ON THE SEMI-CONTINUITY OF DOUBLE INTEGRALS

BY

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Introduction

0. In a paper published in 1942, Radó has extended the scope of known results on the semi-continuity of double integrals in parametric form to a certain degree of finality (Radó [2])(1). Scott significantly improved one of the key lemmas in Radó's work, which permits the various results of Radó to be included as special cases of a more general theorem (Scott, Theorem). The basic class of surfaces in their work is the class $o\mathfrak{C}$ of oriented continuous surfaces oS (see §18) which possess representations (T, B) for which the following conditions are satisfied.

0.1. The ordinary jacobians for the three plane projections of the representation (T, B) exist almost everywhere and are summable in the interior B^0 of B.

0.2. The lebesgue area of oS is given by the classical formula—that is, by the lebesgue integral over B^0 of the square root of the sum of the squares of the three jacobians.

Recently Cesari has established a result which implies that the class $o\mathfrak{C}$ includes all oriented continuous surfaces having finite lebesgue area (Cesari [1, 2]). It is the purpose of this note to replace in the theory of Radó and Scott the ordinary jacobians by the essential generalized jacobians (see §5). By doing this, the need for an assumption like 0.1 is eliminated—if the lebesgue area of an oriented continuous surface is finite then for any representation (T, B) of that surface the three essential generalized jacobians exist almost everywhere and are summable in B^0 (see §5). The basic class of surfaces in the present work is the class of all oriented continuous surfaces having essentially absolutely continuous representations. By the result of Cesari just mentioned, the class of oriented continuous surfaces to which this theory applies is identical with the class $o\mathfrak{C}$ used by Radó and Scott. However, every representation which satisfies conditions 0.1 and 0.2 is essentially absolutely continuous, but the converse is not true. Thus the present theory will apply to a wider class of representations for the oriented continuous surfaces than any of the earlier theories, and include all the results in those theories (see §§18, 20). The treatment in the present paper follows the pattern set in Radó [2]. To assist the reader in following the reasoning, the notations of that paper are generally adopted. New facts necessary to build the present theory are set forth in the following sections.

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

(Added in March 1949.) During the war years, L. Cesari in Italy developed independently a theory for the area of continuous surfaces which parallels that of Radó and his students in this country. After this note was submitted for publication, the attention of the writer was called to a paper by Cesari in which he defines an integral of the Weierstrass type for an admissible function over a surface given in parametric representation (Cesari [3]). For eAC representations his integral can be shown to be equal to the one defined in this paper. However, the methods used by Cesari differ greatly from those developed in this note, and he does not discuss the lower semicontinuity properties of his integral.

Preliminaries

1. There is needed the following result which is an easy consequence of standard theorems on the lebesgue integral (McShane [3]). Let S be a measurable set in euclidean space. Suppose that f_n, $n=0, 1, 2, \cdots$, is a sequence of functions each defined almost everywhere in S, measurable, and summable in S. For each n let E_n^+ denote the set of points of S where f_n is defined and positive, let E_n^- denote the set of points of S where f_n is defined and negative. Then E_n^+ and E_n^- are disjoint measurable sets for every n. A necessary and sufficient condition that there exist a sequence of measurable subsets V_n of S such that $\lim_{n\to\infty} \int_{V_n} f_n = \int_S f_0$ is that both of the following relations hold:

$$\liminf_{n\to\infty} \int_{E_n^+} f_n \geqq \int_S f_0 \quad \text{and} \quad \limsup_{n\to\infty} \int_{E_n^-} f_n \leqq \int_S f_0.$$

Of course, one of these relations is always trivial. If $\int_S f_0$ is non-negative then the second relation is obvious, and if the first relation holds, the measurable sets V_n may be chosen as subsets of E_n^+. An analogous statement may be made when $\int_S f_0$ is non-positive.

