent with $B\times F$ if and only if its principal fibre bundle $\mathfrak{F}^*=\tau\mathfrak{F}$ is parallelisable; therefore, the problem posed above leads to the parallelisability of a given principal fibre bundle. For the notions and the notations used in the present paper, one may refer to those given by Shiing-shen Chern and Yi-fone Sun in the first section of their recent paper [1].

Throughout the present paper, let $\mathfrak{F}=\{G, G; X, B; \psi, \phi_U\}$ be a given principal fibre bundle in which the base space is a finite polyhedron. Let B be given a triangulation such that the closure of every simplex is contained in some coordinate neighborhood U.

For each integer n ($0\leqq n\leqq \dim B$), let B^n be the n-dimensional skeleton of B, that is, the set of simplexes of B with dimensions not exceeding n. A principal fibre bundle $\mathfrak{F}$ is said to be *n-parallelisable* if there exists a mapping $f: B^n\to X$ such that $\psi f(b)=b$ for each point b of B^n. In this case, f is called an *n-lifting* of the principal fibre bundle $\mathfrak{F}$. Every principal fibre bundle $\mathfrak{F}$ is obviously 0-parallelisable. Hence the problem posed above can be considered as solved if one has found a necessary and sufficient condition for an $(n-1)$-parallelisable principal fibre bundle $\mathfrak{F}$ to be n-parallelisable ($0<n\leqq\dim B$). The object of the present paper is to give such a condition in terms of some cohomology invariants for the case that the reference group G is pathwise connected.

2. **Orientability of fibre bundles.** A fibre bundle is said to be *orientable* if its principal fibre bundle is 1-parallelisable. For a pathwise connected reference group G, every fibre bundle is orientable. This is an immediate consequence of the following statement.

(2.1) *If the reference group G is pathwise connected, then every principal fibre bundle $\mathfrak{F}=\{G, G; X, B; \psi, \phi_U\}$ is 1-parallelisable.*

Proof. Let $f: B^0\to X$ be an arbitrary 0-lifting of $\mathfrak{F}$, the existence of which is obvious. According to our hypothesis with respect to the triangulation of B made in §1, for a given 1-simplex σ of B, there exists a coordinate neighborhood U which contains the closure of σ. Let a and b be the two vertices of σ. Since $f(a)\in\psi^{-1}(a)$ and $f(b)\in\psi^{-1}(b)$, we might define two points $p, q\in G$ by taking

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(1) The numbers in brackets denote the references in the bibliography at the end of the paper.

$$p = \overset{-1}{\phi}_{U,a} f(a), \qquad q = \overset{-1}{\phi}_{U,b} f(b).$$

Since G is pathwise connected, there exists a mapping θ: Cl $\sigma \to G$ such that $\theta(a) = p$ and $\theta(b) = q$. Extend the mapping f into the interior of σ by taking

$$f(x) = \phi_{U,x}\theta(x) \qquad (x \in \sigma).$$

Since $\psi\phi_{U,x}\theta(x) = x$ $(x \in \sigma)$, this gives us a 1-lifting f: $B^1 \to X$ of $\mathfrak{F}$ and (2.1) is proved.

3. **The cocycle $w^n(f)$ of an $(n-1)$-lifting f: $B^{n-1} \to X$.** For the remainder of the paper, let n be an integer such that $2 \leqq n \leqq \dim B$ and assume that the principal fibre bundle $\mathfrak{F}$ be $(n-1)$-parallelisable. Now, let f: $B^{n-1} \to X$ be an arbitrary $(n-1)$-lifting of $\mathfrak{F}$. We define a cochain $w^n(f)$ of B with coefficients in the $(n-1)$th homotopy group $\pi_{n-1}(G)$ of the reference group G as follows.

Let σ_i^n be an arbitrary n-simplex of B, then f is defined on the boundary sphere $\partial\sigma_i^n$ of σ_i^n. According to our hypothesis concerning the triangulation of B, there exists a coordinate neighborhood U which contains the closure Cl σ_i^n of σ_i^n. Since $f(b) \in \psi^{-1}(b)$ for each point b of $\partial\sigma_i^n$, we might define a mapping θ: $\partial\sigma_i^n \to G$ by taking

$$\theta(b) = \overset{-1}{\phi}_{U,b} f(b) \qquad (b \in \partial\sigma_i^n).$$

Since G is $(n-1)$-simple [3, p. 69], θ determines a unique element $a_i \in \pi_{n-1}(G)$.

