

ON THE DISTRIBUTION OF THE CHARACTERISTIC VALUES AND SINGULAR VALUES OF LINEAR INTEGRAL EQUATIONS

BY
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Introduction. The order of magnitude of the characteristic values $\mu_h [K]$ of the kernel $K(x, y)$ of a linear integral equation has been discussed by many writers, including Fredholm [4]⁽¹⁾, Mercer [11], Schur [12], Weyl [17, 18], Lalesco [9], Mazurkiewicz [10], Carleman [1, 2], Gheorghiu [5], and Hille and Tamarkin [8]. The order of magnitude of the singular values $\lambda_h [K]$, that is, E. Schmidt's characteristic values for non-symmetric kernels, was first discussed by Smithies [13].

It is natural to ask whether there is any relation between the orders of magnitude of the characteristic values and the singular values of a non-symmetric kernel when both sets of values are infinite in number.

We recall that μ is a characteristic value of the real L^2 kernel $K(x, y)$ if there is a real non-null L^2 function $\phi(x)$ such that

$$\phi(x) = \mu \int_a^b K(x, y)\phi(y)dy,$$

and that λ is a singular value of the same kernel if there exist real non-null L^2 functions $\phi(x), \psi(x)$ such that

$$\phi(x) = \lambda \int_a^b K(x, y)\psi(y)dy, \quad \psi(x) = \lambda \int_a^b K(y, x)\phi(y)dy.$$

When $K(x, y)$ is complex, the functions $\phi(x)$ and $\psi(x)$ may be complex, and the last equation above is replaced by

$$\psi(x) = \lambda \int_a^b \overline{K(y, x)}\phi(y)dy.$$

In a forthcoming paper, I shall show that for normal kernels, that is, L^2 kernels satisfying the condition

$$K\overline{K}' = \int_a^b K(x, s)\overline{K(y, s)}ds = \int_a^b \overline{K(s, x)}K(s, y)ds = \overline{K}'K,$$

the characteristic and singular values, when arranged in the usual way, satisfy

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(¹) Numbers in brackets refer to the references cited at the end of the paper.

$$|\mu_h| = |\lambda_h| \quad (h = 1, 2, \dots).$$

The present paper consists of two parts. In part I, the following three theorems are proved:

THEOREM 1. *Let $\{\mu_h[K]\}$ and $\{\lambda_h[K]\}$ be the sequences of characteristic and singular values, respectively, of the L^2 kernel $K(x, y)$, so that*

$$\int_a^b \int_a^b |K(x, y)|^2 dx dy < \infty.$$

Then the convergence of the series

$$(1) \quad \sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^\tau}$$

for a positive value of τ implies the convergence of the series

$$(2) \quad \sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|^\tau}$$

for the same value of τ .

THEOREM 2. *The converse of Theorem 1 is not true in general; in other words there exist L^2 kernels $K(x, y)$ for which (2) is convergent but (1) is divergent.*

THEOREM 3. *Suppose $K(x, y)$ is a real L^2 kernel, and*

$$D_{KK'}(\lambda) = \sum_{n=0}^{\infty} (-1)^n C_n \lambda^n$$

is the Fredholm determinant of

$$KK'(x, y) = \int_a^b K(x, s)K(y, s)ds;$$

a necessary and sufficient condition for the convergence of the series

$$(1') \quad \sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^\rho},$$

where $0 < \rho < 2$, is that the series

$$(3) \quad \sum_{n=1}^{\infty} |C_n|^{\rho/2n}$$

be convergent.

In part II of the paper, I apply the same methods to a different problem, obtaining some interesting results about composite kernels, that is, kernels

of the form $K(x, y) = K_1K_2(x, y)$, where we write

$$K_1K_2(x, y) = \int_a^b K_1(x, s)K_2(s, y)ds;$$

we also write $K_1K_2K_3(x, y) = (K_1K_2)K_3(x, y)$, and so on. In an earlier paper [3], I showed that a sufficient condition for an L^2 kernel $K(x, y)$ to be expressible in the form $K_1K_2 \cdots K_m(x, y)$, where K_1, \dots, K_m are L^2 kernels, is that

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < +\infty.$$

I can now show that this condition is also necessary. In other words we have:

THEOREM 4. *Suppose $K(x, y) = K_1K_2 \cdots K_m(x, y)$, where each $K_i \in L^2$ ($i = 1, 2, \dots, m$). Let $\{\lambda_h[K]\}$ be the sequence of singular values of $K(x, y)$; then*

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < \infty.$$

As a corollary of Theorems 1 and 4, we obtain:

THEOREM 5. *If $\{\mu_h[K]\}$ is the sequence of characteristic values of a kernel $K(x, y)$ satisfying the conditions of Theorem 4, then*

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|^{2/m}} < \infty.$$

This is a generalization of Lalesco's result [9] that for any composite kernel

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|} < \infty.$$

Lalesco's proof was not applicable to L^2 kernels, for which the result was proved by Gheorghiu [5] and Hille and Tamarkin [8]. We also prove:

THEOREM 6. *If $K(x, y)$ is a Pell kernel, that is,*

$$K(x, y) = \int_a^b Q(x, s)L(s, y)ds,$$

where $Q(x, y)$ is a semi-definite continuous symmetric kernel, and $L(x, y)$ is a symmetric L^2 kernel, then

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/3}} < \infty, \quad \sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|^{2/3}} < \infty.$$

More generally, the conclusions still hold if $L(x, y)$ is an arbitrary L^2 kernel.

Other applications will be found in the Appendix.

I wish to thank Dr. F. Smithies for valuable criticism and encouragement throughout this work.

PART I. GENERAL KERNELS

1.1. Preliminary lemmas.

LEMMA 1. Let $f(z)$ be an integral function whose zeros (repeated according to their multiplicities) are z_1, z_2, \dots , where $|z_h| = r_h, 0 < r_1 \leq r_2 \leq \dots$. Let $n(r)$ be the number of zeros of $f(z)$ such that $r_h < r$ and write

$$M(r) = M(r; f) = \max_{|z|=r} |f(z)|.$$

Then:

(i) If $\tau > 0$, the convergence of the integral

$$(4) \quad \int_0^\infty \frac{n(r)}{r^{1+\tau}} dr$$

is necessary and sufficient for the convergence of the series

$$(5) \quad \sum_{h=1}^\infty r_h^{-\tau}.$$

(ii) If $\tau > 0$, the convergence of the integral

$$(6) \quad \int_\alpha^\infty \frac{\log \mu(r)}{r^{1+\tau}} dr,$$

where $\alpha > 0$, is a sufficient condition for the convergence of the integral (4).

(iii) If the order ρ of the integral function $f(z)$ is not an integer, the convergence of the series (5) for $\tau = \rho$ is necessary and sufficient for the convergence of the integral (6) for $\tau = \rho$.

In order to state the remainder of this lemma, we require some definitions. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be the Maclaurin series of $f(z)$; we write $\log |a_n| = -g_n$. Then $\lim_{n \rightarrow \infty} |a_n|^{1/n} = 0$, so that

$$(7) \quad \lim_{n \rightarrow \infty} (g_n/n) = +\infty.$$

Plot the points A_n with coordinates (n, g_n) in the (x, y) -plane. By (7), we can construct a Newton polygon⁽²⁾ having certain of the points A_n as its vertices whilst the remainder lie either on or above it. We denote this polygon by $\pi(f)$. Let G_n be the ordinate of the point of abscissa n on the curve $\pi(f)$; the

⁽²⁾ J. Hadamard [6, p. 174].

ratio $R_n = e^{g_n - g_{n-1}}$ is then called the *rectified ratio*⁽³⁾ of $|a_{n-1}|$ to $|a_n|$.

We can now complete the statement of Lemma 1 as follows:

(iv) *If $f(z)$ is an integral function of finite order, and $\tau > 0$, a necessary and sufficient condition for the convergence of the integral (6) is that the series*

$$(8) \quad \sum_{n=1}^{\infty} \frac{1}{R_n^\tau}$$

be convergent.

(v) *If $f(z)$ is an integral function whose order ρ is not an integer, a necessary and sufficient condition for $f(z)$ to be of convergent class, that is, for the series (5) to be convergent when $\tau = \rho$, is that the series*

$$(8') \quad \sum_{n=1}^{\infty} \frac{1}{R_n^\rho}$$

be convergent.