2. Lemma. *Given a finitely connected jordan region* $\mathfrak{R}$ *in the* $u=(u^1, u^2)$-*plane, let* T_n: $\xi=\xi_n(u)$, $u\in\mathfrak{R}$, *for* $n=0, 1, 2, \cdots$ *be a sequence of continuous transformations* (Radó [3, IV.1.2]) *satisfying the following conditions*: (i) *the distance* (Radó [3, IV.1.1]) $\rho(T_n, T_0, \mathfrak{R})$ *of* T_0 *and* T_n *on* $\mathfrak{R}$ *converges to zero as* n *tends to infinity*; (ii) T_0 *is eBV in* $\mathfrak{R}^0$ (Radó [3, IV.4.1]); (iii) T_n *is eAC in* $\mathfrak{R}^0$ *for* $n=1, 2, \cdots$. *Then the essential generalized jacobian* $\mathcal{J}_e(u, T_n)$ *exists almost everywhere in* $\mathfrak{R}^0$ *and is summable in* $\mathfrak{R}^0$ *for* $n=0, 1, 2, \cdots$ (Radó [3, IV.3.21, 3.25, 3.13]); *suppose that*

$$\iint_{\mathfrak{R}^0} \mathcal{J}_e(u, T_0) \geqq 0. \tag{2.1}$$

Then there exists a measurable subset V_n *of the set* $E^+(T_n, \mathfrak{R}^0)$ *of points in* $\mathfrak{R}^0$ *where* $\mathcal{J}_e(u, T_n)$ *exists and is positive for* $n=1, 2, \cdots$ *such that*

$$\lim_{n\to\infty} \iint_{V_n} \mathcal{J}_e(u, T_n) = \iint_{\mathfrak{R}^0} \mathcal{J}_e(u, T_0).$$

An analogous conclusion is valid if the opposite inequality holds in (2.1).

Proof. In view of the general theorem stated in §1, it is clearly sufficient to show that

$$\liminf_{n\to\infty} \iint_{E^+(T_n,\mathfrak{R}^0)} \mathcal{J}_e(u, T_n) \geqq \iint_{\mathfrak{R}^0} \mathcal{J}_e(u, T_0). \tag{2.2}$$

Obviously

$$\iint_{E^+(T_n,\mathfrak{R}^0)} \mathcal{J}_e(u, T_n) = \iint_{\mathfrak{R}^0} [\mathcal{J}_e(u, T_n) + | \mathcal{J}_e(u, T_n) |]/2 \quad \text{for } n = 1, 2, \cdots.$$

From Reichelderfer §§22–25 and condition (iii) it follows that

$$\iint_{\mathfrak{R}^0} [\mathcal{J}_e(u, T_n) + | \mathcal{J}_e(u, T_n) |]/2 = \iint \kappa^+(\xi, T_n, \mathfrak{R}^0) \qquad \text{for } n = 1, 2, \cdots.$$

In view of conditions (i) and (ii) it follows from Reichelderfer §§16, 22, 23 and the lemma of Fatou that

$$\liminf_{n\to\infty} \iint \kappa^+(\xi, T_n, R^0) \geqq \iint \kappa^+(\xi, T_0, R^0) \geqq \iint_{\mathfrak{R}^0} \mathcal{J}_e(u, T_0).$$

The above relations clearly yield (2.2), and the lemma is established.

3. Lemma. *Let* T_n: $\xi = \xi_n(u)$, $u \in \mathfrak{D}_n$ *for* $n = 0, 1, 2, \cdots$, *be a sequence of bounded continuous transformations* (Radó [3, IV.1.42]) *satisfying the following conditions*: (i) *if F is any closed set in $\mathfrak{D}_0$ then F is a subset of $\mathfrak{D}_n$ for all n sufficiently large and the distance $\rho(T_n, T_0, F)$ between T_n and T_0 on F converges to zero as n tends to infinity*; (ii) *T_0 is eBV in $\mathfrak{D}_0$*; (iii) *T_n is eAC in $\mathfrak{D}_n$ for* $n = 1, 2, \cdots$. *If*

$$\iint_{\mathfrak{D}_0} \mathcal{J}_e(u, T_0) \geqq 0 \tag{3.1}$$

then there exists a measurable subset V_n of the set $E^+(T_n, \mathfrak{D}_n)\mathfrak{D}_0$ for $n = 1, 2, \cdots$ such that

$$\lim_{n\to\infty} \iint_{V_n} \mathcal{J}_e(u, T_n) = \iint_{\mathfrak{D}_0} \mathcal{J}_e(u, T_0).$$

An analogous conclusion is valid if the opposite inequality holds in (3.1).

