ON THE DISTRIBUTION OF THE CHARACTERISTIC VALUES AND SINGULAR VALUES OF LINEAR INTEGRAL EQUATIONS

BY

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Introduction. The order of magnitude of the characteristic values $\mu_h[K]$ of the kernel $K(x, y)$ of a linear integral equation has been discussed by many writers, including Fredholm [4](1), Mercer [11], Schur [12], Weyl [17, 18], Lalesco [9], Mazurkiewicz [10], Carleman [1, 2], Gheorghiu [5], and Hille and Tamarkin [8]. The order of magnitude of the singular values $\lambda_h[K]$, that is, E. Schmidt's characteristic values for non-symmetric kernels, was first discussed by Smithies [13].

It is natural to ask whether there is any relation between the orders of magnitude of the characteristic values and the singular values of a non-symmetric kernel when both sets of values are infinite in number.

We recall that μ is a characteristic value of the real L^2 kernel $K(x, y)$ if there is a real non-null L^2 function $\phi(x)$ such that

$$\phi(x) = \mu \int_a^b K(x, y)\phi(y)dy,$$

and that λ is a singular value of the same kernel if there exist real non-null L^2 functions $\phi(x)$, $\psi(x)$ such that

$$\phi(x) = \lambda \int_a^b K(x, y)\psi(y)dy, \qquad \psi(x) = \lambda \int_a^b K(y, x)\phi(y)dy.$$

When $K(x, y)$ is complex, the functions $\phi(x)$ and $\psi(x)$ may be complex, and the last equation above is replaced by

$$\psi(x) = \lambda \int_a^b \overline{K(y, x)}\phi(y)dy.$$

In a forthcoming paper, I shall show that for normal kernels, that is, L^2 kernels satisfying the condition

$$K\overline{K}' = \int_a^b K(x, s)\overline{K(y, s)}ds = \int_a^b \overline{K(s, x)}K(s, y)ds = \overline{K}'K,$$

the characteristic and singular values, when arranged in the usual way, satisfy

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(1) Numbers in brackets refer to the references cited at the end of the paper.

$$|\mu_h| = |\lambda_h| \qquad (h = 1, 2, \cdots).$$

The present paper consists of two parts. In part I, the following three theorems are proved:

THEOREM 1. *Let $\{\mu_h[K]\}$ and $\{\lambda_h[K]\}$ be the sequences of characteristic and singular values, respectively, of the L^2 kernel $K(x, y)$, so that*

$$\int_a^b \int_a^b |K(x, y)|^2 dxdy < \infty.$$

Then the convergence of the series

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^\tau} \tag{1}$$

for a positive value of τ implies the convergence of the series

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|^\tau} \tag{2}$$

for the same value of τ.

THEOREM 2. *The converse of Theorem 1 is not true in general; in other words there exist L^2 kernels $K(x, y)$ for which (2) is convergent but (1) is divergent.*

THEOREM 3. *Suppose $K(x, y)$ is a real L^2 kernel, and*

$$D_{KK'}(\lambda) = \sum_{n=0}^{\infty} (-1)^n C_n \lambda^n$$

is the Fredholm determinant of

$$KK'(x, y) = \int_a^b K(x, s)K(y, s)ds;$$

a necessary and sufficient condition for the convergence of the series

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^\rho}, \tag{1'}$$

where $0<\rho<2$, is that the series

$$\sum_{n=1}^{\infty} | C_n |^{\rho/2n} \tag{3}$$

be convergent.

In part II of the paper, I apply the same methods to a different problem, obtaining some interesting results about composite kernels, that is, kernels of the form $K(x, y) = K_1K_2(x, y)$, where we write

$$K_1K_2(x, y) = \int_a^b K_1(x, s)K_2(s, y)ds;$$

we also write $K_1K_2K_3(x, y) = (K_1K_2)K_3(x, y)$, and so on. In an earlier paper [3], I showed that a sufficient condition for an L^2 kernel $K(x, y)$ to be expressible in the form $K_1K_2 \cdots K_m(x, y)$, where $K_1, \cdots, K_m$ are L^2 kernels, is that

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < +\infty.$$

I can now show that this condition is also necessary. In other words we have:

THEOREM 4. *Suppose* $K(x, y) = K_1K_2 \cdots K_m(x, y)$, *where each* $K_i \in L^2$ $(i = 1, 2, \cdots, m)$. *Let* $\{\lambda_h[K]\}$ *be the sequence of singular values of* $K(x, y)$; *then*

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < \infty.$$

As a corollary of Theorems 1 and 4, we obtain:

THEOREM 5. *If* $\{\mu_h[K]\}$ *is the sequence of characteristic values of a kernel* $K(x, y)$ *satisfying the conditions of Theorem* 4, *then*

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|^{2/m}} < \infty.$$

This is a generalization of Lalesco's result [9] that for any composite kernel

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|} < \infty.$$

Lalesco's proof was not applicable to L^2 kernels, for which the result was proved by Gheorghiu [5] and Hille and Tamarkin [8]. We also prove:

THEOREM 6. *If* $K(x, y)$ *is a Pell kernel, that is,*

$$K(x, y) = \int_a^b Q(x, s)L(s, y)ds,$$

where $Q(x, y)$ *is a semi-definite continuous symmetric kernel, and* $L(x, y)$ *is a symmetric* L^2 *kernel, then*

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/3}} < \infty, \qquad \sum_{h=1}^{\infty} \frac{1}{|\mu_h[K]|^{2/3}} < \infty.$$

More generally, the conclusions still hold if $L(x, y)$ is an arbitrary L^2 kernel.

Other applications will be found in the Appendix.

I wish to thank Dr. F. Smithies for valuable criticism and encouragement throughout this work.

PART I. GENERAL KERNELS

1.1. Preliminary lemmas.

LEMMA 1. *Let $f(z)$ be an integral function whose zeros (repeated according to their multiplicities) are $z_1, z_2, \cdots$, where $|z_h| = r_h$, $0 < r_1 \leqq r_2 \leqq \cdots$. Let $n(r)$ be the number of zeros of $f(z)$ such that $r_h < r$ and write*

$$M(r) = M(r; f) = \max_{|z|=r} |f(z)|.$$

Then:

(i) *If $\tau > 0$, the convergence of the integral*

$$\int_0^\infty \frac{n(r)}{r^{1+\tau}}\, dr \tag{4}$$

is necessary and sufficient for the convergence of the series

$$\sum_{h=1}^\infty r_h^{-\tau}. \tag{5}$$

(ii) *If $\tau > 0$, the convergence of the integral*

$$\int_\alpha^\infty \frac{\log \mu(r)}{r^{1+\tau}}\, dr, \tag{6}$$

where $\alpha > 0$, is a sufficient condition for the convergence of the integral (4).

(iii) *If the order ρ of the integral function $f(z)$ is not an integer, the convergence of the series* (5) *for $\tau = \rho$ is necessary and sufficient for the convergence of the integral* (6) *for $\tau = \rho$.*

In order to state the remainder of this lemma, we require some definitions. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be the Maclaurin series of $f(z)$; we write $\log |a_n| = -g_n$. Then $\lim_{n\to\infty} |a_n|^{1/n} = 0$, so that

$$\lim_{n\to\infty} (g_n/n) = +\infty. \tag{7}$$

Plot the points A_n with coordinates (n, g_n) in the (x, y)-plane. By (7), we can construct a Newton polygon[2] having certain of the points A_n as its vertices whilst the remainder lie either on or above it. We denote this polygon by $\pi(f)$. Let G_n be the ordinate of the point of abscissa n on the curve $\pi(f)$; the

[2] J. Hadamard [6, p. 174].

ratio $R_n = e^{G_n - G_{n-1}}$ is then called the *rectified ratio*[3] of $|a_{n-1}|$ to $|a_n|$.

We can now complete the statement of Lemma 1 as follows:

(iv) *If $f(z)$ is an integral function of finite order, and $\tau > 0$, a necessary and sufficient condition for the convergence of the integral* (6) *is that the series*

$$\sum_{n=1}^{\infty} \frac{1}{R_n^{\tau}} \tag{8}$$

be convergent.

(v) *If $f(z)$ is an integral function whose order ρ is not an integer, a necessary and sufficient condition for $f(z)$ to be of convergent class, that is, for the series* (5) *to be convergent when $\tau = \rho$, is that the series*

$$\sum_{n=1}^{\infty} \frac{1}{R_n^{\rho}} \tag{8$'$}$$

be convergent.

Proof. See Valiron [14, pp. 258–265].

REMARK. If the coefficients of the integral function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy the condition

$$\frac{|a_1|}{|a_0|} \geqq \frac{|a_2|}{|a_1|} \geqq \cdots, \tag{9}$$

the rectified ratio of $|a_{n-1}|$ to $|a_n|$ is given by

$$R_n = e^{g_n - g_{n-1}} = \frac{|a_{n-1}|}{|a_n|}.$$

For, if (9) holds, we have

$$\log \frac{|a_{m-1}|}{|a_{m-2}|} \geqq \log \frac{|a_m|}{|a_{m-1}|},$$

whence

$$g_m - g_{m-1} \geqq g_{m-1} - g_{m-2},$$

that is, the slope of the joins of successive points A_n increases with n. It follows without difficulty that all the points A_n actually lie on the Newton polygon, whence the result follows.