Proof. See Valiron [14, pp. 258–265].

REMARK. If the coefficients of the integral function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy the condition

$$(9) \quad \frac{|a_1|}{|a_0|} \geq \frac{|a_2|}{|a_1|} \geq \cdots,$$

the rectified ratio of $|a_{n-1}|$ to $|a_n|$ is given by

$$R_n = e^{g_n - g_{n-1}} = \frac{|a_{n-1}|}{|a_n|}.$$

For, if (9) holds, we have

$$\log \frac{|a_{m-1}|}{|a_{m-2}|} \geq \log \frac{|a_m|}{|a_{m-1}|},$$

whence

$$g_m - g_{m-1} \geq g_{m-1} - g_{m-2},$$

that is, the slope of the joins of successive points A_n increases with n . It follows without difficulty that all the points A_n actually lie on the Newton polygon, whence the result follows.

LEMMA 2. *If $f(z)$ is an integral function of genus zero, and $0 < \tau < 1$, a necessary and sufficient condition for the convergence of the series (5) is the convergence of the integral (6).*

⁽³⁾ G. Valiron [15, p. 30].

Sufficiency follows at once from Lemma 1, (i) and (ii).

If (5) is convergent, the exponent of convergence ρ of $f(z)$ satisfies $\rho \leq \tau < 1$. Since the genus of $f(z)$ is 0, $f(z)$ is a canonical product, and we have

$$f(z) = \prod_{h=1}^{\infty} \left(1 - \frac{z}{z_h}\right).$$

By Borel's inequality⁽⁴⁾,

$$\log M(r; f) \leq Ar \int_{r_1}^{\infty} \frac{n(x)dx}{x(x+r)},$$

where A is a positive constant; hence

$$(10) \quad \int_{\alpha}^r \frac{\log M(x; f)}{x^{1+\tau}} dx \leq A \int_{\alpha}^r \frac{dy}{r^{\tau}} \int_{r_1}^{\infty} \frac{n(x)dx}{x(x+y)}.$$

Since the integrands on the right-hand side in (10) are non-negative, we can invert the order of integration; putting $y=tx$, we then obtain

$$\int_{\alpha}^r \frac{\log M(x)dx}{x^{1+\tau}} \leq A \int_{r_1}^{\infty} \frac{n(x)U(x)dx}{x^{1+\tau}},$$

where

$$U(x) = \int_{\alpha/x}^{r/x} \frac{dt}{t^{\tau}(1+t)} < \int_0^{\infty} \frac{dt}{t^{\tau}(1+t)} = \frac{\pi}{\sin \tau\pi} < \infty.$$

Since (5) is convergent, we can use Lemma 1 (i); hence

$$\int_{\alpha}^r \frac{\log M(x)dx}{x^{1+\tau}} \leq \frac{A\pi}{\sin \tau\pi} \int_{r_1}^{\infty} \frac{n(x)dx}{x^{1+\tau}},$$

a finite number independent of r . The integral (6) is therefore convergent.

COROLLARY. *If $f(z)$ is an integral function of genus zero, and $0 < \tau < 1$, the series (5) converges if and only if the series (8) converges.*

Proof. This is proved by Lemma 1 (iv) and Lemma 2.

LEMMA 3. *Let $K(x, y)$ be a real L^2 kernel, let*

$$K^2(x, y) = \int_a^b K(x, s)K(s, y)ds, \quad KK'(x, y) = \int_a^b K(x, s)K(y, s)ds,$$

and let $D_{K^2}(\lambda)$ and $D_{KK'}(\lambda)$ denote the Fredholm determinants of $K^2(x, y)$ and $KK'(x, y)$, respectively. Write

⁽⁴⁾ G. Valiron [15, p. 53].

$$M(r; K^2) = \max_{|\lambda|=r} |D_{K^2}(\lambda)|, \quad M(r; KK') = \max_{|\lambda|=r} |D_{KK'}(\lambda)|.$$

Then

$$M(r; K^2) \leq M(r; KK').$$

Proof. See Hille and Tamarkin [8, p. 36, Lemma 6.7].