Proof. If the equality sign holds in (3.1) the conclusion is obvious, since each V_n may be taken to be empty. So assume that the inequality sign holds in (3.1). Let ϵ be any positive number smaller than the integral in (3.1). Then there exists a jordan region $\mathfrak{R}$ in $\mathcal{D}_0$ such that

$$\left| \iint_{\mathfrak{R}^0} \mathcal{J}_e(u, T_0) - \iint_{\mathcal{D}_0} \mathcal{J}_e(u, T_0) \right| < \epsilon.$$

From condition (i) it follows that there exists an n_ϵ such that $\mathfrak{R}$ is a subset of $\mathcal{D}_n$ for all n exceeding n_ϵ. The continuous transformations $T_n: \xi = \xi_n(u)$, $u \in \mathfrak{R}$ for $n=0$, $n > n_\epsilon$, clearly satisfy all the hypotheses of the lemma in §2 (Radó [3, IV.4.1]). Hence there are measurable subsets $V_{n\epsilon}$ of the sets $E^+(T_n, \mathfrak{R}^0) \subset E^+(T_n, \mathcal{D}_n)\mathcal{D}_0$ for $n > n_\epsilon$, such that

$$\lim_{n\to\infty} \iint_{V_{n\epsilon}} \mathcal{J}_e(u, T_n) = \iint_{\mathfrak{R}^0} \mathcal{J}_e(u, T_0).$$

By the diagonal process one may obtain a sequence of measurable sets V_n as described in the lemma.

4. The above lemma offers a partial solution to the problem proposed in Radó [3, IV.4.42]. There it is shown that if a sequence of continuous transformations $T_n: \xi = \xi_n(u)$, $u \in \mathcal{D}_n$, for $n = 0, 1, 2, \cdots$ satisfies the following conditions: (i) condition 3 (i); (ii) the ordinary jacobian $J(u, T_0)$ exists almost everywhere in $\mathcal{D}_0$ and is summable in $\mathcal{D}_0$; (iii) condition 3 (iii); then there is a sequence of measurable subsets V_n in $\mathcal{D}_n$ such that

$$\lim_{n\to\infty} \iint_{V_n} \mathcal{J}_e(u, T_n) = \iint_{\mathcal{D}_0} J(u, T_0).$$

Radó remarks that it would be very desirable to replace in his result the ordinary jacobian $J(u, T_0)$ by the essential generalized jacobian $\mathcal{J}_e(u, T_0)$. To achieve a complete solution to the problem he proposes, condition 3 (ii) should be replaced by

(ii*) the essential generalized jacobian $\mathcal{J}_e(u, T_0)$ exists almost everywhere in $\mathcal{D}_0$ and is summable in $\mathcal{D}_0$.

This can be done. Moreover, it can be shown that the result of Radó follows as a special application of the resulting theorem. However, considerable extensions of the present results in transformation theory seem to be necessary to achieve this complete solution. Since these extensions are not needed to establish the lower semi-continuity theorems in the present note, it is felt better to defer the complete solution of the problem proposed by Radó to a later occasion.

5. Given a simply connected jordan region B in the u-plane, a triple of functions $x(u) = [x^1(u), x^2(u), x^3(u)]$ defined and continuous on B establishes a continuous mapping T from B into x-space (Radó [2, §1.6]). Suppose that (T, B) is eBV (Radó [3, V.1.1, 1.15]); then each of the essential generalized

jacobians $\mathcal{J}_e(u, T^i)$ for $i=1, 2, 3$ exists almost everywhere in B^0 and is summable in B^0. Let $X_e(u, T)$ denote the vector with components $\mathcal{J}_e(u, T^1)$, $\mathcal{J}_e(u, T^2)$, $\mathcal{J}_e(u, T^3)$ at every point u where all three essential generalized jacobians exist. Then $X_e(u, T)$ is defined almost everywhere in B^0 and $\|X_e(u, T)\|$ is summable in B^0. Suppose that $f(x, X)$ is an admissible integrand (Radó [2, §§1.1, 1.2]). Then $f[x(u), X_e(u)]$ is defined almost everywhere in B^0, is measurable and summable in B^0 by an argument analogous to that in Radó [2, §1.9]. Let $I(T, B, f)=\iint_{B^0} f[x(u), X_e(u)]du$.