LEMMA 2. *If $f(z)$ is an integral function of genus zero, and $0 < \tau < 1$, a necessary and sufficient condition for the convergence of the series* (5) *is the convergence of the integral* (6).

[3] G. Valiron [15, p. 30].

Sufficiency follows at once from Lemma 1, (i) and (ii).

If (5) is convergent, the exponent of convergence ρ of $f(z)$ satisfies $\rho \leqq \tau < 1$. Since the genus of $f(z)$ is 0, $f(z)$ is a canonical product, and we have

$$f(z) = \prod_{h=1}^{\infty}\left(1 - \frac{z}{z_h}\right).$$

By Borel's inequality[4],

$$\log M(r; f) \leqq Ar \int_{r_1}^{\infty} \frac{n(x)dx}{x(x + r)},$$

where A is a positive constant; hence

$$\int_{\alpha}^{r} \frac{\log M(x; f)}{x^{1+\tau}} dx \leqq A \int_{\alpha}^{r} \frac{dy}{r^{\tau}} \int_{r_1}^{\infty} \frac{n(x)dx}{x(x + y)}. \tag{10}$$

Since the integrands on the right-hand side in (10) are non-negative, we can invert the order of integration; putting $y = tx$, we then obtain

$$\int_{\alpha}^{r} \frac{\log M(x)dx}{x^{1+\tau}} \leqq A \int_{r_1}^{\infty} \frac{n(x)U(x)dx}{x^{1+\tau}},$$

where

$$U(x) = \int_{\alpha/x}^{r/x} \frac{dt}{t^{\tau}(1 + t)} < \int_{0}^{\infty} \frac{dt}{t^{\tau}(1 + t)} = \frac{\pi}{\sin \tau\pi} < \infty.$$

Since (5) is convergent, we can use Lemma 1 (i); hence

$$\int_{\alpha}^{r} \frac{\log M(x)dx}{x^{1+\tau}} \leqq \frac{A\pi}{\sin \tau\pi} \int_{r_1}^{\infty} \frac{n(x)dx}{x^{1+\tau}},$$

a finite number independent of r. The integral (6) is therefore convergent.

COROLLARY. *If $f(z)$ is an integral function of genus zero, and $0 < \tau < 1$, the series* (5) *converges if and only if the series* (8) *converges.*

Proof. This is proved by Lemma 1 (iv) and Lemma 2.

LEMMA 3. *Let $K(x, y)$ be a real L^2 kernel, let*

$$K^2(x, y) = \int_a^b K(x, s)K(s, y)ds, \qquad KK'(x, y) = \int_a^b K(x, s)K(y, s)ds,$$

and let $D_{K^2}(\lambda)$ and $D_{KK'}(\lambda)$ denote the Fredholm determinants of $K^2(x, y)$ and $KK'(x, y)$, respectively. Write

(4) G. Valiron [15, p. 53].

$$M(r; K^2) = \max_{|\lambda|=r} |D_{K^2}(\lambda)|, \qquad M(r; KK') = \max_{|\lambda|=r} |D_{KK'}(\lambda)|.$$

Then

$$M(r; K^2) \leqq M(r; KK').$$

Proof. See Hille and Tamarkin [8, p. 36, Lemma 6.7].

LEMMA 4. *Suppose that*

$$x^n + c_1 x^{n-1} + \cdots + c_n = \prod_{h=1}^{n} (x + a_h)$$

where $a_1, a_2, \cdots, a_n$ are positive numbers. Then

$$c_{r-1} \cdot c_{r+1} \leqq c_r^2. \tag{11}$$

Proof. See Hardy, Littlewood, and Pólya [7, pp. 51–55].

COROLLARY 1. *If $a_1, a_2, \cdots, a_n$ are positive, and $\sum a_{h_1} a_{h_2} \cdots a_{h_r}$ is the elementary symmetric function of order r of $a_1, a_2, \cdots, a_n$, then*

$$\left(\sum a_{h_1} a_{h_2} \cdots a_{h_{r-1}}\right)\left(\sum a_{h_1} a_{h_2} \cdots a_{h_{r+1}}\right) \leqq \left(\sum a_{h_1} a_{h_2} \cdots a_{h_r}\right)^2,$$

$$\left(\sum \frac{1}{a_{h_1} a_{h_2} \cdots a_{h_{r-1}}}\right)\left(\sum \frac{1}{a_{h_1} a_{h_2} \cdots a_{h_{r+1}}}\right) \leqq \left(\sum \frac{1}{a_{h_1} a_{h_2} \cdots a_{h_r}}\right)^2.$$