LEMMA 4. Suppose that

$$x^n + c_1x^{n-1} + \dots + c_n = \prod_{h=1}^n (x + a_h)$$

where a_1, a_2, \dots, a_n are positive numbers. Then

$$(11) \quad c_{r-1} \cdot c_{r+1} \leq c_r^2.$$

Proof. See Hardy, Littlewood, and Pólya [7, pp. 51–55].

COROLLARY 1. If a_1, a_2, \dots, a_n are positive, and $\sum a_{h_1}a_{h_2} \dots a_{h_r}$ is the elementary symmetric function of order r of a_1, a_2, \dots, a_n , then

$$\begin{aligned} & (\sum a_{h_1}a_{h_2} \dots a_{h_{r-1}})(\sum a_{h_1}a_{h_2} \dots a_{h_{r+1}}) \leq (\sum a_{h_1}a_{h_2} \dots a_{h_r})^2, \\ & \left(\sum \frac{1}{a_{h_1}a_{h_2} \dots a_{h_{r-1}}}\right) \left(\sum \frac{1}{a_{h_1}a_{h_2} \dots a_{h_{r+1}}}\right) \leq \left(\sum \frac{1}{a_{h_1}a_{h_2} \dots a_{h_r}}\right)^2. \end{aligned}$$

COROLLARY 2. If a_1, a_2, \dots, a_n are positive, and

$$f(x) = 1 - p_1x + p_2x^2 - \dots + (-1)^n p_n x^n = \prod_{i=1}^n \left(1 - \frac{x}{a_i}\right),$$

then

$$(11') \quad p_{r-1}p_{r+1} - p_r^2 \leq 0.$$

LEMMA 5. Let $K(x, y)$ be a real L^2 kernel with infinitely many singular values $\lambda_h[K]$ ($h = 1, 2, \dots$), and let

$$D_{KK'}(\lambda) = \sum_{n=0}^{\infty} (-1)^n c_n \lambda^n$$

be the Fredholm determinant of the symmetric kernel

$$KK'(x, y) = \int_a^b K(x, s)K(y, s)ds.$$

Then the coefficients c_n are all positive, and satisfy the inequality

$$(12) \quad c_m c_{m-2} \leq c_{m-1}^2 \quad (m = 2, 3, \dots).$$

It is known⁽⁶⁾ that each singular value λ_h is real and that

$$(13) \quad D_{KK'}(\lambda) = \prod_{h=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_h^2}\right).$$

Hence, for every positive integer r ,

$$c_r = \sum_{h_1, h_2, \dots, h_r} \frac{1}{\lambda_{h_1}^2 \lambda_{h_2}^2 \dots \lambda_{h_r}^2} > 0.$$

To prove (12), we consider the function

$$f_N(\lambda) = \prod_{h=1}^N \left(1 - \frac{\lambda}{\lambda_h^2}\right) = 1 - A_1\lambda + A_2\lambda^2 - \dots + (-1)^N A_N \lambda^N.$$

By Lemma 4, Corollary 2,

$$A_m A_{m-2} \leq A_{m-1}^2 \quad (m = 2, 3, \dots),$$

where we write $A_0 = 1$, and $A_m = 0$ when $m > N$. Since the infinite product (13) is uniformly convergent in any bounded portion of the λ -plane, $A_m \rightarrow c_m$ when $N \rightarrow \infty$; hence,

$$c_m c_{m-2} \leq c_{m-1}^2 \quad (m = 2, 3, \dots),$$

the required result.

1.2. Proof of Theorem 1. When $\tau \geq 2$, the series (1) and (2) are both known to be convergent. We therefore need consider only the case when $0 < \tau < 2$.

By the set of characteristic values $\{\mu_h[K]\}$ of $K(x, y)$ we mean the zeros, repeated according to their multiplicities, of the Fredholm determinant $D_K(\lambda)$ of $K(x, y)$, arranged so that $|\mu_1| \leq |\mu_2| \leq \dots$.

By a known result⁽⁶⁾,

$$\mu_h[K^2] = (\mu_h[K])^2 \quad (h = 1, 2, \dots).$$

Also, if $\{\lambda_h[K]\}$ is the set of singular values of $K(x, y)$, arranged in the same way, then

$$(\lambda_h[K])^2 = \mu_h[KK'].$$

Now suppose that (1) is convergent, where $0 < \tau < 2$, and write $t = \tau/2$; then

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[KK']|^t} < +\infty,$$

⁽⁶⁾ G. Vivanti [16, pp. 192-193] or Hille and Tamarkin [8, p. 29].