6. Lemma. *Given a continuous eAC triple* (T, B) (Radó [3, V.1.15]). *Suppose that* $\bar{u}=\tau(u)$, $u\in B$, $u=\sigma(\bar{u})$, $\bar{u}\in\overline{B}$ *is a bimeasurable topological map between* B *and* $\overline{B}$ (Radó [3, IV.4.62]). *Consider the continuous triple* $\overline{T}: \bar{x}(\bar{u}) \equiv x[\sigma(\bar{u})]$, $\bar{u}\in\overline{B}$; $(\overline{T}, \overline{B})$ *is an eAC triple* (Radó [3, IV.4.65]). *If* f *is any admissible integrand it follows that* $I(T, B, f)=I(\overline{T}, \overline{B}, f)$ *if* τ *is sense-preserving, and* $I(T, B, f)=I(\overline{T}, \overline{B}, f^*)$ *where* $f^*(x, X)=f(x, -X)$ *if* τ *is sense-reversing* (Radó [2, §1.10]).

Proof. Observe that f^* is an admissible integrand (Radó [2, §1.2]). Since the topological transformation $u=\sigma(\bar{u})$, $\bar{u}\in\overline{B}$, is eAC in $\overline{B}^0$ (Radó [3, IV.4.52, IV.4.55]) its essential generalized jacobian $\mathcal{J}_e(\bar{u}, \sigma)$ exists almost everywhere in $\overline{B}^0$ and is summable in $\overline{B}^0$; moreover, it is non-negative almost everywhere in $\overline{B}^0$ if τ is sense-preserving; and non-positive almost everywhere in $\overline{B}^0$ if τ is sense-reversing. From §5 and Radó [3, IV.4.64] it follows that $X_e(\bar{u}, \overline{T}) = X_e[\sigma(\bar{u}), T]\mathcal{J}_e(\bar{u}, \sigma)$ almost everywhere in $\overline{B}^0$. There follow from Radó [3, IV.4.58] the relations

$$
\begin{aligned}
I(T, B, f) &= \iint_{B^0} f[x(u), X_e(u, T)]du \\
&= \iint_{\overline{B}^0} f[x(\sigma(\bar{u})), X_e(\sigma(\bar{u}), T)] \,|\, \mathcal{J}_e(\bar{u}, \sigma) \,|\, d\bar{u} \\
&= \iint_{\overline{B}^0} f[\bar{x}(\bar{u}), X_e(\sigma(\bar{u}), T)\cdot \,|\, \mathcal{J}_e(\bar{u}, \sigma) \,|\,]d\bar{u} \\
&= \begin{cases} \displaystyle\iint_{\overline{B}^0} f[\bar{x}(\bar{u}), X_e(\bar{u}, \overline{T})]d\bar{u} = I(\overline{T}, \overline{B}, f) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \tau \text{ is sense-preserving,} \\ \displaystyle\iint_{\overline{B}^0} f[\bar{x}(\bar{u}), -X_e(\bar{u}, \overline{T})]d\bar{u} = I(\overline{T}, \overline{B}, f^*) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \tau \text{ is sense-reversing.} \end{cases}
\end{aligned}
$$