COROLLARY 2. *If $a_1, a_2, \cdots, a_n$ are positive, and*

$$f(x) = 1 - p_1 x + p_2 x^2 - \cdots + (-1)^n p_n x^n = \prod_{i=1}^{n} \left(1 - \frac{x}{a_i}\right),$$

then

$$p_{r-1} p_{r+1} - p_r^2 \leqq 0. \tag{11$'$}$$

LEMMA 5. *Let $K(x, y)$ be a real L^2 kernel with infinitely many singular values $\lambda_h[K]$ $(h = 1, 2, \cdots)$, and let*

$$D_{KK'}(\lambda) = \sum_{n=0}^{\infty} (-1)^n c_n \lambda^n$$

be the Fredholm determinant of the symmetric kernel

$$KK'(x, y) = \int_a^b K(x, s) K(y, s) ds.$$

Then the coefficients c_n are all positive, and satisfy the inequality

$$c_m c_{m-2} \leqq c_{m-1}^2 \qquad (m = 2, 3, \cdots). \tag{12}$$

It is known(5) that each singular value λ_h is real and that

$$D_{KK'}(\lambda) = \prod_{h=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_h^2}\right). \tag{13}$$

Hence, for every positive integer r,

$$c_r = \sum_{h_1, h_2, \cdots, h_r} \frac{1}{\lambda_{h_1}^2 \lambda_{h_2}^2 \cdots \lambda_{h_r}^2} > 0.$$

To prove (12), we consider the function

$$f_N(\lambda) = \prod_{h=1}^{N}\left(1 - \frac{\lambda}{\lambda_h^2}\right) = 1 - A_1\lambda + A_2\lambda^2 - \cdots + (-1)^N A_N \lambda^N.$$

By Lemma 4, Corollary 2,

$$A_m A_{m-2} \leqq A_{m-1}^2 \qquad (m = 2, 3, \cdots),$$

where we write $A_0 = 1$, and $A_m = 0$ when $m > N$. Since the infinite product (13) is uniformly convergent in any bounded portion of the λ-plane, $A_m \rightarrow c_m$ when $N \rightarrow \infty$; hence,

$$c_m c_{m-2} \leqq c_{m-1}^2 \qquad (m = 2, 3, \cdots),$$

the required result.

1.2. **Proof of Theorem 1.** When $\tau \geqq 2$, the series (1) and (2) are both known to be convergent. We therefore need consider only the case when $0 < \tau < 2$.

By the set of characteristic values $\{\mu_h[K]\}$ of $K(x, y)$ we mean the zeros, repeated according to their multiplicities, of the Fredholm determinant $D_K(\lambda)$ of $K(x, y)$, arranged so that $|\mu_1| \leqq |\mu_2| \leqq \cdots$.

By a known result(6),

$$\mu_h[K^2] = (\mu_h[K])^2 \qquad (h = 1, 2, \cdots).$$

Also, if $\{\lambda_h[K]\}$ is the set of singular values of $K(x, y)$, arranged in the same way, then

$$(\lambda_h[K])^2 = \mu_h[KK'].$$

Now suppose that (1) is convergent, where $0 < \tau < 2$, and write $t = \tau/2$; then

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[KK']|^t} < +\infty,$$

(5) G. Vivanti [16, pp. 192–193] or Hille and Tamarkin [8, p. 29].

(6) Hille and Tamarkin [8, p. 37].

where $0<t<1$. Hence, by Lemma 2,

$$\int_\alpha^\infty \frac{\log M(r; KK')}{r^{1+t}} dr < +\infty,$$

and so, by Lemma 3,

$$\int_\alpha^\infty \frac{\log M(r; K^2)}{r^{1+t}} dr < +\infty.$$

Applying Lemma 2 again, we have

$$\sum_{h=1}^\infty \frac{1}{|\mu_h[K^2]|^t} < +\infty,$$

that is,

$$\sum_{h=1}^\infty \frac{1}{|\mu_h[K]|^\tau} < +\infty,$$

the required result.