(3.1) *The element $a_i \in \pi_{n-1}(G)$ depends only on the $(n-1)$-lifting f: $B^{n-1} \to X$ and the n-simplex $\sigma_i^n \in B$; hence the correspondence $\sigma_i^n \to a_i$ defines a cochain $w^n(f)$ of B which depends only on f.*

Proof. Let V be another coordinate neighborhood which contains Cl σ_i^n and let θ': $\partial\sigma_i^n \to G$ be the mapping defined by

$$\theta'(b) = \overset{-1}{\phi}_{V,b} f(b) \qquad (b \in \partial\sigma_i^n).$$

It follows from the Paste Condition [1, §1] for a principal fibre bundle that there exists an element $g \in G$ such that

$$\theta'(b) = g \cdot \theta(b) \qquad (b \in \partial\sigma_i^n).$$

Since G is pathwise connected, there exists a path s: $I \to G$ joining from the identity e of G to the element g, that is, $s(0) = e$ and $s(1) = g$. Define a homotopy θ_t: $\partial\sigma_i^n \to G$ $(0 \leqq t \leqq 1)$ by taking

$$\theta_t(b) = s(t) \cdot \theta(b) \qquad (b \in \partial\sigma_i^n,\ 0 \leqq t \leqq 1).$$

Then $\theta_0 = \theta$ and $\theta_1 = \theta'$. Since G is $(n-1)$-simple, θ' determines the same element $a_i \in \pi_{n-1}(G)$ as θ does. This completes the proof of (3.1).

(3.2) *The cochain $w^n(f)$ is a cocycle.*

Proof. Let σ^{n+1} be an arbitrary $(n+1)$-simplex of B. It need only be shown that

$$(\delta w^n(f))(\sigma^{n+1}) = 0,$$

where $\delta w^n(f)$ denotes the coboundary of $w^n(f)$. According to our hypothesis regarding the triangulation of B, there exists a coordinate neighborhood U which contains the closure $\mathrm{Cl}\ \sigma^{n+1}$ of σ^{n+1}. Let

$$A^{n-1} = B^{n-1} \cap \mathrm{Cl}\ \sigma^{n+1},$$

and define a mapping $\xi: A^{n-1} \to G$ by taking

$$\xi(b) = \overset{-1}{\phi}_{U,b} f(b) \qquad (b \in A^{n-1}).$$

According to S. Eilenberg [2, p. 237], ξ determines an n-cocycle $c^n(\xi)$ of $\mathrm{Cl}\ \sigma^{n+1}$ with coefficients in $\pi_{n-1}(G)$. Clearly we have

$$c^n(\xi) = w^n(f) \mid \mathrm{Cl}\ \sigma^{n+1};$$

and hence

$$(\delta w^n(f))(\sigma^{n+1}) = (\delta c^n(\xi))(\sigma^{n+1}) = 0,$$

that is, $w^n(f)$ is a cocycle. q.e.d.

4. **The characteristic coset $W^n(\mathfrak{F})$.** According to (3.1) and (3.2), every $(n-1)$-lifting $f: B^{n-1} \to X$ of $\mathfrak{F}$ determines an n-cocycle $w^n(f)$ of B and hence an element $\omega^n(f)$ of the cohomology group $H^n(B, \pi_{n-1}(G))$, called an *n-dimensional obstruction element* of the $(n-1)$-parallelisable principal fibre bundle $\mathfrak{F}$. The object of the present section is to prove that the n-dimensional obstruction elements of $\mathfrak{F}$ form a coset of the presentable subgroup $P^n(B, \pi_{n-1}(G))$ in the cohomology group $H^n(B, \pi_{n-1}(G))$ [4, §3].

(4.1) *Every pair of $(n-1)$-liftings $f, g: B^{n-1} \to X$ of $\mathfrak{F}$ determines a unique mapping $\mu: B^{n-1} \to G$ denoted by $\mu = f^{-1} \cdot g$.*

Proof. The required mapping $\mu: B^{n-1} \to G$ is defined as follows: For an arbitrary point $b \in B^{n-1}$, choose a coordinate neighborhood U containing b and define

$$\mu(b) = (\overset{-1}{\phi}_{U,b} f(b))^{-1} \cdot (\overset{-1}{\phi}_{U,b} g(b)).$$

To justify this definition, let V be another coordinate neighborhood which contains b. By the aid of the Paste Condition [1, §1], that $\phi_{V,b}^{-1}\phi_{U,b}$ is a left translation of G determined by some element ξ of G, one may easily verify that

$$\begin{aligned}(\overset{-1}{\phi}_{V,b} f(b))^{-1} \cdot (\overset{-1}{\phi}_{V,b} g(b)) &= (\xi \cdot \overset{-1}{\phi}_{U,b} f(b))^{-1} (\xi \cdot \overset{-1}{\phi}_{U,b} g(b)) \\ &= (\overset{-1}{\phi}_{U,b} f(b))^{-1} \cdot \xi^{-1} \cdot \xi \cdot (\overset{-1}{\phi}_{U,b} g(b)) = \mu(b).\end{aligned}$$

Hence the transformation μ is uniquely defined. The continuity of μ follows from the fact that μ is continuous in every coordinate neighborhood U. This completes the proof.