7. Let (T, B) be any continuous triple. Then (T, B) represents a continu-

ous surface S (Radó [2, §1.21]) and an oriented continuous surface oS (Radó [2, §1.23]). An easy reasoning left for the reader shows that the lebesgue area $L(S)$ of the surface S (Radó [2, §3.13]) is equal to the lebesgue area $L(oS)$ of the oriented surface oS (Radó [2, §3.12]). Thus results entirely analogous to those for the lebesgue area of a continuous surface given in Radó [3, V] are available for the lebesgue area of an oriented continuous surface. For example, a necessary and sufficient condition that an oriented continuous surface oS have finite lebesgue area $L(oS)$ is that oS possess an eBV representation (Radó [3, V.1.16, 2.65, 4.8]). If oS has finite lebesgue area $L(oS)$, and if (T, B) is any representation for oS—which is then necessarily eBV—it is true that

$$L(oS) \geqq \iint_{B^0} \|X_e(u, T)\| du = I(T, B, \|X\|). \tag{7.1}$$

A necessary and sufficient condition that the sign of equality hold in (7.1) is that the representation (T, B) be eAC.

8. According to Radó [2, §1.8] a continuous triple (T, B) is of class K_1 if the first partial derivatives of each of its functions exist almost everywhere in B^0 and the three ordinary jacobians associated with the triple are summable in B^0. In this note, the class of triples (T, B) which are eBV has replaced the class of triples (T, B) which belong to the class K_1 used by Radó. Observe that a continuous triple (T, B) may be eBV without belonging to the class K_1; indeed, every representation (T, B) for a continuous surface having finite lebesgue area is eBV (see §7), but the ordinary jacobians for such a representation may not exist at a single point. On the other hand, a continuous triple (T, B) may belong to the class K_1 of Radó without being eBV. In fact, Youngs has shown that every continuous surface has a representation of class K_1 (see Youngs, Theorem), and thus any class K_1 representation of a continuous surface with infinite lebesgue area is not eBV.

Nevertheless, the results cited in §7 and Radó [3, V.2.64] for continuous surfaces having eBV representations clearly imply the theorems in Radó [2, §§3.14, 3.18], since a K_1 representation which is not eBV must represent a surface having infinite lebesgue area.

9. According to Radó [2, §1.19], a continuous triple (T, B) is of class K_2 if (T, B) is of class K_1 and there is a sequence of quasi-linear triples (T_n, B_n) whose oriented distances $od[(T_n, B_n), (T, B)]$ from (T, B) converge to zero for which

$$\lim_{n\to\infty} \iint_{B_n^0} \|X(u, T_n)\| du = \iint_{B^0} \|X(u, T)\| du,$$

where $X(u, T)$ is the vector whose components are the ordinary jacobians associated with (T, B). In view of the theorems in Radó [2, §§3.15, 3.19] and in Radó [3, V.2.64] it is clear that the class K_2 of Radó may be characterized as the class of all eAC triples (T, B) for which the ordinary jacobians $X(u, T)$ exist almost everywhere in B^0. However, a continuous triple (T, B) may be eAC and fail to have ordinary jacobians.

10. LEMMA. *Let* (T_n, B_n), *for* $n=0, 1, 2, \cdots$, *be a sequence of continuous triples satisfying the following conditions*: (i) *the transformations* T_n^i (Radó [3, V.1.1]) *satisfy condition* 3(i) *for* $i=1, 2, 3$; (ii) (T_0, B_0) *is eBV*; (iii) (T_n, B_n) *is eAC for* $n=1, 2, \cdots$. *If* a^1, a^2, a^3 *are three real constants there exist measurable subsets* V_n *of* $B_n^0 B_0^0$ *such that*

$$\lim_{n\to\infty} \iint_{V_n} a^j \mathcal{J}_e(u, T_n^j) = \iint_{B_0^0} a^j \mathcal{J}_e(u, T_0^j).$$

Moreover, if

$$\iint_{B_0^0} a^j \mathcal{J}_e(u, T_0^j) \geqq 0, \tag{10.1}$$

then the sets V_n *may be chosen as subsets of* $B_n^0 B_0^0$ *on which* $a^j \mathcal{J}_e(u, T_n^j) > 0$. *An analogous conclusion is valid if the opposite inequality holds in* (10.1).