⁽⁶⁾ Hille and Tamarkin [8, p. 37].

where $0 < t < 1$. Hence, by Lemma 2,

$$\int_{\alpha}^{\infty} \frac{\log M(r; KK')}{r^{1+t}} dr < + \infty,$$

and so, by Lemma 3,

$$\int_{\alpha}^{\infty} \frac{\log M(r; K^2)}{r^{1+t}} dr < + \infty.$$

Applying Lemma 2 again, we have

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K^2]|^t} < + \infty,$$

that is,

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|^r} < + \infty,$$

the required result.

1.3. Proof of Theorem 2. Consider the kernel

$$K(x, y) \sim \sum_{h=1}^{\infty} \frac{\cos 2hx \cdot \cos 2hy}{4h^2} + \sum_{h=1}^{\infty} \frac{\cos (2h + 1)x \cdot \sin (2h + 1)y}{2h + 1},$$

where the symbol \sim indicates that the series on the right are convergent in mean (with index 2). Evidently $K(x, y)$ is an L^2 kernel; we write it in the form

$$K(x, y) = A(x, y) + B(x, y),$$

where

$$A(x, y) \sim \sum_{h=1}^{\infty} \frac{\cos 2hx \cdot \cos 2hy}{4h^2},$$

$$B(x, y) \sim \sum_{h=1}^{\infty} \frac{\cos (2h + 1)x \cdot \sin (2h + 1)y}{2h + 1}.$$

Then $AB(x, y) = 0, BA(x, y) = 0,$ and $B^2(x, y) = 0$. It follows that $B(x, y)$ has no characteristic values, and that

$$\mu_h[K] = \mu_h[A] = 4h^2 \quad (h = 1, 2, \dots).$$

Hence

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|} = \sum_{h=1}^{\infty} \frac{1}{4h^2} < + \infty.$$

On the other hand, the set of singular values is

$$3, 2^2, 5, 7, \dots, 15, 4^2, 17, 19, \dots, 35, 6^2, 37, \dots$$

and the series $\sum_{h=1}^{\infty} 1/|\lambda_h[K]|$ is clearly divergent. Our theorem is therefore proved.

1.4. Proof of Theorem 3. By Lemma 5, $c_n > 0$ for all n , and the sequence $\{c_n/c_{n-1}\}$ is monotone decreasing; hence by the remark to Lemma 1, the rectified ratio of $|c_{n-1}|$ to $|c_n|$ is

$$R_n = \left| \frac{c_{n-1}}{c_n} \right| = \frac{c_{n-1}}{c_n}.$$

Now, by (13), $D_{KK'}(\lambda)$ is of genus zero; consequently, by the corollary to Lemma 2, the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{R_n^{\rho/2}} = \sum_{n=1}^{\infty} \left(\frac{c_n}{c_{n-1}} \right)^{\rho/2}$$

is a necessary and sufficient condition for the convergence of the series

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[KK']|^{\rho/2}} = \sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{\rho}},$$

provided that $0 < \rho/2 < 1$, that is, $0 < \rho < 2$.

By Carleman's inequality⁽⁷⁾,

$$\sum_{n=1}^{\infty} \left(\frac{c_n}{c_0} \right)^{\rho/2n} < e \sum_{n=1}^{\infty} \left(\frac{c_n}{c_{n-1}} \right)^{\rho/2};$$

on the other hand, since $\{c_n/c_{n-1}\}$ is a decreasing sequence,

$$\left(\frac{c_1}{c_0} \cdot \frac{c_2}{c_1} \cdot \dots \cdot \frac{c_n}{c_{n-1}} \right)^{\rho/2n} \geq \left(\frac{c_n}{c_{n-1}} \right)^{\rho/2},$$

that is, since $c_0 = 1$,

$$c_n^{\rho/2n} \geq \left(\frac{c_n}{c_{n-1}} \right)^{\rho/2}.$$

The series (3) and

$$(14) \quad \sum_{n=1}^{\infty} \left(\frac{c_n}{c_{n-1}} \right)^{\rho/2}$$

therefore converge or diverge together. Thus (1') converges if and only if (3) converges; this completes the proof.