(4.2) *Given an $(n-1)$-lifting $f: B^{n-1} \to X$ of $\mathfrak{F}$ and a mapping $\mu: B^{n-1} \to G$,* *there exists a unique* $(n-1)$*-lifting* $g: B^{n-1}\to X$ *of* $\mathfrak{F}$ *such that* $f^{-1}\cdot g=\mu$.

Proof. The required $(n-1)$-lifting $g: B^{n-1}\to X$ is defined as follows: For an arbitrary point $b\in B^{n-1}$, choose a coordinate neighborhood U which contains b and define

$$g(b) = \phi_{U,b}(\overset{-1}{\phi}_{U,b}f(b)\cdot\mu(b)).$$

To justify this definition, let V be another coordinate neighborhood which contains b; then we have

$$\begin{aligned}\phi_{V,b}(\overset{-1}{\phi}_{V,b}f(b)\cdot\mu(b)) &= \phi_{U,b}\overset{-1}{\phi}_{U,b}\phi_{V,b}(\overset{-1}{\phi}_{V,b}\phi_{U,b}\overset{-1}{\phi}_{U,b}f(b)\cdot\mu(b))\\ &= \phi_{U,b}(\xi^{-1}\cdot(\xi\cdot\overset{-1}{\phi}_{U,b}f(b)\cdot\mu(b)) = g(b),\end{aligned}$$

where $\xi\in G$ has the same meaning as in the proof of (4.1). Hence $g(b)$ is uniquely defined. The continuity of g follows from the fact that g is continuous in every coordinate neighborhood U. Further, clearly we have

$$\psi g(b) = \psi\phi_{U,b}(\overset{-1}{\phi}_{U,b}f(b)\cdot\mu(b)) = b;$$

hence g is an $(n-1)$-lifting of $\mathfrak{F}$. That $f^{-1}\cdot g=\mu$ is obvious. This completes the proof of (4.2).

(4.3) THEOREM I. *The totality of the n-dimensional obstruction elements of an* $(n-1)$*-parallelisable principal fibre bundle* $\mathfrak{F}$ *forms a coset* $W^n(F)$ *of the presentable subgroup* $P^n(B, \pi_{n-1}(G))$ *in the cohomology group* $H^n(B, \pi_{n-1}(G))$, *that is,* $W^n(\mathfrak{F})$ *is an element of the quotient group*

$$Q^n(B, \pi_{n-1}(G)) = H^n(B, \pi_{n-1}(G))/P^n(B, \pi_{n-1}(G)).$$

Proof. Let $W^n(\mathfrak{F})$ denote the totality of the n-dimensional obstruction elements of $\mathfrak{F}$. First, let f, g: $B^{n-1}\to X$ be two arbitrary $(n-1)$-liftings of $\mathfrak{F}$ and let $\mu=f^{-1}\cdot g$. μ presents a presentable element [4, §3] of $H^n(B, \pi_{n-1}(G))$ represented by the cocycle $c^n(\mu)$, introduced by S. Eilenberg [2, p. 237]. Let σ^n be an arbitrary n-simplex of B. According to our hypothesis concerning the triangulation of B, there exists a coordinate neighborhood U which contains the closure Cl σ^n of σ^n. Then, by the construction given in the proof of (4.1), we have

$$\mu(b) = (\overset{-1}{\phi}_{U,b}f(b))^{-1}\cdot(\overset{-1}{\phi}_{U,b}g(b)) \qquad (b\in\partial\sigma^n).$$

Hence, it follows from a homotopy property of a topological group [3] that

$$c^n(\mu)\cdot\sigma^n = w^n(g)\cdot\sigma^n - w^n(f)\cdot\sigma^n.$$

Hence it follows that

$$w^n(g) = w^n(f) + c^n(\mu),$$

and it implies that the obstruction elements $\omega^n(f)$ and $\omega^n(g)$ are contained in the same coset of $P^n(B, \pi_{n-1}(G))$ in $H^n(B, \pi_{n-1}(G))$. This proves that $W^n(\mathfrak{F})$ is contained in a single coset of the presentable subgroup $P^n(B, \pi_{n-1}(G))$.