Proof. If the a^i are all zero for $i=1, 2, 3$, the proof is trivial. If the a^i are not all zero, there is clearly no loss of generality in assuming that $a^j a^j = 1$. Then one may choose real numbers a_{ij} for $i=1, 2$; $j=1, 2, 3$ so that the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a^1 & a^2 & a^3 \end{pmatrix}$$

is normal and orthogonal with determinant $+1$ (McShane [1, §4]). Consider the plane transformations

$$\overline{T}_n : \bar{x}_n(u) = [a_{1j} x_n^j(u), a_{2j} x_n^j(u)], \quad u \in B_n^0, \qquad \text{for } n = 0, 1, 2, \cdots.$$

From Reichelderfer §4

$$\mathcal{J}_e(u, \overline{T}_n) = a^j \mathcal{J}_e(u, T_n^j) \text{ a.e. in } B_n^0 \qquad \text{for } n = 0, 1, 2, \cdots. \tag{10.2}$$

Clearly the $\overline{T}_n$ for $n=0, 1, 2, \cdots$ satisfy the hypotheses of the lemma in §3. In view of (10.2), the proof of this lemma follows at once from the conclusions in the lemma of §3.

11. In this work the lemma in the preceding section will play the role of the generalization of the lemma of McShane by Radó (McShane [1, §4], McShane [2, §12], Radó [1], Radó [2, §§1.28–1.30]). In its present form the above lemma does not imply the result of Radó since Radó merely requires

(T_0, B_0) to be of class K_1 (see §8); however, it is quite adequate for the purposes of the present note. When the lemma in §3 has been extended by replacing condition 3 (ii) by condition (ii*), as discussed in §4, it is possible to strengthen the lemma in §10 so that the generalization of the lemma of McShane by Radó follows as a special case.

12. Let (T_0, B_0) be a continuous triple which is eBV. Given an admissible integrand f (Radó [2, §1.2]), (T_0, B_0) is said to satisfy condition (eC) with respect to f if (1) there exists in x-space a closed bounded set A such that the set (Radó [2, §1.6]) $\Sigma(T_0, B_0)$ is contained in its interior A^0 and $f(x, X)$ is non-negative for x in A and every vector X; (2) for almost every u in B_0^0 such that $X_e(u, T_0)$ exists and is not zero, one has $E[x_0(u), X_e(u, T_0), \overline{X}]$ non-negative for every $\overline{X}$ different from zero (Radó [2, §1.3]). If (T, B) is any eBV triple and f is any admissible integrand it is easy to verify that there is a positive real number H such that (T, B) satisfies condition (eC) with respect to the admissible integrand $H\|X\|+f$ (Radó [2, §§1.11–1.16]).

Main Results

13. Lemma. *Given an admissible integrand f and a sequence of continuous triples* (T_n, B_n), $n=0, 1, 2, \cdots$, *satisfying the following conditions*: (i) (T_0, B_0) *is eBV*; (ii) (T_n, B_n) *is eAC for* $n=1, 2, \cdots$; (iii) (T_0, B_0) *satisfies condition* (eC) *relative to f*; (iv) *for every closed square q in* B_0^0 *there exists an integer* $N(q)$ *such that q is contained in* B_n^0 *for n exceeding* $N(q)$; (v) *on every closed square q in* B_0^0, $x_n(u)$ *converges to* $x_0(u)$ *uniformly*; (vi) *the oriented distances* $od[(T_0, B_0), (T_n, B_n)]$ *converge to zero as n tends to infinity. Then*

$$\liminf_{n\to\infty} I(T_n, B_n, f) \geqq I(T_0, B_0, f).$$

This is the analogue of the fundamental lemma in Radó [2, §2.1], in which the ordinary jacobians are replaced by the essential generalized jacobians. For the purpose of this note it would be sufficient to require that the (T_n, B_n) for $n=1, 2, \cdots$ are quasi-linear, as is done in Radó [2], but no extra work is required to prove the lemma under the less restrictive hypothesis (ii), and the proof of the theorem in §19 is simplified. A proof may be constructed by paralleling the reasoning in Radó [2, §§2.2–2.5], and using Scott, Lemma 2 instead of Radó [2, §1.5, §10].