1.3. **Proof of Theorem 2.** Consider the kernel

$$K(x, y) \sim \sum_{h=1}^\infty \frac{\cos 2hx\cdot\cos 2hy}{4h^2} + \sum_{h=1}^\infty \frac{\cos(2h+1)x\cdot\sin(2h+1)y}{2h+1},$$

where the symbol $\sim$ indicates that the series on the right are convergent in mean (with index 2). Evidently $K(x, y)$ is an L^2 kernel; we write it in the form

$$K(x, y) = A(x, y) + B(x, y),$$

where

$$A(x, y) \sim \sum_{h=1}^\infty \frac{\cos 2hx\cdot\cos 2hy}{4h^2},$$

$$B(x, y) \sim \sum_{h=1}^\infty \frac{\cos(2h+1)x\cdot\sin(2h+1)y}{2h+1}.$$

Then $AB(x, y)=0$, $BA(x, y)=0$, and $B^2(x, y)=0$. It follows that $B(x, y)$ has no characteristic values, and that

$$\mu_h[K] = \mu_h[A] = 4h^2 \qquad (h = 1, 2, \cdots).$$

Hence

$$\sum_{h=1}^\infty \frac{1}{|\mu_h[K]|} = \sum_{h=1}^\infty \frac{1}{4h^2} < +\infty.$$

On the other hand, the set of singular values is

$$3,\ 2^2,\ 5,\ 7,\ \cdots,\ 15,\ 4^2,\ 17,\ 19,\ \cdots,\ 35,\ 6^2,\ 37,\ \cdots$$

and the series $\sum_{h=1}^{\infty} 1/|\lambda_h[K]|$ is clearly divergent. Our theorem is therefore proved.

1.4. **Proof of Theorem 3.** By Lemma 5, $c_n>0$ for all n, and the sequence $\{c_n/c_{n-1}\}$ is monotone decreasing; hence by the remark to Lemma 1, the rectified ratio of $|c_{n-1}|$ to $|c_n|$ is

$$R_n = \left|\frac{c_{n-1}}{c_n}\right| = \frac{c_{n-1}}{c_n}.$$

Now, by (13), $D_{KK'}(\lambda)$ is of genus zero; consequently, by the corollary to Lemma 2, the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{R_n^{\rho/2}} = \sum_{n=1}^{\infty}\left(\frac{c_n}{c_{n-1}}\right)^{\rho/2}$$

is a necessary and sufficient condition for the convergence of the series

$$\sum_{h=1}^{\infty} \frac{1}{|\mu_h[KK']|^{\rho/2}} = \sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{\rho}},$$

provided that $0<\rho/2<1$, that is, $0<\rho<2$.

By Carleman's inequality(7),

$$\sum_{n=1}^{\infty}\left(\frac{c_n}{c_0}\right)^{\rho/2n} < e\sum_{n=1}^{\infty}\left(\frac{c_n}{c_{n-1}}\right)^{\rho/2};$$

on the other hand, since $\{c_n/c_{n-1}\}$ is a decreasing sequence,

$$\left(\frac{c_1}{c_0}\cdot\frac{c_2}{c_1}\cdot\ \cdots\ \cdot\frac{c_n}{c_{n-1}}\right)^{\rho/2n} \geqq \left(\frac{c_n}{c_{n-1}}\right)^{\rho/2},$$

that is, since $c_0=1$,

$$c_n^{\rho/2n} \geqq \left(\frac{c_n}{c_{n-1}}\right)^{\rho/2}.$$

The series (3) and

$$\sum_{n=1}^{\infty}\left(\frac{c_n}{c_{n-1}}\right)^{\rho/2} \tag{14}$$

therefore converge or diverge together. Thus (1′) converges if and only if (3) converges; this completes the proof.

(7) Hardy, Littlewood, and Pólya [7, p. 249].

PART II. COMPOSITE KERNELS

2.1. Preliminary lemmas.

LEMMA 1. *Suppose that* $K(x, y) = K_1K_2(x, y)$, *where* K_1 *and* K_2 *are* L^2 *kernels, and* $D_{KK'}(\lambda) = \sum_{n=0}^{\infty} (-1)^n c_n \lambda^n$ *be the Fredholm determinant of* $KK'(x, y)$. *Then the series*

$$\sum_{n=1}^{\infty} |c_n|^{1/2n}, \tag{15}$$

$$\sum_{n=1}^{\infty} \left(\frac{c_n}{c_{n-1}}\right)^{1/2}, \tag{16}$$

are both convergent.

The series $\sum_{h=1}^{\infty} 1/|\lambda_h[K_1K_2]|$ is known to be convergent(8); the convergence of (15) then follows from Theorem 3, and the convergence of (16) from the fact that (3) and (14) converge or diverge together.