Conversely, let $f: B^{n-1} \to X$ be a given $(n-1)$-lifting of $\mathfrak{F}$ and let α be an arbitrary presentable element of $H^n(B, \pi_{n-1}(G))$. According to the definition of presentable elements [4, §3], there is a mapping $\mu: B^{n-1} \to G$ such that the cocycle $c^n(\mu)$ represents α. Let $g: B^{n-1} \to X$ be the $(n-1)$-lifting of $\mathfrak{F}$ constructed in (4.2). Then it follows just as above that

$$w^n(g) = w^n(f) + c^n(\mu).$$

This implies that $\omega^n(g) = \omega^n(f) + \alpha$. Hence, every element of the coset $\omega^n(f) + P^n(B, \pi_{n-1}(G))$ is an obstruction element of $\mathfrak{F}$. This completes the proof of Theorem I.

5. n-**Parallelisability theorems.** We are now in a position to prove the main result of this paper.

(5.1) THEOREM II. *An $(n-1)$-parallelisable principal fibre bundle $\mathfrak{F}$ is n-parallelisable if and only if $W^n(\mathfrak{F})$ is the presentable subgroup $P^n(B, \pi_{n-1}(G))$ of the cohomology group $H^n(B, \pi_{n-1}(G))$.*

Proof. *Necessity.* Suppose $\mathfrak{F}$ to be a n-parallelisable. Then there is an n-lifting $f^*: B^n \to X$ of $\mathfrak{F}$. Let $f = f^* | B^{n-1}$, then f is an $(n-1)$-lifting with $w^n(f) = 0$. It follows from (4.3) that $W^n(\mathfrak{F}) = P^n(B, \pi_{n-1}(G))$.

Sufficiency. Suppose that $W^n(\mathfrak{F}) = P^n(B, \pi_{n-1}(G))$. Then there exists an $(n-1)$-lifting $f: B^{n-1} \to X$ of $\mathfrak{F}$ such that its obstruction element $\omega^n(f) = 0$, that is, the cocycle $w^n(f)$ is a coboundary. According to S. Eilenberg [2, (11.6)], there exists a mapping $\mu: B^{n-1} \to G$ such that $c^n(\mu) = -w^n(f)$. Let $g: B^{n-1} \to X$ be the $(n-1)$-lifting given in (4.2); then we have

$$w^n(g) = w^n(f) + c^n(\mu) = w^n(f) - w^n(f) = 0.$$

Let σ_i^n be an arbitrary n-simplex of B. Choose a coordinate neighborhood U which contains Cl σ_i^n. Define a mapping $\theta_i: \partial\sigma_i^n \to G$ by taking

$$\theta_i(b) = \overset{-1}{\phi}_{U,b} g(b) \qquad (b \in \partial \overset{n}{\sigma_i}).$$

Since $w^n(g) = 0$, θ_i has an extension θ_i^*: Cl $\sigma_i^n \to G$. Define a mapping h_i: Cl $\sigma_i^n \to X$ by taking

$$h_i(b) = \phi_{U,b}\theta_i^*(b) \qquad (b \in \mathrm{Cl}\ \overset{n}{\sigma_i}).$$

Then $h_i(b) = g(b)$ for each $b \in \partial\sigma_i^n$. Define a mapping $g^*: B^n \to X$ by taking

$$g^*(b) = \begin{cases} g(b) & (b \in B^{n-1}), \\ h_i(b) & (b \in \overset{n}{\sigma_i}). \end{cases}$$

Clearly, g^* is an n-lifting of $\mathfrak{F}$. This completes the proof of Theorem II.

As an alternative form of Theorem II, we give the following statement.

(5.2) *Let* $\mathfrak{F}$ *be an* $(n-1)$*-parallelisable principal fibre bundle and* $f: B^{n-1} \rightarrow X$ *be a given* $(n-1)$*-lifting of* $\mathfrak{F}$. $\mathfrak{F}$ *is n-parallelisable if and only if the cocycle* $w^n(f)$ *be presentable.*

The following theorem is an immediate consequence of Theorem II.

(5.3) THEOREM III. *If G is pathwise connected and*

$$H^n(B, \pi_{n-1}(G)) = P^n(B, \pi_{n-1}(G))$$

for each $2 \leqq n \leqq \dim B$, *then every fibre bundle* $\mathfrak{F} = \{F, G; X, B; \psi, \phi_U\}$ *is equivalent with the product bundle* $B \times F$.

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ACADEMIA SINICA,
NANKING, CHINA.