14. Lemma. *Given an admissible integrand f and a sequence of continuous triples* (T_n, B_n), $n=0, 1, 2, \cdots$, *satisfying the following conditions*: (i) (T_0, B_0) *is eBV*; (ii) (T_n, B_n) *is eAC for* $n=1, 2, \cdots$; (iii) (T_0, B_0) *satisfies condition* (eC) *relative to f*; (iv) *the oriented distances* $od[(T_0, B_0), (T_n, B_n)]$ *converge to zero as n tends to infinity. Then*

$$\liminf_{n\to\infty} I(T_n, B_n, f) \geqq I(T_0, B_0, f).$$

The type of reasoning used in Radó [2, §2.7] gives a proof at once, providing the lemma in §13 is substituted for the lemma in Radó [2, §2.1], and the results in §6 and Franklin and Weiner are used instead of Radó [2, §1.27].

15. LEMMA. *If* (T_0, B_0) *and* $(\overline{T}_0, \overline{B}_0)$ *are two eAC triples representing the same oriented continuous surface* oS_0 *then*

$$I(T_0, B_0, \|X\|) = I(\overline{T}_0, \overline{B}_0, \|X\|).$$

This is an immediate consequence of the fact that both members of this relation are equal to the lebesgue area $L(oS)$ (see §7, Radó [2, §2.8]).

16. LEMMA. *Given an admissible integrand f and a sequence of continuous triples* (T_n, B_n), $n=0, 1, 2, \cdots$, *satisfying the following conditions*: (i) (T_n, B_n) *is eAC for* $n=0, 1, 2, \cdots$; (ii) *the oriented distances* $od[(T_0, B_0), (T_n, B_n)]$ *converge to zero as n tends to infinity*; (iii) *the integrals* $I(T_n, B_n, \|X\|)$ *converge to* $I(T_0, B_0, \|X\|)$. *Then*

$$\lim_{n\to\infty} I(T_n, B_n, f) = I(T_0, B_0, f).$$

The reasoning in Radó [2, §2.9] yields this proof (see §§12, 14).

17. THEOREM. *If f is an admissible integrand, and* (T_0, B_0), *and* $(\overline{T}_0, \overline{B}_0)$ *are two eAC triples representing the same oriented continuous surface* oS_0, *then*

$$I(T_0, B_0, f) = I(\overline{T}_0, \overline{B}_0, f).$$

Proof. From §15 it follows that

$$L(oS_0) = I(T_0, B_0, \|X\|) = I(\overline{T}_0, \overline{B}_0, \|X\|).$$

But there exists a sequence of oriented continuous surfaces oS_n having quasi-linear representations (T_n, B_n) for $n=1, 2, \cdots$ such that

$$\lim_{n\to\infty} od[(\overline{T}_0, \overline{B}_0), (T_n, B_n)] = 0 \quad \text{and} \quad \lim_{n\to\infty} L(oS_n) = L(oS_0).$$

It was noted in §7 that [Radó [3, V.2.12])

$$L(oS_n) = \iint_{B_n}{}^{0} \|X_e(u, T_n)\| du = I(T_n, B_n, \|X\|) \qquad \text{for } n = 1, 2, \cdots.$$

Thus the argument used in Radó [2, §2.10] leads to the desired conclusion.

18. If f is an admissible integrand, and if oS is any oriented continuous surface having an eAC representation (T, B), the preceding theorem shows that the value of $I(T, B, f)$ does not depend upon the choice of the eAC representation (T, B) for oS—hence it may be denoted by $I(oS, f)$. Let $oe\mathfrak{C}$ be the class of all oriented continuous surfaces oS which possess an eAC representation. Then $I(oS, f)$ is defined for $oS \in oe\mathfrak{C}$. In §9, it was noted that every class K_2 representation for oS according to Radó is an eAC representation,

but the converse is false. Moreover, if (T, B) is a representation of class K_2 for oS, then

$$I(oS, f) = \iint_{B^0} f[x(u), X_e(u, T)]du = \iint_{B^0} f[x(u), X(u, T)]du.$$

Thus the class $o\mathfrak{S}$ of Radó is a subclass of $oe\mathfrak{S}$, and the results in Radó [2, §§3.1–3.4] are properly contained in the present results (see §0).