⁽⁷⁾ Hardy, Littlewood, and Pólya [7, p. 249].

PART II. COMPOSITE KERNELS

2.1. Preliminary lemmas.

LEMMA 1. Suppose that $K(x, y) = K_1K_2(x, y)$, where K_1 and K_2 are L^2 kernels, and $D_{KK'}(\lambda) = \sum_{n=0}^{\infty} (-1)^n c_n \lambda^n$ be the Fredholm determinant of $KK'(x, y)$. Then the series

$$(15) \quad \sum_{n=1}^{\infty} |c_n|^{1/2n},$$

$$(16) \quad \sum_{n=1}^{\infty} \left(\frac{c_n}{c_{n-1}} \right)^{1/2},$$

are both convergent.

The series $\sum_{n=1}^{\infty} 1/|\lambda_n[K_1K_2]|$ is known to be convergent⁽⁸⁾; the convergence of (15) then follows from Theorem 3, and the convergence of (16) from the fact that (3) and (14) converge or diverge together.

LEMMA 2. Let P_i denote the vector

$$P_i = P_i^{(n)} = (s_1^{(i)}, s_2^{(i)}, \dots, s_n^{(i)}) \quad (i = 1, 2, \dots, m),$$

and D_i the domain defined by

$$a \leq s_j^{(i)} \leq b \quad (j = 1, 2, \dots, n).$$

Write $dP_i = ds_1^{(i)} ds_2^{(i)} \dots ds_n^{(i)}$. Let $K_i(P_i, P_{i+1}) \in L^2(P_i, P_{i+1})$ ($i = 1, 2, \dots, m$), that is,

$$\int_{D_i} \int_{D_{i+1}} |K_i(P_i, P_{i+1})|^2 dP_i dP_{i+1} = \|K_i(P_i, P_{i+1})\|^2 < \infty,$$

where $P_{m+1} = P_1$ and $D_{m+1} = D_1$; then

$$(17) \quad \int_{D_1} \int_{D_2} \dots \int_{D_m} |K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)| \dots |K_m(P_m, P_1)| dP_1 dP_2 \dots dP_m \leq \prod_{i=1}^m \|K_i(P_i, P_{i+1})\|.$$

We prove the result for $m = 3$; a similar proof holds for larger values of m . By Schwarz's inequality,

$$\int_{D_2} |K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)| dP_2 \leq F(P_1)G(P_3),$$

⁽⁸⁾ S. H. Chang [3, pp. 185-189].

where

$$F^2(P_1) = \int_{D_2} |K_1(P_1, P_2)|^2 dP_2,$$

$$G^2(P_3) = \int_{D_2} |K_2(P_2, P_3)|^2 dP_2.$$

Hence

$$\begin{aligned} & \int_{D_2} \int_{D_3} |K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)| \cdot |K_3(P_3, P_1)| dP_2 dP_3 \\ & \leq \int_{D_3} F(P_1) \cdot G(P_3) |K_3(P_3, P_1)| dP_3 \\ & \leq F(P_1) \left\{ \int_{D_3} |G(P_3)|^2 dP_3 \right\}^{1/2} \cdot \left\{ \int_{D_3} |K_3(P_3, P_1)|^2 dP_3 \right\}^{1/2} \\ & = F(P_1) \cdot \|K_2(P_2, P_3)\| \left\{ \int_{D_3} |K_3(P_3, P_1)|^2 dP_3 \right\}^{1/2}. \end{aligned}$$

Integrating both sides with respect to P over the domain D , we obtain

$$\begin{aligned} & \int_{D_1} \int_{D_2} \int_{D_3} |K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)| \cdot |K_3(P_3, P_1)| dP_1 dP_2 dP_3 \\ & \leq \|K_2(P_2, P_3)\| \int_{D_1} F(P_1) \left\{ \int_{D_3} |K_3(P_3, P_1)|^2 dP_3 \right\}^{1/2} dP_1 \\ & \leq \|K_2(P_2, P_3)\| \cdot \|K_3(P_3, P_1)\| \left\{ \int_{D_1} |F(P_1)|^2 dP_1 \right\}^{1/2} \\ & = \|K_1(P_1, P_2)\| \cdot \|K_2(P_2, P_3)\| \cdot \|K_3(P_3, P_1)\|, \end{aligned}$$

the required result.