LEMMA 2. *Let* P_i *denote the vector*

$$P_i = P_i^{(n)} = (s_1^{(i)}, s_2^{(i)}, \cdots, s_n^{(i)}) \qquad (i = 1, 2, \cdots, m),$$

and D_i *the domain defined by*

$$a \leqq s_j^{(i)} \leqq b \qquad (j = 1, 2, \cdots, n).$$

Write $dP_i = ds_1^{(i)} ds_2^{(i)} \cdots ds_n^{(i)}$. *Let* $K_i(P_i, P_{i+1}) \in L^2(P_i, P_{i+1})$ $(i = 1, 2, \cdots, m)$, *that is,*

$$\int_{D_i}\int_{D_{i+1}} |K_i(P_i, P_{i+1})|^2 dP_i dP_{i+1} = \|K_i(P_i, P_{i+1})\|^2 < \infty,$$

where $P_{m+1} = P_1$ *and* $D_{m+1} = D_1$; *then*

$$\begin{aligned}\int_{D_1}\int_{D_2} \cdots \int_{D_m} &|K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)| \\ &\cdots |K_m(P_m, P_1)|\, dP_1 dP_2 \cdots dP_m \leqq \prod_{i=1}^{m} \|K_i(P_i, P_{i+1})\|.\end{aligned} \tag{17}$$

We prove the result for $m = 3$; a similar proof holds for larger values of m. By Schwarz's inequality,

$$\int_{D_2} |K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)|\, dP_2 \leqq F(P_1)G(P_3),$$

(8) S. H. Chang [3, pp. 185–189].

where

$$F^2(P_1) = \int_{D_2} |K_1(P_1, P_2)|^2 dP_2,$$

$$G^2(P_3) = \int_{D_2} |K_2(P_2, P_3)|^2 dP_2.$$

Hence

$$\int_{D_2}\int_{D_3} |K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)| \cdot |K_3(P_3, P_1)| \, dP_2 dP_3$$

$$\leqq \int_{D_3} F(P_1)\cdot G(P_3) |K_3(P_3, P_1)| \, dP_3$$

$$\leqq F(P_1)\left\{\int_{D_3} |G(P_3)|^2 dP_3\right\}^{1/2} \cdot \left\{\int_{D_3} |K_3(P_3, P_1)|^2 dP_3\right\}^{1/2}$$

$$= F(P_1)\cdot \|K_2(P_2, P_3)\| \left\{\int_{D_3} |K_3(P_3, P_1)|^2 dP_3\right\}^{1/2}.$$

Integrating both sides with respect to P over the domain D, we obtain

$$\int_{D_1}\int_{D_2}\int_{D_3} |K_1(P_1, P_2)| \cdot |K_2(P_2, P_3)| \cdot |K_3(P_3, P_1)| \, dP_1 dP_2 dP_3$$

$$\leqq \|K_2(P_2, P_3)\| \int_{D_1} F(P_1)\left\{\int_{D_3} |K_3(P_3, P_1)|^2 dP_3\right\}^{1/2} dP_1$$

$$\leqq \|K_2(P_2, P_3)\| \cdot \|K_3(P_3, P_1)\| \left\{\int_{D_1} |F(P_1)|^2 dP_1\right\}^{1/2}$$

$$= \|K_1(P_1, P_2)\| \cdot \|K_2(P_2, P_3)\| \cdot \|K_3(P_3, P_1)\|,$$

the required result.

LEMMA 3 (GENERALIZED CARLEMAN THEOREM). *If $m>1$, and each of the functions $K_1(x, y)$, $K_2(x, y)$, $\cdots$, $K_m(x, y)$ is a real L^2 kernel, the Fredholm determinant of the composite kernel*

$$K(x, y) = K_1 K_2 \cdots K_m(x, y)$$

is

$$(18)\quad D_K(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n!)^m} \int_{D_1}\int_{D_2} \cdots \int_{D_m} K_1(P_1^{(n)}, P_2^{(n)}) K_2(P_2^{(n)}, P_3^{(n)}) \cdots K_m(P_m^{(n)}, P_1^{(n)}) \cdot dP_1^{(n)} dP_2^{(n)} \cdots dP_m^{(n)},$$

where

$$K_i(P_i^{(n)}, P_{i+1}^{(n)}) = \begin{vmatrix} K_i(s_1^{(i)}, s_1^{(i+1)}), \cdots, K_i(s_1^{(i)}, s_n^{(i+1)}) \\ \cdots\cdots\cdots\cdots\cdots \\ K_i(s_n^{(i)}, s_1^{(i+1)}), \cdots, K_i(s_n^{(i)}, s_n^{(i+1)}) \end{vmatrix} \tag{19}$$

$$(i = 1, 2, \cdots, m),$$

and $P_{m+1}^{(n)} = P_1^{(n)} = (s_1^{(1)}, s_2^{(1)}, \cdots, s_n^{(1)})$, and so on.