19. THEOREM. *For a fixed admissible integrand f, the functional $I(oS, f)$ is lower semi-continuous at every oriented continuous surface oS_0 in $oe\mathfrak{S}$ having an eAC representation (T_0, B_0) which satisfies condition (eC) relative to f.*

Proof. Let oS_n be any sequence of oriented surfaces in $oe\mathfrak{S}$ such that $od(oS_0, oS_n)$ converges to zero as n tends to infinity. Choose eAC representations (T_n, B_n) of oS_n for $n=1, 2, \cdots$. Since $od(oS_0, oS_n)=od[(T_0, B_0), (T_n, B_n)]$ and $I(oS_n, f)=I(T_n, B_n, f)$ for $n=0, 1, 2, \cdots$ (see §18), it is clear that the assumptions of the lemma in §14 are fulfilled, whence it follows that

$$\liminf_{n\to\infty} I(oS_n, f) \geqq I(oS_0, f).$$

20. In view of the remarks in §18, the results in Radó [2, §§3.5–3.8] and Scott, Theorem, are special consequences of the theorem in §19. In case the integrand f is admissible and satisfies the condition $f(x, X)=f(x, -X)$ for all x, X, one obtains by similar arguments the following theorems which contain those in Radó [2, §3.9] as special cases (see §6).

THEOREM. *Let $e\mathfrak{S}$ be the class of all continuous surfaces S which possess an eAC representation (T, B). If f is an admissible integrand such that $f(x, X)=f(x, -X)$ for all x, X, then the value of $I(T, B, f)$ is independent of the choice of an eAC representation for S—hence it may be denoted by $I(S, f)$.*

THEOREM. *For a fixed admissible integrand f such that $f(x, X)=f(x, -X)$ for all x, X, the functional $I(S, f)$ is lower semi-continuous at every continuous surface S_0 in $e\mathfrak{S}$ having an eAC representation (T_0, B_0) which satisfies condition (eC) relative to f.*

BIBLIOGRAPHY

G. A. BLISS

Lectures on the calculus of variations, Chicago, 1946.

L. CESARI

1. *Rappresentazione quasi conforme delle superficie continue*, Rendiconti dell' Accademia Nazionale dei Lincei (8) vol. 1 (1946) pp. 509–515.
2. *Parametrizzazione delle superficie continue di area finita secondo Lebesgue*, Annali di Matematica pura ed applicata (4) vol. 26 (1947) pp. 301–375.
3. *La nozione di integrale sopra una superficie in forma parametrica*, Annali della Scuola Normale Superiore di Pisa (2) vol. 13 (1944) pp. 78–117.

P. Franklin and N. Wiener

Analytic approximations to topological transformations, Trans. Amer. Math. Soc. vol. 28 (1926) pp. 762–785.

E. J. McShane

1. *On the semi-continuity of double integrals in the calculus of variations*, Ann. of Math. (2) vol. 33 (1932) pp. 460–484.
2. *Integrals over surfaces in parametric form*, Ann. of Math. (2) vol. 34 (1933) pp. 815–838.
3. *Integration*, Princeton University Press, 1944.

T. Radó

1. *On the lemma of McShane*, Ann. of Math. vol. 42 (1941) pp. 73–83.
2. *On the semi-continuity of double integrals in parametric form*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361.
3. *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, 1948.
4. *Convergence in area*, Duke Math. J. vol. 16 (1949) pp. 61–71.

P. V. Reichelderfer

Law of transformation for essential generalized jacobians, Duke Math. J. vol. 16 (1949) pp. 73–83.

W. Scott

A note on the lower semi-continuity of double integrals in parametric form, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 763–767.

J. W. T. Youngs

On surfaces of class K_1, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 669–673.

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Columbus, Ohio.