LEMMA 3 (GENERALIZED CARLEMAN THEOREM). *If $m > 1$, and each of the functions $K_1(x, y), K_2(x, y), \dots, K_m(x, y)$ is a real L^2 kernel, the Fredholm determinant of the composite kernel*

$$K(x, y) = K_1 K_2 \cdots K_m(x, y)$$

is

$$(18) \quad D_K(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n!)^m} \int_{D_1} \int_{D_2} \cdots \int_{D_m} K_1(P_1^{(n)}, P_2^{(n)}) K_2(P_2^{(n)}, P_3^{(n)}) \cdots K_m(P_m^{(n)}, P_1^{(n)}) dP_1^{(n)} dP_2^{(n)} \cdots dP_m^{(n)},$$

where

$$\begin{aligned}
 D_{KK'}(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n!)^m} \int_{D_1} \int_{D_2} \cdots \int_{D_m} A_1(P_1^{(n)}, P_2^{(n)}) A_2(P_2^{(n)}, P_3^{(n)}) \\
 &\quad \cdots A_m(P_m^{(n)}, P_1^{(n)}) dP_1^{(n)} dP_2^{(n)} \cdots dP_m^{(n)} \\
 &= \sum_{n=0}^{\infty} (-1)^n \lambda^n c_n(A_1 A_2 \cdots A_m),
 \end{aligned}$$

say, where $A_i(P_i^{(n)}, P_{i+1}^{(n)})$ is defined as in (19).

By Lemma 2 we have

$$\begin{aligned}
 |c_n(KK')| &= |c_n(A_1 A_2 \cdots A_m)| \leq \frac{1}{(n!)^m} \prod_{i=1}^m \|A_i(P_i^{(n)}, P_{i+1}^{(n)})\| \\
 &= \prod_{i=1}^m \left\{ \frac{1}{(n!)^2} \|A_i(P_i^{(n)}, P_{i+1}^{(n)})\|^2 \right\}^{1/2} \\
 &= \prod_{i=1}^m |c_n(A_i A'_i)|^{1/2},
 \end{aligned}$$

where we are using, in general, $c_n(K)$ to denote the coefficient of $(-1)^n \lambda^n$ in the power series expansion of the Fredholm determinant $D_K(\lambda)$ of $K(x, y)$. Hence

$$\sum_{n=1}^{\infty} |c_n(KK')|^{1/mn} \leq \sum_{n=1}^{\infty} \prod_{i=1}^m |c_n(A_i A'_i)|^{1/2mn}.$$

Consequently, by a well known inequality⁽¹¹⁾,

$$\sum_{n=1}^{\infty} |c_n(KK')|^{1/mn} \leq \prod_{i=1}^m \left\{ \sum_{n=1}^{\infty} |c_n(A_i A'_i)|^{1/2n} \right\}^{1/m}.$$

Now, by Lemma 1,

$$\sum_{n=1}^{\infty} |c_n(A_i A'_i)|^{1/2n} < +\infty \quad (i = 1, 2, \dots, m).$$

Consequently,

$$\sum_{n=1}^{\infty} |c_n(KK')|^{1/mn} < +\infty,$$

and therefore, by Theorem 3,

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < +\infty,$$

⁽¹¹⁾ Hardy, Littlewood, and Pólya [7, p. 22].

as we wished to prove.

2.3. Proof of Theorem 5. This now follows at once from Theorem 4 and Theorem 1 with $\tau = 2/m$.

2.4. Proof of Theorem 6. It is known⁽¹²⁾ that if $Q(x, y)$ is a continuous semidefinite symmetric kernel, then there exists a symmetric L^2 kernel $A(x, y)$ such that

$$Q(x, y) = A^2(x, y) = \int_a^b A(x, s)A(s, y)ds.$$

Hence any Pell kernel or, more generally, any kernel of the form $K(x, y) = QL(x, y)$, where $Q(x, y)$ has the above properties, can be expressed in the form $K(x, y) = A^2L(x, y)$. The result now follows by taking $m = 3$ in Theorems 4 and 5.