Carleman(9) has proved the formula (18) in the case when $K_1, \cdots, K_m$ are all bounded in $a \leqq x \leqq b$, $a \leqq y \leqq b$. In the general case, we note that $K(x, y) = K_1K_2 \cdots K_m(x, y)$ is of the form $K(x, y) = AB(x, y)$, where $A(x, y)$ and $B(x, y)$ are both L^2 kernels; hence, by Schwarz's inequality,

$$|AB(x, x)| \leqq \left\{\int_a^b |A(x, y)|^2 dy\right\}^{1/2} \left\{\int_a^b |B(y, x)|^2 dy\right\}^{1/2},$$

so that

$$\int_a^b |AB(x, x)| \, dx \leqq \|A(x, y)\| \cdot \|B(y, x)\| < +\infty.$$

Hence, by a known result(10), $D_K(\lambda)$ exists and is given by (18).

COROLLARY. *If $K(x, y)$ is a real L^2 kernel, the Fredholm determinant of $KK'(x, y)$ is*

$$D_{KK'}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n!)^2} \int_{D_1}\int_{D_2} |K(P_1^{(n)}, P_2^{(n)})|^2 dP_1^{(n)} dP_2^{(n)}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{(n!)^2} \|K(P_1^{(n)}, P_2^{(n)})\|^2.$$

In particular, the integral

$$\|K(P_1^{(n)}, P_2^{(n)})\|^2 = \int_{D_1}\int_{D_2} |K(P_1^{(n)}, P_2^{(n)})|^2 dP_1^{(n)} dP_2^{(n)}$$

always exists.

2.2. **Proof of Theorem 4.** The Fredholm determinant of

$$KK'(x, y) = K_1K_2 \cdots K_m(K_1K_2 \cdots K_m)'(x, y)$$
$$= (K_1K_2)(K_3K_4) \cdots (K_4'K_3')(K_2'K_1')(x, y)$$
$$= A_1A_2 \cdots A_m(x, y),$$

say, is given by

(9) T. Carleman [1, p. 213].
(10) T. Carleman [1, p. 198].

$$D_{KK'}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n\lambda^n}{(n!)^m} \int_{D_1}\int_{D_2} \cdots \int_{D_m} A_1(P_1^{(n)}, P_2^{(n)})A_2(P_2^{(n)}, P_3^{(n)})$$

$$\cdots A_m(P_m^{(n)}, P_1^{(n)})dP_1^{(n)}dP_2^{(n)} \cdots dP_m^{(n)}$$

$$= \sum_{n=0}^{\infty} (-1)^n\lambda^n c_n(A_1A_2 \cdots A_m),$$

say, where $A_i(P_i^{(n)}, P_{i+1}^{(n)})$ is defined as in (19).

By Lemma 2 we have

$$|c_n(KK')| = |c_n(A_1A_2 \cdots A_m)| \leqq \frac{1}{(n!)^m} \prod_{i=1}^{m} \|A_i(P_i^{(n)}, P_{i+1}^{(n)})\|$$

$$= \prod_{i=1}^{m} \left\{ \frac{1}{(n!)^2} \|A_i(P_i^{(n)}, P_{i+1}^{(n)})\|^2 \right\}^{1/2}$$

$$= \prod_{i=1}^{m} |c_n(A_iA_i')|^{1/2},$$

where we are using, in general, $c_n(K)$ to denote the coefficient of $(-1)^n\lambda^n$ in the power series expansion of the Fredholm determinant $D_K(\lambda)$ of $K(x, y)$. Hence

$$\sum_{n=1}^{\infty} |c_n(KK')|^{1/mn} \leqq \sum_{n=1}^{\infty} \prod_{i=1}^{m} |c_n(A_iA_i')|^{1/2mn}.$$

Consequently, by a well known inequality[11],

$$\sum_{n=1}^{\infty} |c_n(KK')|^{1/mn} \leqq \prod_{i=1}^{m} \left\{ \sum_{n=1}^{\infty} |c_n(A_iA_i')|^{1/2n} \right\}^{1/m}.$$

Now, by Lemma 1,

$$\sum_{n=1}^{\infty} |c_n(A_iA_i')|^{1/2n} < +\infty \qquad (i = 1, 2, \cdots, m).$$

Consequently,

$$\sum_{n=1}^{\infty} |c_n(KK')|^{1/mn} < +\infty,$$

and therefore, by Theorem 3,

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < +\infty,$$

(11) Hardy, Littlewood, and Pólya [7, p. 22].

as we wished to prove.

2.3. **Proof of Theorem 5.** This now follows at once from Theorem 4 and Theorem 1 with $\tau = 2/m$.

2.4. **Proof of Theorem 6.** It is known[12] that if $Q(x, y)$ is a continuous semidefinite symmetric kernel, then there exists a symmetric L^2 kernel $A(x, y)$ such that

$$Q(x, y) = A^2(x, y) = \int_a^b A(x, s)A(s, y)ds.$$

Hence any Pell kernel or, more generally, any kernel of the form $K(x, y) = QL(x, y)$, where $Q(x, y)$ has the above properties, can be expressed in the the form $K(x, y) = A^2L(x, y)$. The result now follows by taking $m = 3$ in Theorems 4 and 5.

Appendix

We recall[13] that a necessary and sufficient condition for a real L^2 kernel $K(x, y)$ to have a canonical decomposition into m factors is that

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[K]|^{2/m}} < +\infty.$$

We therefore have:

Theorem 7. *If a real L^2 kernel $K(x, y)$ has a decomposition $K(x, y) = K_1K_2 \cdots K_m(x, y)$ into m L^2 factors, then it has a canonical decomposition into m factors.*

Many results can also be proved showing that the smoother a kernel $K(x, y)$ is, the greater is the number of factors into which it can be decomposed. For instance, we have:

Theorem 8. *If $K(x, y)$ is a real symmetric kernel, continuous in $a \leqq x \leqq b$, $a \leqq y \leqq b$, and*

$$\frac{\partial^r K(x, y)}{\partial x^r}$$

is continuous in the same square, then $K(x, y)$ has a decomposition into at least $2r$ factors, so that we can write

$$K(x, y) = K_1K_2 \cdots K_{2r}(x, y).$$

For, by Weyl's theorem[14],

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(12) S. H. Chang [3, p. 189, Corollary 4.]
(13) S. H. Chang [3].
(14) H. Weyl [17].

$$\lim_{n\to\infty} \frac{n^{r+1/2}}{|\lambda_n[K]|} = 0.$$

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n[K]|^{2/m}}$$

is thus convergent provided that $(2/m)(r+1/2)>1$, that is, $m<2r+1$. We can therefore take $m=2r$, and the result follows.

All the above results can be extended to complex-valued kernels if we define singular values and singular functions by the equations

$$\phi_h(x) = \lambda_h \int_a^b K(x, y)\psi_h(y)dy, \qquad \psi_h(x) = \lambda_h \int_a^b \overline{K(y, x)}\phi_h(y)dy,$$

replace the kernel $K'(x, y) = K(y, x)$ by the kernel $K^*(x, y) = \overline{K(y, x)}$, and use Hermitian kernels instead of real symmetric kernels.

The proof of Theorem 4 could also be carried out by using Hille and Tamarkin's formulae[15]:

$$D_{K_1\cdots K_m}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n\lambda^n}{(n!)^m} \sum_{(i_1,\cdots,i_m;n)} \overset{(n)}{\Delta}_{i_1i_2}(K_1)\overset{(n)}{\Delta}_{i_2i_3}(K_2)\cdots\overset{(n)}{\Delta}_{i_mi_1}(K_m),$$

$$D_{KK^*}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n\lambda^n}{(n!)^2} \sum_{(i,j;n)} \left|\overset{(n)}{\Delta}_{ij}(K)\right|^2,$$

where

$$\overset{(n)}{\Delta}_{ij}(K) = \begin{vmatrix} k_{i_1j_1}(K), & \cdots, & k_{i_1j_n}(K) \\ \cdot & \cdots & \cdot \\ k_{i_nj_1}(K), & \cdots, & k_{i_nj_n}(K) \end{vmatrix},$$

$$k_{ij}(K) = \int_a^b \cdot \int_a^b K(s, t)\overline{u_i(s)}u_j(t)dsdt,$$

$\{u_i(x)\}$ being an arbitrary complete orthonormal set of functions for the interval (a, b). We then use the inequality for series corresponding to (17).

This allows us to replace the hypothesis of Theorem 6 by the condition that $Q(x, y)$ is an Hermitian semi-definite L^2 kernel such that

$$\sum_{i=1}^{\infty} |k_{ii}(Q)| = \sum_{i=1}^{\infty} \left| \int_a^b \int_a^b Q(s, t)\overline{u_i(s)}u_i(t)dsdt \right| < +\infty.$$

For the series $\sum_{h=1}^{\infty} 1/|\lambda_h[Q]|$ is then still convergent[16], and therefore $Q(x, y)$ is still expressible in the form $Q(x, y) = A^2(x, y)$.

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[15] Hille and Tamarkin [8, p. 33, Lemmas 6.2 and 6.3].

[16] Hille and Tamarkin [8, p. 29, Theorem 5.1].

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