APPENDIX

We recall⁽¹³⁾ that a necessary and sufficient condition for a real L^2 kernel $K(x, y)$ to have a canonical decomposition into m factors is that

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < +\infty.$$

We therefore have:

THEOREM 7. *If a real L^2 kernel $K(x, y)$ has a decomposition $K(x, y) = K_1K_2 \cdots K_m(x, y)$ into m L^2 factors, then it has a canonical decomposition into m factors.*

Many results can also be proved showing that the smoother a kernel $K(x, y)$ is, the greater is the number of factors into which it can be decomposed. For instance, we have:

THEOREM 8. *If $K(x, y)$ is a real symmetric kernel, continuous in $a \leq x \leq b$, $a \leq y \leq b$, and*

$$\frac{\partial^r K(x, y)}{\partial x^r}$$

is continuous in the same square, then $K(x, y)$ has a decomposition into at least $2r$ factors, so that we can write

$$K(x, y) = K_1K_2 \cdots K_{2r}(x, y).$$

For, by Weyl's theorem⁽¹⁴⁾,

⁽¹²⁾ S. H. Chang [3, p. 189, Corollary 4.]

⁽¹³⁾ S. H. Chang [3].

⁽¹⁴⁾ H. Weyl [17].

$$\lim_{n \rightarrow \infty} \frac{n^{r+1/2}}{|\lambda_n[K]|} = 0.$$

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n[K]|^{2/m}}$$

is thus convergent provided that $(2/m)(r+1/2) > 1$, that is, $m < 2r+1$. We can therefore take $m=2r$, and the result follows.

All the above results can be extended to complex-valued kernels if we define singular values and singular functions by the equations

$$\phi_h(x) = \lambda_h \int_a^b K(x, y)\psi_h(y)dy, \quad \psi_h(x) = \lambda_h \int_a^b \overline{K(y, x)}\phi_h(y)dy,$$

replace the kernel $K'(x, y) = K(y, x)$ by the kernel $K^*(x, y) = \overline{K(y, x)}$, and use Hermitian kernels instead of real symmetric kernels.

The proof of Theorem 4 could also be carried out by using Hille and Tamarkin’s formulae⁽¹⁵⁾:

$$D_{K_1 \dots K_m}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n!)^m} \sum_{(i_1, \dots, i_m; n)} \Delta_{i_1 i_2}^{(n)}(K_1) \Delta_{i_2 i_3}^{(n)}(K_2) \dots \Delta_{i_m i_1}^{(n)}(K_m),$$

$$D_{KK^*}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n!)^2} \sum_{(i, j; n)} |\Delta_{ij}^{(n)}(K)|^2,$$

where

$$\Delta_{ij}^{(n)}(K) = \begin{vmatrix} k_{i_1 j_1}(K), & \dots, & k_{i_1 j_n}(K) \\ \dots & \dots & \dots \\ k_{i_n j_1}(K), & \dots, & k_{i_n j_n}(K) \end{vmatrix},$$

$$k_{ij}(K) = \int_a^b \int_a^b K(s, t) \overline{u_i(s)} u_j(t) ds dt,$$

$\{u_i(x)\}$ being an arbitrary complete orthonormal set of functions for the interval (a, b) . We then use the inequality for series corresponding to (17).

This allows us to replace the hypothesis of Theorem 6 by the condition that $Q(x, y)$ is an Hermitian semi-definite L^2 kernel such that

$$\sum_{i=1}^{\infty} |k_{ii}(Q)| = \sum_{i=1}^{\infty} \left| \int_a^b \int_a^b Q(s, t) \overline{u_i(s)} u_i(t) ds dt \right| < + \infty.$$

For the series $\sum_{n=1}^{\infty} 1/|\lambda_n[Q]|$ is then still convergent⁽¹⁶⁾, and therefore $Q(x, y)$ is still expressible in the form $Q(x, y) = A^2(x, y)$.

⁽¹⁵⁾ Hille and Tamarkin [8, p. 33, Lemmas 6.2 and 6.3].

⁽¹⁶⁾ Hille and Tamarkin [8, p. 29, Theorem 5.1].

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